Complex Analysis: Some Highlights

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Complex Anaylsis

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Consider functions $f : \mathbb{C} \mapsto \mathbb{C}$. We will study the calculus of such functions, e.g., limits, continuity, differentiability, integration, power series. What's similar and what's different?

Cool Formulas

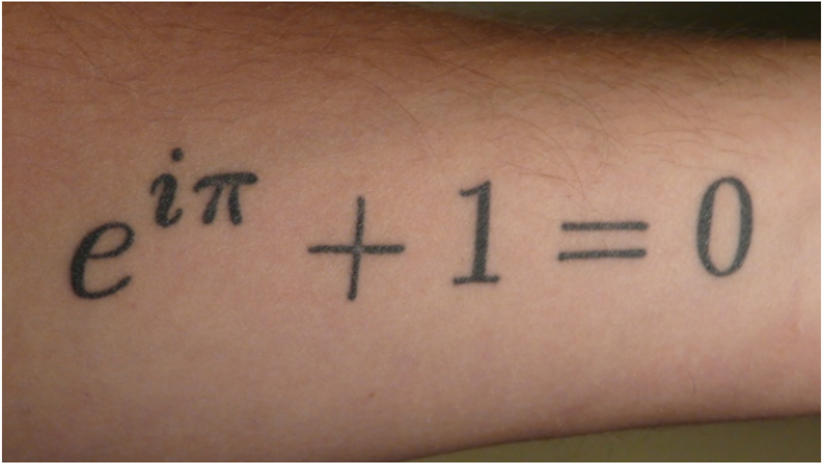


Figure : Math pride!

Cool Graphs

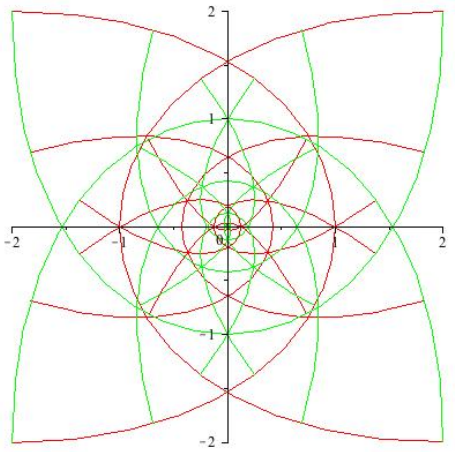


Figure : How do we visualize complex functions?

More Cool Graphs

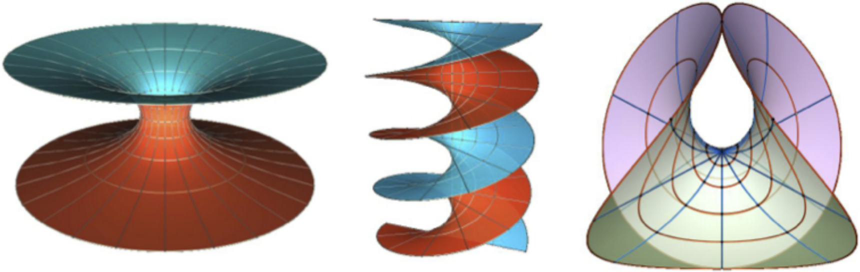
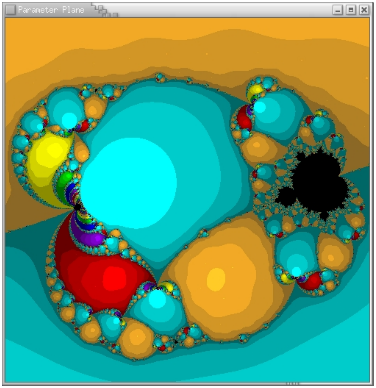


Figure : Some minimal surfaces: catenoid (left), helicoid (center), Enneper surface (right). Nature forms these surfaces to minimize energy (soap film).

Even Cooler Graphs



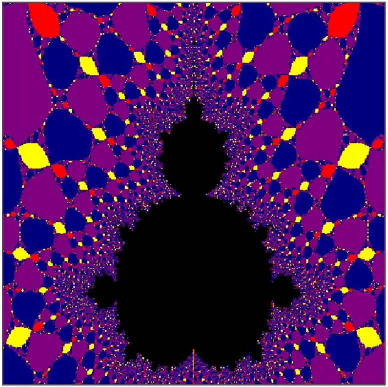


Figure : Some fractals in the complex plane created by some past research students. Both figures concern the use of Newton's method to find the roots of a complex polynomial, an example of a dynamical system. Figures by Gabe Weaver (left) and Trevor O'Brien (right).

Cool Theorems

Theorem (The Fundamental Theorem of Algebra)

Any complex polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *with* $a_n \neq 0$ *has at least one root* $z_0 \in \mathbb{C}$ *(i.e.,* $p(z_0) = 0$*).*

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This in turn implies that any polynomial can be completely factored into a product of linear terms. We say that $\mathbb{C}$ is an algebraically closed field. This is not the case for $\mathbb{R}$.

$$f(x) = x^2 + 3$$

has no solutions in $\mathbb{R}$.

Cool Applications

- Heat equation: $u_t = k\nabla^2 u$, where $u = u(x, y, z, t)$ measures temperature at point (x, y, z) at time t

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- Complex Analysis Fun Fact: Suppose that $f(z)$ is a differentiable function. Then the real and imaginary parts of f each satisfy Laplace's equation.