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Applications of Wolfram Mathematica in the Theoretical Mechanics

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Abstract. Wolfram Mathematica® is one of the Computer Algebra Systems (CAS), a type of programs that allow solving problems in symbolic way. In this paper, applications of this program in the Theoretical Mechanics are presented, namely the Lagrange equations of the first kind and some concepts how to use CAS in seminars of this lecture.

Introduction

The development of Computer Algebra Systems (CAS) started about 30 years ago. During this period of time, possibilities of CAS like Mathematica® were expanded to the whole new level of usage and this also changed the point of view to those programs. Nowadays, they are used not only in science and engineering but also in social sciences, finance and even in education. In this article, applications of this program in the Theoretical Mechanics are presented.

Computer Algebra Systems and other Programs used in Physics

Many programs are used in physics for different purposes—creating plots, numerical calculation, etc. But Computer algebra systems like Mathematica® try to put the potential of many of those programs together into one complex system (Figure 1). This complexity has many advantages, e.g., uniform syntax and all calculations and results being in one place.

Computer Algebra Systems in the Theoretical Mechanics

Understanding of theoretical mechanics is crucial for understanding to following courses of modern physics like quantum theory and relativity theory. For students, it is the first opportunity to see the Lagrangian formalism and the Hamilton's principle. With Mathematica®, it is possible to solve common problems used for demonstration of those formalisms, but also to solve more difficult tasks. Because these programs can solve the problem symbolically, the usual definitions, which stay unchanged in more tasks, may be written once only, and it is still possible to use them again and again (Figure 2).

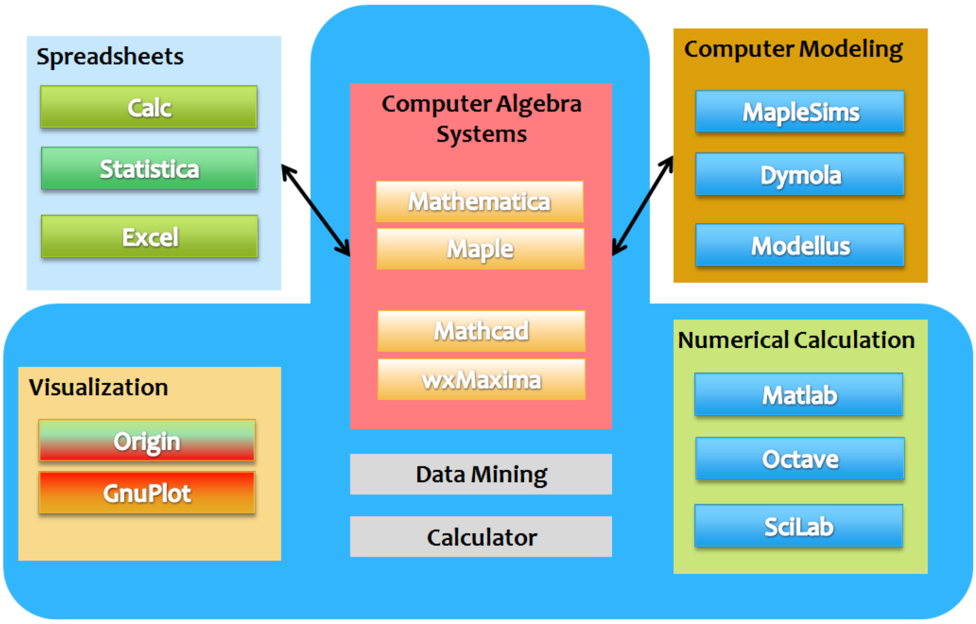

Figure 1. Connection and context of CAS and other programs used in physics.

Application 1—The Ball on a Wire

In Figure 2, the solution of a motion of a ball on a spinning wire with given angular velocity ω and with fixed angle α between the wire and z-axis (Figure 3) is demonstrated. This task is one of the first applications of Lagrange equations of the first kind in the seminars of the theoretical mechanics. This task is done faster by the paper and pencil method, but when another task focused on Lagrange equations of the first kind has to be solved, it becomes faster for Mathematica® user, because only the description of movement, kinetic and potential energy is different in this task. The other parts of the solution are the same.

Application 2—Two Planes

More advanced task is to solve the motion of a ball along an intersection of two planes (Figure 4). This task doesn't have an analytical solution for the general time dependency. Thus, after obtaining the differential equation describing the problem we calculate the solution for a special case $\beta = \gamma t$. Obtaining the solution by paper and pencil method takes about 8 to 16 hours (depending on students skills). By Mathematica®, it is about 1 to 4 hours depending of student's knowledge of Mathematica®. On the other hand, the Mathematica user has to think about the assumptions for given variables e.g. non-zero mass etc. (Figure 5 left top). Otherwise, *m* doesn't represent the mass but only a general (complex) variable during calculations. This part is really important because during paper and pencil calculations, students usually use assumptions without thinking, whereas here, it has to be given to Mathematica®.

```
In[2]:= x = r[t] * Sin[α] * Sin[ω * t]
        y = r[t] * Sin[α] * Cos[ω * t]
        z = r[t] * Cos[α]
Out[2]= r[t] Sin[α] Sin[t ω]
Out[3]= Cos[t ω] r[t] Sin[α]
Out[4]= Cos[α] r[t]

In[5]:= rychlost = D[{x, y, z}, t]
Out[5]= {ω Cos[t ω] r[t] Sin[α] + Sin[α] Sin[t ω] r′[t],
         -ω r[t] Sin[α] Sin[t ω] + Cos[t ω] Sin[α] r′[t],
         Cos[α] r′[t]}

In[6]:= T = 1 / 2 * m * rychlost.rychlost // FullSimplify
Out[6]= 1/2 m (ω^2 r[t]^2 Sin[α]^2 + r′[t]^2)

In[7]:= V = m * g * z
Out[7]= g m Cos[α] r[t]
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In[8]:= L = T - V
Out[8]= -g m Cos[α] r[t] + 1/2 m (ω^2 r[t]^2 Sin[α]^2 + r′[t]^2)

In[9]:= -D[L, r[t]]
Out[9]= g m Cos[α] - m ω^2 r[t] Sin[α]^2

In[10]:= lagrange = D[D[L, D[r[t], t]], t] - D[L, r[t]]
Out[10]= g m Cos[α] - m ω^2 r[t] Sin[α]^2 + m r″[t]

In[11]:= FullSimplify[DSolve[lagrange == 0, r[t], t]]
Out[11]= {{r[t] → e^(t Sqrt[ω^2 Sin[α]^2]) C[1] +
          e^(-t Sqrt[ω^2 Sin[α]^2]) C[2] + g Cot[α] Csc[α] / ω^2}}
```

Figure 2. Task Solved by Mathematica®. The left hand side of the picture is different for each task, because it describes the movement of an object and its kinetic and potential energy. On the other hand, the right hand side is the same for all tasks in one variable.

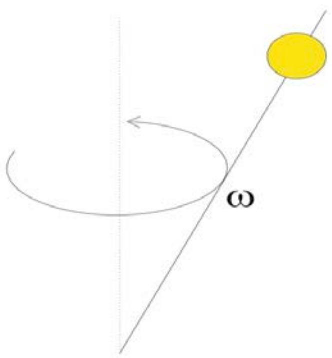

Figure 3. Ball on Wire—ball on spinning wire with angular velocity ω and with fixed angle α between wire and z-axis. Ball can freely move on wire.

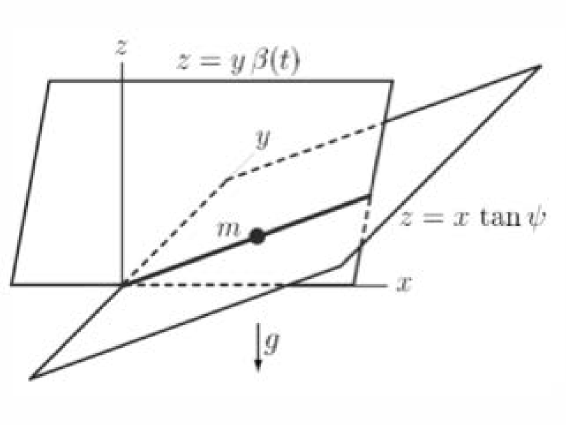

Figure 4. Two planes—One of the planes is fixed, the other is moving with given time dependency.

Figure 5. Solution of two planes and the ball.

One of the advantages of the Mathematica® is mixing steps together. In this example, the student has to eliminate λ_1, λ_2 and later on also x, z (common way to solve Lagrange equations of the first kind). It takes plenty of time and it is a lot of operations going on. But in Mathematica®, it is just about to write down command Eliminate and to use a command for simplification—FullSimplify.

Methods of using of Computer algebra systems

From the point of view of education, programs like Mathematica® offer many possible ways how to be used in theoretical mechanics. To show different approaches to this field of theoretical mechanics, many books on this topic has been written. Some books with some basic examples can be found in, e.g., *Boccara* [2007] (this book is generally focused on Mathematica® and its examples) an there are also whole textbooks of theoretical mechanics done in Mathematica® by *Baumann* [2005a, 2005b]. But there is a question for education how to use them. Thus, we are concentrated to the methods as an essential part of education. The importance of methods is shown by, e.g., *Kendal*

[2001]. Methods (strategies) presented below are supposed to be taken rather like a recommendation how to use Mathematica®. These methods are suggested ones and it will be tested on students of physics and physics education in the next two years.

Time Scale usage

First point of view in those methods is in using the time scale. In simple way—at which time in the seminar or a lesson the teacher is going to use Mathematica® or generally CAS.

Table 1. Time Scale usage of CAS.

Time Scale Usage	Description of Method
Using CAS before manual calculations	First CAS is used to solve the task, then the paper and pencil method is used.
Using CAS during manual calculations	The task is solved by hand and solutions are being checked using CAS.
	The task is solved as much as possible by hand, then the difficult part of it is solved by CAS and then again by hand (e.g. the equation of a harmonic oscillator). The task has an analytical solution, but it takes plenty of time to calculate it.
	The task is solved as much as possible by hand. Then because it doesn't have an analytical solution, an approximation is used or the task is solved numerically by CAS and then both solutions are compared.
Using CAS after manual calculations	First, the task is solved by hand then CAS is used to verify the solutions.

Degree of involvement

Other point of view (but not independent) is to consider the degree of involvement. In simple way—how much is the teacher going to use Mathematica® of generally CAS in the lessons or seminars.

Table 2. Degree of involvement of CAS in seminars.

Degree of Involvement	Description of Method
None	CAS hasn't been used.
Calculation of numerical values (with units)	Only numerical values are solved by CAS. It's possible to use in CAS numerical values with units.
Calculation of some parts of the task	Student chooses parts solved by CAS. Calculation of some parts of the task. Parts of task don't have to be difficult.
Calculation of difficult parts of the task	E.g., in the movement, CAS solves the differential equation, then student continues manually.
Calculation of whole task	Whole task is solved by CAS.
General solution	General method(s) to solve similar type of tasks (e.g. motions by Newton's second law with different kinds of forces), a general solution of task—the same task with different parameters.

Discussion and Conclusion

In this article, it has been shown that Mathematica® could be very useful for solving tasks in the theoretical mechanics. Its use can reduce the time constraint in solving the tasks enabling to offer more time for different interpretations of solution(s). But to use this program wisely, it requires thinking deeply about methods and methodology how to use them. Methods above mentioned are supposed to be taken more as recommendation how to use Mathematica® and general CAS.

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References

Baumann, G. (2005a). Mathematica for Theoretical Physics—Classical Mechanics and Nonlinear Dynamics, 2nd ed. Springer.
Baumann, G. (2005b). Mathematica for Theoretical Physics—Electrodynamics, Quantum Mechanics, General Relativity and Fractals, 2nd ed. Springer.
Boccara, N. (2007). Essentials of Mathematica (With Applications to Mathematics and Physics). Springer.
Kendal, M. (2001). The Impact of Teacher Privileging on Learning Differentiation with Technology. Int. J. of Computers for Mathematical Learning (6), pp. 143–165.