EQUATION OF STATE

Consider elementary cell in a phase space with a volume

$$\Delta x \, \Delta y \, \Delta z \, \Delta p_x \, \Delta p_y \, \Delta p_z = h^3, \tag{st.1}$$

where $h = 6.63 \times 10^{-27}$ erg s is the Planck constant, $\Delta x \, \Delta y \, \Delta z$ is volume in ordinary space measured in cm^3, and $\Delta p_x \, \Delta p_y \, \Delta p_z$ is volume in momentum space measured in (g cm s^{-1})^3. According to quantum mechanics there is enough room for approximately one particle of any kind within any elementary cell. More precisely, an average number of particles per cell is given as

$$n_{av} = \frac{g}{e^{(E-\mu)/kT} \pm 1}, \tag{st.2}$$

with a "+" sign for fermions, and a "−" sign for bosons. The corresponding distributions are called Fermi-Dirac and Bose-Einstein, respectively. Particles with a spin of 1/2 are called fermions, while those with a spin 0, 1, 2... are called bosons. Electrons and protons are fermions, photons are bosons, while larger nuclei or atoms may be either fermions or bosons, depending on the total spin of such a composite particle. In the equation (st.2) E is the particle energy, $k = 1.38 \times 10^{-16}$ erg K^{-1} is the Boltzman constant, T is temperature, μ is chemical potential, and g is a number of different quantum states a particle may have within the cell. The meaning of temperature is obvious, while chemical potential will become more familiar later on. In most cases it will be close to the rest mass of a particle under consideration. If there are anti-particles present in equilibrium with particles, and particles have chemical potential μ then antiparticles have chemical potential $\mu - 2m$, where m is their rest mass.

For free particles their energy is a function of their momentum only, with the total momentum p given as

$$p^2 = p_x^2 + p_y^2 + p_z^2. \tag{st.3}$$

The number density of particle in a unit volume of 1 cm^3, with momenta between p and $p + dp$ is given as

$$n(p) \, dp = \frac{g}{e^{(E-\mu)/kT} \pm 1} \frac{4\pi p^2}{h^3} dp, \tag{st.4}$$

because the number of elementary cells within 1 cm^3 and a momentum between p and $p + dp$, i.e. within a spherical shell with a surface $4\pi p^2$ and thickness dp is equal to $4\pi p^2 \, dp \, h^{-3}$. The number density of particles with all momenta, contained within 1 cm^3 is

$$n = \int_0^\infty n(p) \, dp, \qquad \rho = nm \tag{st.5}$$

where ρ is the mass density of gas. Note, that if we know density and temperature, then we can calculate chemical potential with the eqs. (st.4) and (st.5), provided we know how particle energy depends on its momentum, i.e. the function $E(p)$ is known.

In the equation (st.4) g, k, T, μ, π, h are all constant, and the energy E depends on the momentum p only. As our particles may be relativistic as well as non-relativistic, we have to use a general formula for the relation between E and p. We have

$$E \equiv E_{total} = E_0 + E_k, \tag{st.6}$$

where the rest mass $E_0 = mc^2$, $c = 3 \times 10^{10}$ cm s^{-1} is the speed of light, and E_k is the kinetic energy of a particle. For a particle moving with arbitrary velocity there is a special relativistic relation:

$$E_{total}^2 = \left(mc^2\right)^2 + (pc)^2 . \tag{st.7}$$

Combining the last two equations we obtain

$$E = mc^2 \left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2}, \qquad E_k = mc^2 \left[\left(1 + \left(\frac{p}{mc}\right)^2\right)^{1/2} - 1\right]. \tag{st.8}$$

The last equation has two simple asymptotic limits:

$$\begin{aligned} E_k &\approx \frac{p^2}{2m}, &&\text{for } \ p \ll mc, && (\text{ non-relativistic limit}), \\ E_k &\approx pc, &&\text{for } \ p \gg mc, && (\text{ ultra-relativistic limit}). \end{aligned} \tag{st.9}$$

Velocity of a particle is given in general as

$$v = \frac{dE}{dp} = \frac{p}{m}\left[1 + \left(\frac{p}{mc}\right)^2\right]^{-1/2} . \tag{st.10}$$

Again, we have two simple asymptotic limits:

$$\begin{aligned} v &\approx \frac{p}{m}, &&\text{for } \ p \ll mc, && (\text{ non-relativistic limit}), \\ v &\approx c, &&\text{for } \ p \gg mc, && (\text{ ultra-relativistic limit}), \end{aligned} \tag{st.11}$$

Kinetic energy of all particles in a unit volume of $1\,\mathrm{cm}^3$ may be calculated as

$$U = \int_0^\infty E_k(p)\, n(p)\, dp, \qquad [\mathrm{erg\ cm}^{-3}]. \tag{st.12}$$

We are interested in isotropic gas, so the velocity and momentum vectors of every particle are parallel to each other. Velocities of different particles are pointing in different directions, and their angular distribution is isotropic. For a given absolute value of velocity and momentum we may calculate the dot-product of the two vectors, and average it over all angles:

$$\mathbf{vp} = vp = v_x p_x + v_y p_y + v_z p_z . \tag{st.13a}$$

$$\langle v_x p_x \rangle = \langle v_y p_y \rangle = \langle v_z p_z \rangle = \frac{1}{3} vp. \tag{st.13b}$$

Pressure is defined as a flux of momentum across a unit surface of $1\,\mathrm{cm}^2$, integrated over all particles. For an isotropic gas we may select the unit surface to be perpendicular to the "x" axis, and we may calculate pressure as

$$P = \int_0^\infty < v_x p_x > n(p)\, dp = \frac{1}{3}\int_0^\infty v(p)\, p\ n(p)\, dp, \tag{st.14}$$

where as before the averaging is done over particles moving in all directions.

We shall consider now some special, and simple, but important cases. First, let us consider any non-relativistic gas, with arbitrary momentum distribution function $n(p)$, such that for a vast majority of particles a condition $p \ll mc$ is satisfied. In this case we have $v \approx p/m$, $E_k \approx p^2/2m$, and

$$P \approx \frac{1}{3}\int_0^\infty \frac{p^2}{m} n(p)\, dp, \qquad U \approx \int_0^\infty \frac{p^2}{2m} n(p)\, dp \approx \frac{3}{2} P, \qquad (\mathrm{non-relativistic}). \tag{st.15}$$

Second, there is a case that most particles are ultra-relativistic, with $p \gg mc$, $v \approx c$, and $E_k \approx pc$ for most of them. Now, pressure and kinetic energy density are given as

$$P \approx \frac{1}{3} \int_0^\infty pc\, n(p)\, dp, \qquad U \approx \int_0^\infty pc\, n(p)\, dp \approx 3P, \qquad (\mathrm{ultra-relativistic}). \tag{st.16}$$

Notice, that the relations $U \approx 3P/2$ and $U \approx 3P$ for the two limits are very general, and do not depend on the details of the distribution function $n(p)$.

Our formulae allow us to calculate, at least in principle, pressure and kinetic energy density for a gas made of identical particles in a local thermodynamic equilibrium (LTE) . We made an implicit assumption that particles are interacting so weakly that energy of their interaction may be neglected, but that there is enough interaction between them to establish equilibrium distribution. If we have a mixture of few different types of particles, for example electrons, protons, and photons, we make the same assumptions about their interactions: it is small enough that we may assume the total pressure and kinetic energy density of the composite gas is the sum of pressures and kinetic energy densities of the components, but it is strong enough to maintain equilibrium distributions with the same temperature for all types of particles. Now, we shall calculate in detail some examples of the distribution functions, and the corresponding thermodynamic quantities.

Maxwell distribution

This is the case when

$$mc^2 \gg E - \mu \gg kT, \tag{st.17}$$

i.e. the particles are non-relativistic, and the exponential term in the denominator of equation (st.2) is very large, $n_{av} \ll 1$, i.e. there are very few particles per elementary cell in phase space. It is clear that the ± 1 term is negligible, and therefore the distribution will be the same for fermions and for bosons. Within the approximation following from equation (st.17) we have

$$n(p) = \left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT} \right] e^{-p^2/2mkT} p^2, \tag{st.18a}$$

$$n = \left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT} \right] \int_0^\infty e^{-p^2/2mkT} p^2 dp = \tag{st.18b}$$

$$\left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT} \right] (2mkT)^{1.5} \frac{1}{2} \int_0^\infty e^{-x} x^{0.5} dx =$$

$$\left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT} \right] (2mkT)^{1.5} \frac{\pi^{0.5}}{4},$$

$$P = \frac{1}{3} \left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT} \right] \int_0^\infty \frac{p^2}{m} e^{-p^2/2mkT} p^2 dp = \tag{st.18c}$$

$$\left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT} \right] (2mkT)^{2.5} \frac{1}{3m} \frac{1}{2} \int_0^\infty e^{-x} x^{1.5} dx =$$

$$\left[\frac{4\pi g}{h^3} e^{\left(\mu - mc^2\right)/kT} \right] (2mkT)^{2.5} \frac{1}{3m} \frac{1.5\pi^{0.5}}{4} = nkT,$$

$$U = 1.5nkT, \tag{st.18d}$$

where we substituted $x = p^2/2mkT$.

In practice the Maxwell distribution is used for atoms, ions and electrons with a relatively low density. The relation between the number density of particles $n[cm^{-3}]$ and physical density $\rho[\mathrm{g\ cm}^{-3}]$ depends on chemical composition. It is customary to use X for the fractional abundance of hydrogen (by mass fraction) , Y for the abundance of helium, and Z for the combined abundance of all heavier elements. Frequently, all those heavier elements are described in astrophysical literature as "metals", even though the most common among them are carbon, nitrogen and oxygen. Of course, we have $X + Y + Z = 1$. A typical chemical composition of an interstellar medium, or the main sequence stars is: $X = 0.70$, $Y = 0.28$, $Z = 0.02$, i.e. there is mostly hydrogen, and only very small fraction of heavy elements. These abundances are by mass fraction, so the fractional **number** of heavy atoms is approximately 0.002, or even less.

In most stellar applications it is safe to assume that all atoms are fully ionized. The only exceptions are stellar atmospheres and sub-atmospheric layers. A convenient unit of mass is $H = 1.67 \times 10^{-24}\,\mathrm{g}$, i.e. mass of a single hydrogen atom. Mass of a helium atom is $4H$, and mass of an element with a nuclear charge Z^* is approximately $2Z^*$. An average charge of all heavier elements is approximately $< Z^* > \approx 8$. In one gram of matter there are X/H hydrogen nuclei, $Y/4H$ helium nuclei, and approximately $Z/(2 < Z^* > H)$ heavier nuclei. Therefore, the number of nuclei per gram of matter, i.e. the number density of ions is

$$n_i = \frac{\rho}{H}\left[X + \frac{Y}{4} + \frac{Z}{2 < Z^* >}\right] \approx \frac{\rho}{H}\left(X + \frac{Y}{4} + \frac{Z}{16}\right). \tag{st.19}$$

Fully ionized hydrogen provides 1 electron per nucleus, i.e. 1 electron per nucleon. Helium provides 2 electrons per nucleus, i.e. 1 electron per 2 nuclei. A heavier element with a charge Z^* provides Z^* electrons per nucleon, which in a typical case has $A \approx 2Z^*$ nucleons, i.e. we get 1 electron per 2 nucleons, just like for helium. The number density of electrons may be calculated as

$$n_e = \frac{\rho}{H}\left(X + \frac{Y}{2} + \frac{Z}{2}\right) = \frac{\rho}{H}\frac{1+X}{2}. \tag{st.20}$$

The number density of all particles is given as

$$n = n_i + n_e \approx \frac{\rho}{H}\left(2X + 0.75Y + 0.5Z\right). \tag{st.21}$$

It is customary to define mean molecular weight μ, mean molecular weight per ion μ_i, and mean molecular weight per electron μ_e as

$$\mu \equiv \frac{\rho}{nH} \approx \frac{1}{2X + 0.75Y + 0.5Z}, \tag{st.22a}$$

$$\mu_i \equiv \frac{\rho}{n_i H} = \frac{1}{X + Y/4 + Z/16}, \tag{st.22b}$$

$$\mu_e \equiv \frac{\rho}{n_e H} = \frac{2}{1+X}. \tag{st.22c}$$

With the new definitions of mean molecular weights (not to be confused with a chemical potential, which is also written as μ) we may write the equation of state as

$$P_e = \frac{k}{\mu_e H}\rho T, \tag{st.23a}$$

$$P_i = \frac{k}{\mu_i H}\rho T, \tag{st.23b}$$

$$P_g = P_e + P_i = \frac{k}{\mu H}\rho T, \tag{st.23c}$$

where P_e, P_i, and P_g are the electron pressure, the ion pressure, and the gas pressure, respectively. We also have the corresponding kinetic energy densities given as

$$U_e = 1.5P_e, \qquad U_i = 1.5P_i, \qquad U = 1.5P. \tag{st.24}$$

Planck distribution

This is the distribution of photons under LTE (local thermodynamic equilibrium) conditions. It is customary to characterize photons with the frequency of their oscillations ν. Their wavelength is given as $\lambda = c/\nu$, energy as $E = h\nu$, and momentum as $p = h\nu/c$. Photons always move with the speed of light, have zero rest mass, and are their own antiparticles. Therefore, under LTE conditions their chemical potential is zero. Their spin is 1, so they are bosons. A photon has two quantum states with the same energy; they correspond to two different polarization states. Therefore $g = 2$ for photons, and we have

$$n\left(p\right) dp = \frac{8\pi}{h^3} \frac{p^2}{e^{pc/kT} - 1} dp, \tag{st.25a}$$

or, in a more customary form

$$n_\nu d\nu = \frac{8\pi}{c^3} \frac{\nu^2}{e^{h\nu/kT} - 1} d\nu. \tag{st.25b}$$

The total energy associated with radiation is

$$U_r = \int\limits_0^\infty n_\nu h\nu \; d\nu = \frac{8\pi h}{c^3} \int\limits_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int\limits_0^\infty \frac{x^3 dx}{e^x - 1}, \tag{st.26a}$$

where we substituted $x = h\nu/kT$. The last integral has a value of $\frac{\pi^4}{15} \approx 6.088$, and we may write

$$U_r = aT^4, \qquad \text{where} \quad a = \frac{8\pi^5}{15} \frac{k^4}{h^3 c^3} \approx 7.565 \times 10^{-15} \, \text{erg cm}^{-3} \, \text{K}^{-4}. \tag{st.26b}$$

Radiation pressure is given as

$$P_r = \frac{1}{3} U_r = \frac{a}{3} T^4, \tag{st.27}$$

because photons are ultra-relativistic.

It is customary to describe radiation with its intensity $B_\nu(T)$, and

$$B\left(T\right) = \int\limits_0^\infty B_\nu\left(T\right) d\nu. \tag{st.28}$$

In LTE radiation intensity is the same in all directions, over the whole solid angle $\Omega = 4\pi$. The radiation energy density may be calculated as

$$U_r = \frac{1}{c} \int B\left(T\right) d\Omega = \frac{4\pi}{c} B\left(T\right), \tag{st.29}$$

where the integration covers all directions; in our case radiation is isotropic, i.e. $B(T)$ is the same in all directions. We may combine equations (st.26b) and (st.29) to obtain

$$B\left(T\right) = \frac{c}{4\pi} U_r = \frac{ac}{4\pi} T^4, \tag{st.30}$$

and also

$$B_\nu\left(T\right) d\nu = \frac{c}{4\pi} U_\nu d\nu = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu. \tag{st.31}$$

$B_\nu(T)$ is called the **Planck function**. It gives the intensity of radiation of a black body at a given temperature T.

Let us calculate the flux of radiation emitted from a surface of a black body which has a temperature T. The intensity of radiation integrated over all frequencies is given as $B(T)$ and it is the same in any direction pointing out from the surface. Let us introduce angle θ, between the normal to the surface and the direction of a light ray, and an azimuthal angle ϕ. An element of a solid angle is given as $d\Omega = \sin\theta d\phi d\theta$. The flux of black body radiation comming in the direction normal to the surface may be calculated integrating over all angles the component of $B(T)$ orthogonal to the surface, i.e. $B(T)\cos\theta$. We have:

$$F_{BB} = \int\limits_0^{\pi/2} \left[\int\limits_0^{2\pi} B\left(T\right)\cos\theta\sin\theta d\phi \right] d\theta = 2\pi B\left(T\right) \int\limits_0^{\pi/2} \cos\theta\sin\theta d\theta = \pi B\left(T\right). \qquad \text{(st.32)}$$

Combining the last equation with (st.30) we obtain the flux of radiation from a unit surface of a black body

$$F_{_{BB}} = \sigma T^4, \qquad \sigma = \frac{ac}{4} = 5.67 \times 10^{-5}\,\text{erg cm}^{-2}\,\text{s}^{-1}\,\text{K}^{-4}, \qquad \text{(st.33)}$$

where σ is the Stefan-Boltzman constant.

Real astronomical objects radiate in a more complicated way than a black body of any temperature. However, it is useful to define an effective temperature as the temperature of a black body that would radiate out energy at the rate that the star does. If stellar luminosity is L, and its radius is R, then its surface radiates at a rate of $F = L/4\pi R^2$. The effective temperature is defined with a relation:

$$\sigma T_{eff}^4 \equiv \frac{L}{4\pi R^2}. \qquad \text{(st.34)}$$

If we have a mixture of gas and radiation then total pressure and total kinetic energy density are given as

$$P = P_g + P_r = \frac{k}{\mu H}\rho T + \frac{a}{3}T^4, \qquad \text{(st.35a)}$$

$$U = U_g + U_r = 1.5\frac{k}{\mu H}\rho T + aT^4, \qquad \text{(st.35b)}$$

It is useful to define the ratio of gas pressure to total pressure

$$\beta \equiv P_g/P, \qquad 0 < \beta < 1. \qquad \text{(st.36)}$$

At a low temperature gas pressure dominates, while at a high temperature radiation pressure dominates. The two contributions are equal, i.e. we have $P_g = P_r$ when

$$\frac{k}{\mu H}\rho T = \frac{a}{3}T^4. \qquad \text{(st.37)}$$

For a standard chemical composition, i.e. for $X = 0.70$, $Y = 0.28$, $Z = 0.02$, the mean molecular weight is $\mu^{-1} = 1.62$, and the gas pressure constant is $k/\mu H = 5.1 \times 10^7\,\text{erg g}^{-1}\,\text{K}^{-1}$. The density where gas pressure equals radiation pressure is

$$\rho \approx 5 \times 10^{-23}\, T^3 \left[\text{g cm}^{-3}\,\text{K}^{-3}\right] = \left(\frac{T}{3 \times 10^7\,\text{K}}\right)^3 \left[\text{g cm}^{-3}\right]. \qquad \text{(st.38)}$$

Fermi-Dirac distribution

Now, we shall consider a distribution of electrons, i.e. spin 1/2 particles, when density is high. An electron may be in two spin states: $+1/2$ and $-1/2$, so the number of different quantum states per unit cell in a phase space is g $= 2$, and the average number of electrons per unit cell is

$$n_{av} = \frac{2}{e^{(E-\mu)/kT} + 1}, \tag{st.39a}$$

and the number density of electrons is given as

$$n_e = \int_0^\infty n_e\left(p\right) dp = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{e^{(E-\mu)/kT} + 1}, \tag{st.39b}$$

with electron energy E being a function of its momentum (cf. equation st.8) . The density, pressure, and kinetic energy density are given as (cf. equations: st.12, st.14, st.22c)

$$\rho = n_e \mu_e H = n_e \frac{2H}{1+X}, \tag{st.40a}$$

$$P_e = \frac{1}{3} \int_0^\infty v\left(p\right) p \; n_e\left(p\right) dp, \tag{st.40b}$$

$$U_e = \int_0^\infty E_k\left(p\right) n_e\left(p\right) dp. \tag{st.40c}$$

The chemical potential which appears in equations (st.39) is approximately equal to the electron rest mass: $\mu = mc^2 + E_F$, where E_F is Fermi energy - its physical meaning will become apparent shortly. The $E-\mu$ term in the exponent in the denominator in equations (st.39) is equal to $E_k - E_F$, where E_k is a kinetic energy of an electron. If $E_F \ll 0$ the distribution function for electrons is Maxwellian, as the exponential term is much larger than 1 even for electrons with zero kinetic energy. Here we shall be interested in another limit: $E_F \gg kT$. Now, the exponential term is very small for $E_k < E_F$, it is equal 1 for $E_k = E_F$, and it is very large for $E_k > E_F$. The transition from being much less than 1 to much larger than 1 takes place over a relatively small change in kinetic energy, $\Delta E \approx 2kT \ll E_F$. Therefore, in the limit when $kT \ll E_F$ the occupation number n_{av} as given with equation (st.39a) becomes almost a step function of electron's kinetic energy:

$$n_{av} \approx 2, \qquad \text{for} \qquad E_k < E_F, \quad kT \ll E_F, \tag{st.41a}$$

$$n_{av} \approx 0, \qquad \text{for} \qquad E_k > E_F, \quad kT \ll E_F, \tag{st.41b}$$

This result may be interpreted as follows. When temperature drops the electrons occupy the lowest energy states available. However, as they are fermions they have to obey Pauli exclusion principle: there cannot be more then one electron in the same quantum state. An electron can be in two different spin states. This means there can be at most two electrons in any elementary cell of a phase space. It is the Heisenberg uncertainty principle that is responsible for the size of those cells. So, as the temperature drops the electrons fill up the lowest energy cells all the way up to Fermi energy, and all higher energy states are empty. Such gas is called **degenerate**. The value of Fermi energy depends on the density of electrons in ordinary space. This is obvious upon evaluating the integral (es.39) , subject to the conditions (st.41) . We obtain

$$n_e = \frac{8\pi}{h^3} \int\limits_0^{p_F} p^2 dp = \frac{8\pi}{3} \left(\frac{p_F}{h} \right)^3 , \tag{st.42}$$

where p_F is Fermi momentum related to Fermi energy by equation (st.8) . That relation may be written as

$$\left(\frac{E_F}{mc^2} + 1 \right)^2 = \left(\frac{p_F}{mc} \right)^2 + 1. \tag{st.43}$$

We may express density, pressure and kinetic energy density in terms of Fermi momentum, and we shall write equations (st.40) as

$$\rho = \mu_e H \frac{8\pi}{3h^3} p_F^3 , \tag{st.44a}$$

$$P_e = \frac{8\pi}{3mh^3} \int\limits_0^{p_F} \frac{p^4 \, dp}{\left[1 + \left(\frac{p}{mc} \right)^2 \right]^{1/2}} , \tag{st.44b}$$

$$U_e = \frac{8\pi mc^2}{h^3} \int\limits_0^{p_F} \left[\left(1 + \left(\frac{p}{mc} \right)^2 \right)^{1/2} - 1 \right] p^2 dp, \tag{st.44c}$$

where the dependence of kinetic energy and velocity on momentum was given with equations (st.8) and (st.10) . It is convenient to introduce dimensionless variables:

$$x \equiv p/mc, \qquad\qquad x_F \equiv p_F/mc, \tag{st.45}$$

and to rewrite equations (st.44) as

$$\rho = A \mu_e x_F^3 , \tag{st.46a}$$

$$P_e = B \int\limits_0^{x_F} \frac{x^4 \ dx}{\left(1 + x^2 \right)^{1/2}} , \tag{st.46b}$$

$$U_e = 3B \int\limits_0^{x_F} \left[\left(1 + x^2 \right)^{1/2} - 1 \right] x^2 dx = 3B \int\limits_0^{x_F} \frac{x^4 \ dx}{\left(1 + x^2 \right)^{1/2} + 1} . \tag{st.46c}$$

With the electron mass $m = 9.11 \times 10^{-28}$ g the constants are

$$A \equiv \frac{8\pi}{3} \left(\frac{mc}{h} \right)^3 H = 0.981 \times 10^6 \qquad [\text{g cm}^{-3}], \tag{st.46d}$$

$$B \equiv \frac{8\pi}{3} \left(\frac{mc}{h} \right)^3 mc^2 = 4.80 \times 10^{23} \qquad [\text{erg cm}^{-3}]. \tag{st.46e}$$

The integrals in equations (st.45) can be evaluated analytically, but the results are too complicated to be of much practical value. It is much more useful to consider two limiting cases. We shall first consider $x_F \ll 1$, i.e. the electron gas will be **degenerate non-relativistic**. For $x \ll 1$ the leading term in the expansion under the integrals (st.45) is x^4, and we obtain

$$P_e = \frac{1}{5} B x_F^5 = 0.991 \times 10^{13} \left(\rho / \mu_e \right)^{5/3} , \qquad x_F \ll 1, \tag{st.47a}$$

$$U_e = \frac{3}{10} B x_F^5 = \frac{3}{2} P_e , \qquad x_F \ll 1. \tag{st.47b}$$

The relation between P_e and U_e is just that expected for any non-relativistic gas (cf. equation st.15) .

The second limiting case is $x_F \gg 1$, i.e. **ultra-relativistic degeneracy**. Following the previous procedure we find that the leading term under the integrals (st.46b,c) is x^3, and we obtain

$$P_e = \frac{1}{4} B x_F^4 = 1.231 \times 10^{15} \left(\rho / \mu_e \right)^{4/3}, \qquad x_F \gg 1, \tag{st.48a}$$

$$U_e = \frac{3}{4} B x_F^4 = 3 P_e, \qquad x_F \gg 1, \tag{st.48b}$$

Again, the relation between P_e and U_e is that expected for any ultra-relativistic gas.

The transition between equations (st.47) and (st.48) should be for $x_F \approx 1$, i.e. at $\rho \approx 10^6$ g cm $^{-3}$. In the transition region one may either use the exact and complicated analytical formula for the integral (st.44b) , or the following approximate formula, which has accuracy better than about 2%. We may write the equations (st.47a) and (st.48a) as

$$P_{e,nr} = K_1 \rho^{5/3}, \qquad K_1 = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m \left(H \mu_e \right)^{5/3}} = 0.991 \times 10^{13} \mu_e^{-5/3}, \tag{st.49a}$$

$$P_{e,r} = K_2 \rho^{4/3}, \qquad K_2 = \frac{1}{8} \left(\frac{3}{\pi} \right)^{1/3} \frac{hc}{\left(H \mu_e \right)^{4/3}} = 1.231 \times 10^{15} \mu_e^{-4/3}, \tag{st.49b}$$

$$P_{e,d} = \left[P_{e,nr}^{-2} + P_{e,r}^{-2} \right]^{-1/2}, \tag{st.49c}$$

where $P_{e,d}$ stands for pressure of degenerate electron gas. The interpolation formula (st.49c) selects the smaller of the two limiting case formulae.

In the non-degenerate and non-relativistic limit, i.e. for $mc^2 \gg E - \mu \gg kT$ the integrals (st.39) and (st.40) can be evaluated as in the case of Maxwell distribution function, and we obtain

$$P_{e,nd} = \frac{k}{\mu_e H} \rho T, \qquad U_{e,nd} = \frac{3}{2} P_{e,nd}, \tag{st.50}$$

where the subscript ”e,nd” stands for **non-degenerate electron gas**.

In the non-relativistic partly degenerate region, i.e. when $E - \mu \approx kT$, there is no exact analytical formula for the integrals (st.39) and (st.40) , but there is an analytical interpolation formula that is accurate to about 2%. Pressure of partly degenerate electron gas P_e, may be calculated as

$$P_e = \left[P_{e,nd}^2 + P_{e,d}^2 \right]^{1/2}. \tag{st.51}$$

The last formula selects the larger of the two limiting case formulae.

Let us now find the region in the density - temperature plane where there is a transition from non-degenerate to degenerate electron gas. This may be estimated setting $P_{e,nd} = P_{e,d} \approx P_{e,nr}$, which gives

$$\frac{k}{\mu_e H} \rho T = K_1 \rho^{5/3} \qquad \Longleftrightarrow \qquad T \approx 10^5 \rho^{2/3}. \tag{st.52}$$

Electron-positron pairs

The final formula (st.51) for pressure of electron gas has a very large range of applicability. There is no limit at very low densities. At very high densities the only correction required is in the evaluation of the mean number of nucleons per electron, μ_e. At very low temperature gas is no longer fully ionized, and again the evaluation of the number of nucleons per free electron becomes complicated. At very high temperature, when $kT >= mc^2$, electrons become relativistic. It becomes energetically possible to form electron - positron pairs, i.e. to create new particles. Now we have to write down the distribution functions for both types of electrons. As one is antiparticle of another their chemical potential differ by two rest mass energies, i.e. we have

$$\mu^- = \mu \approx mc^2, \qquad \mu^+ = \mu - 2mc^2 \approx -mc^2, \tag{st.53}$$

and the average number of particles per unit cell is

$$n_{av}^- = \frac{2}{e^{(E-\mu)/kT} + 1}, \qquad n_{av}^+ = \frac{2}{e^{(E-\mu+2mc^2)/kT} + 1}, \tag{st.54}$$

and the number density per 1 cm^3 is given as

$$n_e^- = \int_0^\infty n_e^- (p)\, dp = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{e^{(E-\mu)/kT} + 1}, \tag{st.55a}$$

$$n_e^+ = \int_0^\infty n_e^+ (p)\, dp = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{e^{(E-\mu+2mc^2)/kT} + 1}, \tag{st.55b}$$

where superscripts ”-” and ”+” refer to electrons and positrons, respectively.

There is a constraint on the total number of particles, as the difference between the number density of electrons and positrons must be equal to the number of electrons that came from ionizing all the ions, i.e.

$$n_e = n_e^- - n_e^+ = \frac{\rho}{\mu_e H}, \qquad \mu_e = \frac{2}{1+X}, \tag{st.56}$$

where X is hydrogen abundance by mass fraction.

Given the distribution functions we may calculate electron pressure end electron energy density. In the latter we shall include not only the kinetic energy of all electrons and positrons, but also the annihilation energy of all electron - positron pairs, but not the rest mass energy of the ”original” electrons. We may write

$$P_e = P_{e^-} + P_{e^+} = \frac{1}{3} \int_0^\infty v(p)\, p\, n_e^- (p)\, dp + \frac{1}{3} \int_0^\infty v(p)\, p\, n_e^+ (p)\, dp, \tag{st.57a}$$

$$U_e = U_{e^-} + U_{e^+} + 2n_e^+ mc^2 = \tag{st.57b}$$

$$\int_0^\infty E_k (p)\, n_e^- (p)\, dp + \int_0^\infty E_k (p)\, n_e^+ (p)\, dp + 2n_e^+ mc^2.$$

As usual, the integrals are too complicated to evaluate them analytically, but there are simple limiting cases.

First, we shall consider a case of a relatively low temperature and low density: $mc^2 \gg E - \mu \gg kT$, just like the Maxwell distribution, but we shall look for a small correction in order to estimate the number of pairs. We have, under this approximation:

$$e^{(E-\mu)/kT} \gg 1, \qquad e^{\left(E-\mu+2mc^2\right)/kT} \gg 1, \qquad E \approx mc^2 + \frac{p^2}{2m}, \tag{st.58}$$

and therefore the number density of electrons and positrons may be calculated as

$$n_e^- \approx \frac{8\pi}{h^3} e^{\mu/kT} \int_0^\infty p^2 e^{-E/kT} dp \approx \tag{st.59a}$$

$$\frac{8\pi}{h^3}e^{\left(\mu-mc^2\right)/kT}\int\limits_0^\infty p^2e^{-p^2/2mkT}dp \approx n_e,$$

and

$$n_e^+ \approx \frac{8\pi}{h^3}e^{\left(\mu-2mc^2\right)/kT}\int\limits_0^\infty p^2e^{-E/kT}dp \approx \tag{st.59b}$$

$$\frac{8\pi}{h^3}e^{\left(\mu-3mc^2\right)/kT}\int\limits_0^\infty p^2e^{-p^2/2mkT}dp \approx n_e e^{-2mc^2/kT}.$$

Finally, the number ratio is just what we might expect from a Boltzman formula:

$$\frac{n_e^+}{n_e^-} \approx e^{-2mc^2/kT}, \tag{st.60}$$

It is obvious how to calculate small corrections to electron pressure and electron energy density. Notice, that the corrections to energy density are relatively larger than the corrections to pressure because the rest mass energy of a small number of pairs is much larger than their thermal energy.

Another simple case is when $kT \gg mc^2$, and therefore most electrons and positrons are ultra-relativistic, their velocities are approximately equal c, and their energies $E \approx pc \gg mc^2 \approx \mu$. In this limit we have

$$n_e^- \approx n_e^+ \approx \frac{8\pi}{h^3}\int\limits_0^\infty \frac{p^2dp}{e^{pc/kT}+1} \gg n_e, \tag{st.61}$$

i.e. there are many more pairs than original electrons. The energy density due to all these pairs may be calculated as

$$U_e \approx \frac{16\pi}{h^3}\int\limits_0^\infty \frac{pc\;p^2dp}{e^{pc/kT}+1} = 16\pi\frac{(kT)^4}{(hc)^3}\int\limits_0^\infty \frac{x^3dx}{e^x+1} = \tag{st.62}$$

$$\frac{14\pi^5}{15}\frac{(kT)^4}{(hc)^3} = \frac{7}{4}U_r,$$

where we substituted $x = pc/kT$, the last integral in equation (st.62) has a value $7\pi^4/120$, and U_r is the radiation energy density. As the pairs are ultra-relativistic in this limit, their thermodynamic properties are very much like those of radiation. In particular, in this limit $P_e = U_e/3 = 7P_r/4$.

Summary

In a large region in a density - temperature plane the equation of state, and all important thermodynamic quantities may be calculated analytically with a reasonable precision, better than 2%. The region is limited at very low density by the mass density associated with radiation energy density becoming dominant, at low temperature by partial recombination of helium and hydrogen, at high density by crystallization of ions, and at high temperature by creation of electron-positron pairs. As long as we stay within this very large region the following prescription can be adopted to calculate equation of state.

Let X, Y, and Z be the abundances of hydrogen, helium and heavy elements, respectively, all by mass fraction. Of course, we have $X+Y+Z=1$. Various mean weights can be calculated as follows:

$$\frac{1}{\mu_i} \equiv \frac{n_i H}{\rho} = X + \frac{Y}{4} + \frac{Z}{16}, \qquad \frac{1}{\mu_e} \equiv \frac{n_e H}{\rho} = \frac{1+X}{2}, \tag{st.63}$$

$$\frac{1}{\mu} \equiv \frac{nH}{\rho} = \frac{1}{\mu_i} + \frac{1}{\mu_e} \approx 2X + 0.75Y + 0.5Z, \qquad n = n_i + n_e,$$

where n, n_i, and n_e, is a number density of all particles, ions, and electrons, respectively, and $H = 1.673 \times 10^{-24}\,\mathrm{g}$ is the mass of a hydrogen atom.

In the whole region of interest the ion and radiation pressure can be calculated according to

$$P_i = \frac{k}{\mu_i H}\rho T, \qquad \frac{k}{H} = 0.825 \times 10^8, \quad [\mathrm{erg\ g^{-1}\,K^{-1}}], \tag{st.64}$$

$$P_r = \frac{1}{3}U_r = \frac{a}{3}T^4, \qquad a = \frac{8\pi^5}{15}\frac{k^4}{h^3c^3} \approx 7.565 \times 10^{-15} \quad [\mathrm{erg\ cm^{-3}\,K^{-4}}]. \tag{st.65}$$

The electron pressure can be calculated according to the following set of formulae:

$$P_{e,nd} = \frac{k}{\mu_e H}\rho T, \tag{st.66a}$$

$$P_{e,nr} = K_1 \rho^{5/3}, \qquad K_1 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m\,(H\mu_e)^{5/3}} = 0.991 \times 10^{13} \mu_e^{-5/3}, \tag{st.66b}$$

$$P_{e,r} = K_2 \rho^{4/3}, \qquad K_2 = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{(H\mu_e)^{4/3}} = 1.231 \times 10^{15} \mu_e^{-4/3}, \tag{st.66c}$$

$$P_{e,d} = \left[P_{e,nr}^{-2} + P_{e,r}^{-2}\right]^{-1/2}, \qquad P_e = \left[P_{e,nd}^2 + P_{e,d}^2\right]^{1/2}. \tag{st.66d}$$

The derivatives $(\partial P_i/\partial T)_\rho$, $(\partial P_r/\partial T)_\rho$, $(\partial P_e/\partial T)_\rho$, $(\partial P_i/\partial \rho)_T$, $(\partial P_r/\partial \rho)_T$, $(\partial P_e/\partial \rho)_T$, can be calculated analytically from the above expressions, as well as the derivatives $(\partial u_i/\partial T)_\rho$, $(\partial u_r/\partial T)_\rho$, noticing that

$$U_i = u_i \rho = 1.5 P_i, \qquad U_r = u_r \rho = 3 P_r. \tag{st.67}$$

The derivative $(\partial u_e/\partial T)_\rho$, can be calculated with the following approximate formula:

$$\left(\frac{\partial u_e}{\partial T}\right)_\rho = \frac{1+b}{2+b}\,\frac{3}{\rho}\left(\frac{\partial P_e}{\partial T}\right)_\rho, \qquad b = \left(\frac{K_1}{K_2}\right)^2 = 6.48 \times 10^{-5}\left(\frac{\rho}{\mu_e}\right)^{2/3}, \tag{st.68}$$

The combined pressure, as well as the combined derivatives may be calculated according to

$$P = P_r + P_i + P_e, \tag{st.69a}$$

$$\left(\frac{\partial P}{\partial T}\right)_\rho = \left(\frac{\partial P_r}{\partial T}\right)_\rho + \left(\frac{\partial P_i}{\partial T}\right)_\rho + \left(\frac{\partial P_e}{\partial T}\right)_\rho, \tag{st.69b}$$

$$\left(\frac{\partial P}{\partial \rho}\right)_T = \left(\frac{\partial P_r}{\partial \rho}\right)_T + \left(\frac{\partial P_i}{\partial \rho}\right)_T + \left(\frac{\partial P_e}{\partial \rho}\right)_T, \tag{st.69c}$$

$$\left(\frac{\partial u}{\partial T}\right)_\rho = \left(\frac{\partial u_r}{\partial T}\right)_\rho + \left(\frac{\partial u_i}{\partial T}\right)_\rho + \left(\frac{\partial u_e}{\partial T}\right)_\rho. \tag{st.69d}$$