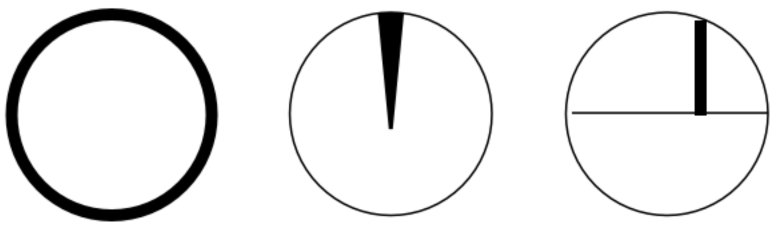
Area of a circle by integration



Integration is used to compute areas and volumes (and other things too) by adding up lots of little pieces. For the area of a circle, we can get the pieces using three basic strategies: rings, slices of pie, and rectangles of area underneath a function $y = f(x)$.

In the first approach (left panel), we imagine a series of concentric rings of radius r where r varies from 0 at the origin to R at the outside of the circle. The area of each ring is its circumference, $2\pi r$, times the little slice of radius dr. This view is much different than our first uses of integration: the pieces of area are no longer rectangles but circles. But it poses most clearly the question we are trying to answer, "how does area vary with r"? It varies like $2\pi r$!

The equation is

$$A = \int_0^R 2\pi r \ dr = 2\pi \ \frac{1}{2} r^2 \ \Big|_{r=0}^{r=R} = \pi R^2$$

In the second method, we need to first find the area of a wedge. For a thin enough slice, this is a triangle, with a similar formula

$$\frac{1}{2} R \ R \ d\theta$$

The factor of $R \ d\theta$ is the length of the base of the triangle, the piece of arc on the circle. So we have for the area

$$A = \int_{\theta=0}^{\theta=2\pi} R^2 \ d\theta = \frac{1}{2} R^2 \theta \ \Big|_{\theta=0}^{\theta=2\pi} = \pi R^2$$

The third view is the most familiar, but has a somewhat harder calculation. We will find the area under the positive square root in the equation for a circle, between the limits $x = 0, x = R$ and multiply by 4 at the end to get the whole area.

$$x^2 + y^2 = R^2, \quad y = \sqrt{R^2 - x^2}$$

We use a trigonometric substitution

$$x = R\ sin\theta, \quad y = R\ cos\theta, \quad dx = R\ cos\theta\ d\theta$$

$$\int \sqrt{R^2 - x^2}\ dx = \int R\ cos\theta\ R\ cos\theta\ d\theta = R^2 \int cos^2\theta\ d\theta$$

We have worked this integral out elsewhere (or you can solve it by substituting from the double angle formula). We obtain

$$R^2 \int cos^2\theta\ d\theta = \frac{1}{2} R^2 \int (1 + cos\ 2\theta)\ d\theta = \frac{1}{2} R^2\ [\theta + \frac{1}{2} sin\ 2\theta]$$

The limits for the integral are

$$x = 0, \quad x = R$$

which after the substitution become

$$\theta = \frac{\pi}{2}, \quad \theta = 0$$

But there's a subtlety lurking here. If we integrate from $\theta > 0$ to $\theta = 0$, the area will be negative. So we must reverse the order of the limits. With an upper limit equal to $\pi/2$, and 0 as the lower limit, the term in brackets is

$$(\frac{\pi}{2} + \frac{1}{2}\ sin\ \pi) - (0 + \frac{1}{2} sin\ 0) = \frac{\pi}{2}$$

$$A = \frac{1}{4} \pi R^2$$

This is one-fourth of the total, hence $A = \pi R^2$.