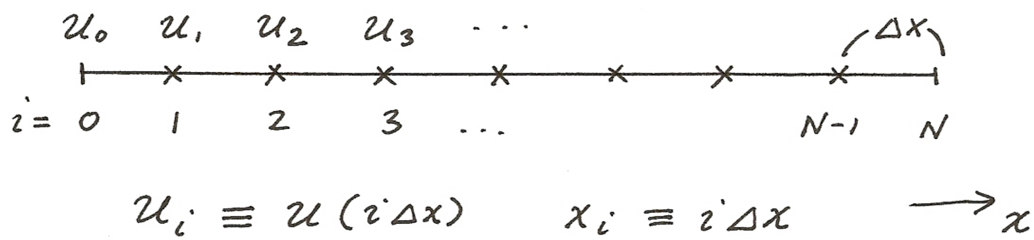
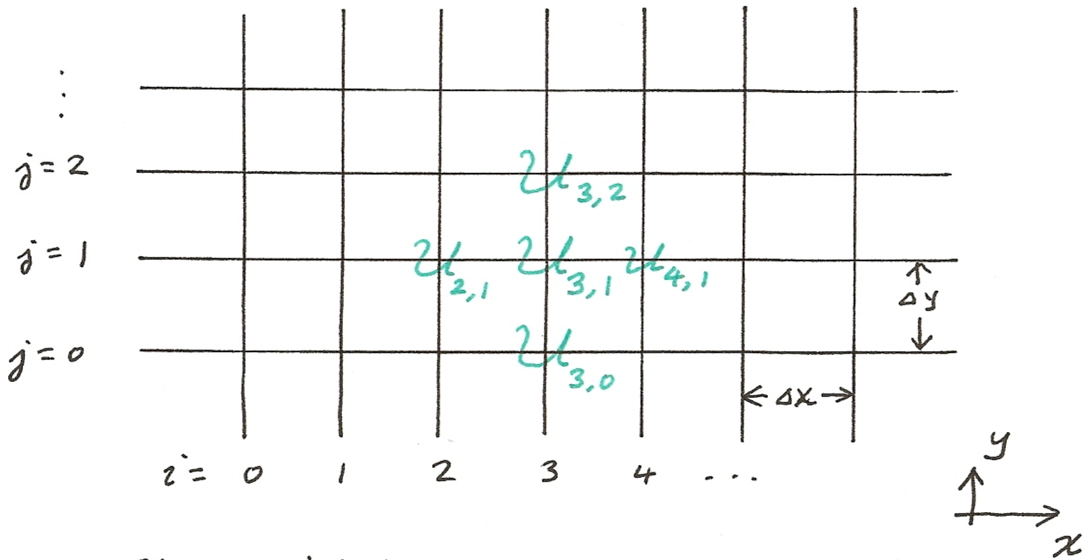
Numerical methods for Laplace's equation

Discretization: From ODE to PDE

For an ODE for $u(x)$ defined on the interval, $x \in [a, b]$, and consider a uniform grid with $\Delta x = (b-a)/\mathrm{N}$, discretization of x, u, and the derivative(s) of u leads to N equations for u_i, $i = 0, 1, 2, \ldots, \mathrm{N}$, where $u_i \equiv u(i\Delta x)$ and $x_i \equiv i\Delta x$. (See illustration.)



The idea for PDE is similar. The diagram in next page shows a typical grid system for a PDE with two variables x and y. Two indices, i and j, are used for the discretization in x and y. We will adopt the convention, $u_{i,j} \equiv u(i\Delta x, j\Delta y)$, $x_i \equiv i\Delta x$, $y_j \equiv j\Delta y$, and consider Δx and Δy constants (but generally allow Δx to differ from Δy).



$$x_i \equiv i\Delta x \qquad y_j \equiv j\Delta y$$

$$u_{i,j} \equiv u(i\Delta x, j\Delta y)$$

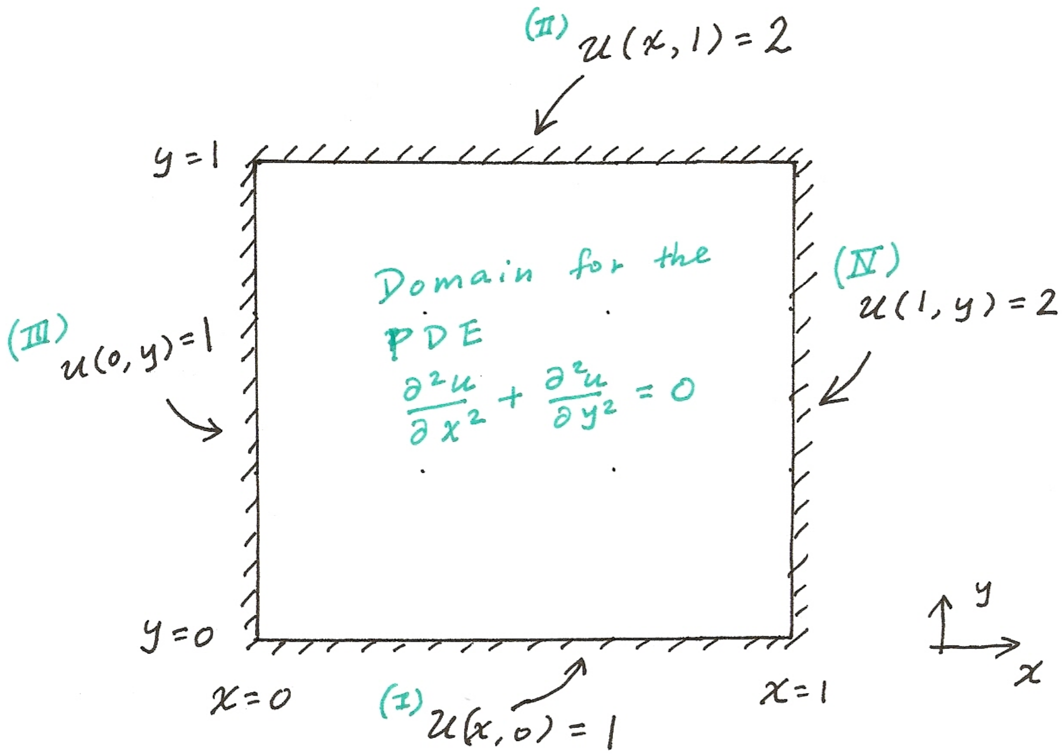
For a boundary value problem with a 2nd order ODE, the two b.c.'s would reduce the degree of freedom from N to N–2; We obtain a system of N–2 linear equations for the interior points that can be solved with typical matrix manipulations. For an initial value problem with a 1st order ODE, the value of u_0 is given. Then, u_1, u_2, u_3, ..., are determined successively using a finite difference scheme for du/dx, and so on. We will extend the idea to the solution for Laplace's equation in two dimensions.

Laplace equation

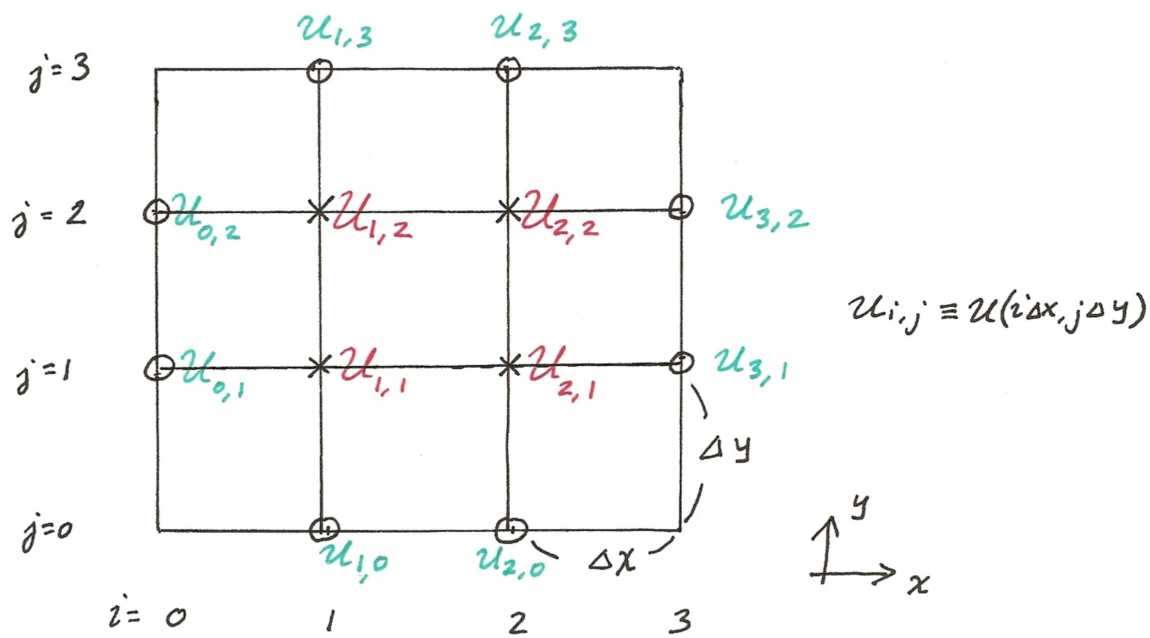
Example 1: Solve the discretized form of Laplace's equation, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, for $u(x,y)$ defined within the domain of $0 \le x \le 1$ and $0 \le y \le 1$, given the boundary conditions

(I) $u(x, 0) = 1$ (II) $u(x,1) = 2$ (III) $u(0,y) = 1$ (IV) $u(1,y) = 2$.

The domain for the PDE is a square with 4 "walls" as illustrated below. The four boundary conditions are imposed to each of the four walls.



Consider a "toy" example with just a few grid points (with Δx = Δy = 1/3):



In the preceding diagram, the values of the variables in green are already given by the boundary conditions. The only unknowns are the red $u_{i,j}$ at the interior points. We have 4 unknowns, need 4 equations to determine their values. Let us first approximate the second partial derivatives in the PDE by a *2nd order centered difference scheme*,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} \quad , \tag{1}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2} \quad . \tag{2}$$

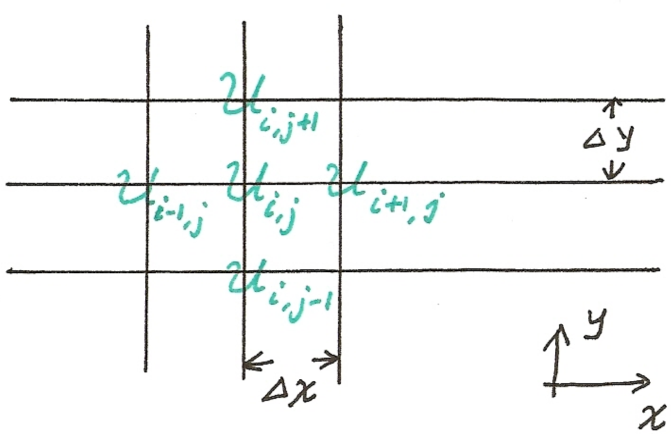
(The formula in (1) or (2) can be readily derived by Taylor series expansion. See undergraduate textbooks on numerical methods.)

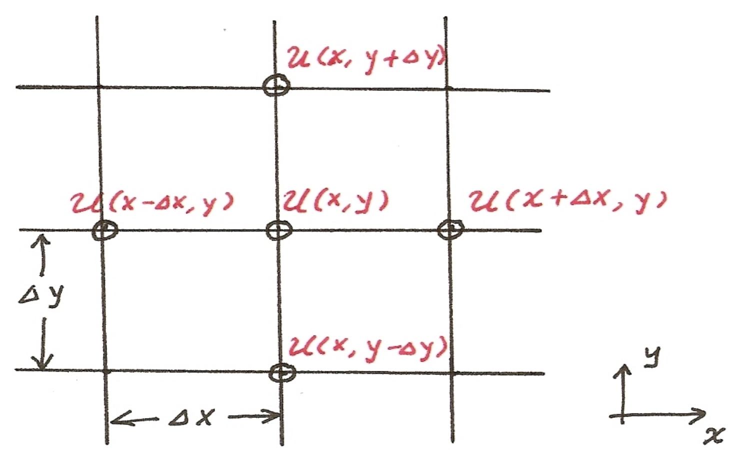
Equations (1) and (2) are the same as those for the ordinary 2nd derivatives, d^2u/dx^2 and d^2u/dy^2, only that in Eq. (1) y is held constant (all terms in Eq. (1) have the same j) and in Eq. (2) x is held constant (all terms have the same i). For those who are not familiar with the index notation, Eqs. (1) and (2) are equivalent to

$$\left(\frac{\partial^2 u}{\partial x^2}\right) \approx \frac{u(x-\Delta x, y) - 2u(x, y) + u(x+\Delta x, y)}{(\Delta x)^2} \quad , \tag{1a}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right) \approx \frac{u(x, y-\Delta y) - 2u(x, y) + u(x, y+\Delta y)}{(\Delta y)^2} \quad . \tag{2a}$$

The correspondence between the two set of notations is illustrated in the following.





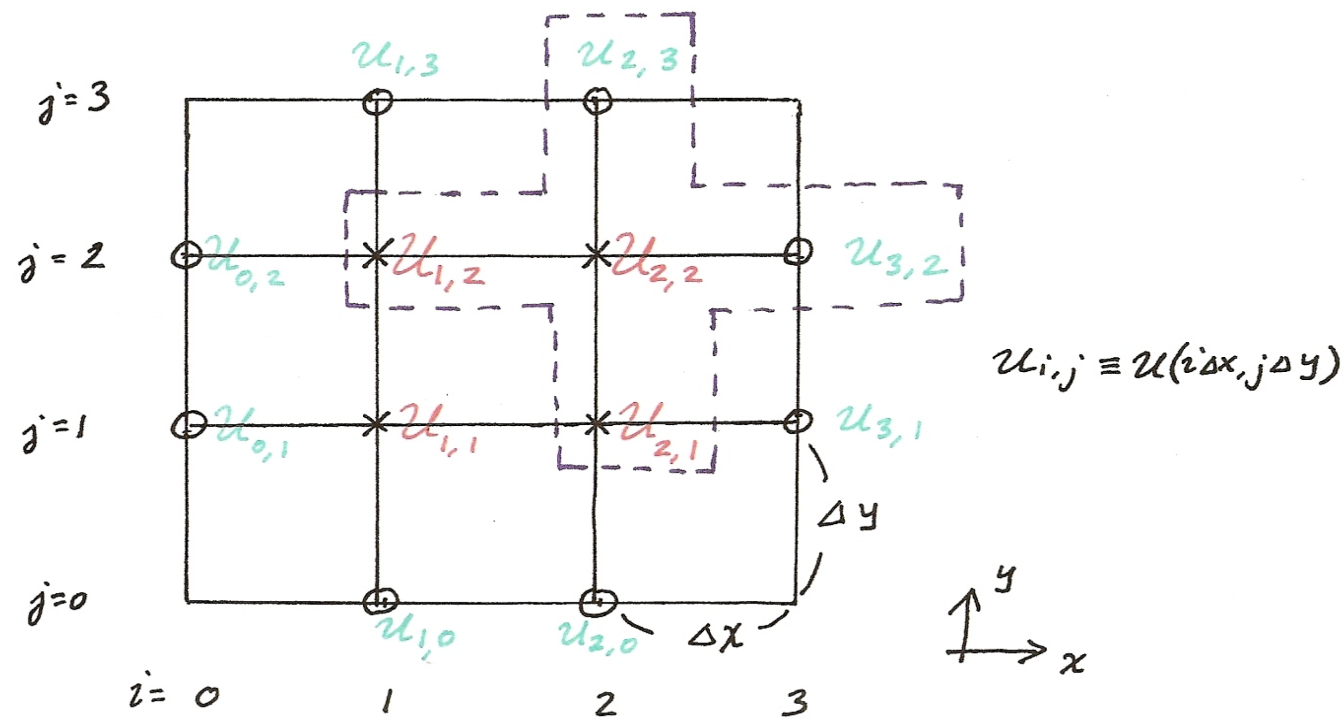
Plugging Eqs. (1) and (2) into the original Laplace's equation, we obtain

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{(\Delta x)^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{(\Delta y)^2}=0 \quad \text{, at the grid point } (i,j)\,. \qquad (3A)$$

When $\Delta x = \Delta y$, this equation can be rearranged into

$$-4u_{i,j}+u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}=0 \quad \text{, at the grid point } (i,j)\,. \qquad (3)$$

The key insight here is that the partial derivatives, $\partial^2 u/\partial x^2+\partial^2 u/\partial y^2$, at the grid point (i, j) can be evaluated by Eq. (3) using the discrete values of u at (i, j) itself (with weight of – 4) and those at its 4 neighboring points - at left, right, top, and bottom. The diagram in the next page illustrates how this fits into the grid system of our problem. For example, at the grid point, $(i, j) = (2,2)$, the terms in Eq. (3) are $u_{2,2}$ at center and $u_{2,3}$, $u_{2,1}$, $u_{1,2}$, and $u_{3,2}$ at top, bottom, left, and right of the grid point. The relevant grid points form a "cross" pattern.



$$-4u_{2,2} + u_{1,2} + u_{2,1} + \underbrace{u_{2,3} + u_{3,2}}_{\text{known from B.C.}} = 0$$

Using Eq. (3), we can now write the equations for $u_{i,j}$ at the four interior points,

$$
\begin{array}{rrrrrl}
-4\,u_{1,1} & +\ u_{1,2} & & +\ u_{2,1} & +\,u_{0,1}+u_{1,0} & = 0 \\
u_{1,1} & -4\,u_{1,2} & +\ u_{2,2} & & +\,u_{0,2}+u_{1,3} & = 0 \\
 & u_{1,2} & -4\,u_{2,2} & +\ u_{2,1} & +\,u_{2,3}+u_{3,2} & = 0 \\
u_{1,1} & + & u_{2,2} & -4\,u_{2,1} & +\,u_{2,0}+u_{3,1} & = 0
\end{array} \quad . \tag{4}
$$

See the preceding diagram for the locations of the red and green variables. The red symbols correspond to the unknown $u_{i,j}$ at the interior points. The green ones are known values of $u_{i,j}$ given by the boundary conditions,

(I) Bottom: $u_{1,0} = 1$, $u_{2,0} = 1$ (II) Top: $u_{1,3} = 2$, $u_{2,3} = 2$

(III) Left: $u_{0,1} = 1$, $u_{0,2} = 1$ (IV) Right: $u_{3,1} = 2$, $u_{3,2} = 2$ (5)

Moving the green symbols in Eq. (4) to the right hand side and replacing them with the known values given by the b.c. in Eq. (5), we have

$$
\begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 0 & 1 & -4 \end{pmatrix}
\begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,2} \\ u_{2,1} \end{pmatrix}
=
\begin{pmatrix} -2 \\ -3 \\ -4 \\ -3 \end{pmatrix} \quad , \tag{6}
$$

which can be readily solved to obtain the final solution, $(u_{1,1}, u_{1,2}, u_{2,2}, u_{2,1}) = (1.25, 1.5, 1.75, 1.5)$.

The solution is illustrated below. The behavior of the solution is well expected: Consider the Laplace's equation as the governing equation for the steady state solution of a 2-D heat equation, the "temperature", *u*, should decrease from the top right corner to lower left corner of the domain. Note that while the matrix in Eq. (6) is not strictly tridiagonal, it is sparse. The situation will remain so when we improve the grid resolution (i.e., using more grid points for the domain): The matrix will become bigger but it will have many zero elements. If the matrix is small enough, solution by a direct inversion of the matrix or classical direct method (such as Gauss elimination) will work. Otherwise, iterative methods (e.g., Jacobi's or Gauss-Seidel methods) can be adopted to solve the matrix problem.