The University of Georgia

Mathematics Education
EMAT 4680/6680 Mathematics with Technology
Jim Wilson, Instructor

Polar Roses of Sine and Cosine

by

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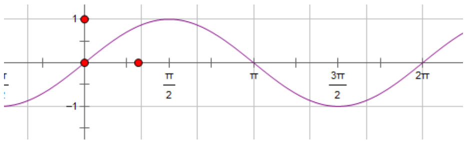

Figure 1: Sin(x) in the coordinate plane

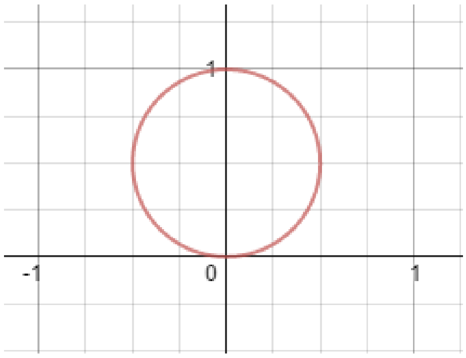

Figure 2: Sin(θ) in the polar plane

Let's begin by considering a function that we are all familiar with: the sine function in the coordinate plane, seen in figure 1. Sine has a markedly different appearance in the polar plane, particularly when changes in amplitude, periodicity and the starting constant take place. Sine in the polar plane is shown in figure 2.

The shape of sine in polar coordinates is somewhat intuitive. We know sine starts at zero, and then grows until the function reaches a height of one at $\pi/2$. As the function approaches π, the value reduces back to zero. We see this general pattern in the circle of figure 2. We will call figure a petal, and while petals are curved loops, they are not usually circles. Note that there is no second petal, even though our function has a non-zero value from π to 2π. This is because sine is negative at these values, so the second petal actually overlaps the first petal, making our rose a very simple one.

This estabilishes why the graph of sine in figure 2 is a reasonable figure ang gives an intuitive sense of the graph. A more rigorous analysis of our graph can be obtained from converting r=sin(θ) into rectangular coordinates. r represents the distance from the origin, and is therefore equal to $x^2 + y^2$. Tangent(θ) would be equal to y/x, which is made apparent by figure. So $\theta = arctan(y/x)$.

Since $r^2 = x^2 + y^2, r = \sqrt{x^2 + y^2}$. $Sin(\theta) = sin(arctan(y/x))$. We can view sin(arctan(y/x) as asking what the ratio of the opposite of theta and the hypoteneuse of a triangle is, given that the ratio of the opposite and adjacent sides is y/x. One triangle we could use to

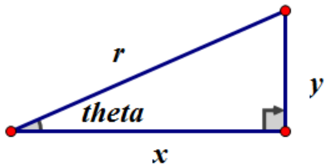

Figure 3: Relationship between coordinate plane and polar plane

determine this is shown in figure 3. Since r is equal to $\sqrt{x^2+y^2}$, our ratio must be y/$\sqrt{x^2+y^2}$. Therefore, in rectangular coordinates, r=sin(θ) is written as $\sqrt{x^2+y^2}$=y/$\sqrt{x^2+y^2}$. Let's multiply both sides by $\sqrt{x^2+y^2}$ to have $x^2+y^2=y$. We recall that the equation for a circle is $(x-a)^2+(x-b)^2=(radius)^2$, so we will match this form. subtract y from both sides to recieve $x^2+y^2-y=0$. If we add 1/4 to both sides, then we can complete the square for the y terms. This will result in the equation: $x^2+y^2-y+1/4=1/4$. So, we see that our equation is that of a circle in rectangular coordinates: $x^2+(y-1/2)^2=(1/2)^2$. So, we have a circle of radius 1/2 with the center at the point (0,1/2). This clearly shows that our petal should be of the form that it is.

Transformation of sin(theta).

We will now analyze how changes in amplitude, periodicity and adding constants will affect our graph. First let's consider changes in peroidicity. Let's consider the graph of sin(2θ), seen in figure 4. We see that in this figure, there are four petals. Normally, a 'loop', which was circle in our first polar graph, occurs during half the period of sine, ie. from zero to π or from π to 2π. In the graph of $sin(2\theta)$, these loops happen entirely in the arc of a single quadrant, due to the decreased size of the period. Note that when we get to the second quadrant, sin(2θ) is negative. So, its loop actually appears inside the fourth quadrant, and the reverse is true of the fourth quadrant. This explains why there are four times as many petals than in the graph of sin(θ), the even petals do not overlap with the odd petals.

Consider the graph of sin($/theta$) shown in figure 5. We see that the first loop takes place in half a period as expected, from zero to $\pi/3$. Then, the next petal does not show up until $2\pi/3$. This petal actually does exist, but since sine is negative during every other loop, it overlaps the petal facing directly down on the interval from $\pi/3$ to $2\pi/3$. Generally speaking, for the function sin(kθ), every

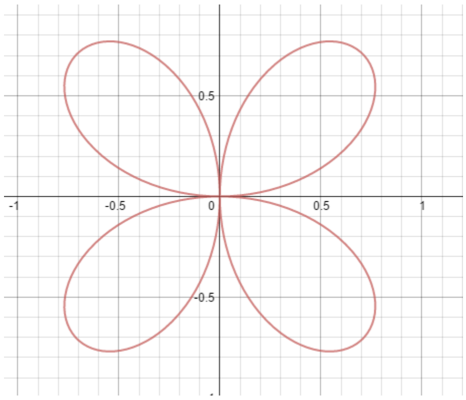

Figure 4: $\sin(2\theta)$

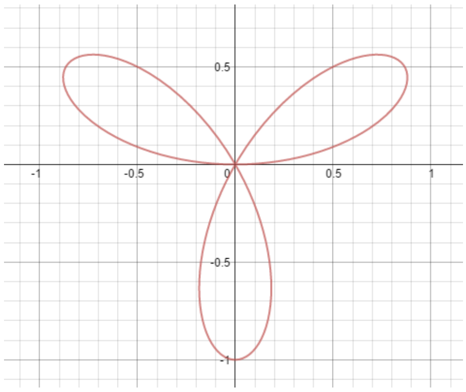

Figure 5: $\sin(3\theta)$

odd petal overlaps when k is odd, but do not when k is even. This is because there are always 2k petals if we count the petals that overlap. To go from one petal to the 'opposite petal', ie made from the same angle arc plus π, we need to count k petals. Since the petals alternate from positive to negative, that means that if k is odd, the petal on the opposite side will be negative and therefore overlap. If k is even, then either both petals will be negative or both petals will be positive, so they will not overlap.

Now consider $2\sin(4\theta)$ shown in figure six. Note that this further demonstrates how roses with an even value of k for $\sin(k\theta)$ have 2k petals. It also demonstrates that the distance of the tip of each petal to the origin will be the value of the lead coeficient, at least in the absence of an added constant. This is similar to the idea of amplitude found in the sine function in the rectangular coordinate system.

Now we consider $2\sin(4\theta)+1$ shown in figure 7. We see that some

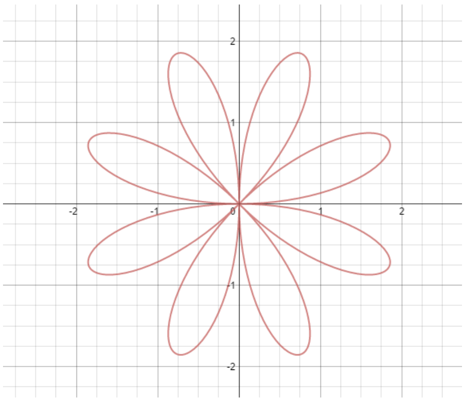

Figure 6: 2sin(4θ)

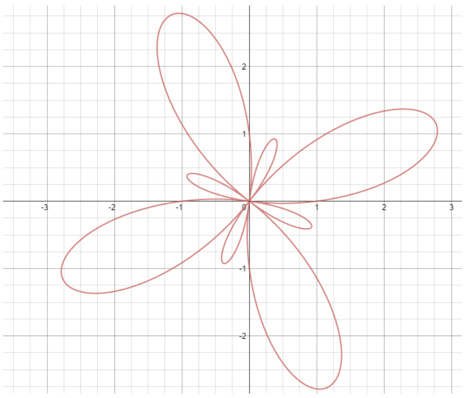

Figure 7: 2sin(4θ)+1

petals have grown and some petals have gotten smaller. Recall since each petal is actually alternating from positive to negative, some of the petal lenghts are increasing from lengtht two to three, and others are going from length 2 to 1. Predictably, a similar pattern occurs when k is even as demonstrated in $2sin(5\theta)+1$ shown in figure 8. In this case, smaller petals seem to be incased in the new petals. This is because the petals that come from positive and negative petals overlap instead of alternate in graphs where k is odd.

Now consider when k is not an integer such as with 2sin(2.7θ)+1 shown in figuree 9. We observe that despite 2.7 being a relatively low number, there are a large number of petals. Specifically, there are 32.4 petals shown in the graph, some big and some small. Consider that the domain of these polar sine functions are not necessarily restricted to (0,2π). In this graph, the domain is (0, 12π). The other graphs did not have such a large number of petals because every petal after 2π overlapped another petal. Since 2.7 isn't an

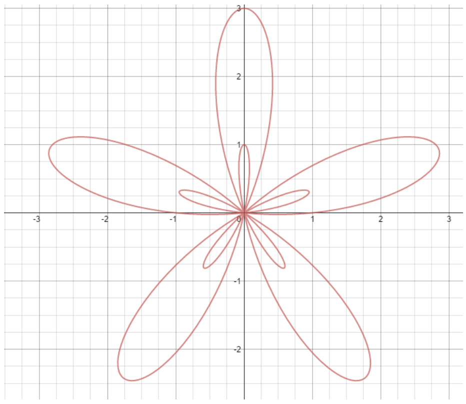

Figure 8: 2sin(5θ)+1

integer, the petal does not end at exactly 2π, so this overlapping does not happen. In fact, if the domain continued on, it would have a total of 54 petals. This is because the 'length' of each petal is $\pi/2.7$, so we will require 27 petals before a petal ends on at the end of a 2π interval. However, it will end on an odd, and therefore positive value petal, meaning that the next 'negative valued petal' will not overlap with our first petal. So, we need to do another 27 petals so that we end a 2π interval with a negative valued petal. This brings our petal count up to 54. Notice that if k is a rational number of the form c/d, then petal number 2c will be a negative valued petal that ends at a 2π. If c is even, then petal number a will be a negative valued petal at the end of a π interval. It will not be at the end of a 2π interval because d be odd, since only one of c or d could be even. So, petals do not begin to overlap until there have been 2c petals in this case too.

So, we see that our description of the number of petals can be generalized. When c and d are integers with a gcd of one, then asin($\frac{c}{(d}\theta$)+b, $a > b > 0$. Then our rose will have 2c petals. Notice that since $b > 0$, there is no chance for negative valued petals to overlap positive valued petals.

Now let's take a moment to consider the cosine function, using cos(2θ) shown in figure 10 as an example. We see that this matches the sine function almost exactly. The difference is that when θ is zero, we start at the middle of a petal instead of the beginning of a petal. This is because sin(0)=0 and cos(0)=1. Recall that cosine is just a translation of sine by one fourth of a period in the rectangular coordinate plane, and cosine can be compared to sine with a similar transformation in this polar case.

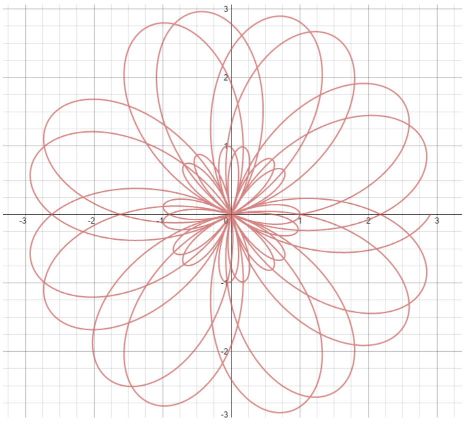

Figure 9: $2\sin(2.7\theta)+1$

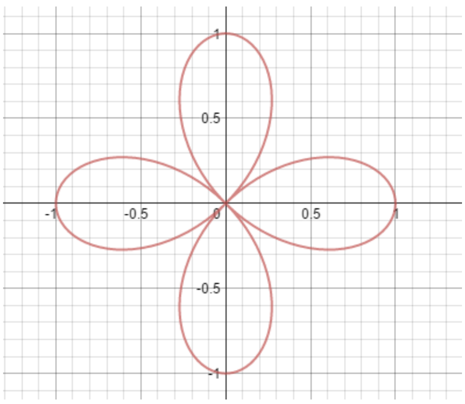

Figure 10: $\cos(2\theta)$

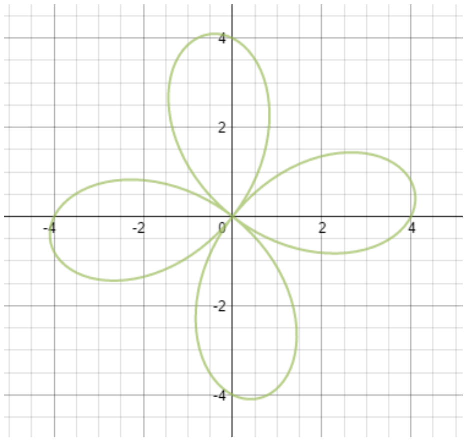

Figure 11: 4cos(2θ)+sin(2θ)

Observe what occurs when we add sin(2θ) to 4cos($2/theta$) shown in figure 11. This is akin to combining our two roses. Recall that the cosine roses has a petal at its peak at cos(0). Since our cosine term was greater, our flower is tilted much in that direction. In addition the length of the petals have increased to between four and five. Since the flowers to do not line up, their lenghts are not simply added to get the length of our new rose's petals.

Now consider the inverse of this equation 1/(4cos(2θ)+sin(2θ)) found in figure 12. At the radian values where we see the tips of the petals, the curve of our inverse function gives a miniature mirror of our original sine and cosine function. Just as the ends of the petals gradually reach their peak and then return, the inverse of this graph gradually reaches it's minimum, but then grows exponentially. Visually, the graph's curves shoot off in a relatively straight direction as an asymptote. This pattern is similar to the recangular coordinate version of the graph, shown in figure 13, where our petals approach their base, they are also approaching zero. Therefore, we see assymptotes occur in the inverses near these radian values. Notice that this behavior is similar to that of secant and cosecant lines.

It is now clear why sine and cosine roses and their inverses have the petals patterns that can be compelling.

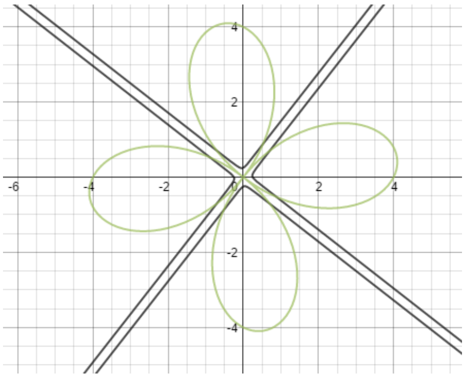

Figure 12: $1/(4\cos(2\theta)+\sin(2\theta))$ in polar coordinates.

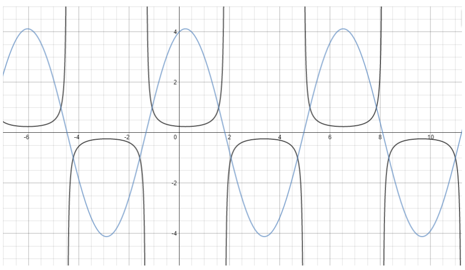

Figure 13: $4\cos(2x)+\sin(2x)$ in rectangular coordinates.