Chapter 1
Linear programming

We start by giving some examples of linear programs and how they are used in practice.

1.1 A healthy and low-priced diet

Imagine you are going on a long vacation and you need to buy food. Your objective is to buy food at minimum price, such that the daily needs of certain vitamins and energy are satisfied. There are three kinds of food. Carrots, white cabbage and oatmeal, each having a certain amount of Vitamin A, Vitamin C and energy per 100g serving. [1]

- 100g carrots contain: 3.5 mg Vitamin A, 6 mg Vitamin C, 50 kcal Energy
- 100g white cabbage contains: 0.1 mg Vitamin A, 30 mg Vitamin C, 70 kcal Energy
- 100g Oatmeal contains: 0.02 mg Vitamin A, 0.04mg Vitamin C and 300 kcal Energy

The prices for 100g of the above are 1 CHF, 0.5 Chf and 3 CHF respectively. Your daily needs are

- Vitamin A: 0.75 mg
- Vitamin C: 0.5 mg
- Energy: 1500 kcal

Your goal is now to come up with the right mix of these dishes, such that all your needs in terms of energy, vitamin A, and vitamin C are satisfied and such that this mix is as cheap as possible.

This is done with a *linear program*, a central object of study in this course. We reserve variables x_1, x_2 and x_3 which is the amount of 100g units of carrots,

[1] Those are fantasy values. We are doing math and no dietary consulting ;)

cabbage and oatmeal respectively that we will eat each day. We want that the cost of a daily serving is minimized, in other words, we want to minimize the following linear function

$$\min 1 \cdot x_1 + 0.5 \cdot x_2 + 3 \cdot x_3.$$

Certain constraints have to be satisfied. The constraint, which tells us that we need at least $0.75mg$ of vitamin A is

$$3.5x_1 + 0.1x_2 + 0.02x_3 \geqslant 0.75.$$

The variables x_1, x_2 and x_3 have to be nonnegative, so all-together, we have to solve the following problem

$$\begin{array}{ll} \min & 1 \cdot x_1 + 0.5 \cdot x_2 + 3 \cdot x_3 \\ \text{subject to} & x_1 \geqslant 0 \\ & x_2 \geqslant 0 \\ & x_3 \geqslant 0 \\ & 3.5x_1 + 0.1x_2 + 0.02x_3 \geqslant 0.75 \\ & 6x_1 + 30x_2 + 0.04x_3 \geqslant 0.5 \\ & 50x_1 + 70x_2 + 300x_3 \geqslant 1500. \end{array}$$

1.2 Linear Programs

We use the following notation. For a matrix $A \in \mathbb{R}^{m \times n}$, $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ we denote the i-th row of A by a_i and the j-th column of A by a^j. With $A(i, j)$ we denote the element of A which is in the i-th row and j-th column of A. For a vector $v \in \mathbb{R}^m$ and $i \in \{1, \ldots, m\}$ we denote the i-th element of v by $v(i)$.

Definition 1.1. Let $A \in \mathbb{R}^{m \times n}$ be a matrix, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ be vectors and $I_{\geqslant}, I_{\leqslant}, I_{=} \subseteq \{1, \ldots, m\}$ and $J_{\geqslant}, J_{\leqslant} \subseteq \{1, \ldots, n\}$ be index sets. A *linear program (LP)* consists of

i) a linear *objective function*:

$$\begin{array}{c} \max c^T x \\ \text{or } \min c^T x \end{array}$$

ii) Linear *constraints*

$$\begin{array}{c} a_i^T x \geqslant b(i), i \in I_{\geqslant} \\ a_j^T x \leqslant b(j), j \in I_{\leqslant} \\ a_k^T x = b(k), k \in I_{=} \end{array}$$

iii) and *bounds on the variables*

$$\begin{array}{c} x(j) \geqslant 0,\ j \in J_{\geqslant} \\ x(j) \leqslant 0,\ j \in J_{\leqslant}. \end{array}$$

Notice that we can re-write the objective function $\min c^T x$ as $\max -c^T x$. Similarly, the constraints $a_i^T x \geqslant b(i), i \in I_{\geqslant}$ are equivalent to the constrains $-a_i^T x \leqslant -b(i), i \in I_{\geqslant}$. Also the constraints $a_k^T x = b(k), k \in I_{=}$ can be replaced by the constraints $a_k^T x \leqslant b(k)$, $-a_k^T x \leqslant -b(k)$, $k \in I_{=}$. A lower bound $x(j) \geqslant 0$ can be written as $-e_j^T x \leqslant 0$, where e_j is the j-th unit vector which has zeroes in every component, except for the j-th component, which is 1. Similarly an upper bound $x(j) \leqslant 0$ can be written as $e_j^T x \leqslant 0$.

All-together, a linear program as in Definition 1.1 can always be written as

$$\max\{c^T x\colon \widetilde{A}x \leqslant \widetilde{b}, x \in \mathbb{R}^n\}$$

with a suitable matrix $\widetilde{A} \in \mathbb{R}^{m \times n}$ and a suitable vector $\widetilde{b} \in \mathbb{R}^m$. This representation has a name.

Definition 1.2. A linear program is in *inequality standard form*, if it is of the form

$$\max\{c^T x\colon Ax \leqslant b, x \in \mathbb{R}^n\}$$

for some matrix $A \in \mathbb{R}^{m \times n}$ and some vector $b \in \mathbb{R}^m$.

Definition 1.3. A point $x^* \in \mathbb{R}^n$ is called *feasible*, if x^* satisfies all constraints and bounds on the variables. If there are feasible solutions of a linear program, then the linear program is called *feasible* itself. A linear program is *bounded* if there exists a constant $M \in \mathbb{R}$ such for all feasible $x^* \in \mathbb{R}^n$ $c^T x^* \leqslant M$, if the linear program is a maximization problem and $c^T x^* \geqslant M$, if the linear program is a minimization problem. A feasible solution x^* is an optimal solution if $c^T x^* \geqslant c^T y^*$ for all feasible y^* if the linear program is a maximization problem and $c^T x^* \leqslant c^T y^*$ if the linear program is a minimization problem.

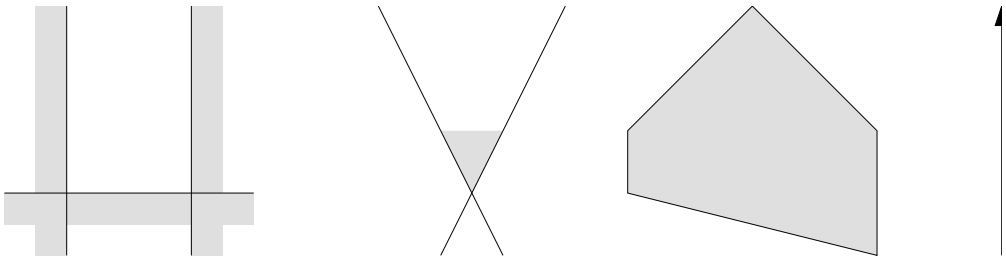

Fig. 1.1: With the objective function being to find the highest point, we have from left-to-right an infeasible linear program, and unbounded linear program and a bounded linear program.

We will see later that a feasible and bounded linear program has an optimal solution.

1.3 Two-variable linear programs

Two-variable linear programs can be solved graphically. Consider for example the linear program

$$\begin{array}{ll} \max x_1 + x_2 & \\ 2x_1 + 3x_2 & \leqslant 9 \\ 2x_1 + x_2 & \leqslant 5 \\ x_1, x_2 \geqslant 0. & \end{array}$$

Figure 1.2 depicts the feasible solutions as the gray area. The red vector is the objective vector $(1,1)$. This linear program is feasible and bounded. The optimal solution is the intersection of the two lines $2x_1 + x_2 = 5$ and $2x_1 + 3x_2 = 9$. This intersection is $x^* = (3/2, 2)$.

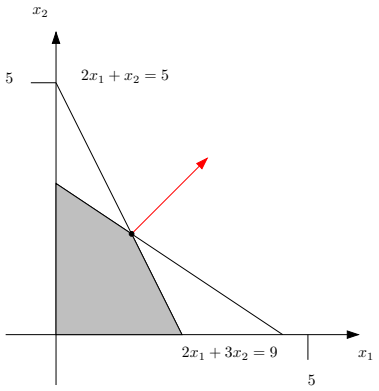

Fig. 1.2: A two-variable linear program

1.4 Fitting a line

The following is an example which is well known in statistics. Suppose that you measure points $(y_i, x_i) \in \mathbb{R}^2$ $i = 1, \ldots, n$ and you are interested in a linear function $y = a \cdot x + b$ that reflects the sample. One way to do that is by minimizing the

expression

$$\sum_{i=1}^{n} (ax_i + b - y_i)^2, \tag{1.1}$$

where $a, b \in \mathbb{R}$ are the parameters of the line that we are looking for. The number $(ax_i + b - y_i)^2$ is the square of the vertical distance of the point x_i, y_i from the line $y = a\,x + b$.

Instead of using the method of least-squares, we could also minimize the following function, see also [13, Chapter 2.4],

$$\sum_{i=1}^{n} |ax_i + b - y_i|. \tag{1.2}$$

This objective has the advantage to be slightly more robust towards outliers. How can we model this as a linear program. The trick is to use an extra variable h_i which models the absolute value of $ax_i + b - y_i$.

$$\begin{array}{lll} \min \sum_{i=1}^{n} h_i & & \\ h_i & \geqslant & ax_i + b - y_i,\ i = 1, \ldots, n \\ h_i & \geqslant & -(ax_i + b - y_i),\ i = 1, \ldots, n \end{array} \tag{1.3}$$

The variables of this linear program are h_i, $i = 1, \ldots, n$, a and b. For a fixed $a \in \mathbb{R}$ and $b \in \mathbb{R}$ the optimal h_i's will be $h_i = |ax_i + b - y_i|$ since the objective minimizes the sum of the h_i's. If one of the was strictly larger than $|ax_i + b - y_i|$, then the objective could be improved by making it smaller.

1.5 Linear Programming solvers and modeling languages

We will demonstrate now how to use a modeling language for linear programming and a linear programming solver to find a fitting line, as described in Section 1.4 for the points

$$(1,3), (2.8,3.3), (4,2), (5.5,2.1), (6,.2), (7,1.3), (7.5,1), (8.5,0.8)$$

There are two popular formats for linear programming problems which are widely used by linear programming solvers, the *lp-format* and the *mps-format*. Both are not easy to read. To facilitate the modeling of a linear program, so-called modeling languages are used. We demonstrate the use of the popular open source modeling software called zimpl [8]. Below you see a way to model our fitting line linear program with zimpl:

```
set I := { 1 to 8};
param X[I] :=  <1> 1, <2> 2.8, <3> 4, <4> 5.5,
               <5> 6, <6> 7, <7> 7.5, <8> 8.5 ;
param Y[I] :=  <1> 3, <2> 3.3, <3> 2, <4> 2.1,
               <5> .2, <6> 1.3, <7> 1, <8> .8 ;
```

```
var h[I] >= -infinity <= infinity;
var a  >= -infinity <= infinity ;
var b  >= -infinity <= infinity ;

minimize cost: sum <i> in I: h[i];

subto c1: forall <i> in I:      h[i] >=   ( a * X[i] + b -Y[i]);
subto c2: forall <i> in I:      h[i] >= - ( a * X[i] + b -Y[i]);
```

Zimpl creates a linear program which is readable by linear programming solvers like QSopt or SoPlex. An optimal fitting-line w.r.t. the distance measure (1.2) is the line $y = -0.293333 \cdot x + 3.293333$. It is depicted in figure 1.2.

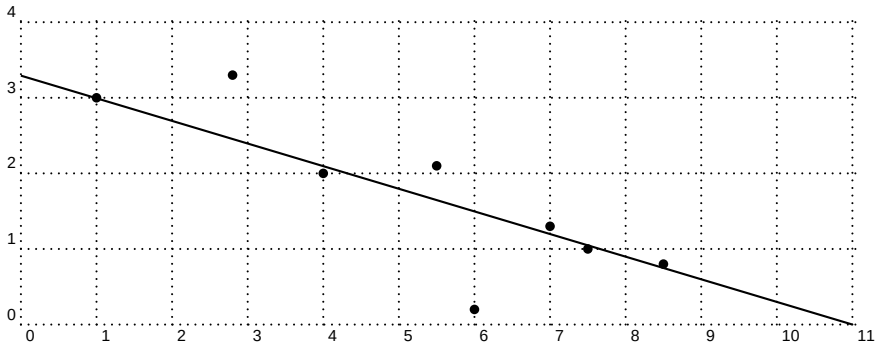

Fig. 1.3: A set of points $\{(1,3),(2.8,3.3),(4,2),(5.5,2.1),(6,.2),(7,1.3),(7.5,1),(8.5,0.8)\}$ and the line determined by linear program (1.3).

1.6 Linear programming for longer OLED-lifetime

Organic Light Emitting Diodes (OLEDs) are considered as the display technology of the future and more and more commercial products are equipped with such displays as shown in Fig. 1.4. However, the cheapest OLED technology suffers from short lifetimes. We will show in this section how linear programming can be used to increase the lifetime of such displays.

A (passive matrix) OLED display has a matrix structure consisting of n rows and m columns. At any crossover between a row and a column there is a vertical diode which works as a pixel. The image itself is given as an integral non-negative $n \times m$ matrix $(r_{ij}) \in [0,\ldots,\varrho]^{n\times m}$ representing its RGB values. Consider the contacts for the rows and columns as switches. For the time the switch of row i and column j is closed, an electrical current flows through the diode of pixel (i,j) and it shines. Hence, we can control the intensity of a pixel by the two quantities *electrical current* and *time*. The value r_{ij} determines the amount of time within the time frame in which the switches i and j have to be simultaneously closed. At a

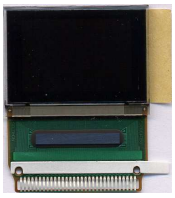

Fig. 1.4: Sample of a commercial OLED device with integrated driver chip

sufficient high frame rate e.g. 50 Hz, the perception by the eye is the average value of the light emitted by the pixel and one sees the image.

The traditional addressing scheme is row-by-row. This means that the switch for the first row is closed for a certain time while the switches for the columns are closed for the necessary amount of time dictated by the entries r_{1j}, $j = 1,\dots,m$. Consequently the first row can be displayed in time $\max\{r_{1j} : j = 1,\dots,m\}$. Then the second row is displayed and so on. With this addressing scheme, the pixels are idle most of the time and then have to shine with very high intensity. This puts the diodes under stress and is a major cause of the short lifetime of the displays.

How can this lifetime problem be dealt with? The main idea is to save time, or equivalently to lower the maximum intensity, by displaying several rows at once.

Consider the schematic image on the left of Fig. 1.5. Let us compute the amount of time which is necessary to display the image with this addressing scheme. The maximum value of the entries in the first row is 238. This is the amount of time which is necessary to display the first row. After that the second row is displayed in time 237. In total the time which is required to display the image is $238+237+234+232+229 = 1170$ time units.

$$\begin{matrix} 109 & 238 & 28 \\ 112 & 237 & 28 \\ 150 & 234 & 25 \\ 189 & 232 & 22 \\ 227 & 229 & 19 \end{matrix} = \begin{matrix} 0 & 82 & 25 \\ 0 & 82 & 25 \\ 0 & 41 & 22 \\ 0 & 41 & 22 \\ 0 & 0 & 0 \end{matrix} + \begin{matrix} 0 & 0 & 0 \\ 112 & 155 & 3 \\ 112 & 155 & 3 \\ 189 & 191 & 0 \\ 189 & 191 & 0 \end{matrix} + \begin{matrix} 109 & 156 & 3 \\ 0 & 0 & 0 \\ 38 & 38 & 0 \\ 0 & 0 & 0 \\ 38 & 38 & 19 \end{matrix}$$

Fig. 1.5: An example decomposition

Now consider the decomposition of the image as the sum of the three images on the right of Fig. 1.5. In the first image, each odd row is equal to its even successor. This means that we can close the switches for rows 1 and 2 simultaneously, and these two equal rows are displayed in 82 time units. Rows 3 and 4 can also be displayed simultaneously which shows that the first image on the right can be displayed in $82+41$ time units. The second image on the right can be displayed in $155+191$ time units while the third image has to be displayed traditionally. In total all three images, and thus the original image on the left via this decomposition,

can be displayed in $82+41+155+191+156+38+38 = 701$ time units. This means that we could reduce the necessary time via this decomposition by roughly 40%. We could equally display the image in the original 1170 time units but reduce the peak intensity, or equally the maximum electrical current through a diode by roughly 40%.

We now show how to model the time-optimal decomposition of an image as a linear program. To decompose R we need to find matrices $F^{(1)} = (f_{ij}^{(1)})$ and $F^{(2)} = (f_{ij}^{(2)})$ where $F^{(1)}$ represents the singleline part and $F^{(2)}$ the two doubleline parts. More precisely, the i-th row of matrix $F^{(2)}$ represents the doubleline covering rows i and $i+1$. Since the overlay (addition) of the subframes must be equal to the original image to get a valid decomposition of R, the matrices $F^{(1)}$ and $F^{(2)}$ must fulfill the constraint $f_{ij}^{(1)} + f_{i-1,j}^{(2)} + f_{ij}^{(2)} = r_{ij}$ for $i = 1,\dots,n$ and $j = 1,\dots,m$, where we now and in the following use the convention to simply omit terms with indices running out of bounds. Since we cannot produce "negative" light we require also non-negativity of the variables $f_{ij}^{(\alpha)} \geqslant 0$. The goal is to find an integral decomposition that minimizes

$$\sum_{i=1}^{n} \max\{f_{ij}^{(1)} : 1 \leqslant j \leqslant m\} + \sum_{i=1}^{n-1} \max\{f_{ij}^{(2)} : 1 \leqslant j \leqslant m\} \ .$$

This problem can be formulated as a linear program by replacing the objective by $\sum_{i=1}^{n} u_i^{(1)} + \sum_{i=1}^{n-1} u_i^{(2)}$ and by adding the constraints $f_{ij}^{(\alpha)} \leqslant u_i^{(\alpha)}$. This yields

$$\begin{aligned}
\min \ & \sum_{i=1}^{n} u_i^{(1)} + \sum_{i=1}^{n-1} u_i^{(2)} & & \\
\text{s.t.} \ & f_{ij}^{(1)} + f_{i-1,j}^{(2)} + f_{ij}^{(2)} = r_{ij} & \text{for all } i,j & \quad (1.4)\\
& f_{ij}^{(\alpha)} \leqslant u_i^{(\alpha)} & \text{for all } i,j,\alpha & \quad (1.5)\\
& f_{ij}^{(\alpha)} \in \mathbb{R}_{\geqslant 0} & \text{for all } i,j,\alpha &
\end{aligned}$$

Note that the objective does not contain the f-variables. By decomposing images like this, the average lifetime of an OLED display can be increased by roughly 100%, see [4].

Exercises

1) A company produces and sells two different products. Our goal is to determine the number of units of each product they should produce during one month, assuming that there is an unlimited demand for the products, but there are some constraints on production capacity and budget.
   There are 20000 hours of machine time in the month. Producing one unit takes 3 hours of machine time for the first product and 4 hours for the second

product. Material and other costs for producing one unit of the first product amount to 3CHF, while producing one unit of the second product costs 2CHF. The products are sold for 6CHF and 5CHF per unit, respectively. The available budget for production is 4000CHF initially. 25% of the income from selling the first product can be used immediately as additional budget for production, and so can 28% of the income from selling the second product.

a. Formulate a linear program to maximize the profit subject to the described constraints.
b. Solve the linear program graphically by drawing its set of feasible solutions and determining an optimal solution from the drawing.
c. Suppose the company could modernize their production line to get an additional 2000 machine hours for the cost of 400CHF. Would this investment pay off?

2) A factory produces two different products. To create one unit of product 1, it needs one unit of raw material A and one unit of raw material B. To create one unit of product 2, it needs one units of raw material B and two units of raw material C. Raw material B needs preprocessing before it can be used, which takes one minute per unit. At most 20 hours of time is available per day for the preprocessing. Raw materials of capacity at most 1200 can be delivered to the factory per day. One unit of raw material A, B and C has size 4, 3 and 2 respectively.
At most 130 units of the first and 100 units of the second product can be sold per day. The first product sells for 6 CHF per unit and the second one for 9 CHF per unit.
Formulate the problem of maximizing turnover as a linear program in two variables and solve it.
3) Prove the following statement or give a counterexample: The set of optimal solutions of a linear program is always finite.
4) Let (1.6) be a linear program in inequality standard form, i.e.

$$\max\{c^T x \mid Ax \leqslant b, x \in \mathbb{R}^n\} \tag{1.6}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.
Prove that there is an equivalent linear program (1.7) of the form

$$\max\{\tilde{c}^T x \mid \tilde{A}x = \tilde{b}, x \geqslant 0, x \in \mathbb{R}^{\tilde{n}}\} \tag{1.7}$$

where $\tilde{A} \in \mathbb{R}^{\tilde{m}\times\tilde{n}}$, $\tilde{b} \in \mathbb{R}^{\tilde{m}}$, and $\tilde{c} \in \mathbb{R}^{\tilde{n}}$ are such that every feasible point of (1.6) corresponds to a feasible point of (1.7) with the same objective function value and vice versa.
Linear programs of the form in (1.7) are said to be in *equality standard form*.

5) Model the linear program (1.4) to decompose the EPFL logo with Zimpl. An incomplete model containing the encoding of the grayscale values of the logo can be found here here[2].
   Use an LP solver library of your choice to compute an optmal solution.
6) Provide an example of a convex and closed set $K \subseteq \mathbb{R}^2$ and a linear objective function $c^T x$ such that $\inf\{c^T x\colon\ x \in K\} > -\infty$ but there does not exist an $x^* \in K$ with $c^T x^* \leqslant c^T x$ for all $x \in K$.

[2] http://disopt.epfl.ch/webdav/site/disopt/users/190205/public/logo_dec.zmpl

Chapter 2
Convex sets

A polyhedron $P \subseteq \mathbb{R}^n$ is a set of the form $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ for some $A \in \mathbb{R}^{m \times n}$ and some $b \in \mathbb{R}^m$. The set of feasible solutions of a linear program $\max\{c^T x \colon Ax \leqslant b\}$ is a polyhedron. Polyhedra are convex sets. Convex sets are the main objects of study of this chapter.

2.1 Linear, affine, conic and convex hulls

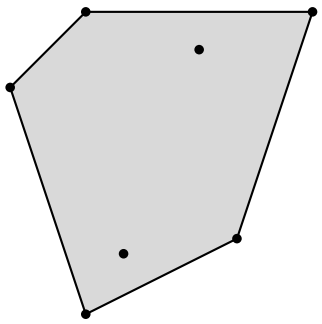

Fig. 2.1: The convex hull of 7 points in $\mathbb{R}^2$.

Let $X \subseteq \mathbb{R}^n$ be a set of n-dimensional vectors. The *linear hull, affine hull, conic hull* and *convex hull* of X are defined as follows.

$$\begin{aligned}
\text{lin.hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 0,\ x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\} && (2.1)\\
\text{affine.hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 1, && (2.2)\\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\}\\
\text{cone}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 0, && (2.3)\\
&\qquad x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geqslant 0}\}\\
\text{conv}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 1, && (2.4)\\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geqslant 0}\}
\end{aligned}$$

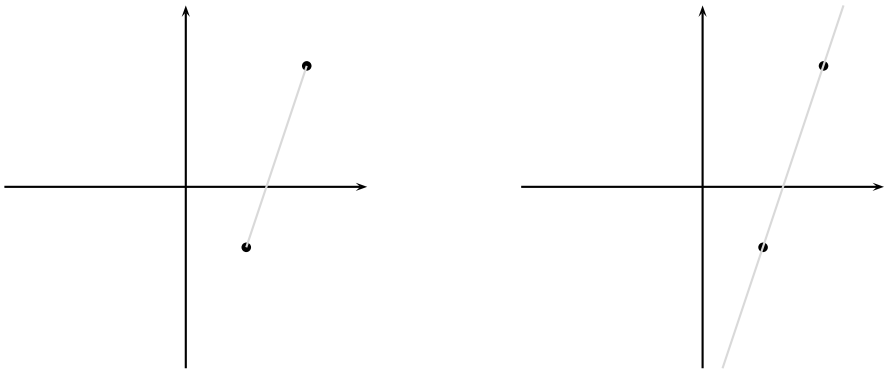

Fig. 2.2: Two points with their convex hull on the left and their affine hull on the right.

Proposition 2.1. *Let $X \subseteq \mathbb{R}^n$ and $x_0 \in X$. One has*

$$\text{affine.hull}(X) = x_0 + \text{lin.hull}(X - x_0),$$

where for $u \in \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$, $u + V$ denotes the set $u + V = \{u + v \mid v \in V\}$.

Proof. We first show that each $x \in \text{affine.hull}(X)$ is also an element of the set $x_0 + \text{lin.hull}(X - x_0)$ and then we show that each point $x \in x_0 + \text{lin.hull}(X - x_0)$ is also an element of $\text{affine.hull}(X)$.

Let $x \in \text{affine.hull}(X)$,i.e., there exists a natural number $t \geqslant 1$ and $\lambda_1, \ldots, \lambda_t \in \mathbb{R}$, with $x = \lambda_1 x_1 + \cdots + \lambda_t x_t$ and $\sum_{i=1}^{t} \lambda_i = 1$. Now

$$\begin{aligned}
x &= x_0 - x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_t x_t\\
&= x_0 - \lambda_1 x_0 - \cdots - \lambda_t x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_t x_t\\
&= x_0 + \lambda_1 (x_1 - x_0) + \cdots + \lambda_t (x_t - x_0),
\end{aligned}$$

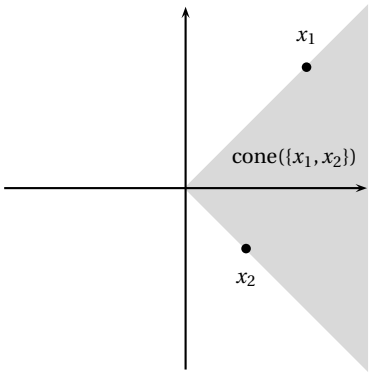

Fig. 2.3: Two points with their conic hull

which shows that $x \in x_0 + \text{lin.hull}(X - x_0)$.

Suppose now that $x \in x_0 + \text{lin.hull}(X - x_0)$. Then there exist $\lambda_1, \ldots, \lambda_t \in \mathbb{R}$ with $x = x_0 + \lambda_1(x_1 - x_0) + \cdots + \lambda_t(x_t - x_0)$. With $\lambda_0 = 1 - \sum_{i=1}^{t} \lambda_i$ one has $\sum_{i=0}^{t} \lambda_i = 1$ and

$$\begin{aligned} x &= x_0 + \lambda_1(x_1 - x_0) + \cdots + \lambda_t(x_t - x_0) \\ &= \lambda_0 x_0 + \cdots + \lambda_t x_t \end{aligned}$$

and thus that $x \in \text{affine.hull}(X)$. □

Definition 2.1. The convex hull of two distinct points $u \neq v \in \mathbb{R}^n$ is called a *line segment* and is denoted by $\overline{uv}$.

Definition 2.2. A set $K \subseteq \mathbb{R}^n$ is *convex* if for each $u \neq v$, the line-segment $\overline{uv}$ is contained in K, $\overline{uv} \subseteq K$.

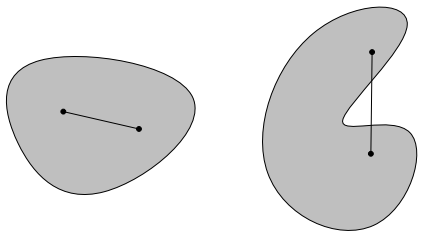

Fig. 2.4: The set on the left is convex, the set on the right is non-convex.

In other words, a set $K \subseteq \mathbb{R}^n$ is convex, if for each $u, v \in K$ and $\lambda \in [0,1]$ the point $\lambda u + (1 - \lambda) v$ is also contained in K.

Theorem 2.1. *Let $X \subseteq \mathbb{R}^n$ be a set of points. The convex hull,* $\mathrm{conv}(X)$*, of X is convex.*

Proof. Let u and v be points in $\mathrm{conv}(X)$. This means that there exists a natural number $t \geqslant 1$, real numbers $\alpha_i, \beta_i \geqslant 0$, and points $x_i \in X$, $i = 1, \ldots, t$ with $\sum_{i=1}^t \alpha_i = \sum_{i=1}^t \beta_i = 1$ with $u = \sum_{i=1}^t \alpha_i x_i$ and $v = \sum_{i=1}^t \beta_i x_i$. For $\lambda \in [0,1]$ one has $\lambda\alpha_i + (1-\lambda)\beta_i \geqslant 0$ for $i = 1, \ldots, t$ and $\sum_{i=1}^t \big(\lambda\alpha_i + (1-\lambda)\beta_i\big) = 1$. This shows that

$$\lambda u + (1-\lambda)v = \sum \big(\lambda_i\alpha_i + (1-\lambda_i)\beta_i\big) x_i \in \mathrm{conv}(X),$$

and therefore that $\mathrm{conv}(X)$ is convex. □

Theorem 2.2. *Let $X \subseteq \mathbb{R}^n$ be a set of points. Each convex set K containing X also contains* $\mathrm{conv}(X)$.

Proof. Let K be a convex set containing X, and let $x_1, \ldots, x_t \in X$ and $\lambda_i \in \mathbb{R}$ with $\lambda_i \geqslant 0$, $i = 1, \ldots, t$ and $\sum_{i=1}^t \lambda_i = 1$. We need to show that $u = \sum_{i=1}^t \lambda_i x_i$ is contained in K. This is true for $t \leqslant 2$ by the definition of convex sets.

We argue by induction. Suppose that $t \geqslant 3$. If one of the λ_i is equal to 0, then one can represent u as a convex combination of $t-1$ points in X and, by induction, $u \in K$. Since $t \geqslant 3$, each $\lambda_i > 0$ and $\sum_{i=1}^t \lambda_i = 1$ one has $0 < \lambda_i < 1$ for $i = 1, \ldots, t$ and thus we can write

$$u = \lambda_1 x_1 + (1-\lambda_1) \sum_{i=2}^{t} \frac{\lambda_i}{1-\lambda_1} x_i.$$

One has $\lambda_i/(1-\lambda_1) > 0$ and

$$\sum_{i=2}^{t} \frac{\lambda_i}{1-\lambda_1} = 1,$$

which means that the point $\sum_{i=2}^t \frac{\lambda_i}{1-\lambda_1} x_i$ is in K by induction. Again, by the definition of convex sets, we conclude that u lies in K. □

Theorem 2.2 implies that $\mathrm{conv}(X)$ is the intersection of all convex sets containing X, i.e.,

$$\mathrm{conv}(X) = \bigcap_{\substack{K \supseteq X \\ K \text{ convex}}} K.$$

Definition 2.3. A set $C \subseteq \mathbb{R}^n$ is a *cone,* if it is convex and for each $c \in C$ and each $\lambda \in \mathbb{R}_{\geqslant 0}$ one has $\lambda \cdot c \in C$.

Similarly to Theorem 2.1 and Theorem 2.2 one proves the following.

Theorem 2.3. *For any $X \subseteq \mathbb{R}^n$, the set* $\mathrm{cone}(X)$ *is a cone.*

Theorem 2.4. *Let $X \subseteq \mathbb{R}^n$ be a set of points. Each cone containing X also contains* $\mathrm{cone}(X)$.

These theorems imply that $\operatorname{cone}(X)$ is the intersection of all cones containing X, i.e.,

$$\operatorname{cone}(X) = \bigcap_{\substack{C \supseteq X \\ C \text{ is a cone}}} C.$$

2.2 Radon's lemma and Carathéodory's theorem

Theorem 2.5 (Radon's lemma). *Let $A \subseteq \mathbb{R}^n$ be a set of $n+2$ points. There exist disjoint subsets $A_1, A_2 \subseteq A$ with*

$$\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset.$$

Proof. Let $A = \{a_1, \ldots, a_{n+2}\}$. We embed these points into $\mathbb{R}^{n+1}$ by appending a 1 in the $n+1$-st component, i.e., we construct

$$A' = \left\{ \binom{a_1}{1}, \ldots, \binom{a_{n+2}}{1} \right\} \subseteq \mathbb{R}^{n+1}.$$

The set A' consists of $n+2$ vectors in $\mathbb{R}^{n+1}$. Those vectors are linearly dependent. Let

$$0 = \sum_{i=1}^{n+2} \lambda_i \binom{a_i}{1} \tag{2.5}$$

be a nontrivial linear representation of 0, i.e., not all λ_i are 0. Furthermore, let $P = \{i \colon \lambda_i \geqslant 0,\ i = 1, \ldots, n+2\}$ and $N = \{i \colon \lambda_i < 0,\ i = 1, \ldots, n+2\}$. We claim that

$$\operatorname{conv}(\{a_i \colon i \in P\}) \cap \operatorname{conv}(\{a_i \colon i \in N\}) \neq \emptyset.$$

It follows from (2.5) and the fact that the $n+1$-st component of the vectors is 1 that $\sum_{i \in P} \lambda_i = -\sum_{i \in N} \lambda_i = s > 0$. It follows also from (2.5) that

$$\sum_{i \in P} \lambda_i a_i = \sum_{i \in N} -\lambda_i a_i.$$

The point $u = \sum_{i \in P} (\lambda_i / s) \cdot a_i = \sum_{i \in N} (-\lambda_i / s) a_i$ is contained in $\operatorname{conv}(\{a_i \colon i \in P\}) \cap \operatorname{conv}(\{a_i \colon i \in N\})$, implying the claim. □

Theorem 2.6 (Carathéodory's theorem). *Let $X \subseteq \mathbb{R}^n$, then for each $x \in \operatorname{cone}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most n such that $x \in \operatorname{cone}(\widetilde{X})$. The vectors in $\widetilde{X}$ are linearly independent.*

Proof. Let $x \in \operatorname{cone}(X)$, then there exist $t \in \mathbb{N}_+$, $x_i \in X$ and $\lambda_i \geqslant 0$, $i = 1, \ldots, t$, with $x = \sum_{i=1}^{t} \lambda_i x_i$. Suppose that $t \in \mathbb{N}_+$ is minimal such that x can be represented as above. We claim that $t \leqslant n$. If $t \geqslant n+1$, then the x_i are linearly dependent. This means that there are $\mu_i \in \mathbb{R}$, not all equal to 0 with

$$\sum_{i=1}^{t} \mu_i x_i = 0. \tag{2.6}$$

By multiplying each μ_i in (2.6) with -1 if necessary, we can assume that at least one of the μ_i is strictly larger than 0. One has for each $\varepsilon \in \mathbb{R}$

$$x = \sum_{i=1}^{t} (\lambda_i - \varepsilon \cdot \mu_i) x_i. \tag{2.7}$$

What is the largest $\varepsilon^* > 0$ that we can pick for ε such that (2.7) is still a conic combination? We need to have

$$\lambda_i - \varepsilon \cdot \mu_i \geqslant 0, \text{ for each } i \in \{1, \ldots, t\}. \tag{2.8}$$

Let J be the set of indices $J = \{j \colon j \in \{1, \ldots, t\}, \mu_j > 0\}$. We observed that we can assume $J \neq \emptyset$. We have (2.8) as long as

$$\varepsilon \leqslant \lambda_j / \mu_j \text{ for each } j \in J. \tag{2.9}$$

This means that $\varepsilon^* = \min\{\lambda_j / \mu_j \colon j \in J\}$. Let $j^* \in J$ be an index where this minimum is attained. Since $\lambda_i - \varepsilon^* \cdot \mu_i \geqslant 0$ for all $i = 1, \ldots, t$ and since $\lambda_{j^*} - \varepsilon^* \cdot \mu_{j^*} = 0$, we have $x \in \operatorname{cone}(\{x_1, \ldots, x_t\} \setminus \{x_{j^*}\})$, which is a contradiction to the minimality of t. □

Corollary 2.1 (Carathéodory's theorem for convex hulls). *Let $X \subseteq \mathbb{R}^n$, then for each $x \in \operatorname{conv}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most $n+1$ such that $x \in \operatorname{conv}(\widetilde{X})$.*

2.3 Separation theorem and Farkas' lemma

We recall a basic fact from analysis, see, e.g. [12, Theorem 4.4.1].

Theorem 2.7. *Let $X \subseteq \mathbb{R}^n$ be compact and $f : X \to \mathbb{R}$ be continuous. Then f is bounded and there exist points $x_1, x_2 \in X$ with $f(x_1) = \sup\{f(x) \colon x \in X\}$ and $f(x_2) = \inf\{f(x) \colon x \in X\}$.*

Theorem 2.8. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $x^* \in \mathbb{R}^n \setminus K$, then there exists an inequality $a^T x \geqslant \beta$ such that $a^T y > \beta$ holds for all $y \in K$ and $a^T x^* < \beta$.*

Proof. Since the mapping $f(x) = \|x^* - x\|$ is continuous and since for any $k \in K$, $K \cap \{x \in K \colon \|x^* - x\| \leqslant \|x^* - k\|\}$ is compact, there exists a point $k^* \in K$ with minimal distance to x^*. Consider the midpoint $m = 1/2(k^* + x^*)$ on the line-segment $\overline{k^* x^*}$ and the hyperplane $a^T x = \beta$ with $\beta = a^T m$ and $a = (k^* - x^*)$. Clearly, $a^T x^* = \beta - 1/2\|k^* - x^*\|^2$ and $a^T k^* = \beta + 1/2\|k^* - x^*\|^2$. Suppose that there exists a $k' \in K$ with $a^T k' \leqslant \beta$. The points $\lambda k^* + (1-\lambda)k'$, $\lambda \in [0,1]$ are in K by the convexity of K, thus we can also assume that k' lies on the hyperplane, i.e., $a^T k' = \beta$. This means that there exists a vector x' which is orthogonal to a and $k' = m + x'$. The distance squared of a point $\lambda k^* + (1-\lambda)k'$ with $\lambda \in [0,1]$ to m is, by Pythagoras equal to

$$\lambda^2 \| \frac{1}{2} a \|^2 + (1-\lambda)^2 \| x' \|^2.$$

As a function of λ, this is increasing at at $\lambda = 1$. Thus there exists a point on the line-segment $\lambda x^* + (1-\lambda)k$ which is closer to m than k^*. This point is also closer to x^* than k^*, which is a contradiction. Therefore $a^T k > \beta$ for each $k \in K$. □

Theorem 2.9 (Farkas' lemma). *Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax = b$, $x \geqslant 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geqslant 0$ one has $\lambda^T b \geqslant 0$.*

Proof. Suppose that $x^* \in \mathbb{R}^n_{\geqslant 0}$ satisfies $Ax^* = b$ and let $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A \geqslant 0$. Then $\lambda^T b = \lambda^T A x^* \geqslant 0$, since $\lambda^T A \geqslant 0$ and $x^* \geqslant 0$.

Now suppose that $Ax = b$, $x \geqslant 0$ does not have a solution. Then, with $X \subseteq \mathbb{R}^n$ being the set of column vectors of A, b is not in $\text{cone}(X)$. The set $\text{cone}(X)$ is convex and closed, see exercise 5. Theorem 2.8 implies that there is an inequality $\lambda^T x \geqslant \beta$ such that $\lambda^T y > \beta$ for each $y \in \text{cone}(X)$ and $\lambda^T b < \beta$. Since for each $a \in X$ and each $\mu \geqslant 0$ one has $\mu \cdot a \in \text{cone}(X)$ and thus $\lambda^T(\mu \cdot a) > \beta$, it follows that $\lambda^T a \geqslant 0$ for each $a \in X$. Furthermore, since $0 \in \text{cone}(X)$ it follows that $0 \geqslant \beta$ and thus that $\lambda^T b < 0$.

Exercises

1) Let $\{C_i\}_{i \in I}$ be a family of convex subsets of $\mathbb{R}^n$. Show that the intersection $\bigcap_{i \in I} C_i$ is convex.
2) Show that the set of feasible solutions of a linear program is convex.
3) Prove Carathéodory's Theorem for convex hulls, Corollary 2.1.
4) Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix and let $a_1, \ldots, a_n \in \mathbb{R}^n$ be the columns of A. Show that $\text{cone}(\{a_1, \ldots, a_n\})$ is the polyhedron $P = \{y \in \mathbb{R}^n \colon A^{-1} y \geqslant 0\}$. Show that $\text{cone}(\{a_1, \ldots, a_k\})$ for $k \leqslant n$ is the set $P_k = \{y \in \mathbb{R}^n \colon a_i^{-1} x \geqslant 0, i = 1, \ldots, k,\ a_i^{-1} x = 0, i = k+1, \ldots, n\}$, where a_i^{-1} denotes the i-th row of A^{-1}.
5) Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\text{cone}(X)$ is closed and convex. *Hint: Use Carathéodory's theorem and exercise 4.*
6) Find a countably infinite set $X \subset \mathbb{R}^2$ such that $\text{cone}(X)$ is not closed. Are there any cones that are open?
7) Prove Theorem 2.3.
8) Prove Theorem 2.4.
9) Let $f : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map.
   a) Show that $f(K) = \{f(x) \colon x \in K\}$ is convex if K is convex. Is the reverse also true?
   b) For $X \subseteq \mathbb{R}^n$ arbitrary, prove that $\text{conv}(f(X)) = f(\text{conv}(X))$.
10) Using Theorem 2.9, prove the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leqslant b$, $x \in \mathbb{R}^n$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.

11) Provide an example of a convex and closed set $K \subseteq \mathbb{R}^2$ and a linear objective function $c^T x$ such that $\min\{c^T x \colon x \in K\} > -\infty$ but there does not exist an $x^* \in K$ with $c^T x^* \leqslant c^T x$ for all $x \in K$.
12) Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $A = \{x_1, \dots, x_5\}$. Find two disjoint subsets $A_1,\ A_2 \subseteq A$ such that

$$\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset.$$

Hint: Recall the proof of Radon's lemma
13) Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 14 \\ 25 \end{pmatrix}$$

is a conic combination of the x_i.
Write v as a conic combination using only three vectors of the x_i.
Hint: Recall the proof of Carathéodory's theorem

Chapter 3
The simplex method

In this chapter, we describe the simplex method. The task is to solve a linear program

$$\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant b\} \tag{3.1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. We make the following assumption.

> FULL-RANK ASSUMPTION
> The matrix $A \in \mathbb{R}^{m\times n}$ has full column-rank. In other words, the columns of A are linearly independent

We will see later that this assumption can be made without loss of generality.

3.1 Roofs

Roofs are linear programs originating from (3.1) by selecting a subset of the inequalities only. A roof should provide an upper bound on the optimal value of the linear program (3.1) and at the same time consist of n "linearly independent constraints". Here is the definition of a roof.

Definition 3.1. Consider the linear program (3.1) and let $B \subseteq \{1,\dots,m\}$ be a subset of the row-indices. This set B is a *roof* if

i) $|B| = n$,
ii) The rows a_i, $i \in B$ are linearly independent, and
iii) The linear program

$$\max\{c^T x\colon\ a_i^T x \leqslant b(i),\ i \in B\} \tag{3.2}$$

is bounded.

What is the optimal solution of a linear program (3.2) defined by a roof? This question is answered in the next lemma.

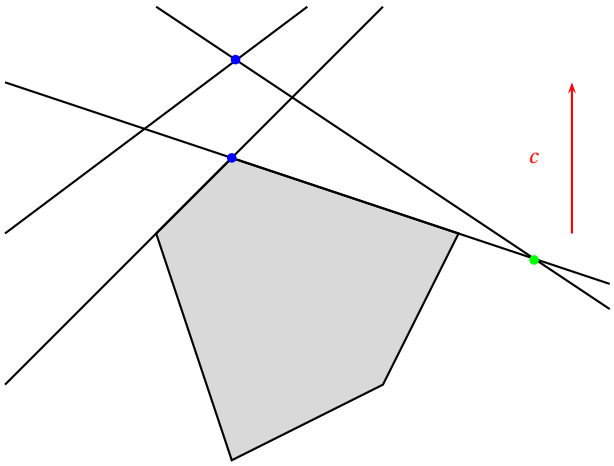

Fig. 3.1: A linear program; the objective function vector c is pointing vertically upwards. The blue dots mark two roofs. Notice that the lowest roof is the optimum of the linear program. The green point marks a non-roof. The two constraints satisfy i) and ii) but not iii).

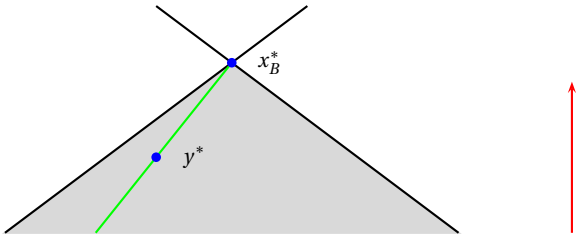

Fig. 3.2: An illustration for the proof of Lemma 3.1. The green ray illustrates the set $\{x_B^* + \lambda(y^* - x_B^*) : \lambda \in \mathbb{R}_{\geqslant 0}\}$.

Lemma 3.1. *Let $B \subseteq \{1,\ldots,m\}$ be a roof of the linear program* (3.1) *and let x_B^* be the unique solution of the linear system*

$$a_i\, x = b(i),\ i \in B,$$

then x_B^* is an optimal solution of the roof-linear program (3.2).

Proof. Suppose that y^* is a feasible solution of the roof-linear program with $c^T y^* > c^T x_B^*$. We now show that each $x_B^* + \lambda(y^* - x_B^*)$ for $\lambda \geqslant 0$ is feasible. One

has $a_i\,(x_B^* + \lambda(y^* - x_B^*)) = b(i) + \lambda \cdot (a_i\,y^* - b(i)) \leqslant b(i)$ for each $i \in B$ and thus $x_B^* + \lambda(y^* - x_B^*)$ is feasible for each $\lambda \geqslant 0$.

The objective function value of such a point is $c^T x_B^* + \lambda(c^T y^* - c^T x_B^*)$ which, for $\lambda \to \infty$ tends to infinity. This is a contradiction to B being a roof (condition iii). Thus x_B^* must be an optimal solution to the roof-linear program (3.2). □

Now that we know that a roof-linear program has an optimal solution, we can define the value of a roof B.

Definition 3.2. The *value* of a roof B is the optimum value $c^T x_B^*$ of the roof-linear program

$$\max\{c^T x\colon\ a_i\,x \leqslant b(i),\ i \in B\}.$$

The next theorem is very simple, but in fact very important. It states that the value of a roof is an upper bound on the optimum value of a linear program $\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant b\}$.

Theorem 3.1 (Weak duality). *The value of a roof is an upper bound on the objective function value of any feasible point of the linear program.*

Proof. Let B be a roof of the linear program $\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant b\}$. Any feasible point x^* of this linear program is also a feasible point of the roof-linear program $\max\{c^T x\colon\ a_i\,x \leqslant b(i)\}$. Therefore $c^T x_B^* \geqslant c^T x^*$ and the claim follows. □

When is an index-set $B \subseteq \{1,\ldots,m\}$ satisfying i) and ii) a roof? Consider the example in see Figure 3.3. The objective is to maximize $2x_1 + x_2$ and the two roof-constraints are $x_1 + x_2 \leqslant 5$ and $x_1 \leqslant 6$. From the picture, it is clear that the objective function vector is in the cone of the two constraint vectors. In fact, this is the characterization that holds in any dimension as we now show.

Lemma 3.2. *Let $B \subseteq \{1,\ldots,m\}$ satisfy i) and ii). Then B is a roof, if and only if $c \in \operatorname{cone}\{a_i\colon\ i \in B\}$.*

Proof. Suppose that $c \in \operatorname{cone}\{a_i\colon\ i \in B\}$. Thus there exist $\lambda_i \geqslant 0$, $i \in B$ with $c = \sum_{i\in B} \lambda_i \cdot a_i$. The unique solution x_B^* to the system

$$a_i\,x = b(i),\ i \in B \tag{3.3}$$

is an optimal solution to $\max\{c^T x\colon\ a_i\,x \leqslant b(i)\}$. Because if $\widetilde{x}$ is another feasible solution, then $c^T \widetilde{x} = \sum_{i\in B} \lambda_i \cdot a_i\,\widetilde{x}$. Since $\lambda \geqslant 0$ and $a_i\,\widetilde{x} \leqslant b(i)$ we can write

$$\begin{aligned}
c^T \widetilde{x} &= \sum_{i\in B} \lambda_i \cdot a_i\,\widetilde{x} && (3.4)\\
&\leqslant \sum_{i\in B} \lambda_i \cdot b(i) && (3.5)\\
&= \sum_{i\in B} \lambda_i \cdot a_i\,x_B^* && (3.6)\\
&= (\sum_{i\in B} \lambda_i \cdot a_i)\,x_B^* && (3.7)\\
&= c^T x_B^*. && (3.8)
\end{aligned}$$

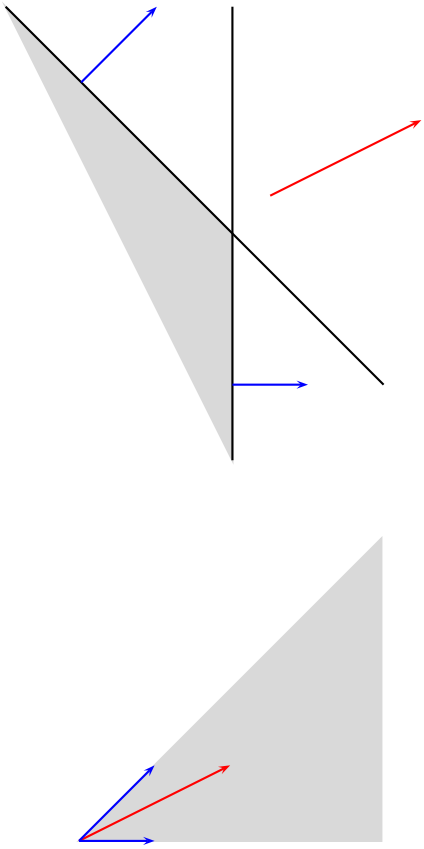

Fig. 3.3: A roof that is defined by the two constraints $x_1 \leqslant 2$ and $x_1 + x_2 \leqslant 3$. The objective function vector is $(2,1)^T$. Indeed $(1,1)^T = 1 \cdot (1,1) + 1 \cdot (1,0)$ which shows that it is in the cone generated by the constraint-defining vectors.

Thus B is a roof.

Suppose on the other hand that B is a roof. Then, since $a_i, i \in B$ is a basis of $\mathbb{R}^n$, there exist $y_i \in \mathbb{R}$, $i \in B$ with $c = \sum_{i \in B} y_i \cdot a_i$. If all $y_i \geqslant 0$, $i \in B$, then it follows that $c \in \text{cone}\{a_i : i \in B\}$ and we are done. Suppose therefore that there exists an index $j \in B$ with $y_j < 0$. We will derive a contradiction.

Consider the system of linear equations

$$a_j\, x = -1, a_i\, x = 0,\ i \in B \setminus \{j\}. \tag{3.9}$$

This system (3.9) has a unique solution $0 \neq v \in \mathbb{R}^n$. Let x^* be a feasible solution to the roof-linear program. Clearly $x^* + \lambda \cdot v$ is also feasible for each $\lambda > 0$ (Exercise 2). But $c^T(x^* + \lambda v) = c^T x^* + \lambda \cdot \sum_{i=1}^n y(i) a_i\, v = c^T x^* + \lambda \cdot y_j \cdot a_j\, v$. This increases with λ

since $y_j < 0$ and $a_j\, v < 0$. This contradicts the fact B is a roof, since the roof-linear program is unbounded. □

Definition 3.3. Let B be a roof of the linear program (3.1). The unique solution x^* of the system

$$a_i^T\, x = b(i),\ i \in B, \tag{3.10}$$

is the *vertex* of the roof.

Similarly one can prove the following fact.

Proposition 3.1. *Let B be a roof of the linear program* (3.1)*. The vertex of a roof is the* unique optimal solution *of the roof-linear program* (3.2) *if and only if c is a* strictly positive *conic combination of the normal-vectors $a_i, i \in B$.*

3.2 The simplex algorithm

We now sketch one iteration of the simplex algorithm. Our task is to solve a linear program (3.1) and we assume that we have a roof B to start with.

i) Compute the vertex x_B^* of the roof B.
ii) Find an index $i \in \{1,\dots,m\} \setminus B$ with $a_i\, x_B^* > b(i)$. If there does not exist such an index, then x_B^* is an optimal solution of the linear program (3.1).
iii) Determine an index $j \in B$ such that

   a) $B' = B \cup \{i\} \setminus \{j\}$ is a roof, and
   b) The vertex $x_{B'}^*$ of B' is feasible for B.

   If such an index does not exist, then the linear program (3.1) is infeasible.

The simplex algorithm iterates these steps until it has found an optimal solution, or asserts that the linear program (3.1) is infeasible. The big questions are how to determine an index j such that iiia) and iiib) hold in step iii) and that the algorithm is correct. Furthermore, we want to understand whether the simplex method eventually terminates.

3.2.1 Termination and degeneracy

Definition 3.4 (Degenerate roof and linear program). A roof B of a linear program (3.1) is *degenerate* if the optimum solution of the roof-linear program (3.2) is not unique. A linear program is called degenerate, if it has degenerate roofs.

We now argue that the simplex algorithm terminates if the linear program is non-degenerate.

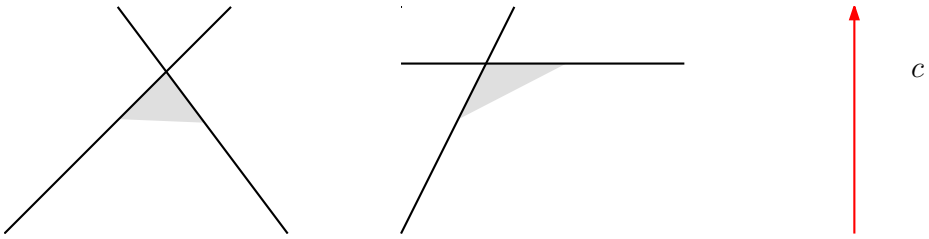

Fig. 3.4: A non-degenerate and a degenerate roof.

Theorem 3.2. *If the linear program* (3.1) *is non-degenerate, then the simplex algorithm terminates.*

Proof. The important observation is that the simplex method makes progress from iteration to iteration because of the non-degeneracy of the roofs. If B' is the roof computed in step iii), then, since $x^*_{B'}$ is contained in the feasible region of the roof B, and since B is non-degenerate, we have $c^T x^*_B > c^T x^*_{B'}$. Since there is only a finite number of roofs, the algorithm thus terminates. □

3.2.2 Implementing step iii)

The situation is as follows. We are having a roof B and its vertex x^*_B and an index $i \in \{1,\dots,m\}$ with $a_i x^*_B > b(i)$. We now want to bring i into the new roof and we have to determine a $j \in B$ that is supposed to leave the roof. The idea is very similar now to the proof of Carathéodory's theorem.

Consider the systems of equations

$$\sum_{k\in B} a_k z_k = c^T \tag{3.11}$$

$$\sum_{k\in B} a_k y_k = -a_i \tag{3.12}$$

with variables z_k, $k \in B$ and y_k, $k \in B$.

Compute solutions $z^* \in \mathbb{R}^n$ of (3.11) and $y^* \in \mathbb{R}^n$ of (3.12). Now we have for any $\lambda \in \mathbb{R}$

$$\sum_{k\in B} a_k(z^*_k + \lambda\, y^*_k) + \lambda a_i = c^T \tag{3.13}$$

Notice that $z^* \geqslant 0$. To bring the index i into the roof, we want to increase $\lambda = 0$ until some other component of $z^* + \lambda y^*$, component j lets say, becomes zero. So in virtue of finding an index which drops out of B, we have to determine the largest $\lambda^* \in \mathbb{R}_{\geqslant 0}$ such that all components of $x^* + \lambda\, y^*$ are nonnegative. This is done as follows.

Compute the index set $J = \{k \in B\colon y^*_k < 0\}$. Those are the indices we have to worry about, since only those components can become negative with increasing λ. Still, how large can λ^* be? We have to ensure that

$$z^*(k) + \lambda^* y^*(k) \geqslant 0 \text{ for all } k \in J. \tag{3.14}$$

In other words we have to ensure

$$\lambda^* \leqslant -\frac{z^*(k)}{y^*(k)} \text{ for all } k \in J. \tag{3.15}$$

If $J \neq \emptyset$, we pick

$$\lambda^* = \min_{k \in J} -\frac{z^*(k)}{y^*(k)}. \tag{3.16}$$

We choose an index $j \in J$ for which this minimum is achieved. This index j is the one which leaves the roof.

Lemma 3.3. *The index set $B' = B\backslash\{i\} \cup \{j\}$ is a roof and the new vertex $x^*_{B'}$ is contained in the feasible set of the roof B.*

Proof. By construction, c is a nonnegative linear combination of the vectors a_k, $k \in B'$. Thus in order to conclude that B' is a roof, we need to show that the a_k, $k \in B'$ are linearly independent. The component y^*_j is nonzero. Since y^* is a solution of equation (3.12) it follows that a_j is a linear combination of the normal-vectors of B'. Thus the a_k, $k \in B'$ are a basis of $\mathbb{R}^n$ and since $|B'| = n$ they are linearly independent.

Let x^*_B be the vertex of B and let $w \in \mathbb{R}^n$ be a solution to the system

$$a_j\, w = -1, a_k\, w = 0,\ k \in B \setminus \{j\}. \tag{3.17}$$

The half-line $l(x^*_B, w) = \{x^* + \lambda\, w \mid \lambda \in \mathbb{R}_{\geqslant 0}\}$ is feasible for B. We have the equation

$$a_i = -\sum_{k \in B} y^*_k a_k \tag{3.18}$$

where $y^*_j < 0$. Thus

$$a_i\, w = -\sum_{k \in B} y^*_k a_k\, w \tag{3.19}$$

$$= y^*_k \tag{3.20}$$

$$< 0. \tag{3.21}$$

Therefore the hal-fline $l(x^*, w)$ enters at some point, $x' \in \mathbb{R}^n$ say, the halfspace $a_i\, x \leqslant b(i)$. Clearly, this x' is the vertex $x^*_{B'}$ of B'. □

Example 3.1. Consider the linear program $\max\{x_2\colon x \in \mathbb{R}^n, (-1,1)x \leqslant 1, (2,1)x \leqslant 1, (1,2)x \leqslant 1\}$. We start with the roof $B = \{1,2\}$ that consists of the first two inequalities see Figure 3.5.

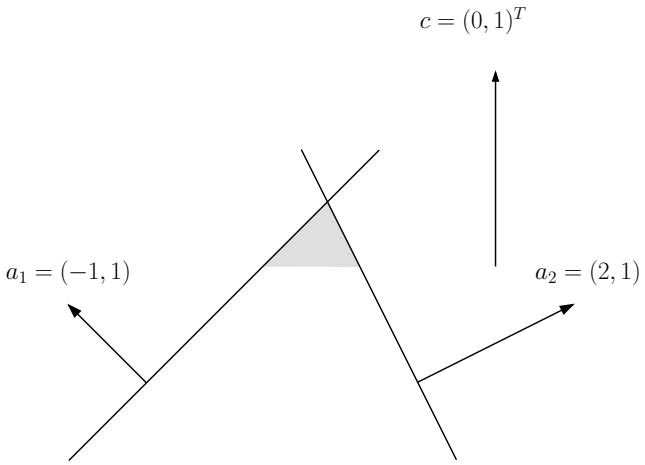

Fig. 3.5: The initial roof of Example 3.1.

We compute first the vertex x_B^* which is the solution to the system

$$\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.22}$$

Thus the vertex is the vector $x_B^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Next we find that the halfspace $(1,2)x \leqslant 1$ is not satisfied by $x_B^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We want to bring this index into the new roof B'.
Step 3: Now we compute the solution to the system

$$\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.23}$$

and find

$$z^* = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}.$$

Next we find a solution to the system

$$\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} y = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.24}$$

and find

$$y^* = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The index set $J = \{1,2\}$ is not empty. The minimum (3.16) is achieved at $j = 2$. So the halfspace $(2,1)x \leqslant 1$ will leave the roof and $B' = \{1,3\}$. This is also what we immediately see by looking at Figure 3.6.

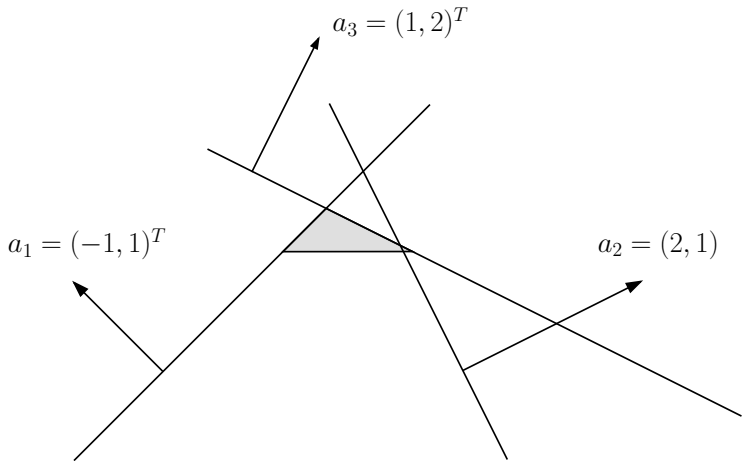

Fig. 3.6: The new roof from Example 3.1.

If $J = \emptyset$, we assert that the linear program is infeasible based on the following result.

Proposition 3.2. *The half-spaces $a_k\, x \leqslant b(k)$, $k \in B$ and $a_i\, x \leqslant b(i)$ define together an infeasible system if and only if $J = \emptyset$.*

Proof. If $J \neq \emptyset$, then Lemma 3.3 implies that the half-spaces $a_k\, x \leqslant b(k)$, $k \in B$ and $a_i\, x \leqslant b(i)$ define a feasible system.

The index set J is empty if and only if $y^* \geqslant 0$. We now show that, if $y^* \geqslant 0$, then the half-spaces $a_k\, x \leqslant b(k)$, $k \in B$ and $a_i\, x \leqslant b(i)$ define an infeasible system. Since $\sum_{k\in B} y_k^* a_k + a_i = 0$ this assertion follows, once we have shown that $\sum_{k\in B} y_k b(k) + b(i) < 0$. But

$$\begin{aligned}
\sum_{k\in B} y_k b(k) + b(i) &= \sum_{k\in B} y_k^* a_k\, x_B^* + b(i) \\
&< \sum_{k\in B} y_k^* a_k\, x_B^* + a_i\, x_B^* \\
&= \left(\sum_{k\in B} y_k^* a_k + a_i \right) x_B^* \\
&= 0^T x_B^* \\
&= 0.
\end{aligned}$$

3.2.3 The degenerate case

The termination argument for the non-degenerate case was that the value of the new roof is strictly dropping and thus, that a roof can never be revisited. Since there are only a finite number of roofs, this implies that the simplex algorithm terminates.

In the degenerate case, roofs could be revisited. This phenomenon is called *cycling* and you are asked to construct such an example in the exercises. What can we do about it? The idea is to change the objective vector $c \in \mathbb{R}^n$ a little bit and turn it into a vector c_ϵ such that the following conditions hold.

1) The linear program
$$\max\{c_\epsilon^T x \colon x \in \mathbb{R}^n, Ax \leqslant b\} \tag{3.25}$$
has a roof.
2) Each non-roof of the linear program (3.1) is a non-roof of the linear program (3.25).
3) No roof of (3.25) is degenerate.

Suppose we start with an initial roof R at the beginning of the simplex algorithm and let $A_R \in \mathbb{R}^{n\times n}$ be the matrix whose rows are the vectors a_i, $i \in R$. Notice that we implicitly assume that the set R is ordered. We adhere to the following notation. For $i \in R$, the function $f_R(i)$ denotes the position of i in the ordered set R. For example, if $R = \{5,2,9\}$, then $f_R(5) = 1$ and $f_R(9) = 3$.

The system ${A_R}^T y = c$ has a solution $y^* \geqslant 0$, where some components of y^* are zero if and only if R is degenerate. This is undesirable and we wish that y^* is replaced by
$$y^* + \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^n \end{pmatrix} \tag{3.26}$$
for some $\epsilon > 0$. Later it will become clear why we add the vector $(\epsilon,\ldots,\epsilon^n)^T$ instead of the vector $(\epsilon,\ldots,\epsilon)^T$. Now the vector (3.26) becomes a solution if we perturb c and consider the vector
$$c_\epsilon = c + {A_R}^T \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^n \end{pmatrix} \tag{3.27}$$
instead, see figure 3.7. If $\epsilon > 0$, then R is a non-degenerate roof of the linear program (3.25). Thus condition 1) holds for any $\epsilon > 0$. In the sequel, we make ϵ smaller and smaller, such that also the conditions 2) and 3) will be satisfied.

Let us first deal with condition 2). Let B be a set of linear indices such that the vectors a_i, $i \in B$ are a basis of $\mathbb{R}^n$ and suppose that B is not a roof. We have to guarantee that B is not a roof of the perturbed linear program.

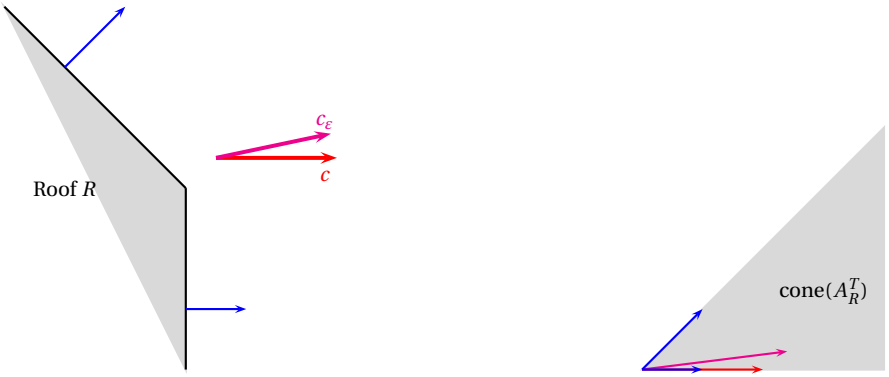

Fig. 3.7: An example of perturbation.

Let $A_B \in \mathbb{R}^{n\times n}$ be the sub-matrix of A that is defined by the rows of A indexed by B. Since B is not a roof, the vector ${A_B}^{-T} c$ has a strictly negative component. Suppose that this component is the i-th component $(A_B^{-T} c)(i) < 0$. By choosing $\epsilon > 0$ sufficiently small, we guarantee that

$$(A_B^{-T} c_\epsilon)(i) = \left(A_B^{-T} c + A_B^{-T} A_R^T \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^n \end{pmatrix} \right)(i) < 0. \tag{3.28}$$

Thus condition 2) is satisfied by choosing $\epsilon > 0$ sufficiently small.

For condition 3) we have to work only a little harder, and in fact, this is why we add the vector $(\epsilon, \ldots, \epsilon^n)$ instead of $(\epsilon, \ldots, \epsilon)$. Let B be a roof of (3.25) and let $A_B \in \mathbb{R}^{n\times n}$ be again the above described sub-matrix. We now argue that, if ϵ is sufficiently small, then $A_B^{-T} c_\epsilon$ does not have any component equal to zero. We are done then in showing that, for ϵ sufficiently small, the linear program (3.25) is non-degenerate. Because any roof of (3.25) will then be non-degenerate.

So let us inspect the vector

$$A_B^{-T} c_\epsilon = A_B^{-T} c + A_B^{-T} \cdot {A_R}^T \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon^n \end{pmatrix}. \tag{3.29}$$

Each component of (3.29) is a nonzero polynomial with variable ϵ. Nonzero, because A_B and A_R are non-singular matrices. It is well known, that a nonzero polynomial of degree n has at most n roots. Thus, for $\epsilon > 0$, sufficiently small, no component of (3.29) will be zero.

So the conditions 1), 2) and 3) hold for $\epsilon > 0$ sufficiently small. Thus we can modify a degenerate linear program into an equivalent non-degenerate linear program and apply the simplex algorithm to it.

Theorem 3.3. *Let*

$$\max\{c^T x\colon x \in \mathbb{R}^n,\ Ax \leqslant b\} \tag{3.30}$$

be a linear program. The simplex method terminates on the perturbed linear program (3.25)*. It either returns a roof B of* (3.30) *and* (3.25) *whose vertex x_B^* is an optimal solution of* (3.30) *or it asserts that* (3.30) *is infeasible.*

Proof. The simplex method terminates on (3.25) since this linear program is non-degenerate. If it asserts that (3.27) is infeasible, then it also follows that (3.30) is infeasible, since the perturbation only changes the objective-function vector.

If it returns a roof B of (3.25), then this is also a roof of (3.30) by condition 2). Furthermore, the vertex x_B^* is feasible for (3.30). It follows from weak duality (Theorem 3.1) that x_B^* is an optimal solution of (3.30). □

3.2.4 The lexicographic pivot rule

We now show that we do not have to physically perform the perturbation which sends c to c_ϵ, but instead describe a rule to choose the leaving index from the possible candidates for which the minimum in (3.16) is attained. Rules for entering and exiting indices are called *pivoting rules*. Recallthat for $i \in B$, the function $f_B(i)$ denotes the position of i in the ordered set B. For example, if $B = \{5,2,9\}$, then $f_B(5) = 1$ and $f_B(9) = 3$. We need to define the lexicographic order. A vector $u \in \mathbb{R}^n$ is lexicographically smaller than a vector $v \in \mathbb{R}^n$ if $u \neq v$ and the first nonzero component of $v - u$ is positive. We write $u <_{lex} v$ and if $u <_{lex} v$ of $u = v$ we write $u \leqslant_{lex} v$.

Algorithm 3.1. Simplex algorithm with lexicographic pivoting rule

Input: $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ $R \subseteq \{1,\dots,n\}$ initial roof
Output: B optimal roof or assert that LP is infeasible

$B := R$
while $\exists i \in \{1,\dots,n\}$ with $a_i x_B^* > b_i$
    compute solution y^* of
        $\Sigma_{k\in B} a_k y_k = -a_i$
    Compute $J = \{k \in B\colon y_k^* < 0\}$
    if $J = \emptyset$ assert **LP not feasible**
    else
        Choose *unique* $j \in J$ for which the vector
        $\frac{(A_B^{-T}[c|A_R{}^T])_{f_B(j)}}{-y_j^*}$ is *lexicographically minimal*
        $B := B\backslash\{j\} \cup \{i\}$

Change T to R

A close inspection reveals that this algorithm only mimics the simplex algorithm on the perturbed problem (3.25) if ε is arbitrarily close to 0. This is because the first component of the vector $(A_B^{-T}[c \mid A_R{}^T])_{f_B(j)}$ is z_j^*, where z^* is the solution of (3.11) and if one replaces c in (3.11) with c_ε, the corresponding z_j^* is simply

$$(A_B^{-T}[c \mid A_R{}^T])_{f_B(j)} \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^n \end{pmatrix}. \tag{3.31}$$

3.3 Phase I, finding an initial roof

So far, we always started with an initial roof. Where do we get it from? This is where Phase I of the simplex method is put to work. Above, we described Phase II. First we prove a little lemma.

Lemma 3.4. *If the linear program* (3.1) *is feasible and bounded, then it has an optimal roof. In particular, a feasible and bounded linear program has an optimal solution.*

Proof. We change the linear program (3.1) by adding the additional constraint $c^T x \leqslant M+1$, where $c^T x \leqslant M$ is valid for all feasible solutions. We then have a (degenerate) roof by choosing this inequality together with any subset of $n-1$ constraints whose normal-vectors together with c form a basis of $\mathbb{R}^n$. The simplex algorithm will find an optimal roof. □

We now form an auxiliary linear program. Observe that the linear program

$$\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant b\} \tag{3.32}$$

has a roof if and only if the linear program $\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant 0\}$ has a roof. The latter linear program is feasible since 0 is a feasible solution. Furthermore, 0 is an optimal solution of this linear program if and only if it has a roof. This is what we check with the simplex algorithm on the auxiliary program

$$\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant 0,\ c^T x \leqslant 1\} \tag{3.33}$$

and we start with a (possibly) degenerate roof involving the inequality $c^T x \leqslant 1$ and $n-1$ of the constraints of $Ax \leqslant 0$ whose normal-vectors, together with c form a basis of $\mathbb{R}^n$. The simplex algorithm terminates with an optimal roof. If the roof has vertex 0, then we have found a roof of the original linear program (3.32) and can start the simplex algorithm for it. If the roof of Phase I still contains the in-

equality $c^T x \leqslant 1$, then the original linear program (3.32) does not have any roof. This either means that the program is infeasible or unbounded.

3.4 The full column-rank assumption

There is one thing that we still have to deal with. The simplex algorithm is based on the assumption that the columns of A are linearly independent. We now argue that this assumption can be made without loss of generality.

Suppose that A can be written as $[A_1 \mid A_2]$ with $A_1 \in \mathbb{R}^{m\times k}$ and $A_2 \in \mathbb{R}^{m\times(n-k)}$ where the columns of A_1 are linearly independent and each column of A_2 is a linear combinations of the columns of A_1. We also write $c = \binom{c_1}{c_2}$ with $c_1 \in \mathbb{R}^k$ and $c_2 \in \mathbb{R}^{n-k}$ and consider the linear program

$$\max\{c_1^T x_1 : x_1 \in \mathbb{R}^k, A_1 x_1 \leqslant b\} \tag{3.34}$$

Since each column of A_2 is a linear combination of the columns of A_1, there exists a uniquely determined matrix $U \in \mathbb{R}^{k\times(n-k)}$ with $A_2 = A_1 \cdot U$.

Lemma 3.5. *The linear program* (3.34) *is feasible if and only if the linear program* (3.1) *is feasible.*

Proof. Suppose that $x^* = \binom{x_1^*}{x_2^*}$ is a feasible solution of (3.1), i.e., $A_1 x_1^* + A_2 x_2^* \leqslant b$. But $A_1 x_1^* + A_2 x_2^* = A_1(x_1^* + U \cdot x_2^*)$ which yields a feasible solution of (3.34). Likewise we see that any solution x_1^* of $A_1 x_1 \leqslant b_1$ can be extended to a feasible solution $\binom{x_1^*}{0}$ of (3.1). □

Lemma 3.6. *If* (3.34) *is feasible and if* $c_2^T \neq c_1^T \cdot U$*, then* (3.1) *is unbounded.*

Proof. Since (3.34) is feasible, then also (3.1) is feasible. Let x^* thus be a feasible solution of (3.1). Then $x^* + \mu\binom{-Uv}{v}$ is feasible for any $v \in \mathbb{R}^{n-k}$ and $\mu \in \mathbb{R}$. Let v satisfy $c_2^T v \neq 0$, then $c_1^T(-U)v + c_2^T v \neq 0$ which implies that (3.1) is unbounded. □

The idea is thus to re-order the columns of A such that the first k columns are linearly independent and the last $n-k$ columns are linear combinations of the first k columns. This also yields a matrix U and we now solve the linear program (3.34) with the simplex algorithm. If it is infeasible or unbounded, then so is the linear program (3.1). Otherwise the simplex algorithm finds an optimal solution x_1^* of (3.34). If $c_2^T = c_1^T \cdot U$ then this optimal solution $\binom{x_1^*}{0}$ is also an optimal solution of the linear program (3.34). This is because a feasible solution $\binom{x_1^*}{x_2^*}$ of (3.1) yields a feasible solution $x_1^* + U \cdot x_2^*$ of (3.34) with same objective value $c_1^T(x_1^* + Ux_2^*) = c_1^T x_1^* + c_2^T x_2^*$.

Exercises

1) Show that the linear program (3.2) that is defined by a roof is always feasible.
2) Let B be a roof of the linear program (3.1) and consider the system of linear equations
$$a_j^T x = -1, a_i^T x = 0,\ i \in B \setminus \{j\}$$
for an index $j \in B$. Let x^* be a feasible solution to the roof-linear program. Show that $x^* + \lambda \cdot v$ is also feasible for each $\lambda > 0$.
3) A polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ *contains a line*, if there exists a nonzero $v \in \mathbb{R}^n$ and an $x^* \in R^n$ such that for all $\lambda \in \mathbb{R}$, the point $x^* + \lambda \cdot v \in P$. Show that a nonempty polyhedron P contains a line if and only if A does not have full column-rank.
4) Prove Proposition 3.1.
5) Let B be a roof with vertex x_B^*. Show that the set of feasible points $\{x \in \mathbb{R}^n \colon a_i^T x \leqslant b(i),\ i \in B\}$ of the roof is of the form $x_B^* + \text{cone}\{r_1, \ldots, r_n\}$ for some suitable vectors $r_i \in \mathbb{R}^n$, $i = 1, \ldots, n$.
6) A cone $C \subseteq \mathbb{R}^n$ is *pointed* if it does not contain a line: There are no vectors $x \in C$, $v \in \mathbb{R}^n$ such that $x + \lambda v \in C$ for all $\lambda \in \mathbb{R}$.
Prove the following variant of Carathéodory's theorem. Given some set $X \subseteq \mathbb{R}^n$, $|X| > n$ such that cone(X) is pointed. For any $x \in \text{cone}(X)$, there exist at least two different subsets $X_1, X_2 \subseteq X$ with $|X_1| = |X_2| = n$ such that $x \in \text{cone}(X_1) \cap \text{cone}(X_2)$.
7) Consider the problem
$$\begin{aligned}
\max \quad & z && && (3.35)\\
\text{s.t.} \quad & x + 2y && \leqslant -3 && (3.36)\\
& -2x - 3y && \leqslant 5 && (3.37)\\
& -2x - y + 2z && \leqslant -1 && (3.38)\\
& 3x + y && \leqslant 2 && (3.39)\\
& x && \leqslant 0 && (3.40)\\
& y && \leqslant 0 && (3.41)\\
& z && \leqslant 0. && (3.42)
\end{aligned}$$
Assume that we perform the simplex method, and at some point have the roof given by the rows (3.36), (3.41) and (3.42). Figure 3.8 shows the situation in the 2-dimensional subspace given by the hyperplane $z = 0$.
Show that the simplex algorithm might not terminate, by giving a cycling sequence of roofs that might be selected by the simplex method. Explain why your sequence is valid (it is sufficient to give drawings here, you do not need to compute the roof vertices explicitly).
Hint: Never let (3.42) *leave the roof. Then it is sufficient to consider the subspace as in the illustration.*

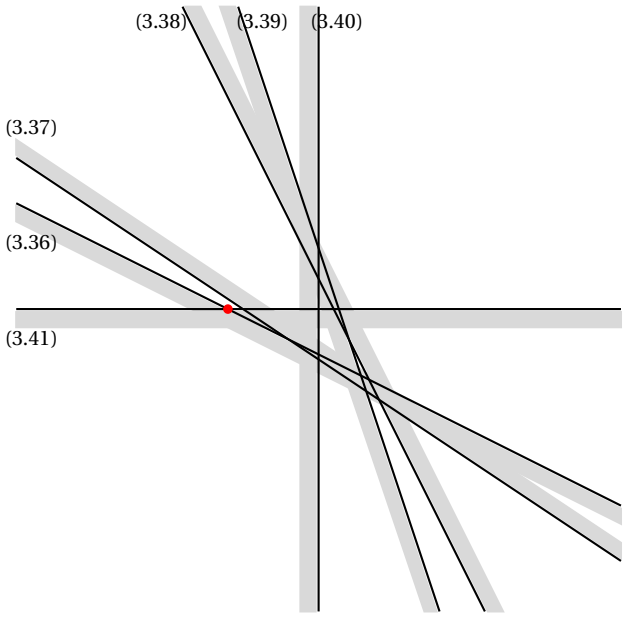

Fig. 3.8: The halfspaces defined by system (3.35) in the subspace $\{(x, y, 0) : x, y \in \mathbb{R}\}$.

Chapter 4
Strong Duality

Via the termination argument by perturbation, we we can now prove the duality theorem. We are given a linear program

$$\max\{c^T x\colon x \in \mathbb{R}^n,\, Ax \leqslant b\}, \tag{4.1}$$

called the *primal* and its *dual*

$$\min\{b^T y\colon y \in \mathbb{R}^m,\, A^T y = c,\, y \geqslant 0\}. \tag{4.2}$$

We again formulate the theorem of weak duality in this setting.

Theorem 4.1 (Weak duality). *If x^* and y^* are primal and dual feasible respectively, then $c^T x^* \leqslant b^T y^*$.*

Proof. We have $c^T x^* = {y^*}^T A x^* \leqslant {y^*}^T b$. □

The strong duality theorem tells us that if there exist feasible primal and dual solutions, then there exist feasible primal and dual solutions which have the same objective value. We can prove it with the simplex algorithm.

Theorem 4.2. *If the primal linear program is feasible and bounded, then so is the dual linear program. Furthermore in this case, both linear programs have an optimal solution and the optimal values coincide.*

Proof. Suppose first that A has full column rank. The simplex method finds a roof B of (4.1) with x^*_B being an optimal feasible solution. At the same time, the roof is a conic combination of the normal-vectors in the roof. This means that there exists a $y^* \in \mathbb{R}^m_{\geqslant 0}$ with $y^*(i) = 0$ for all $i \notin B$ such that ${y^*}^T A = c$. But $b^T y^* = \sum_{i\in B} b(i) y^*(i) = \sum_{i\in B} a_i^T x^*_B y^*(i) = (\sum_{i\in B} y^*(i) a_i)^T x^*_B = c^T x^*_B$. Thus y^* is an optimal solution of the dual and the objective function values coincide.

Suppose now that A does not have full column rank and we can write $A = [A_1 \mid A_2]$ where A_1 has full column rank and $A_2 = A_1 \cdot U$ with some matrix U as in Section 3.4. Again, as in Section 3.4 we define the linear program

$$\max\{c_1^T x_1 \colon x_1 \in \mathbb{R}^k,\ A_1 x_1 \leqslant b\}. \tag{4.3}$$

This linear program has an optimal solution x_1^* and $c_2^T = c_1^T \cdot U$. By what we have proved above, there exists a $y^* \in \mathbb{R}^m_{\geqslant 0}$ such that $b^T y^* = c_1^T x_1^*$ and ${y^*}^T A_1 = c_1^T$ and since $c_2^T = c_1^T \cdot U$ and $A_2 = A_1 \cdot U$ we have ${y^*}^T [A_1 \mid A_2] = [c_1^T \mid c_2^T]$. □

We can formulate dual linear programs also if the linear program is not in inequaliy standard form. The procedure above can be described as follows. We transform a linear program into a linear program in inequality standard form and construct its dual linear program. This dual is then transformed into an equivalent linear program again which is conveniently described.

Let us perform such operations on the dual linear program

$$\min\{b^T y \colon y \in \mathbb{R}^m,\ A^T y = c,\ y \geqslant 0\}$$

of the primal $\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b\}$. We transform it into inequality standard form

$$\begin{aligned} \max -b^T y & \\ A^T y &\leqslant c \\ -A^T y &\leqslant -c \\ -Iy &\leqslant 0. \end{aligned}$$

The dual linear program of this is

$$\begin{aligned} \min c^T x_1 - c^T x_2 & \\ Ax_1 - Ax_2 - x_3 &= -b \\ x_1, x_2, x_3 &\geqslant 0 \end{aligned}$$

This is equivalent to

$$\begin{aligned} \max c^T (x_2 - x_1) & \\ A(x_2 - x_1) + x_3 &= b \\ x_1, x_2, x_3 &\geqslant 0 \end{aligned}$$

which is equivalent to the primal linear program

$$\begin{aligned} \max c^T x & \\ Ax &\leqslant b. \end{aligned}$$

Loosely formulated one could say that "The dual of the dual is the primal". But this, of course, is not to be understood as a mathematical statement. In any case we can state the following corollary.

Corollary 4.1. *If the dual linear program has an optimal solution, then so does the primal linear program and the objective values coincide.*

We present another example of duality that we will need later on. Consider a linear program

$$\begin{array}{rcl} \max\ c^T x & & \\ Bx & = & b \\ Cx & \leqslant & d. \end{array} \tag{4.4}$$

After re-formulation, we obtain

$$\begin{array}{rcl} \max\ c^T x & & \\ Bx & \leqslant & b \\ -Bx & \leqslant & -b \\ Cx & \leqslant & d \end{array}$$

We can form the dual of the latter problem and obtain

$$\begin{array}{rcl} \min\ b^T y_1 - b^T y_2 + d^T y_3 & & \\ B^T y_1 - B^T y_2 + C^T y_3 & = & c \\ y_1, y_2, y_3 & \geqslant & 0. \end{array}$$

But this linear program is equivalent to the linear program

$$\begin{array}{rcl} \min\ b^T y_1 + d^T y_2 & & \\ B^T y_1 + C^T y_2 & = & c \\ y_2 & \geqslant & 0. \end{array} \tag{4.5}$$

This justifies to say that (4.5) is the dual of (4.4).

4.1 Zero sum games

Consider the following two-player game defined by a matrix $A \in \mathbb{R}^{m\times n}$. The *row-player* chooses a row $i \in \{1,\dots,m\}$ and the column-player chooses a column $j \in \{1,\dots,n\}$. Both players make this choice at the same time. The *payoff* for the row-player is then the matrix-element $A(i,j)$ whereas $A(i,j)$ also determines the *loss* of the column player. In other words, the column player pays $A(i,j)$ to the row-player. If this number is negative, then the row-player actually pays the absolute value of $A(i,j)$ to the column player.

Consider for example the matrix

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 3 & 2 & 4 \\ -3 & 0 & 1 \end{pmatrix}. \tag{4.6}$$

If the row-player chooses the second row and the column player chooses the second-column, then the payoff for the row-player is 2, whereas this is the loss of the column player.

The row-player is now interested in finding a strategy that maximizes his *guaranteed* payoff. For example, if he chooses row 1, then the best choice of the column player would be column 2, since the second element of the first row is the

smallest element of that row. Thus the strategy that maximizes the minimal possible payoff would be to choose row 2. In other words

$$\max_i \min_j A(i,j) = 2.$$

What would be the column-player's best hedging strategy? He wants to choose a column such that the largest element in this column is minimized. This column would be the second one. In other words

$$\min_j \max_i A(i,j) = 2.$$

Is it always the case that $\max_i \min_j A(i,j) = \min_j \max_i A(i,j)$? The next example shows that the answer is no:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.7}$$

Here we have $\max_i \min_j A(i,j) = -1$ and $\min_j \max_i A(i,j) = 1$. This can be interpreted as follows. If the column player knows beforehand, the row to be chosen by the row-player, then he would choose a column that results in a gain for him. Similarly, if the row-player knows beforehand the column to be chosen by the column-player, then he can guarantee him a gain of one.

The idea is thus not to stick with a *pure* strategy, but to play with a *random* or *mixed* strategy. If the row-player chooses each of the two rows above uniformly at random, then his expected payoff is zero. Similarly, if the column player chooses each of his two columns with probability 1/2, then his expected payoff is zero as well.

Definition 4.1 (Mixed strategy). Let $A \in \mathbb{R}^{m\times n}$ define a two-player matrix game. A mixed strategy for the row-player is a vector $x \in \mathbb{R}^m_{\geqslant 0}$ with $\sum_{i=1}^m x(i) = 1$. A mixed strategy for the column player is a vector $y \in \mathbb{R}^n_{\geqslant 0}$ with $\sum_{j=1}^n y(i) = 1$.

Such mixed strategies define a probability distribution on the row and column indices respectively. If the row-player and column-player choose a row and column according to this distribution respectively, then the *expected payoff* for the row-player is

$$E[\text{Payoff}] = x^T A y. \tag{4.8}$$

For the game defined by (4.7) and $x^T = (1/2, 1/2)$ and $y^T = (1/2, 1/2)$ the expected payoff is 0.

Lemma 4.1. *Let $A \in \mathbb{R}^{m\times n}$, then*

$$\max_{x\in X} \min_{y\in Y} x^T A y \leqslant \min_{y\in Y} \max_{x\in X} x^T A y,$$

where X and Y denote the set of mixed row and column-strategies respectively.

Proof. Let x' and y' be some fixed mixed strategies of the row and column-player respectively. Clearly

$$\min_y x'^T Ay \leqslant x'^T Ay' \leqslant \max_x x^T Ay',$$

which implies the assertion. □

The next theorem is one of the best-known results in the field of *game theory*. It states that there are mixed strategies x' and y' from above such that equality holds. It is proved with the theorem of strong duality.

Theorem 4.3 (Minimax-Theorem).

$$\max_{x\in X}\min_{y\in Y} x^T Ay = \min_{y\in Y}\max_{x\in X} x^T Ay,$$

where X and Y denote the set of mixed row and column-strategies respectively.

Proof. Let us inspect the value $\max_{x\in X}\min_{y\in Y} x^T Ay$. This can be understood as to maximize the function

$$f(x) = \min\{(x^T A)\cdot y\colon \sum_{j=1}^{n} y_j = 1,\ y \geqslant 0\}.$$

Thus the value $f(x)$ is the optimal solution of a bounded and feasible linear program. The dual of this linear program (for fixed x) has only one variable x_0 and reads

$$\max\{x_0\colon x_0 \in \mathbb{R},\ \mathbb{1}x_0 \leqslant A^T x\}.$$

But this shows that the maximum value of $f(x)$, where x ranges over all mixed row-strategies is the linear program

$$\begin{aligned} \max x_0 & \\ \mathbb{1}x_0 - A^T x &\leqslant 0 \\ \textstyle\sum_{i=1}^{m} x_i &= 1 \\ x &\geqslant 0. \end{aligned} \tag{4.9}$$

Let us now inspect the value $\min_{y\in Y}\max_{x\in X} x^T Ay$. Again, by applying duality this can be computed with the linear program

$$\begin{aligned} \min y_0 & \\ \mathbb{1}y_0 - Ay &\geqslant 0 \\ \textstyle\sum_{j=1}^{n} y_j &= 1 \\ y &\geqslant 0. \end{aligned} \tag{4.10}$$

It follows from the duality of (4.5) and (4.4) that the linear programs (4.9) and (4.10) are duals of each other. This proves the Minimax-Theorem. □

4.2 The classical simplex algorithm

I have chosen to explain the simplex algorithm as a method to obtain primal feasibility of the linear program

$$\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant b\} \tag{4.11}$$

via improving the roofs. By duality, our description can be re-interpreted as solving the linear program

$$\min\{b^T y\colon\ y \in \mathbb{R}^m,\ A^T y = c,\ y \geqslant 0\} \tag{4.12}$$

where $A^T \in \mathbb{R}^{n\times m}$ has now full row-rank. Recall that a roof is set of row-indices B of A such that the corresponding rows are a basis of $\mathbb{R}^n$ and c is a conic combination of these rows. The translation of a roof to the classical setting (4.12) is a set of n linearly independent columns such that c is a conic combination of these columns. The corresponding *basic feasible solution* is the vector x^*, where $x^*_B = A_B^{T^{-1}} c$ and $x^*_{\overline{B}} = 0$. The simplex method can now be interpreted as a method that improves the basic-feasible solution by letting a new index enter the basis while another index leaves the basis. More on this classical simplex versus the roof-interpretation can be found in exercise 4

4.3 A proof of the duality theorem via Farkas' lemma

Remember Farkas' lemma (Theorem 2.9) which states that $Ax = b, x \geqslant 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geqslant 0$ one also has $\lambda^T b \geqslant 0$. In fact the duality theorem follows from this. First, we derive another variant of Farkas' lemma.

Theorem 4.4 (Second variant of Farkas' lemma). *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system $Ax \leqslant b$ has a solution if and only if for all $\lambda \geqslant 0$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.*

Proof. Necessity is clear: If x^* is a feasible solution, $\lambda \geqslant 0$ and $\lambda^T A = 0$, then $\lambda^T A x^* \leqslant \lambda^T b$ implies $0 \leqslant \lambda^T b$.

On the other hand, $Ax \leqslant b$ has a solution if and only if

$$Ax^+ - Ax^- + z = b,\ x^+, x^-, z \geqslant 0 \tag{4.13}$$

has a solution. So, if $Ax \leqslant b$ does not have a solution, then also (4.13) does not have a solution. By Farkas' lemma, there exists a $\lambda \in \mathbb{R}^m$ with $\lambda^T [A \mid -A \mid I_m] \geqslant 0$ and $\lambda^T b < 0$. For this λ one also has $\lambda^T A = 0$ and $\lambda \geqslant 0$. □

We are now ready to prove the theorem of strong duality via the second variant of Farkas' lemma.

Proof (of strong duality via Farkas' lemma). Let δ be the objective function value of an optimal solution of the dual $\max\{b^T y\colon\ y \in \mathbb{R}^m,\ A^T y \leqslant c\}$. For all $\varepsilon > 0$, the system $A^T y \leqslant c, -b^T y \leqslant -\delta - \varepsilon$ does not have a solution. By the second variant of Farkas' lemma, there exists a $\lambda \geqslant 0$ with $\lambda^T \binom{-b^T}{A^T} = 0$ and $\lambda^T \binom{-\delta-\varepsilon}{c} < 0$. Write λ as $\lambda = \binom{\lambda_1}{\lambda'}$ with $\lambda' \in \mathbb{R}^n$. If λ_1 were zero, we could apply the second variant of Farkas' lemma to the system $A^T y \leqslant c$ and λ', since we know that $A^T y \leqslant c$ has a solution. Therefore, we can conclude $\lambda_1 > 0$. Furthermore, by scaling, we can assume $\lambda_1 = 1$. One has $\lambda'^T A^T = b^T$ and $\lambda'^T c < \delta + \varepsilon$. The first equation implies that λ' is a feasible solution of the primal (recall $\lambda' \geqslant 0$). The second equation shows that the objective function value of λ' is less than $\delta + \varepsilon$. This means that the optimum value of the primal linear program is also δ, since the primal has an optimal solution and ε can be chosen arbitrarily small. □

Exercises

1. Formulate the dual linear program of

$$\begin{aligned} \max 2x_1 + 3x_2 - 7x_3 & \\ x_1 + 3x_2 + 2x_3 &= 4 \\ x_1 + x_2 &\leqslant 8 \\ x_1 - x_3 &\geqslant -15 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

2. Consider the following linear program

$$\begin{aligned} \max x_1 + x_2 & \\ 2x_1 + x_2 &\leqslant 6 \\ x_1 + 2x_2 &\leqslant 8 \\ 3x_1 + 4x_2 &\leqslant 22 \\ x_1 + 5x_2 &\leqslant 23 \end{aligned}$$

   Show that $(4/3, 10/3)$ is an optimal solution by providing a suitable feasible dual solution.
3. Show that for $A \in \mathbb{R}^{m \times n}$, one has

$$\max_i \min_j A(i,j) \leqslant \min_j \max_i A(i,j).$$

4. In the lecture you have seen the simplex algorithm for linear programs of the form

$$\max\{c^T x\ :\ Ax \leqslant b\}.$$

   We will now derive a simplex algorithm for linear programs of the form

$$\min\{c^T x \,:\, Ax = b,\ x \geqslant 0\} \tag{4.14}$$

with $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Throughout the exercise we assume that (4.14) is feasible and bounded, and that A has full row rank.
For $i \in \{1,\dots,n\}$ we define A_i as the i-th column of A. Moreover, for some subset $B \subseteq \{1,\dots,n\}$, A_B is the matrix A restricted to the columns corresponding to elements of B.
A subset $B \subseteq \{1,\dots,n\}$ with $|B| = m$ such that A_B has full rank is called a *basis*. The vector $x \in \mathbb{R}^n$ defined as $x_i := 0$ for all $i \notin B$ and $x_B := A_B^{-1} b$ is called the *basic solution* associated to B. Note that x is a feasible solution to (4.14) if and only if $x \geqslant 0$.
Given a basis B and let $j \in \{1,\dots,n\}$, $j \notin B$. The vector $d \in \mathbb{R}^n$ defined as $d_j = 1$, $d_i = 0$ for all $i \notin B$ and $d_B := -A_B^{-1} A_j$ is called the *j-th basic direction*.
Assume that the solution x associated to B is feasible. Moreover assume that $x_B > 0$.

a. Show that there is a $\theta > 0$ such that $x + \theta d$ is a feasible solution. Give a formula to compute the largest θ such that $x + \theta d$ is feasible.
b. Let θ^* be maximal. Show that there is a basis B' such that $x + \theta^* d$ is the basic solution associated to B'.
c. Let $x' = x + \theta d$. Show that the objective value of x' changes by $\theta\left(c_j - c_B^T A_B^{-1} A_j\right)$.
d. Consider a basis B with basic feasible solution x. Show that if $c - c_B^T A_B^{-1} A \geqslant 0$, then x is an optimal solution to (4.14).

This suggests the following algorithm: Start with some basis B whose associated basic solution is feasible. Compute $\bar{c} := c - c_B^T A_B^{-1} A$. If $\bar{c} \geqslant 0$, we have an optimal solution (see 4d). Otherwise, let j be such that $\bar{c}_j < 0$. Part 4b and 4c show that if we change the basis, we find a feasible solution with an improved objective value. We repeat these steps until the vector $\bar{c}$ is nonnegative.
This is the way the simplex algorithm usually is introduced in the literature. This algorithm is exactly the same as the one you learned in the lecture. To get an intuition why this is true, show the following:

a. Given a basis B, show that its associated basic solution is feasible if and only if B is a *roof* of the LP dual to (4.14).
b. Consider a basis B and its associated feasible basic solution x. As seen before, B is also a roof in the dual LP. Let y be the vertex of that roof. Show that for any $j \in \{1,\dots,n\}$ we have $\bar{c}_j < 0$ if and only if $A_j^T y > c_j$.

Chapter 5
Integer Programming

An *integer program* is a problem of the form

$$\begin{aligned} &\max c^T x \\ &Ax \leqslant b \\ &x \in \mathbb{Z}^n, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

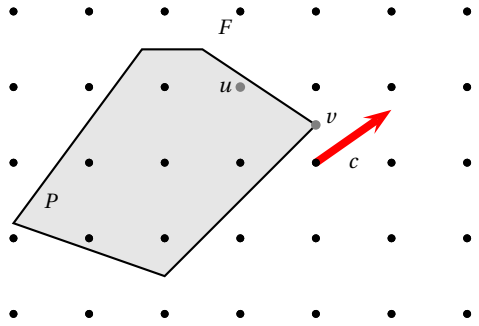

Fig. 5.1: This picture illustrates a polyhedron P an objective function vector c and optimal points u, v of the integer program and the relaxation respectively.

The difference to linear programming is the *integrality constraint* $x \in \mathbb{Z}^n$. This powerful constraint allows to model discrete choices but, at the same time, makes an integer program much more difficult to solve than a linear program. In fact one can show that integer programming is NP-hard, which means that it is *in theory* computationally intractable. However, integer programming has nowadays become an important tool to solve difficult industrial optimization problems efficiently. In this chapter, we characterize some integer programs which are easy to

solve, since the *linear programming relaxation* $\max\{c^T x\colon Ax \leqslant b\}$ yields already an optimal integer solution. The following observation is crucial.

Theorem 5.1. *Suppose that x^* is an integral optimum solution of the linear programming relaxation $\max\{c^T x\colon Ax \leqslant b\}$ is integral, i.e., $x^* \in \mathbb{Z}^n$, then x^* is also an optimal solution to the integer programming problem $\max\{c^T x\colon Ax \leqslant b, x \in \mathbb{Z}^n\}$*

Before we present an example for the power of integer programming we recall the definition of an undirected graph.

Definition 5.1 (Undirected graph, matching). An *undirected graph* is a tuple $G = (V, E)$ where V is a finite set, called the *vertices* and $E \subseteq \binom{V}{2}$ is the set of *edges* of G. A *matching* of G is a subset $M \subseteq E$ such that for all $e_1 \neq e_2 \in M$ one has $e_1 \cap e_2 = \emptyset$.

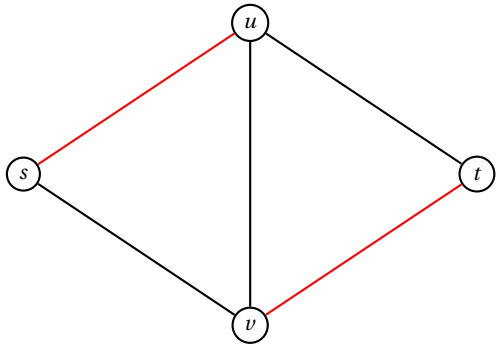

Fig. 5.2: A graph with 4 nodes $V = \{s, u, v, t\}$ and 5 edges $E = \{\{s, u\}, \{s, v\}, \{u, v\}, \{u, t\}, \{v, t\}\}$. The red edges are a matching of the graph

We are interested in the solution of the following problem, which is called *maximum weight matching* problem. Given a graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$, compute a matching with maximum weight $w(M) = \sum_{e \in M} w(e)$.

For a vertex $v \in V$, the set $\delta(v) = \{e \in E\colon v \in e\}$ denotes the *incident* edges to v. The maximum weight matching problem can now be modeled as an integer program as follows.

$$\begin{gathered} \max \textstyle\sum_{e \in E} w(e)x(e) \\ v \in V : \textstyle\sum_{e \in \delta(v)} x(e) \leqslant 1 \\ e \in E\colon x(e) \geqslant 0 \\ x \in \mathbb{Z}^{|E|}. \end{gathered} \tag{5.1}$$

Clearly, if an integer vector $x \in \mathbb{Z}^n$ satisfies the constraints above, then this vector is the *incidence vector* of a matching of G. In other words, the integral solutions to the constraints above are the vectors $\{\chi^M\colon M \text{ matching of } G\}$, where $\chi^M(e) = 1$ if $e \in M$ and $\chi^M(e) = 0$ otherwise.

5.1 Integral Polyhedra

In this section we derive sufficient conditions on an integer program to be solved easily by an algorithm for linear programming. A central notion is the one of an integral polyhedron.

Definition 5.2 (Valid inequality, face, vertex). Let $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ be a polyhedron. An inequality $c^T x \leqslant \beta$ is *valid* for P if $c^T x^* \leqslant \beta$ for all $x^* \in P$. A *face* of P is a set of the form $P \cap \{x \in \mathbb{R}^n \colon c^T x = \beta\}$ for a valid inequality $c^T x \leqslant \beta$ of P. If a face consist of one point, then it is called a *vertex* of P.

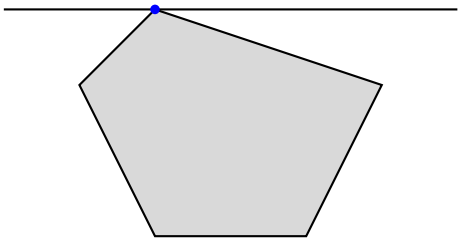

Fig. 5.3: A polyhedron with a valid inequality defining a vertex.

Definition 5.3. A rational polyhedron is called *integral* if each nonempty face of P contains an integer vector.

Lemma 5.1. *Let $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ be an integral polyhedron with $A \in \mathbb{R}^{m \times n}$ full-column rank. If the linear program*

$$\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b\} \tag{5.2}$$

is feasible and bounded, then the simplex method computes an optimal integral solution to the linear program.

Proof. Recall that the simplex method finds an optimal roof $B \subseteq \{1, \ldots, m\}$ of (5.2) and the vertex of the roof x_B^* is an optimal solution to the linear program (5.2). We have to show that x_B^* is integral. This will follow from the fact that $\{x_B^*\}$ is a face of P.

Proposition 3.1 implies that x_B^* is the unique optimum solution of the linear program $\max\{\widetilde{c}^T x \colon x \in R^n,\ a_i^T x \leqslant b(i),\ i \in B\}$, where $\widetilde{c} = \sum_{i \in B} a_i$. Consequently x_B^* is the unique solution of the linear program

$$\max\{\widetilde{c}^T x \colon x \in P\}$$

which implies that $\{x_B^*\}$ is a face defined by the valid inequality $\widetilde{c}^T x \leqslant \widetilde{c}^T x_B^*$. □

Lemma 5.2. *Let $A \in \mathbb{Z}^{n\times n}$ be an integral and invertible matrix. One has $A^{-1}b \in \mathbb{Z}^n$ for each $b \in \mathbb{Z}^n$ if and only if* $\det(A) = \pm 1$.

Proof. Recall Cramer's rule which says $A^{-1} = 1/\det(A)\widetilde{A}$, where $\widetilde{A}$ is the adjoint matrix of A. Clearly $\widetilde{A}$ is integral. If $\det(A) = \pm 1$, then A^{-1} is an integer matrix.

If $A^{-1}b$ is integral for each $b \in \mathbb{Z}^n$, then A^{-1} is an integer matrix. We have $1 = \det(A\cdot A^{-1}) = \det(A)\cdot\det(A^{-1})$. Since A and A^{-1} are integral it follows that $\det(A)$ and $\det(A^{-1})$ are integers. The only divisors of one in the integers are ± 1. □

An integral matrix $A \in \{0,\pm 1\}^{m\times n}$ is called *totally unimodular* if each of its square sub-matrices has determinant $0,\pm 1$.

Theorem 5.2 (Hoffman-Kruskal Theorem). *Let $A \in \mathbb{Z}^{m\times n}$ be an integral matrix. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b,\, x \geqslant 0\}$ is integral for each integral $b \in \mathbb{Z}^m$ if and only if A is totally unimodular.*

Proof. Let $A \in \mathbb{Z}^{m\times n}$ be totally unimodular and $b \in \mathbb{Z}^m$. Let x^* be vertex of P and suppose that this vertex is defined by the valid inequality $c^T x \leqslant \delta$. Notice that the matrix $\binom{A}{-I}$ has full column-rank. If one applies the simplex algorithm to the problem

$$\max\{c^T x\colon x \in \mathbb{R}^n, \tbinom{A}{-I}x \leqslant \tbinom{b}{0}\},$$

it finds an optimal roof $B \subseteq \{1,\dots,m+n\}$ with $x^*_B = x^*$. If A_B denotes the matrix whose rows are those rows of $\binom{A}{-I}$ indexed by B and if b_B denotes the vector whose components are those of $\binom{b}{0}$ indexed by B, then $x^* = A_B^{-1}b_B$. We are done, once we conclude that $\det(A_B) = \pm 1$, since then A_B^{-1} is an integer matrix and since b_B is an integer vector $x^* = A_B^{-1}b$ is integral as well. We can permute the columns of A_B in such a way that one obtains a matrix of the form

$$\begin{pmatrix} \overline{A} & \widetilde{A} \\ 0 & -I_k \end{pmatrix}$$

where $\overline{A}$ is a $(n-k)\times(n-k)$ sub-matrix of A and I_k is the $k\times k$ identity matrix. Here $k = |B \cap \{m+1,\dots,m+n\}$. Clearly $0 \neq \det(A_B) = \pm\det(\overline{A}) = \pm 1$.

For the converse, suppose that A is not totally unimodular. Then there exists an index set $B \subseteq \{1,\dots,m+n\}$ with $|B| = n$ such that the matrix A_B defined as above satisfies $|\det(A_B)| \geqslant 2$. By Lemma 5.2 there exists choices for the components of b_B making $A_B^{-1}b_B$ non-integral. In fact, if we split B into components $L \subseteq B$ corresponding to lines of A and C corresponding to lines of $-I$ we can choose those components of b_B corresponding to L being equal to zero. Now let v be the vector with $v(i) = 1$ for all $i \in L$ and $v(i) = 0$ for all $i \in \{1,\dots,n\}\backslash L$. By choosing $\gamma \in \mathbb{N}$ large enough the point $x^*_B = A_B^{-1}(b_B + \gamma A_B v)$ is non-integral and positive. The set B is a non-degenerate roof of the linear program

$$\max\{\overline{c}^T x\colon x \in \mathbb{R}^n, \tbinom{A}{-I}x \leqslant \tbinom{b}{0}\},$$

where $\overline{c} = \sum_{i\in B} a_i$ and a_i denotes the i-th row of $\binom{A}{-I}$. If we define for $j \in \{1,\dots,m\}\setminus B$, $b(j) = \lceil a_j^T x^*_B\rceil$, then x^*_B is feasible and thus a vertex of P that is non-integral. □

5.2 Applications of total unimodularity

5.2.1 Bipartite matching

A graph is *bipartite*, if V has a partition into sets A and B such that each edge uv satisfies $u \in A$ and $v \in B$. Recall that $\delta(v)$ is the set of edges incident to the vertex $v \in V$, that is $\delta(v) = \{e \in E \mid v \in e\}$.

The *node-edge* incidence matrix of a graph $G = (V, E)$ is the $A \in \{0,1\}^{|V| \times |E|}$ with

$$A(v, e) = \begin{cases} 1, & \text{if } v \in e, \\ 0 & \text{otherwise.} \end{cases}$$

The integer program (5.1) can thus be formulated as

$$\max\{w^T x \colon Ax \leqslant 1,\, x \geqslant 0,\, x \in \mathbb{Z}^E\}. \tag{5.3}$$

The next lemma implies that the simplex algorithm can be used to compute a maximum-weight matching of a bipartite graph.

Lemma 5.3. *If G is bipartite, the node-edge incidence matrix of G is totally unimodular.*

Proof (Proof of Lemma 5.3). Let $G = (V, E)$ be a bipartite graph with bi-partition $V = V_1 \cup V_2$.

Let A' be a $k \times k$ sub-matrix of A. We are interested in the determinant of A. Clearly, we can assume that A does not contain a column which contains no 1 or only one 1, since we simply consider the sub-matrix A'' of A', which emerges from developing the determinant of A' along this column. The determinant of A' would be zero or $\pm 1 \cdot \det(A'')$.

Thus we can assume that each column contains exactly two ones. Now we can order the rows of A' such that the first rows correspond to vertices of V_1 and then follow the rows corresponding to vertices in V_2. This re-ordering only affects the sign of the determinant. By summing up the rows of A' in V_1 we obtain exactly the same row-vector as we get by summing up the rows of A' corresponding to V_2. This shows that $\det(A') = 0$. □

5.2.2 Bipartite vertex cover

A *vertex cover* of a graph $G = (V, E)$ is a subset $C \subseteq V$ of the nodes such $e \cap C \neq \emptyset$ for each $e \in E$. Let us formulate an integer program for the *minimum-weight vertex-cover* problem. Here, one is given a graph $G = (V, E)$ and weights $w \in \mathbb{R}^V$. The goal is to find a vertex cover C with minimum weight $w(C) = \sum_{v \in V} w(v)$.

$$\begin{array}{c} \min \sum_{v \in V} w(v)x(v) \\ uv \in E : x(u) + x(v) \geqslant 1 \\ v \in V : x(v) \geqslant 0 \\ x \in \mathbb{Z}^V. \end{array} \tag{5.4}$$

Clearly, this is the integer program

$$\min\{w^T x \colon A^T x \geqslant 1,\, x \geqslant 0,\, x \in \mathbb{Z}^V\}, \tag{5.5}$$

where A is the node-edge incidence matrix of G. A matrix A is totally unimodular if and only of A^T is totally unimodular. Thus the simplex algorithm can be used to compute a minimum-weight vertex-cover of a bipartite graph. Furthermore we have the following theorem.

Theorem 5.3 (König's theorem). *In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.*

Proof. Let A be the node-edge incidence-matrix of the bipartite graph $G = (V, E)$. The linear programs $\max\{1^T x \colon Ax \leqslant 1,\, x \geqslant 0\}$ and $\min\{1^T x \colon Ax \geqslant 1,\, x \geqslant 0\}$ are duals of each other. Since A is totally unimodular, the value of the linear programs are the cardinality of a maximum matching and minimum vertex-cover respectively. Thus the theorem follows from strong duality. □

5.2.3 Flows

Let $G = (V, A)$ be a directed graph. The *node-edge incidence matrix of a directed graph* is a matrix $A \in \{0, \pm 1\}^{V \times E}$ with

$$A(v, a) = \begin{cases} 1 & \text{if } v \text{ is the starting-node of } a, \\ -1 & \text{if } v \text{ is the end-node of } a, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

A *feasible flow* f of G with capacities u and in-out-flow b is then a solution $f \in \mathbb{R}^A$ to the system $Af = b, 0 \leqslant f \leqslant u$.

Lemma 5.4. *The node-edge incidence matrix A of a directed graph is totally unimodular.*

Proof. Let A' be a $k \times k$ sub-matrix of A. Again, we can assume that in each column we have exactly one 1 and one -1. Otherwise, we develop the determinant along a column which does not have this property. But then, the A' is singular, since adding up all rows of A' yields the 0-vector.

A consequence is that, if the b-vector and the capacities u are integral and an optimal flow exists, then there exists an integer optimal flow.

5.2.4 Doubly stochastic matrices

A matrix $A \in \mathbb{R}^{n\times n}$ is *doubly stochastic* if it satisfies the following linear constraints

$$\begin{array}{rcl} \sum_{i=1}^{n} A(i,j) & = & 1,\ \forall j = 1,\ldots,n \\ \sum_{j=1}^{n} A(i,j) & = & 1,\ \forall i = 1,\ldots,n \\ A(i,j) & \geqslant & 0,\ \forall 1 \leqslant i,j \leqslant n. \end{array} \tag{5.7}$$

A permutation matrix is a matrix which contains exactly one 1 per row and column, where the other entries are all 0.

Theorem 5.4. *A matrix $A \in \mathbb{R}^{n\times n}$ is doubly stochastic if and only if A is a convex combination of permutation matrices.*

Proof. Since a permutation matrix satisfies the constraints (5.7), then so does a convex combination of these constraints.

On the other hand it is enough to show that each vertex of the polytope defined by the system (5.7) is integral and thus a permutation matrix. However, the matrix defining the system (5.7) is the node-edge incidence matrix of the complete bipartite graph having $2n$ vertices. Since such a matrix is totally unimodular, the theorem follows.

5.3 The matching polytope

We now come to a deeper theorem concerning the convex hull of matchings. We mentioned several times in the course that the maximum weight matching problem can be solved in polynomial time. We are now going to show a theorem of Edmonds [2] which provides a complete description of the matching polytope and present the proof by Lovász [11].

Before we proceed let us inspect the symmetric difference $M_1 \Delta M_2$ of two matchings of a graph G. If a vertex is adjacent to two edges of $M_1 \cup M_2$, then one of the two edges belongs to M_1 and one belongs to M_2. Also, a vertex can never be adjacent to three edges in $M_1 \cup M_2$. Edges which are both in M_1 and M_2 do not appear in the symmetric difference. We therefore have the following lemma.

Lemma 5.5. *The symmetric difference $M_1 \Delta M_2$ of two matchings decomposes into node-disjoint paths and cycles, where the edges on these paths and cycles alternate between M_1 and M_2.*

The *Matching polytope* $P(G)$ of an undirected graph $G = (V,E)$ is the convex hull of incidence vectors χ^M of matchings M of G.

The incidence vectors of matchings are exactly the 0/1-vectors that satisfy the following system of equations.

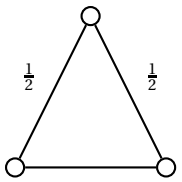

Fig. 5.4: Triangle

$$\begin{aligned} \textstyle\sum_{e\in\delta(v)} x(e) &\leqslant 1 \ \ \forall v \in V \\ x(e) &\geqslant 0 \ \ \forall e \in E. \end{aligned} \tag{5.8}$$

However the triangle (Figure 5.4) shows that the corresponding polytope is not integral. The objective function $\max 1^T x$ has value 1.5. However, one can show that a maximum weight matching of an undirected graph can be computed in polynomial time which is a result of Edmonds [3].

The following (Figure 5.5) is an illustration of an Edmonds inequality. Suppose that U is an odd subset of the nodes V of G and let M be a matching of G. The number of edges of M with both endpoints in U is bounded from above by $\lfloor |U|/2 \rfloor$.

Thus the following inequality is valid for the integer points of the polyhedron defined by (5.8).

$$\sum_{e\in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor, \qquad \text{for each } U \subseteq V, \quad |U| \equiv 1 \pmod 2. \tag{5.9}$$

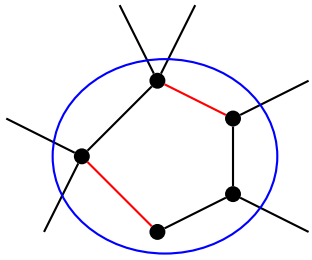

Fig. 5.5: Edmonds inequality.

The goal of this lecture is a proof of the following theorem.

Theorem 5.5 (Edmonds 65). *The matching polytope is described by the following inequalities:*

i) $x(e) \geqslant 0$ *for each* $e \in E$,
ii) $\sum_{e \in \delta(v)} x(e) \leqslant 1$ *for each* $v \in V$,
iii) $\sum_{e \in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ *for each* $U \subseteq V$

Lemma 5.6. *Let* $G = (V, E)$ *be connected and let* $w : E \longrightarrow \mathbb{R}_{>0}$ *be a weight-function. Denote the set of maximum weight matchings of* G *w.r.t.* w *by* $\mathscr{M}(w)$. *Then one of the following statements must be true:*

i) $\exists\, v \in V$ *such that* $\delta(v) \cap M \neq \emptyset$ *for each* $M \in \mathscr{M}(w)$
ii) $|M| = \lfloor |V|/2 \rfloor$ *for each* $M \in \mathscr{M}(w)$ *and* $|V|$ *is odd.*

Proof. Suppose both i) and ii) do not hold. Then there exists $M \in \mathscr{M}(w)$ leaving two exposed nodes u and v. Choose M such that the minimum distance between two exposed nodes u, v is minimized.

Now let t be on shortest path from u to v. The vertex t cannot be exposed.

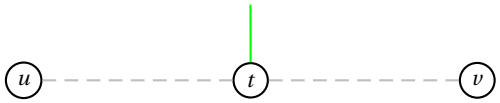

Fig. 5.6: Shortest path between u and v.

Let $M' \in \mathscr{M}(w)$ leave t exposed. Both u and v are covered by M' because the distance to u or v from t is smaller than the distance of u to v.

Consider the symmetric difference $M \triangle M'$ which decomposes into node disjoint paths and cycles. The nodes u, v and t have degree one in $M \triangle M'$. Let P be a path with endpoint t in $M \triangle M'$

Fig. 5.7: Swapping colors.

If we swap colors on P, see Figure 5.7, we obtain matchings $\widetilde{M}$ and $\widetilde{M'}$ with $w(M) + w(M') = w(\widetilde{M}) + w(\widetilde{M'})$ and thus $\widetilde{M} \in \mathscr{M}(w)$.

The node t is exposed in $\widetilde{M}$ and u or v is exposed in $\widetilde{M}$. This is a contradiction to u and v being shortest distance exposed vertices

Proof (Proof of Theorem 5.5).

Let $w^T x \leqslant \beta$ be a *facet* of $P(G)$, we need to show that this facet it is of the form

i) $x(e) \geqslant 0$ for some $e \in E$

ii) $\sum_{e\in\delta(v)} x(e) \leqslant 1$ for some $v \in V$
iii) $\sum_{e\in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ for some $U \in P_{odd}$

To do so, we use the following method: One of the inequalities i), ii), iii) is satisfied with equality by each χ^M, $M \in \mathscr{M}(w)$. This establishes the claim since the matching polytope is full-dimensional and a facet is a maximal face.

If $w(e) < 0$ for some $e \in E$, then each $M \in \mathscr{M}(w)$ satisfies $e \notin M$ and thus satisfies $x(e) \geqslant 0$ with equality.

Thus we can assume that $w \geqslant 0$.

Let $G^* = (V^*, E^*)$ be the graph induced by edges e with $w(e) > 0$. Each $M \in \mathscr{M}(w)$ contains maximum weight matching $M^* = M \cap E^*$ of G^* w.r.t. w^*.

If G^* is not *connected* , suppose that $V^* = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ and $V_1, V_2 \neq \emptyset$ and there is no edge connecting V_1 and V_2, then $w^T x \leqslant \beta$ can be written as the sum of $w_1^T x \leqslant \beta_1$ and $w_2^T x \leqslant \beta_2$, where β_i is the maximum weight of a matching in V_i w.r.t. w_i, $i = 1,2$, see Figure 5.8. This would also contradict the fact that $w^T x \leqslant \beta$ is a facet, since it would follow from the previous inequalities and thus would be a redundant inequality.

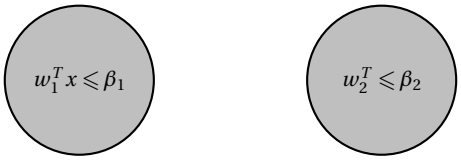

Fig. 5.8: G^* is connected.

Now we can use Lemma 5.6 for G^*.

i) $\exists v$ such that $\delta(v) \cap M = \emptyset$ for each $M \in \mathscr{M}(w)$. This means that each M in $\mathscr{M}(w)$ satisfies
$$\sum_{e\in\delta(v)} x(e) \leqslant 1 \quad \text{with equality}$$
ii) $|M \cap E^*| = \lfloor |V^*|/2 \rfloor$ for each $M \in \mathscr{M}(w)$ and $|V^*|$ is odd. This means that each M in $\mathscr{M}(w)$ satisfies
$$\sum_{e\in E(V^*)} x(e) \leqslant \lfloor |V^*|/2 \rfloor \quad \text{with equality}$$

Exercises

1. Let $M \in \mathbb{Z}^{n\times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well:
   i) M^T

ii) $(M \quad I_n)$
iii) $(M \quad -M)$
iv) $M \cdot (I_n - 2e_j e_j^T)$ for some j

I_n is the $n \times n$ identity matrix, and e_j is the vector having a 1 in the jth component, and 0 in the other components.

2. A family $\mathscr{F}$ of subsets of a finite groundset E is *laminar*, if for all $C, D \in \mathscr{F}$, one of the following holds:

$$(i)\ C \cap D = \emptyset,\ \ (ii)\ C \subseteq D,\ \ (iii)\ D \subseteq C.$$

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two laminar families of the same groundset E and consider its union $\mathscr{F}_1 \cup \mathscr{F}_2$. Define the $|\mathscr{F}_1 \cup \mathscr{F}_2| \times |E|$ adjacency matrix A as follows: For $F \in \mathscr{F}_1 \cup \mathscr{F}_2$ and $e \in E$ we have $A_{F,e} = 1$, if $e \in F$ and $A_{F,e} = 0$ otherwise. Show that A is totally unimodular.

3. Consider the following scheduling problem: Given n tasks with periods $p_1, \ldots, p_n \in \mathbb{N}$, we want to find offsets $x_i \in \mathbb{N}_0$, such that every task i can be executed periodically at times $x_i + p_i \cdot k$ for all $k \in \mathbb{N}_0$. In other words, for all pairs i, j of tasks we require $x_i + k \cdot p_i \neq x_j + l \cdot p_j$ for all $k, l \in \mathbb{N}_0$.
Formulate the problem of finding these offsets as an integer program (with zero objective function).

4. Let $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ be a polyhedron. Show that the following are equivalent for a feasible x^*:

i) x^* is a vertex of P.
ii) There exists a set $B \subseteq \{1, \ldots, m\}$ such that $|B| = n$, A_B is invertible and $A_B x^* = b_B$. Here the matrix A_B and the vector b_B consists of the rows of A indexed by B and the components of b indexed by B respectively.
iii) For every feasible $x_1, x_2 \neq x^* \in P$ one has $x^* \notin \operatorname{conv}\{x_1, x_2\}$.

5. Show the following: A polyhedron $P \subseteq \mathbb{R}^n$ with vertices is integral, if and only if each vertex is integral.
6. Consider the polyhedron $P = \{x \in \mathbb{R}^3 \colon x_1 + 2x_2 + 4x_3 \leqslant 4,\ x \geqslant 0\}$. Show that this polyhedron is integral.
7. Which of these matrices is totally unimodular? Justify your answer.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

8. Consider the complete graph G_n with 3 vertices, i.e., $G = (\{1,2,3\}, \binom{3}{2})$. Is the polyhedron of the linear programming relaxation of the vertex-cover integer program integral?

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Chapter 6
Paths, cycles and flows in graphs

Suppose you want to find a shortest path from a given starting point to a given destination. This is a common scenario in driver assistance systems (GPS) and can be modeled as one of the most basic combinatorial optimization problems, the *shortest path problem*. In this chapter, we introduce directed graphs, shortest paths and flows in networks. We focus in particular on the maximum-flow problem, which is a linear program that we solve with direct methods, versus the simplex method, and analyze the running time of these direct methods.

6.1 Growth of functions

In the analysis of algorithms, it is more appropriate to investigate the asymptotic running time of an algorithm depending on the input and not the precise running time itself. We review the O, Ω and Θ-notation.

Definition 6.1 (O, Ω, Θ-notation).
Let $T, f : \mathbb{N} \to \mathbb{N}$ be two functions

- $T(n)$ is in $O(f(n))$, if there exist positive constants $n_o \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ with $T(n) \leqslant c \cdot f(n)$ for all $n \geqslant n_0$.
- $T(n)$ is in $\Omega(f(n))$, if there exist constants $n_o \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ with $T(n) \geqslant c \cdot f(n)$ for all $n \geqslant n_0$.
- $T(n)$ is in $\Theta(f(n))$ if $T(n)$ is both in $O(f(n))$ and in $\Omega(f(n))$.

Example 6.1. The function $T(n) = 2n^2 + 3n + 1$ is in $O(n^2)$, since for all $x \geqslant 1$ one has $2n^2 + 3n + 1 \leqslant 6n^2$. Here $n_0 = 1$ and $c = 6$.

Similarly $T(n) = \Omega(n^2)$, since for each $n \geqslant 1$ one has $2n^2 + 3n + 1 \geqslant n^2$. Thus $T(n)$ is in $\Theta(n^2)$.

6.2 Graphs

Definition 6.2. A *directed graph* is a tuple $G = (V, A)$, where V is a finite set, called the *vertices* of G and $A \subseteq (V \times V)$ is the set of *arcs* of G. We denote an arc by its two defining nodes $(u, v) \in A$. The nodes u and v are called *tail* and *head* of the arc (u, v) respectively.

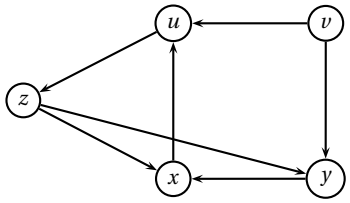


Fig. 6.1: Example of a directed graph with 5 nodes and 7 arcs.

Definition 6.3 (Walk, path, distance). A *walk* is a sequence of the form

$$P = (v_0, a_1, v_1, \ldots, v_{m-1}, a_m, v_m),$$

where $a_i = (v_{i-1}, v_i) \in A$ for $i = 1, \ldots, m$. If the nodes $v_0, \ldots, v_m$ are all different, then P is a *path*. The *length* of P is m. The *distance* of two nodes u and v is the length of a shortest path from u to v. It is denoted by $d(u, v)$.

Example 6.2. The following is a walk and a path of the graph in Figure 6.1.

$$u, (u,z), z, (z,x), x, (x,u), u, (u,z), z, (z,y), y$$
$$u, (u,z), z, (z,y), y$$

6.3 Representing graphs and computing the distance of two nodes

We represent a graph with n vertices $v_1, \ldots, v_n$ as an array $A[v_1, \ldots, v_n]$, where the entry $A[v_i]$ is a pointer to a linked list of vertices, the *neighbors of* v_i. $N(v_i) = \{u \in V : (v_i, u) \in A\}$.

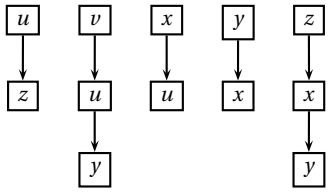


Fig. 6.2: Adjacency list representation of the graph in Figure 6.1.

6.3.1 Graphs in SAGE

In SAGE [1], directed graphs have several representations. We use in the following the dictionary of dictionaries representation. In this representation, the nodes are integers. The graph depicted in figure 6.1 could be represented in SAGE as follows.

```
G = DiGraph('u':['z'], 'v':['u','y'], 'z':['x','y'], 'x':['u'],
                          'y':['x']);
```

We next describe a very basic algorithm that computes the distances from a designated node $s \in V$ to all other nodes. The *distance* from s to v is denoted by $d(s,v)$. It is the smallest integer i such that there exists a path from s to v of length i. If there does not exist such a path, then s and v are *not connected* and we define $d(s,v) = \infty$. For $i \in \mathbb{N}_0$, $V_i \subseteq V$ denotes the set of vertices that have distance i from s. Notice that $V_0 = \{s\}$.

Lemma 6.1. *For $i = 1,\dots,n-1$, the set V_i is equal to the set of vertices $v \in V \setminus (V_0 \cup \cdots \cup V_{i-1})$ such that there exists an arc $(u,v) \in A$ with $u \in V_{i-1}$.*

Proof. Suppose that $v \notin V_0 \cup \cdots \cup V_{i-1}$ and there exists an arc $uv \in A$ with $u \in V_{i-1}$. Since $u \in V_{i-1}$, there exists a path $s, a_1, v_1, a_2, v_2, \dots, a_{i-1}, u$ of length $i-1$ from s to u. The sequence $s, a_1, v_1, a_2, v_2, \dots, a_i, u, uv, v$ is a path of length i from s to v and thus $v \in V_i$.

If, on the other hand, $v \in V_i$, then there exists a path

$$s, a_1, v_1, \dots, a_{i-1}, u, a_i, v$$

of length i from s to v. We need to show that $u \in V_{i-1}$ holds. Clearly, since there exists a path of length $i-1$ from s to u, one has $u \in V_j$ with $j \leqslant i-1$. If $j < i-1$, then there exists a path $s, a'_1, v'_1, \dots, a'_j, u$ of length j which can be extended to a path of length $j+1 < i$ from s to v

$$s, a'_1, v'_1, \dots, a'_j, u, a_i, v$$

which contradicts $v \in V_{i+1}$. □

The *breadth-first search algorithm* is an implementation of Lemma 6.1. The algorithm maintains arrays

$$D[v_1 = s, v_2, \ldots, v_n]$$
$$\pi[v_1 = s, v_2, \ldots, v_n]$$

and a queue Q that contains only s in the beginning. The array D contains at termination of the algorithm the distances from s to all other nodes and is initialized with $[0, \infty, \ldots, \infty]$. The array π contains predecessor information for shortest paths, in other words, when the algorithm terminates, $\pi[v] = u$, where uv is an arc and $D[u] + 1 = D[v]$. The array π is initialized with $[0, \ldots, 0]$.

After this initialization, the algorithm proceeds as follows.

while $Q \neq \emptyset$
    $u := head(Q)$
    for each $v \in N(u)$
        if $(D[v] = \infty)$
            $\pi[v] := u$
            $D[v] := D[u] + 1$
            $enqueue(Q, v)$
    $dequeue(Q)$

Here the function $head(Q)$ returns the next element in the queue and $dequeue(Q)$ removes the first element of Q, while $enqueue(Q, v)$ adds v to the queue as last element.

Lemma 6.2. *The breadth-first search algorithm assigns* distance labels *D correctly.*

Proof. We show the following claim by induction on $i \in \{0, \ldots, n-1\}$.

For each $i \in \{1, \ldots, n-1\}$ there exists a point in time where:

i) Q contains precisely the elements of V_i
ii) for each $v \in V_i$, $D[v] = d(s, v)$
iii) for each $v \in V_i$ one has $\pi[v]v$ is an arc and $\pi[v] \in V_{i-1}$.

Once this claim is shown, the lemma follows, because the labels $D[v]$ and $\pi[v]$ are only changed once, if at all, from ∞ to an integer or a vertex respectively.

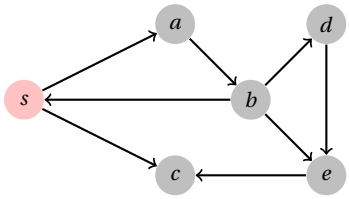
Since $V_0 = \{s\}$ and since $Q = [s]$ and $D[s] = 0$ after the initialization, the claim holds for $i = 0$. Suppose $i > 0$. By the induction hypothesis, there is a point in time, where Q contains precisely V_{i-1}. By Lemma 6.1, after the last element of V_{i-1} is dequeued Q contains precisely the elements in V_i. Also, since $D[u] = d(s, u) = i - 1$ for all $u \in V_{i-1}$ we have for each $v \in V_i$ that $D[v] = d(s, v) = i$. Also $\pi[v]v$ is an arc, by virtue of the algorithm, and $\pi[v] \in V_{i-1}$. □

Definition 6.4 (Tree). A *directed tree* is a directed graph $T = (V, A)$ with $|A| = |V| - 1$ and there exists a node $r \in T$ such that there exists a path from r to all other nodes of T.

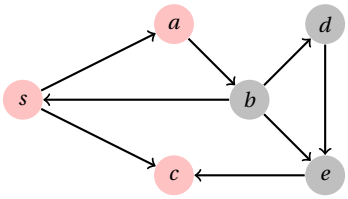
Lemma 6.3. *Consider the arrays D and π after the termination of the breadth-first-search algorithm. The graph $T = (V', A')$ with $V' = \{v \in V : D[v] < \infty\}$ and $A' = \{\pi(v)v : 1 \leqslant D[v] < \infty\}$ is a tree.*

Proof. Clearly, $|A'| = |V'| - 1$. For any $i \in \{1, \ldots, n-1\}$, by backtracking the π-labels from any $v \in V_i$, we will eventually reach s.

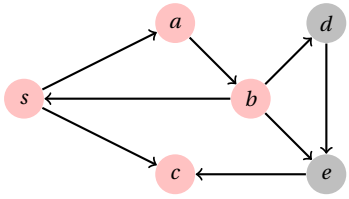
Definition 6.5. The tree T from above is the *shortest-path-tree* of the (unweighted) directed graph $G = (V, A)$.



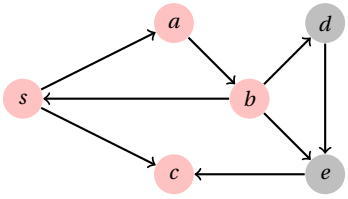
(a) The breadth-first search algorithm starts with the queue $Q = [s]$. The distance labels for $[s, a, b, c, d, e]$ are $[0, \infty, \infty, \infty, \infty, \infty]$ respectively.



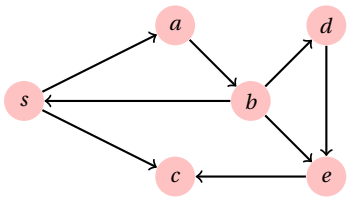
(b) After the first iteration of the **while** loop the queue is $Q = [a, c]$ and the distance labels are $[0, 1, \infty, 1, \infty, \infty]$ respectively.



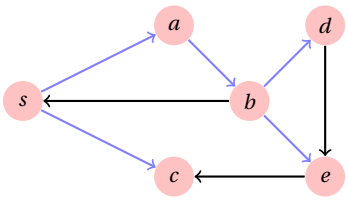
(c) After the second iteration of the **while** loop the queue is $Q = [c, b]$ and the distance labels are $[0, 1, 2, 1, \infty, \infty]$ respectively.



(d) After the third iteration of the **while** loop the queue is $Q = [b]$ and the distance labels are unchanged, since c does not have any neighbors.



(e) After the fourth iteration of the **while** loop the queue is $Q = [d, e]$ and the distance labels are $[0, 1, 2, 1, 3, 3]$ respectively.



(f) After the sixth iteration of the **while** loop the queue is empty $Q = []$ and the distance labels remain unchanged. The blue edges denote the shortest path tree.

Fig. 6.3: An example-run of breadth-first search

Theorem 6.1. *The breath-first-search algorithm runs in time* $O(|V|+|A|)$.

Proof. Each vertex is queued and dequeued at most once. These queuing operations take constant time each. Thus queuing and dequeuing costs $O(|V|)$ in total.

When a vertex u is dequeued, its neighbors are inspected and the operations in the **if** statement cost constant time each. Thus one has an additional cost of $O(|A|)$, since these constant-time operations are carried out for each arc $a \in A$.

□

6.4 Shortest Paths

Definition 6.6 (Cycle). A walk in which starting node and end-node agree is called a *cycle*.

Suppose we are given a directed graph $D = (V, A)$ and a length function $c : A \longrightarrow \mathbb{R}$. The *length* of a walk W is defined as

$$c(W) = \sum_{\substack{a \in A \\ a \in W}} c(a).$$

We now study how to determine a shortest path in the weighted directed graph G efficiently, in case of the absence of cycles of negative length.

Theorem 6.2. *Suppose that each cycle in* D *has non-negative length and suppose there exists an* $s-t$*-walk in* D*. Then there exists a path connecting* s *with* t *which has minimum length among all walks connecting* s *and* t.

Proof. If there exists an $s-t$-walk, then there exists an $s-t$-path. Since the number of arcs in a path is at most $|V|-1$, there must exist a shortest *path* P connecting s and t. We claim that $c(P) \leqslant c(W)$ for all $s-t$-walks W. Suppose that there exists an $s-t$-walk W with $c(W) < c(P)$. Then let W be such a walk with a minimum number of arcs. Clearly W contains a cycle C. Since the cycle has non-negative length, then it can be removed from W to obtain a walk whose length is at most $c(W)$ and whose number of arcs is strictly less than $|C|$. □

We use the notation $|W|, |C|, |P|$ to denote the number of arcs in a walk W a cycle C or a path P.

As a conclusion we can note here:

> If there do not exist negative cycles in D, and s and t are connected, then there exists a shortest walk traversing at most $|V|-1$ arcs.

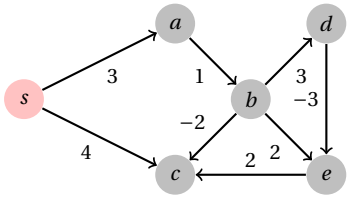
The Bellman-Ford algorithm

Let $n = |V|$. We calculate functions $f_0, f_1, \ldots, f_n : V \longrightarrow \mathbb{R} \cup \{\infty\}$ successively by the following rule.

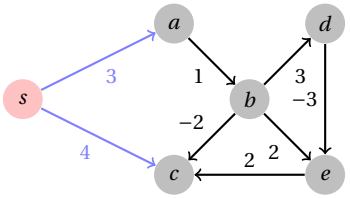
i) $f_0(s) = 0$, $f_0(v) = \infty$ for all $v \neq s$
ii) For $k < n$ if f_k has been found, compute

$$f_{k+1}(v) = \min\{f_k(v), \min_{(u,v)\in A}\{f_k(u) + c(u,v)\}\}$$

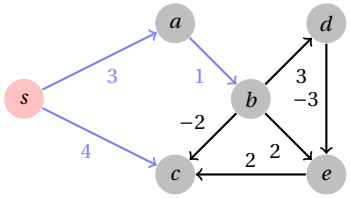
for all $v \in V$.



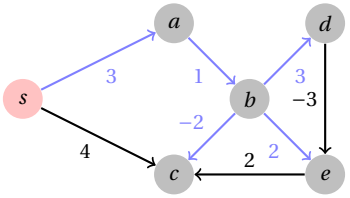
(a) The algorithm is initialized with distance labels for s, a, b, c, d, e being $[0, \infty, \infty, \infty, \infty, \infty]$ respectively



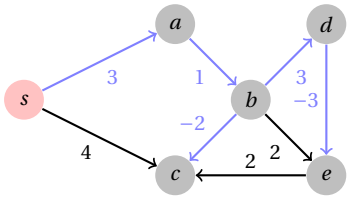
(b) After the first iteration the labels are $[0, 3, \infty, 4, \infty, \infty]$



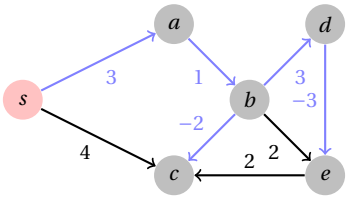
(c) After the second iteration the labels are $[0, 3, 4, 4, \infty, \infty]$



(d) After the third iteration the labels are $[0, 3, 4, 2, 7, 6]$



(e) After the fourth iteration the labels are $[0, 3, 4, 2, 7, 4]$



(f) After the fifth iteration the labels are unchanged. The shortest path distances have been computed.

Fig. 6.4: An example-run of the Bellman-Ford algorithm. The blue edges represent the tree whose paths have the corresponding lengths.

Theorem 6.3. *For each $k = 0, \ldots, n$ and for each $v \in V$*

$$f_k(v) = \min\{c(P) : P \text{ is an } s-v\text{-walk traversing at most } k \text{ arcs}\}.$$

Corollary 6.1. *If $D = (V, A)$ does not contain negative cycles w.r.t. c, then $f_n(v)$ is equal to the length of a shortest $s-v$-path. The numbers $f_n(v)$ can be computed in time $O(|V| \cdot |A|)$.*

Corollary 6.2. *In time $O(|V|^2|A|)$ one can test whether $D = (V, A)$ has a negative cycle w.r.t. c and eventually return one.*

6.5 Maximum $s-t$-flows

We now turn our attention to a linear programming problem which we will solve by direct methods, motivated by the nature of the problem. We often use the following notation. If $f : A \longrightarrow B$ denotes a function and if $U \subseteq A$, then $f(U)$ is defined as $f(U) = \sum_{a \in U} f(a)$.

Definition 6.7 (Network, $s-t$-flow). A network with capacities consists of a directed simple graph $D = (V, A)$ and a *capacity function* $u : A \to \mathbb{R}_{\geqslant 0}$. A function $f : A \to \mathbb{R}_{\geqslant 0}$ is called an *$s-t$-flow*, if

$$\sum_{e \in \delta^{out}(v)} f(e) = \sum_{e \in \delta^{in}(v)} f(e), \text{ for all } v \in V - \{s, t\}, \tag{6.1}$$

where $s, t \in V$. The flow is *feasible*, if $f(e) \leqslant u(e)$ for all $e \in A$. The *value* of f is defined as $value(f) = \sum_{e \in \delta^{out}(s)} f(e) - \sum_{e \in \delta^{in}(s)} f(e)$. The *maximum $s-t$-flow problem* is the problem of determining a maximum feasible $s-t$-flow.

Here, for $U \subseteq V$, $\delta^{in}(U)$ denotes the arcs which are entering U and $\delta^{out}(U)$ denotes the arcs which are leaving U. Arc sets of the form $\delta^{out}(U)$ are called a *cut* of D. The *capacity of a cut* $u(\delta^{out}(U))$ is the sum of the capacities of its arcs.

Thus the maximum flow problem is a linear program of the form

$$\max \sum_{e \in \delta^{out}(s)} x(e) - \sum_{e \in \delta^{in}(s)} x(e) \tag{6.2}$$

$$\sum_{e \in \delta^{out}(v)} x(e) = \sum_{e \in \delta^{in}(v)} x(e), \text{ for all } v \in V - \{s, t\} \tag{6.3}$$

$$x(e) \leqslant u(e), \text{ for all } e \in A \tag{6.4}$$

$$x(e) \geqslant 0, \text{ for all } e \in A \tag{6.5}$$

Definition 6.8 (excess function). For any $f : A \to \mathbb{R}$, the excess function is the function $excess_f : 2^V \to \mathbb{R}$ defined by $excess_f(U) = \sum_{e \in \delta^{in}(U)} f(e) - \sum_{e \in \delta^{out}(U)} f(e)$.

Theorem 6.4. *Let $D = (V, A)$ be a digraph, let $f : A \to \mathbb{R}$ and let $U \subseteq V$, then*

$$excess_f(U) = \sum_{v \in U} excess_f(v). \tag{6.6}$$

Proof. An arc which has both endpoints in U is counted twice with different parities on the right, and thus cancels out. An arc which has its tail in U is subtracted once on the right and once on the left. An arc which has its head in U is added once on the right and once on the left. □

A cut $\delta^{out}(U)$ with $s \in U$ and $t \notin U$ is called an $s-t$-cut.

Theorem 6.5 (Weak duality). *Let f be a feasible $s-t$-flow and let $\delta^{out}(U)$ be an $s-t$-cut, then $value(f) \leqslant u(\delta^{out}(U))$.*

Proof. $value(f) = -excess_f(s) = -excess_f(U) = f(\delta^{out}(U)) - f(\delta^{in}(U)) \leqslant f(\delta^{out}(U)) \leqslant u(\delta^{out}(U))$. □

For an arc $a = (u, v) \in A$ the arc a^{-1} denotes the arc (v, u).

Definition 6.9 (Residual graph). Let $f : A \to \mathbb{R}$, and $u : A \to \mathbb{R}$ where $0 \leqslant f \leqslant u$. Consider the sets of arcs

$$A_f = \{a \mid a \in A,\, f(a) < u(a)\} \cup \{a^{-1} \mid a \in A,\, f(a) > 0\}. \tag{6.7}$$

The digraph $D(f) = (V, A_f)$ is called the *residual graph* of f (for capacities u).

Corollary 6.3. *Let f be a feasible $s-t$-flow and suppose that $D(f)$ has no path from s to t, then f has maximum value.*

Proof. Let U be the set of nodes which are reachable in $D(f)$ from s. Clearly $\delta^{out}(U)$ is an $s-t$-cut. Now $value(f) = f(\delta^{out}(U)) - f(\delta^{in}(U)$. Each arc leaving U is not an arc of $D(f)$ and thus $f(\delta^{out}(U)) = u(\delta^{out}(U))$. Each arc entering U does not carry any flow and thus $f(\delta^{in}(U) = 0$. It follows that $value(f) = u(\delta^{out}(U))$ and f is optimal by Theorem 6.5. □

Definition 6.10 (undirected walk). An *undirected walk* is a sequence of the form $P = (v_0, a_1, v_1, \ldots, v_{m-1}, a_m, v_m)$, where $a_i \in A$ for $i = 1, \ldots, m$ and $a_i = (v_{i-1}, v_i)$ or $a_i = (v_i, v_{i-1})$. If the nodes $v_0, \ldots, v_m$ are all different, then P is an *undirected path*.

Any directed path P in $D(f)$ yields an undirected path in D. Define for such a path P the vector $\chi^P \in \{0, \pm 1\}^A$ as

$$\chi^P(a) = \begin{cases} 1 & \text{if } P \text{ traverses } a, \\ -1 & \text{if } P \text{ traverses } a^{-1}, \\ 0 & \text{if } P \text{ traverses neither } a \text{ or } a^{-1}. \end{cases} \tag{6.8}$$

Theorem 6.6 (max-flow min-cut theorem, strong duality). *The maximum value of a feasible $s-t$-flow is equal to the minimum capacity of an $s-t$ cut.*

Proof. Let f be a maximum $s-t$-flow. Consider the residual graph $D(f)$. If this residual graph contains an $s-t$-path P, then we can route flow along this path. More precisely, there exists an $\epsilon > 0$ such that $f+\epsilon\chi^P$ is feasible. We have $value(f+\epsilon\chi^P) = value(f)+\epsilon$. This contradics the maximality of f thus there exists no $s-t$-path in $D(f)$.

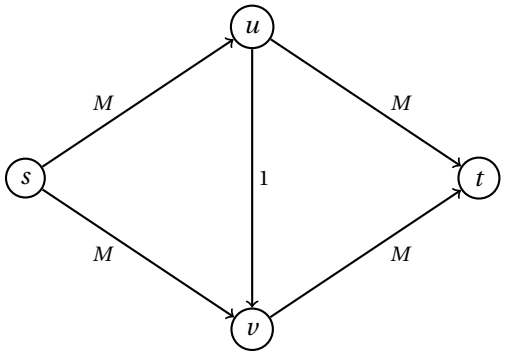
Let U be the nodes reachable from s in $D(f)$. Then $value(f) = u(\delta^{out}(U))$ and $\delta^{out}(U)$ is an $s-t$-cut of minimum capacity by the weak duality theorem.

This suggests the algorithm of Ford and Fulkerson to find a maximum flow. Start with $f = 0$. Next iteratively apply the following *flow augmentation algorithm.*

Let P be a directed $s-t$-path in $D(f)$. Set $f \leftarrow f+\epsilon\chi^P$, where ϵ is as large as possible to maintain $0 \leqslant f \leqslant u$.

Exercise 6.1. Define a *residual capacity* for $D(f)$. Then determine the maximum ϵ such that $0 \leqslant f \leqslant u$.

Theorem 6.7. *If all capacities are rational, this algorithm terminates.*



The example above shows that, if the augmenting paths are chosen in a disadvantageous way, then the Ford-Fulkerson algorithm may take $\Omega(M)$ iterations, where M is the largest capacity in the network. This happens if all augmenting paths use the arc uv or vu respectively in the residual network.

Corollary 6.4 (integrity theorem). *If $u(a) \in \mathbb{N}$ for each $a \in A$, then there exists an integer maximum flow ($f(a) \in \mathbb{N}$ for all $a \in A$).*

Proof. This follows from the fact that the residual capacities remain integral and thus the augmented flow is always integral. □

Theorem 6.8. *If we choose in each iteration a shortest $s-t$-path in $D(f)$ as a flow-augmenting path, the number of iterations is at most $|V| \cdot |A|$.*

Definition 6.11. Let $D = (V, A)$ be a digraph, $s, t \in V$ and let $\mu(D)$ denote the length of a shortest path from s to t. Let $\alpha(D)$ denote the set of arcs contained in at least one shortest $s-t$ path.

Theorem 6.9. *Let $D = (V, A)$ be a digraph and $s, t \in V$. Define $D' = (V, A \cup \alpha(D)^{-1})$. Then $\mu(D) = \mu(D')$ and $\alpha(D) = \alpha(D')$.*

Proof. It suffices to show that $\mu(D)$ and $\alpha(D)$ are invariant if we add a^{-1} to D for one arc $a \in \alpha(D)$. Suppose not, then there is a directed $s-t$-path P_1 traversing a^{-1} of length at most $\mu(D)$. As $a \in \alpha(D)$ there is a path P_2 traversing a of length $\mu(D)$. If we follow P_2 until the tail of a is reached and from thereon follow P_1, we obtain another $s-t$ path P_3 in D. Similarly if we follow P_1 until the head of a is reached and then follow P_2, we obtain a fourth $s-t$ path P_4 in D. However P_3 or P_4 has length less than $\mu(D)$. This is a contradiction. □

Proof (of Theorem 6.8). Let us augment flow f along a shortest $s-t$-path P in $D(f)$ obtaining flow f'. The residual graph $D_{f'}$ is a subgraph of $D' = (V, A_f \cup \alpha(D(f))^{-1})$. Hence $\mu(D_{f'}) \geqslant \mu(D') = \mu(D(f))$. If $\mu(D_{f'}) = \mu(D(f))$, then $\alpha(D_{f'}) \subseteq \alpha(D') = \alpha(D(f))$. At least one arc of P does not belong to $D_{f'}$, (the arc of minimum residual capacity!) thus the inclusion is strict. Since $\mu(D(f))$ increases at most $|V|$ times and, as long as $\mu(D(f))$ does not change, $|\alpha(D(f))|$ decreases at most $2|A|$ times, we have the theorem. □

In the following let $m = |A|$ and $n = |V|$.

Corollary 6.5. *A maximum flow can be found in time $O(n\,m^2)$.*

6.6 Minimum cost network flows, MCNFP

In contrast to the maximum $s-t$-flow problem, the goal here is to route a flow, which comes from several sources and sinks through a network with capacities and *costs* in such a way, that the total cost is minimized.

Example 6.3. Suppose you are given a directed graph width arc weights $D = (V, A)$, $c : A \to \mathbb{R}_{\geqslant 0}$ and your task is to compute a shortest path from a particular node s to all other nodes in the graph and assume that such paths exist. Then one can model this as a MCNFP by sending a flow of value $|V| - 1$ into the source node and by letting a flow of value 1 leave each node. The costs on the arcs are defined by c. The arcs have infinite capacities. We will see later, that this minimum cost network flow problem has an integral solution which corresponds to the shortest paths from s to all other nodes.

Here is a formal definition of a minimum cost network flow problem. In this notation, vertices are indexed with the letters i, j, k and arcs are denoted by their tail and head respectively, for example (i, j) denotes the arc from i to j.

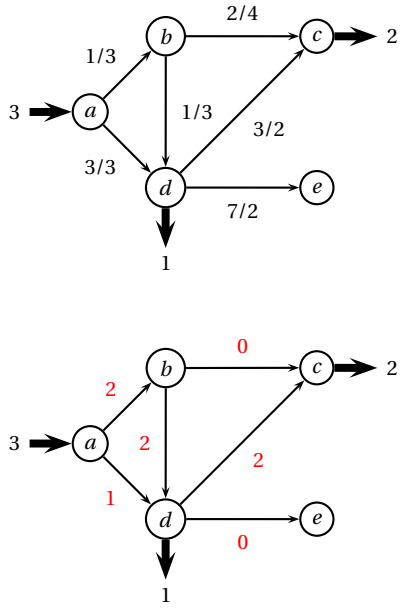


Fig. 6.5: A Network with in/out-flow, costs and capacities and a feasible flow of cost 13.

A *network* is now a directed graph $D = (V, A)$ together with a capacity function $u : A \to \mathbb{Q}_{\geqslant 0}$, a cost function $c : A \to \mathbb{Q}$ and an external flow $b : V \to \mathbb{Q}$. The value of $b(i)$ denotes the amount of flow which comes from the exterior. If $b(i) > 0$, then there is flow from the outside, entering the network through node i. If $b(i) < 0$, there is flow which leaves the network through i.

In the following we often use the notation $f(i, j)$ for the flow-value on the arc (i, j) (instead of $f((i, j))$). Similarly we write $c(i, j)$ and $u(i, j)$.

A *feasible flow* is a function $f : A \to \mathbb{Q}_{\geqslant 0}$ which satisfies the following constraints.

$$\begin{aligned} \textstyle\sum_{e \in \delta^{out}(i)} f(e) - \sum_{j \in \delta^{in}(i)} f(e) &= b_i && \text{for all } i \in V, \\ 0 \leqslant f(e) &\leqslant u(e) && \text{for all } e \in A. \end{aligned}$$

The goal is to find a feasible flow with minimum cost:

$$\begin{aligned} &\text{minimize} && \textstyle\sum_{e \in A} c(e) f(e) \\ &\text{subject to} && \textstyle\sum_{e \in \delta^{out}(i)} f(e) - \sum_{e \in \delta^{in}(i)} f(e) = b(i) && \text{for all } i \in V, \\ & && 0 \leqslant f(e) \leqslant u(e) && \text{for all } (e) \in A \end{aligned}$$

Example 6.4. Imagine you are a pilot and fly a passenger airplane in hops from airport 1 to airport 2 to airport 3 and so on, until airport n. At airport i there are b_{ij} passengers that want to travel to airport j, where $j > i$. You may decide how

many of the b_{ij} passengers you will take on board. Each of the passengers will pay c_{ij} dollars for the trip. The airplane can accommodate p people.

You are a greedy pilot and think of a plan to pick up and deliver passengers on your hop from 1 to n which maximizes your revenue.

Finding this plan can be modeled as a minimum cost network flow problem. Your network has nodes $1,\dots,n$ and arcs $(i,i+1), i = 1,\dots,n-1$ with capacities p and without costs. These nodes do not have in/out-flow from the outside. You furthermore have nodes $i \to j$ for $i < j$ and $i, j \in \{1,\dots,n\}$ which are excess nodes with in-flow b_{ij} from the outside. Each node $i \to j$ is connected to i and to j with a directed arc. The capacities on these arcs are infinite. The cost of the arc $(i \to j, i)$ is $-c_{ij}$. The cost of the arc $(i \to j, j)$ is zero. The outflow on the node j is the total number of passengers that want to fly to node j. An integral optimal flow to this problem is an optimal plan for you.

Throughout this chapter we make the following assumptions.

1. All data (cost, supply, demand and capacity) are integral.
2. The network contains an incapacitated directed path between every pair of nodes.
3. The supplies/demands at the nodes satisfy the condition $\sum_{i\in V} b(i) = 0$ and the MCNFP has a feasible solution.
4. All arc costs are nonnegative.
5. The graph does not contain a pair of reverse arcs.

Exercise 6.2. Show how to transform a MCNFP on a digraph with pairs of reverse arcs into a MCNFP on a digraph with no pairs of reverse arcs. The number of arcs and nodes should asymptotically remain the same.

An *arc-flow* of D is a flow vector, that satisfies the nonnegativity and capacity constraints.

$$\sum_{e\in\delta^{in}(i)} f(e) - \sum_{e\in\delta^{out}(i)} f(e) = e(i) \quad \text{for all } i \in V,$$
$$0 \leqslant f(e) \leqslant u(e) \quad \text{for all } e \in A.$$

- If $e(i) > 0$, then i is an *excess node* (more inflow than outflow).
- If $e(i) < 0$, then i is a *deficit node* (more outflow than inflow).
- If $e(i) = 0$ then i is called *balanced*.

Exercise 6.3. Prove that $\sum_{i\in V} e(i) = 0$ holds and thus that a feasible flow only exists if the sum of the $b(i)$ is equal to zero.

Let $\mathscr{P}$ be the collection of directed paths of D and let $\mathscr{C}$ be the collection of directed cycles of D. A path-flow is a function $\beta : \mathscr{P} \cup \mathscr{C} \to \mathbb{R}_{\geqslant 0}$ which assigns flow values to paths and cycles.

For $(i,j) \in A$ and $P \in \mathscr{P}$ let $\delta_{(i,j)}(P)$ be 1 if $(i,j) \in P$ and 0 otherwise. For $C \in \mathscr{C}$ let $\delta_{(i,j)}(C)$ be 1 if $(i,j) \in C$ and 0 otherwise.

A path-flow β determines a unique arc-flow

$$f(i,j) = \sum_{P \in \mathscr{P}} \delta_{(i,j)}(P)\beta(P) + \sum_{C \in \mathscr{C}} \delta_{(i,j)}(C)\beta(C).$$

Theorem 6.10. *Every path and cycle flow has a unique representation as a nonnegative arc-flow. Conversely, every nonnegative arc flow f can be represented as a path and cycle flow with the following properties:*

1. *Every directed path with positive flow connects a deficit node with an excess node.*
2. *At most $n+m$ paths and cycles have nonzero flow and at most m cycles have nonzero flow.*

If the arc flow f is integral, then so are the path and cycle flows into which it decomposes.

Proof. "$\Rightarrow$" See discussion above.

"$\Leftarrow$"

Let f be an arc flow. Suppose i_0 is a deficit node. Then there exists an incident arc (i_0, i_1) which carries a positive flow. If i_1 is an excess node, we have found a path from deficit to excess node. Otherwise, the flow balance constraint at i_1 implies that there exists an arc (i_1, i_2) with positive flow. Repeating this procedure, we finally must arrive at an excess node or revisit a node. This means that we either have constructed a directed path P from deficit node to excess node or a directed cycle C, both involving only arcs with strictly positive flow.

In the first case, let $P = i_0, \ldots, i_k$ be the directed path from deficit node i_0 to excess node i_k. We set $\beta(P) = \min\{-e_{i_0}, e_{i_k}, \min\{f(i,j) \mid (i,j) \in P\}\}$ and $f(i,j) = f(i,j) - \beta(P)$, $(i,j) \in P$. In the second case, set $\beta(C) = \min\{f(i,j) \mid (i,j) \in C$ and $f(i,j) = f(i,j) - \beta(C)$, $(i,j) \in C$. Repeat this procedure until all node imbalances are zero.

Now find an arc with positive flow and construct a cycle C by following only positive arcs from there. Set $\beta(C) = \min\{f(i,j) \mid (i,j) \in C\}$ and $f(i,j) = f(i,j) - \beta(C)$, $(i,j) \in C\}$. Repeat this process until there are no positive flow-arcs left.

Each time a path or a cycle is identified, the excess/deficit of some node is set to zero or some arc is set to zero. This implies that we decompose into at most $n+m$ paths and cycles. Since cycle detection sets an arc to zero we have at most m cycles. □

An arc flow f with $e(i) = 0$ for each $i \in V$ is called a *circulation*.

Corollary 6.6. *A circulation can be decomposed into at most m cycle flows.*

Let $D = (V, A)$ be a network with capacities $u(i,j)$, $(i,j) \in A$ and costs $c(i,j)$, $(i,j) \in A$ and let f be a feasible flow of the network. The *residual network* $D(f)$ is defined as follows.

- We replace each arc $(i,j) \in A$ with two arcs (i,j) and (j,i).
- The arc (i,j) has cost $c(i,j)$ and *residual capacity* $r(i,j) = u(i,j) - f(i,j)$.

- The arc (j,i) has cost $-c(i,j)$ and residual capacity $r(j,i) = f(i,j)$.
- Delete all arcs which do not have strictly positive residual capacity.

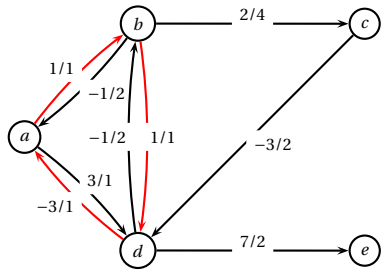


Fig. 6.6: The residual network of the flow in Figure 6.5 and a negative cycle marked by the red edges.

A directed cycle in $D(f)$ is called an *augmenting cycle* of f.

Lemma 6.4. *Suppose that f and f° are feasible flows, then $f - f^\circ$ is a circulation in $D(f^\circ)$. Here $f - f^\circ$ is the flow*

$$(f - f^\circ)(e) = \begin{cases} \max\{0, f(e) - f^\circ(e)\}, & \text{if } e \in A(D) \\ \max\{0, f^\circ(e) - f(e)\}, & \text{if } e^{-1} \in A(D) \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It is very easy to see that the flow $f - f^\circ$ satisfies the capacity constraints. One also has for each $v \in V$

$$\sum e \in \delta^{out}(v)(f(e) - f^\circ(e)) - \sum_{e \in \delta^{in}(v)} (f(e) - f^\circ(e)) = 0.$$

If a term $(f(e) - f^\circ(e))$ is negative, it is replaced by its absolute value and charged as flow on the arc e^{-1} in $D(f^\circ)$ which leaves its contribution to the sum above invariant. □

Theorem 6.11 (Augmenting Cycle Theorem). *Let f and f° be any two feasible flows of a network flow problem. Then f equals f° plus the flow of at most m directed cycles in $D(f^\circ)$. Furthermore the cost of f equals the cost of f° plus the cost of flow on these augmenting cycles.*

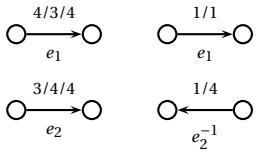


Fig. 6.7: Two arcs $e_1, e_2 \in A$ labeled with $f(e)/f^\circ(e)/u(e)$ and the corresponding flow on these arcs (or their reverse) in $D(f^\circ)$. Arcs in $D(f^\circ)$ are labeled with flow and capacity values respectively.

Proof. This can be seen by applying flow decomposition on the flow $f - f^\circ$ in $D(f^\circ)$. □

Theorem 6.12 (Negative Cycle Optimality Conditions). *A feasible flow f^* is an optimal solution of the minimum cost network flow problem, if and only if it satisfies the negative cycle optimality conditions: The residual network $D(f^*)$ contains no directed cycle of negative cost.*

Proof. "⇒" Suppose that f is a feasible flow and that $D(f)$ contains a negative directed cycle. Then f cannot be optimal, since we can augment positive flow along the corresponding cycle in the network. Therefore, if f^* is an optimal flow, then $D(f^*)$ cannot contain a negative directed cycle.

"⇐" Suppose now that f^* is a feasible flow and suppose that $D(f^*)$ does not contain a negative cycle. Let f° be an optimal flow with $f^\circ \neq f^*$. The vector $f^\circ - f^*$ is a circulation in $D(f^\circ)$ with non-positive cost $c^T(f^\circ - f^*) \leqslant 0$. It follows from Theorem 6.11 that the cost of f° equals the cost of f^* plus the cost of directed cycles in the residual network $D(f^*)$. The cost of these cycles is nonnegative, and therefore $c(f^\circ) \geqslant c(f^*)$ which implies that f^* is optimal. □

Algorithm 6.1 (Cycle Canceling Algorithm).

1. establish a feasible flow f in the network
2. `WHILE` $D(f)$ contains a negative cycle
   a. detect a negative cycle C in $D(f)$
   b. $\delta = \min\{r(i,j) \mid (i,j) \in C\}$
   c. augment δ units of flow along the cycle C
   d. update $D(f)$
3. `RETURN` f

Theorem 6.13. *The cycle canceling algorithm terminates after a finite number of steps if the MCNFP has an optimal solution.*

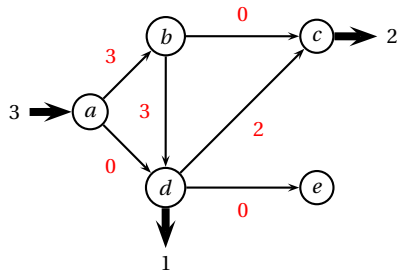


Fig. 6.8: The result of augmenting a flow of one along the negative cycle in Figure 6.6. This flow has cost 12 but is not optimal, since the residual network still contains a negative cycle.

Proof. The cycle canceling algorithm reduces the cost in each iteration. We have assumed that the input data is integral. Thus the cost decreases by at least one unit each iteration. Therefore the number of iterations is finite. □

Corollary 6.7. *If the capacities are integral and if the MCNFP has a optimal flow, then it has an optimal flow with integer values only.*

Let $\pi : V \to \mathbb{R}$ be a function (*node potential*). The *reduced cost* of an arc (i, j) w.r.t. π is $c_\pi((i, j)) = c((i, j)) + \pi(i) - \pi(j)$. The potential π is called *feasible* if $c_\pi((i, j)) \geqslant 0$ for all arcs $(i, j) \in A$.

Lemma 6.5. *Let $D = (V, A)$ be a digraph with arc weights $c : A \to \mathbb{R}$. Then D does not have a negative cycle if and only if there exists a feasible node potential π of D with arc weights c.*

Proof. Consider a directed path $P = i_0, i_1, \ldots, i_k$. The cost of this path is

$$c(P) = \sum_{j=1}^{k} c((i_{j-1}, i_j)).$$

The reduced cost of this path is equal to

$$c_\pi(P) = \sum_{j=1}^{k} c((i_{j-1}, i_j)) + \pi(i_0) - \pi(i_k).$$

If P is a cycle, then i_0 and i_k are equal, which means that its cost and reduced cost coincide. Thus, if there exists a feasible node potential, then there does not exist a negative cycle.

On the other hand, suppose that D, c does not contain a negative cycle. Add a vertex s to D and the arcs (s, i) for all $i \in V$. The weights (costs) of all these new arcs is 0. Notice that in this way, no new cycles are created, thus still there does

not exist a negative cycle. This means we can compute the shortest paths from s to all other nodes $i \in V$. Let π be the function which assigns these shortest paths lengths. Clearly $c_\pi((i,j)) = \pi(i) - \pi(j) + c((i,j)) \geqslant 0$, since the shortest-path length to j is at most the shortest-path length to $i + c((i,j))$. □

This means that we have again a nice way to prove that a flow is optimal. Simply equip the residual network with a feasible node potential.

Corollary 6.8 (Reduced Cost Optimality Condition). *A feasible flow f^* is optimal if and only if there exists a node potential π such that the reduced costs $c_\pi(i,j)$ of each arch (i,j) of $D(f)$ are nonnegative.*

The cycle canceling algorithm is only pseudopolynomial. If we could always chose a minimum cycle (cycle with best improvement) as an augmenting cycle, we would have a polynomial number of iterations. Finding minimum cycles is NP-hard. Instead we augment along *minimum mean cycles*. One can find minimum mean cycles in polynomial time.

The *mean cost* of a cycle $C \in \mathscr{C}$ is the cost of C divided by the number of arcs in C:

$$(\sum_{(i,j)\in C} c(i,j))/|C|.$$

Algorithm 6.2 (Minimum Mean Cycle Canceling, MMCC).

1. establish a feasible flow f in the network
2. `WHILE` $D(f)$ contains a negative cycle
   a. detect a minimum mean cycle C in $D(f)$
   b. $\delta = \min\{r(i,j) \mid (i,j) \in C\}$
   c. augment δ units of flow along the cycle C
   d. update $D(f)$
3. `RETURN` f

We now analyze the MMCC-algorithm. Let $\mu(f)$ denote the minimum mean-weight of a cycle in $D(f)$.

Lemma 6.6 (See Korte & Vygen [9]). *Let $f_1, f_2, \ldots$ be a sequence of feasible flows such that f_{i+1} results from f_i by augmenting flow along C_i, where C_i is a minimum mean cycle of $D(f_i)$, then*

1. *$\mu(f_k) \leqslant \mu(f_{k+1})$ for all k.*
2. *$\mu(f_k) \leqslant \frac{n}{n-1}\mu(f_l)$, where $k < l$ and $C_k \cup C_l$ contains a pair of reversed arcs.*

Proof. 1): Suppose f_k and f_{k+1} are two subsequent flows in this sequence. Consider the multi-graph H which results from C_k and C_{k+1} by deleting pairs of opposing arcs. The arcs of H are a subset of the arcs of $D(f_k)$, since an arc of C_{k+1} which is not in $D(f_k)$ must be a reverse arc of C_k.

Each node in H has even degree. Thus H can be decomposed into cycles, each of mean weight at least $\mu(f_k)$. Thus we have $c(A(H)) \geqslant \mu(f_k)|A(H)|$.

Since the total weight of each reverse pair of arcs is zero we have

$$c(A(H)) = c(C_k) + c(C_{k+1}) = \mu(f_k)|C_k| + \mu(f_{k+1})|C_{k+1}|.$$

Since $|A(H)| \leqslant |C_k| + |C_{k+1}|$ we conclude

$$\begin{aligned} \mu(f_k)(|C_k| + |C_{k+1}|) &\leqslant \mu(f_k)|A(H)| \\ &\leqslant c(A(H)) \\ &= \mu(f_k)|C_k| + \mu(f_{k+1})|C_{k+1}|. \end{aligned}$$

Thus $\mu(f_k) \leqslant \mu(f_{k+1})$.

2): By the first part of the theorem, it is enough to prove the statement for k, l such that $C_i \cup C_l$ does not contain a pair of reverse arcs for each i, $k < i < l$.

Again, consider the graph H resulting from C_k and C_l by deleting pairs of opposing arcs. H is a subgraph of $D(f_k)$, since any arc of C_l which does not belong to $D(f_k)$ must be a reverse arc of $C_k, C_{k+1}, \ldots, C_{l-1}$. But only C_k contains a reverse arc of C_l. So as above we have

$$c(A(H)) = c(C_k) + c(C_l) = \mu(f_k)|C_k| + \mu(f_l)|C_{k+1}|.$$

Since $|A(H)| \leqslant |C_k| + |C_l| - 2$ we have $|A(H)| \leqslant \frac{n-1}{n}(|C_k| + |C_l|)$. Thus we get

$$\begin{aligned} \mu(f_k)\frac{n-1}{n}(|C_k| + |C_l|) &\leqslant \mu(f_k)|A(H)| \\ &\leqslant c(A(H)) \\ &= \mu(f_k)|C_k| + \mu(f_l)|C_l| \\ &\leqslant \mu(f_l)(|C_k| + |C_l|) \end{aligned}$$

This implies that $\mu(f_k) \leqslant \frac{n}{n-1}\mu(f_l)$. □

Corollary 6.9. *During the execution of the MMCC-algorithm, $|\mu(f)|$ decreases by a factor of $1/2$ every $n \cdot m$ iterations.*

Proof. Let $C_1, C_2, \ldots$ be the sequence of augmenting cycles. Every m-th iteration, there must be an arc of the cycle, which is reverse to one of the succeeding $m-1$ cycles, because every iteration, one arc of the residual network will be deleted. Thus after $n\,m$ iterations, the absolute value of μ has dropped by $\left(\frac{n-1}{n}\right)^n \leqslant e^{-1} \leqslant 1/2$. □

Corollary 6.10. *If all data are integral, then the MMCC-algorithm runs in polynomial time.*

Proof.
- A lower bound on μ is the smallest cost c_{min}
- $|\mu|$ drops by $1/2$ every $m\,n$ iterations.

- After $mn\log n|c_{min}|$ iterations, absolute value of minimum mean weight cycle drops below $1/n$, thus is zero.
- **We need to prove that a minimum mean cycle can be found in polynomial time**

□

This is a so-called *weakly polynomial* bound, since the binary encoding length of the numbers in the input (here the costs) influences the running time. We now prove that the MMCC-algorithm is *strongly polynomial*.

Theorem 6.14 (See Korte & Vygen [9]). *The MMCC-algorithm requires $O(m^2\, n\log n)$ iterations (mean weight cycle cancellations).*

Proof. One shows that every $m\,n(\lceil\log n\rceil+1)$ iterations, at least one arc is *fixed*, which means that the flow through this arc does not change anymore.

Let f_1 be some flow at some iteration and let f_2 be the flow $m\,n(\lceil\log n\rceil+1)$ iterations later. It follows from Corollary 6.9 that

$$\mu(f_1) \leqslant 2\,n\,\mu(f_2) \tag{6.9}$$

holds.

Define the costs $c'(e) = c(e) - \mu(f_2)$ for the residual network $D(f_2)$. There exists no negative cycle in $D(f_2)$ w.r.t. this cost c'. (A cycle C has weight $c'(C) = \sum_{e\in C} c(e) - |C|\mu(f_2)$ and thus $c'(C)/|C| = \sum_{e\in C} c(e)/|C| - \mu(f_2) \geqslant 0$). By Lemma 6.5 there exists a feasible node potential π for these weights. One has $0 \leqslant c'_\pi(e) = c_\pi(e) - \mu(f_2)$ and thus

$$c_\pi(e) \geqslant \mu(f_2), \text{ for all } e \in A(D(f_2)). \tag{6.10}$$

Let C be a minimum mean cycle of $D(f_1)$. One has

$$c_\pi(C) = c(C) = \mu(f_1)\,|C| \leqslant 2\,n\,\mu(f_2)|C|. \tag{6.11}$$

It follows that there exists an arc e_0 of C such that

$$c_\pi(e_0) \leqslant 2\,n\,\mu(f_2) \tag{6.12}$$

holds. The inequalities (6.10) imply that $e_0 \notin A(D(f_2))$

We now make the following claim:

Let f' be a feasible flow such that $e_0 \in D(f')$, then $\mu(f') \leqslant \mu(f_2)$.

If we have shown this claim, then it follows from Lemma 6.6 that e_0 cannot be anymore in the residual network of a flow after f_2. Thus the flow along the arc e_0 (or e_0^{-1}) is fixed.

Let f' be a flow such that $e_0 \in A(D(f'))$. Recall that $f' - f_2$ is a circulation in $D(f_2)$ where $e_0 \notin D(f_2)$, $e_0^{-1} \in D(f_2)$ and this circulation sends flow over e_0^{-1}. This circulation can be decomposed into cycles and one of these cycles C contains e_0^{-1}. One has $c_\pi(e_0^{-1}) = -c_\pi(e_0) \geqslant -2\,n\,\mu(f_2)$ (eq. (6.12)). Using (6.10) one obtains

$$\begin{aligned} c(C) &= \sum_{e\in C} c_\pi(e) && (6.13)\\ &\geqslant -2\,n\,\mu(f_2) + (n-1)\mu(f_2) && (6.14)\\ &= -(n+1)\,\mu(f_2) && (6.15)\\ &> -n\,\mu(f_2). && (6.16) \end{aligned}$$

The reverse of C is an augmenting cycle for f' with total weight at most $n\,\mu(f_2)$ and thus with mean weight at most $\mu(f_2)$. Thus $\mu(f') \leqslant \mu(f_2)$. □

6.7 Computing a minimum cost-to-profit ratio cycle

Given a digraph $D = (V, A)$ with costs $c : A \to \mathbb{Z}$ and profit $p : A \to \mathbb{N}_{>0}$, the task is to compute a cycle $C \in \mathscr{C}$ with minimum ratio

$$\frac{c(C)}{p(C)}. \tag{6.17}$$

Notice that this is the largest number $\beta \in \mathbb{Q}$ which satisfies

$$\beta \leqslant \frac{c(C)}{p(C)}, \text{ for all } C \in \mathscr{C}. \tag{6.18}$$

By rewriting this inequality, we understand this to be the largest number $\beta \in \mathbb{Q}$ such that

$$c(C) - \beta\, p(C) \geqslant 0 \text{ for all } C \in \mathscr{C}. \tag{6.19}$$

In other words, we search the largest number $\beta \in \mathbb{Q}$ such that the digraph $D = (V, A)$ with costs $c_\beta : A \to \mathbb{Q}$, where $c_\beta(e) = c(e) - \beta\, p(e)$.

We need a routine to check whether D has a negative cycle for a given weight function c. For this we assume w.l.o.g. that each vertex is reachible from the vertex s, if necessary by intruducing a new vertex s from which there is an arc with cost and profit 0 to all other nodes. The minimum cost-to-profit ration cycle w.r.t. this new graph is then the minimum cost to profit ratio cycle w.r.t. the original graph, since s is not a vertex of any cycle.

Recall the following single-source shortest-path algorithm of Bellman-Ford which we now apply with weights c_β:

Let $n = |V|$. We calculate functions $f_0, f_1, \ldots, f_n : V \longrightarrow \mathbb{R} \cup \{\infty\}$ successively by the following rule.

i) $f_0(s) = 0$, $f_0(v) = \infty$ for all $v \neq s$

ii) For $k < n$ if f_k has been found, compute

$$f_{k+1}(v) = \min\{f_k(v), \min_{(u,v)\in A}\{f_k(u) + c_\beta(u,v)\}$$

for all $v \in V$.

There exists a negative cycle w.r.t. c_β if and only if $f_n(v) < f_k(v)$ for some $v \in V$ and $1 \leqslant k < n$. Thus we can test in $O(m \cdot n)$ steps whether D, c_β contains a negative cycle.

We now apply the following idea to search for the correct value of β. We keep an interval $I = [L, U]$ with the invariant that the value β that we are searching lies in this interval I. As starting values, we can chose $L = c_{min}$ and $U = c_{max}$, where c_{min} and c_{max} are the smallest and largest cost respectively. In one iteration we compute $M = (L + U)/2$. We then check whether D, together with c_M contains a negative cycle. If yes, we know that β is at least M and we set $L \leftarrow M$. If not, then β is at most M and we update the upper bound $U \leftarrow M$.

When can we stop this procedure? We can stop it, if we can assure that only one valid cost-to-profit ratio cycle lies in $[L, U]$. Suppose that C_1 and C_2 have different cost-to-profit ratios. Then

$$|c(C_1)/p(C_1) - c(C_2)/p(C_2)| = \left| \frac{c(C_1)\,p(C_2) - c(C_2)\,p(C_1)}{(p(C_1)\,p(C_2))} \right| \tag{6.20}$$

$$\geqslant 1/(n^2 p_{max}^2). \tag{6.21}$$

Thus we can stop our process, if $U - L < 1/(n^2 p_{max}^2)$, since we know then that there can be only one cycle $c \in \mathscr{C}$ with $c(C)/p(C) \in [L, U]$.

Suppose that $[L, U]$ is the final interval. We know then that

$$L \leqslant c(C)/p(C) \text{ for all } C \in \mathscr{C}$$

and

$$U > c(C)/p(C) \text{ holds for some } C \in \mathscr{C}.$$

Let C be a minimum weight cycle w.r.t. the arc costs c_L. Clearly $U > c(C)/p(C) \geqslant L$ holds and thus C is the minimum cost-to-profit cycle we have been looking for.

Let us analyze the number of required iterations. We need to halve the starting interval-length $2c$, where c is the largest absolute value of a cost, until the length is at most $1/(n^2 p_{max}^2)$. We search the minimal $i \in \mathbb{N}$ such that

$$(1/2)^i c \leqslant 1/(n^2 p_{max}^2). \tag{6.22}$$

This shows us that we need $O(\log(c\, p_{max}^2 n^2))$ iterations which is $O(\log n \log K)$, where K is the largest absolute value of a cost or a profit.

Theorem 6.15 (Lawler [10]). *Let D be a digraph with costs $c : A \to \mathbb{Z}$ and profit $p : A \to \mathbb{N}_{>0}$ an let $K \in \mathbb{N}$ such that $|c(e)| + |p(e)| \leqslant K$ for all $e \in \mathbb{N}$. A minimum cost-to-profit ratio cycle of G can be computed in time $O(m\, n \log n \log K)$.*

But we knew a weakly polynomial algorithm for MCNFP from the exercises. So you surely ask: Can we do better for minimum cost-to-profit cycle computation? The answer is “Yes”!

6.7.1 Parametric search

Let us first roughly describe the idea on how to obtain a strongly polynomial algorithm, see [14]. The Bellman-Ford algorithm tells us whether our current β is too large or too small, depending on whether D with weights c_β contains a negative cycle or not. Recall that the B-F algorithm computes labels $f_i(v)$ for $v \in V$ and $1 \leqslant i \leqslant n$. If these labels are computed with costs c_β, then they are *piecewise linear* functions in β and we denote them by $f_i(v)[\beta]$.

Denote the optimal β that we look for by β^* and suppose that we know an interval I with such that $\beta^* \in I$ and each function $f_i(v)[\beta]$ is linear if it is restricted to this domain I. Then we can determine β^* as follows.

Let $I = [L,U]$ be the interval and remember that we are searching for the largest value of $\beta \in I$ such that $f_n(v)[\beta] = f_{n-1}(v)[\beta]$ holds for each $v \in V$. Clearly this holds for $\beta = L$. Thus we only need to check whether $\beta = U$ by computing the values $f_n(v)[U]$ and $f_{n-1}(v)[U]$ for each $v \in V$ and check whether one of these pairs consists of different numbers.

The idea is now to compute such an interval $I = [L,U]$ in strongly polynomial time.

Consider the function $f_1(v)[\beta]$. Clearly one has

$$f_1(v)[\beta] = \begin{cases} c(s,v) - \beta \cdot p(s,v) & \text{if } (s,v) \in A, \\ \infty & \text{otherwise.} \end{cases}$$

This shows that $f_1(v)[\beta]$ is a linear function in β for each $v \in V$.

Now suppose that $i \geqslant 1$ and that we have computed an interval $I = [L,U]$ with $\beta^* \in I$ and each function $f_i(v)[\beta]$ is a linear function if β is restricted to I.

Now consider the function $f_{i+1}(v)[\beta]$ for a particular $v \in V$. Recall the formula

$$f_{i+1}(v)[\beta] = \min\{f_i(v)[\beta], \min_{(u,v)\in A}\{f_i(u)[\beta] + c(u,v) - \beta \cdot p(u,v)\}\}. \tag{6.23}$$

Each of the functions $f_i(v)[\beta]$ and $f_i(u)[\beta] + c(u,v) - \beta \cdot p(u,v)$ are linear on I. The function $f_i(v)[\beta]$ can be retrieved by computing a shortest path $P_i(v)$ from s to v with arc weights c_β for some β in (L,U) which uses at most i arcs. If β is then allowed to vary, the line which is defined by $f_i(v)[\beta]$ on I is then the length of this path P with parameter β. Similarly we can retrieve the functions (lines) $f_i(u)[\beta] + c(u,v) - \beta \cdot p(u,v)$ for each $(u,v) \in A$. With the Bellman-Ford algorithm, this amounts to a running time of $O(m \cdot n)$.

We now have n lines and can now compute the lower envelope of these lines in time $O(n \log n)$ alternatively we can also compute all intersection points of these lines and sort them w.r.t. increasing β-coordinate. This would amount to $O(n^2 \log n)$. Let $\beta_1, \ldots, \beta_k$ be the sorted list of these β-coordinates. Now $\beta_{trial} := \beta_{\lfloor k/2 \rfloor}$ and check whether $\beta^* > \beta_{trial}$. If yes, we can replace L by β_{trial} and we can delete the numbers $\beta_1, \ldots, \beta_{\lfloor k/2 \rfloor - 1}$. Otherwise, we replace U by β_{trial} and delete $\beta_{\lfloor k/2 \rfloor + 1}, \ldots, b_k$. In any case, we halved the number of possible β-coordinates and continue in this way. Such a check requires a negative cycle test in the graph D

with arc weights β_{trial} and costs $O(m \cdot n)$. In the end we have two consecutive β-coordinates and have an interval $[L,U]$ on which $f_{i+1}(v)[\beta]$ is linear. To find an interval I such that $f_{i+1}(v)[\beta]$ is linear on I and $\beta^* \in I$ costs thus $O(m \cdot n \log n)$ steps.

We now continue to tighten *this interval* such that all functions $f_{i+1}(v)[\beta], v \in V$ are linear on $[L,U]$. Thus in step $i+1$ this amounts to a running time of

$$O\big(n \cdot (m \cdot n \log n)\big).$$

The total running time is thus

$$O(n^3 \cdot m \cdot \log n).$$

Theorem 6.16. *Let $D = (V,A)$ be a directed graph and let $c : A \longrightarrow \mathbb{R}$ and $p : A \longrightarrow \mathbb{R}_{>0}$ be functions. One can compute a cycle C of D minimizing $c(C)/p(C)$ in time $O(n^3 \cdot m \cdot \log n)$.*

6.7.1.1 Exercises

1) Show that there are no two different paths from r to another node in a directed tree $T = (V,A)$.
2) Prove Lemma 6.3.
3) Why can we assume without loss of generality that a minimum cost network has a path from i to j for all $i \neq j \in V$ which is incapacitated?
4) Provide an example of a MCNFP for which the simple cycle-canceling algorithm from above can require an exponential number of cancels, if the cycles are chosen in a disadvantageous way.
5) Provide a proof of Theorem 6.7.
6) Let $Q =< u_1, \ldots, u_k >$ be the queue before an iteration of the **while** loop of the breadth-first-search algorithm. Show that $D[u_i]$ is monotonously increasing and that $D[u_1] + 1 \geqslant D[u_k]$. Conclude that the sequence of assigned labels (over time) is a monotonously increasing sequence.

Chapter 7
The ellipsoid method

It is not known whether the simplex algorithm is an algorithm that runs in polynomial time. For many pivoting rules it was even proved to require an exponential number of iterations [7]. It was long open, whether there exists a polynomial time algorithm for linear programming until Khachiyan [6] showed that the ellipsoid method[17, 15] can solve linear programs in polynomial time. The remarkable fact is that the algorithm is polynomial in the binary encoding length of the linear program. In other words, if the input consists of the problem $\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant b\}$, where $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$, then the algorithm runs in polynomial time in $m+n+s$, where s is the largest binary encoding length of a rational number appearing in A or b. The question, whether there exists an algorithm which runs in time polynomial in $m+n$ and performs arithmetic operations on numbers, whose binary encoding length remains polynomial in $m+n+s$ is one of the most prominent open problems in theoretical computer science and discrete optimization.

Initially, the ellipsoid method can be used to solve the following problem.

> Given a matrix $A \in \mathbb{Z}^{m\times n}$ and a vector $b \in \mathbb{Z}^m$, determine a feasible point x^* in the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ or assert that P is *not full-dimensional* or P is unbounded.

After we understand how the ellipsoid method solves this problem in polynomial time, we discuss why linear programming can be solved in polynomial time.

Clearly, we can assume that A has full column rank. Otherwise, we can find with Gaussian elimination an invertible matrix $U \in \mathbb{R}^{n\times n}$ with $A\cdot U = [A'|0]$ where A' has full column rank. The system $A'x \leqslant b$ is then feasible if and only if $Ax \leqslant b$ is feasible.

Exercise 7.1. Let x' be a feasible solution of $A'x \leqslant b$ and suppose that U from above is given. Show how to compute a feasible solution x of $Ax \leqslant b$. Also vice versa, show how to compute x', if x is given.

The *unit ball* is the set $B = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$ and an *ellipsoid* $E(A,b)$ is the image of the unit ball under a linear map $t : \mathbb{R}^n \to \mathbb{R}^n$ with $t(x) = Ax + b$, where $A \in \mathbb{R}^{n\times n}$ is an invertible matrix and $b \in \mathbb{R}^n$ is a vector. Clearly

$$E(A,b) = \{x \in \mathbb{R}^n \mid \|A^{-1}x - A^{-1}b\| \leqslant 1\}. \tag{7.1}$$

Exercise 7.2. Consider the mapping $t(x) = \left(\begin{smallmatrix} 1 & 3 \\ 2 & 5 \end{smallmatrix}\right)\left(\begin{smallmatrix} x(1) \\ x(2) \end{smallmatrix}\right)$. Draw the ellipsoid which is defined by t. What are the axes of the ellipsoid?

The *volume* of the unit ball is denoted by V_n, where $V_n \sim \frac{1}{\pi n}\left(\frac{2e\pi}{n}\right)^{n/2}$. It follows that the volume of the ellipsoid $E(A,b)$ is equal to $|\det(A)| \cdot V_n$. The next lemma is the key to the development of the ellipsoid method.

Lemma 7.1 (Half-Ball Lemma). *The half-ball* $H = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1, x(1) \geqslant 0\}$ *is contained in the ellipsoid*

$$E = \left\{ x \in \mathbb{R}^n \mid \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \leqslant 1 \right\} \tag{7.2}$$

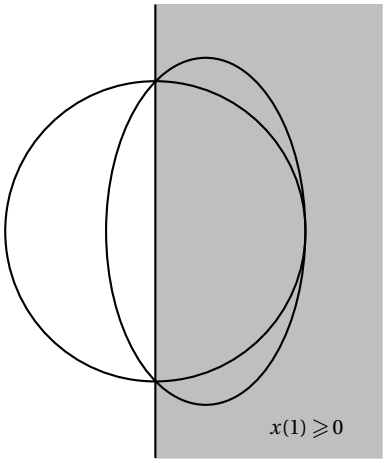


Fig. 7.1: Half-ball lemma.

Proof. Let x be contained in the unit ball, i.e., $\|x\| \leqslant 1$ and suppose further that $0 \leqslant x(1)$ holds. We need to show that

$$\left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \leqslant 1 \tag{7.3}$$

holds. Since $\sum_{i=2}^{n} x(i)^2 \leqslant 1 - x(1)^2$ holds we have

$$\begin{aligned} \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x(i)^2 \\ \leqslant \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(1 - x(1)^2) \end{aligned} \tag{7.4}$$

This shows that (7.3) holds if x is contained in the half-ball and $x(1) = 0$ or $x(1) = 1$. Now consider the right-hand-side of (7.4) as a function of $x(1)$, i.e., consider

$$f(x(1)) = \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(1 - x(1)^2). \tag{7.5}$$

The first derivative is

$$f'(x(1)) = 2 \cdot \left(\frac{n+1}{n}\right)^2 \left(x(1) - \frac{1}{n+1}\right) - 2 \cdot \frac{n^2-1}{n^2} x(1). \tag{7.6}$$

We have $f'(0) < 0$ and since both $f(0) = 1$ and $f(1) = 1$, we have $f(x(1)) \leqslant 1$ for all $0 \leqslant x(1) \leqslant 1$ and the assertion follows.

In terms of a matrix A and a vector b, the ellipsoid E is described as $E = \{x \in \mathbb{R}^n \mid \|A^{-1}x - A^{-1}b\|\}$, where A is the diagonal matrix with diagonal entries

$$\frac{n}{n+1}, \sqrt{\frac{n^2}{n^2-1}}, \ldots, \sqrt{\frac{n^2}{n^2-1}}$$

and b is the vector $b = (1/(n+1), 0, \ldots, 0)$. Our ellipsoid E is thus the image of the unit sphere under the linear transformation $t(x) = Ax + b$. The determinant of A is thus $\frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2}$ which is bounded by

$$e^{-1/(n+1)} e^{(n-1)/(2\cdot(n^2-1))} = e^{-\frac{1}{2(n+1)}}. \tag{7.7}$$

We can conclude the following theorem.

Theorem 7.1. *The half-ball $\{x \in \mathbb{R}^n \mid x(1) \geqslant 0, \|x\| \leqslant 1\}$ is contained in an ellipsoid E, whose volume is bounded by $e^{-\frac{1}{2(n+1)}} \cdot V_n$.*

Recall the following notion from linear algebra. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called positive definite if all its eigenvalues are positive. Recall the following theorem.

Theorem 7.2. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. The following are equivalent.*

i) A is positive definite.
ii) $A = L^T L$, where $L \in \mathbb{R}^{n\times n}$ is a uniquely determined upper triangular matrix.
iii) $x^T Ax > 0$ for each $x \in \mathbb{R}^n \setminus \{0\}$.
iv) $A = Q^T \mathrm{diag}(\lambda_1, \ldots, \lambda_n) Q$, where $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and $\lambda_i \in \mathbb{R}_{>0}$ for $i = 1, \ldots, n$.

It is now convenient to switch to a different representation of an ellipsoid. An ellipsoid $\mathscr{E}(A, a)$ is the set $\mathscr{E}(A, a) = \{x \in \mathbb{R}^n \mid (x-a)^T A^{-1}(x-a) \leqslant 1\}$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $a \in \mathbb{R}^n$ is a vector. Consider the half-ellipsoid $\mathscr{E}(A, a) \cap (c^T x \leqslant c^T a)$.

Our goal is a similar lemma as the half-ball-lemma for ellipsoids. Geometrically it is clear that each half-ellipsoid $\mathscr{E}(A, a) \cap (c^T x \leqslant c^T a)$ must be contained

in another ellipsoid $\mathscr{E}(A', b')$ with $\mathrm{vol}(\mathscr{E}(A', a'))/\mathrm{vol}(\mathscr{E}(A, a)) \leqslant e^{-1/(2n)}$. More precisely this follows from the fact that the half-ellipsoid is the image of the half-ball under a linear transformation. Therefore the image of the ellipsoid E under the same transformation contains the half-ellipsoid. Also, the volume-ratio of the two ellipsoids is invariant under a linear transformation.

We now record the formula for the ellipsoid $\mathscr{E}'(A', a')$. It is defined by

$$a' = a - \frac{1}{n+1} b \tag{7.8}$$

$$A' = \frac{n^2}{n^2-1}\left(A - \frac{2}{n+1} b\, b^T\right), \tag{7.9}$$

where b is the vector $b = Ac/\sqrt{c^T Ac}$. The proof of the correctness of this formula can be found in [5].

Lemma 7.2 (Half-Ellipsoid-Theorem). *The half-ellipsoid $\mathscr{E}(A, b) \cap (c^T x \leqslant c^T a)$ is contained in the ellipsoid $\mathscr{E}'(A', a')$ and one has $\mathrm{vol}(\mathscr{E}')/\mathrm{vol}(\mathscr{E}) \leqslant e^{-1/(2n)}$.*

7.1 The method

Suppose we know the following things of our polyhedron P.

I) We have a number L such that $\mathrm{vol}(P) \geqslant L$ if P is full-dimensional.
II) We have an ellipsoid $\mathscr{E}_{init}$ which contains P if P is bounded.

The ellipsoid method is now easily described.

Algorithm 7.1 (Ellipsoid method exact version).

a) (Initialize): Set $\mathscr{E}(A, a) := \mathscr{E}_{init}$
b) If $a \in P$, then assert $P \neq \emptyset$ and stop
c) If $\mathrm{vol}(\mathscr{E}) < L$, then assert that P is unbounded or P is not full-dimensional
d) Otherwise, compute an inequality $c^T x \leqslant \beta$ which is valid for P and satisfies $c^T a > \beta$ and replace $\mathscr{E}(A, a)$ by $\mathscr{E}(A', a)$ computed with formula (7.8) and goto step b).

Theorem 7.3. *The ellipsoid method computes a point in the polyhedron P or asserts that P is unbounded or not full-dimensional. The number of iterations is bounded by $2 \cdot n \ln(\mathrm{vol}(\mathscr{E}_{init})/L)$.*

Proof. Unless P is unbounded, we start with an ellipsoid which contains P. This then holds for all the subsequently computed ellipsoids. After i iterations one has

$$\mathrm{vol}(\mathscr{E})/\mathrm{vol}(\mathscr{E}_{init}) \leqslant e^{-\frac{i}{2n}}. \tag{7.10}$$

Since we stop when $\mathrm{vol}(\mathscr{E}) < L$, we stop at least after $2 \cdot n \ln(\mathrm{vol}(\mathscr{E}_{init})/L)$ iterations. This shows the claim.

7.1.1 The separation problem

At this point we can already notice a very important fact. Inspect step d of the algorithm. What is required here ? An inequality which is valid for P but not for the center a of $\mathscr{E}(A,a)$. Such an inequality is readily at hand if we have the complete inequality description of P in terms of a system $Cx \leqslant d$. Just pick an inequality which is violated by a. Sometimes however, it is not possible to describe the polyhedron of a combinatorial optimization problem with an inequality system efficiently, simply because the number of inequalities is too large. An example of such a polyhedron is the matching polytope, see Theorem 5.5.

The great power of the ellipsoid method lies in the fact that we do not have to *write down* the polyhedron entirely. We only have to solve the so-called separation problem for the polyhedron, which is defined as follows.

SEPARATION PROBLEM
Given a point $a \in \mathbb{R}^n$ determine, whether $a \in P$ and if not, compute an inequality $c^T x \leqslant \beta$ which is valid for P with $c^T a > \beta$.

Exercise 7.3. We are given an undirected graph $G = (V,E)$. A *spanning tree* T is a subset $T \subseteq E$ of the edges such that T does not contain a cycle and T connects all the vertices V. Consider the following *spanning tree polytope* P_{span}

$$\sum_{e \in E} x(e) = n-1 \tag{7.11}$$

$$\sum_{e \in \delta(U)} x(e) \geqslant 1 \qquad \forall \emptyset \subset U \subset V \tag{7.12}$$

$$x(e) \leqslant 1 \qquad \forall e \in E \tag{7.13}$$

$$x(e) \geqslant 0 \qquad \forall e \in E. \tag{7.14}$$

Let x be an integral solution of P_{span} and define $T = \{e \in E \mid x(e) = 1\}$. The inequality (7.11) ensures that exactly $n-1$ edges are picked. The inequalities (7.12) ensure that T connects the vertices of G. Thus T must be a spanning tree. Clearly, there are exponentially many inequalities of type (7.12). Nevertheless, a fractional solution of this polytope can be computed using the ellipsoid method.

Show that the separation problem for P_{span} can be solved in polynomial time.

Hint: To verify whether a vector $x \in \mathbb{R}^{|E|}_{\geqslant 0}$ fulfills inequalities of type (7.12), it is a good idea to recall the MinCut or MaxFlow problem.

Via binary search even an optimal solution can be computed in polynomial time (in the input length) if we introduce edge costs (you don't have to show that). In the next semester you will see that any optimal basis solution is integral and hence defines an optimal spanning tree w.r.t. the edge costs.

Exercise 7.4. Consider the triangle defined by

$$\begin{aligned} -x(1) - x(2) &\leqslant -2 \\ 3x(1) &\leqslant 4 \\ -2x(1) + 2x(2) &\leqslant 3. \end{aligned}$$

Draw the triangle and simulate the ellipsoid method with starting ellipsoid being the ball of radius 6 around 0. Draw each of the computed ellipsoids with your favorite program (pstrics, maple ...). How many iterations does the ellipsoid method take?

Ignore the occurring rounding errors!

7.2 How to start and when to stop

In our description of the ellipsoid method, we did not explain yet what the initial ellipsoid is and when we can stop with asserting that P is either not full-dimensional or unbounded.

Suppose therefore that $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ is full-dimensional and bounded with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Let B be the largest absolute value of a component of A and b. In this section we will show the following things.

i) The vertices of P are in the box $\{x \in \mathbb{R}^n \mid -n^{n/2}B^n \leqslant x \leqslant n^{n/2}B^n\}$. Thus P is contained in the ball around 0 with radius $n^n B^n$. Observe that the encoding length of this radius is $\text{size}(n^n B^n) = O(n \log n + n \,\text{size}(B))$ which is polynomial in the dimension n and the largest encoding length of a coefficient of A and b.

ii) The volume of P is bounded from below by $1/(n \cdot B)^{3n^2}$.

The following lemma is proved in any linear algebra course.

Lemma 7.3 (Inverse formula and Cramer's rule). *Let $C \in \mathbb{R}^{n \times n}$ be a nonsingular matrix. Then*

$$C^{-1}(j,i) = (-1)^{i+j} \det(C_{ij}) / \det(C),$$

where C_{ij} is the matrix arising from C by the deletion of the i-th row and j-th column. If $d \in \mathbb{R}^n$ is a vector then the j-th component of $C^{-1}d$ is given by $\det(\widetilde{C})/\det(C)$, where $\widetilde{C}$ arises from C be replacing the j-th column with d.

We now define the size of a rational number $r = p/q$ with p and q relatively prime integers, a vector $c \in \mathbb{Q}^n$ and a matrix $A \in \mathbb{Q}^{m \times n}$:

- $\text{size}(r) = 1 + \lceil \log(|p| + 1) \rceil + \lceil \log(|q| + 1) \rceil$
- $\text{size}(c) = n + \sum_{i=1}^{n} \text{size}(c(i))$
- $\text{size}(A) = m \cdot n + \sum_{i=1}^{n} \sum_{j=1}^{m} \text{size}(A(i,j))$

We recall the Hadamard inequality which states that for $A \in \mathbb{R}^{n \times n}$ one has

$$|\det(A)| \leqslant \prod_{i=1}^{n} \|a_i\|, \tag{7.15}$$

where a_i denotes the i-th column of A. In particular, if B is the largest absolute value of an entry in A, then

$$|\det(A)| \leqslant n^{n/2} B^n. \tag{7.16}$$

Now let us inspect the vertices of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$, where A and b are integral and the largest absolute value of any entry in A and b is bounded by B. A vertex is determined as the unique solution of a linear system $A'x = b'$, where $A'x \leqslant b'$ is a subsystem of $Ax \leqslant b$ and A' is invertible. Using Cramer's rule and our observation (7.16) we see that the vertices of P lie in the box $\{x \in \mathbb{R}^n \mid -n^{n/2}B^n \leqslant x \leqslant n^{n/2}B^n\}$. This shows i).

Now let us consider a lower bound on the volume of P. Since P is full-dimensional, there exist $n+1$ affinely independent vertices $v_0, \ldots, v_n$ of P which span a *simplex* in $\mathbb{R}^n$. The volume of this simplex is determined by the formula

$$\frac{1}{n!} \cdot \left| \det \begin{pmatrix} 1 & \cdots & 1 \\ v_0 & \ldots & v_n \end{pmatrix} \right|. \tag{7.17}$$

By Cramer's rule and the Hadamard inequality, the common denominator of each component of v_i can be bounded by $n^{n/2}B^n$. Thus (7.17) is bounded by

$$1/\left(n^n (n^{\frac{n}{2}} \cdot B^n)^{n+1}\right) \geqslant 1/\left(n^{3n^2} B^{2n^2}\right) \geqslant 1/(n \cdot B)^{3 \cdot n^2}, \tag{7.18}$$

which shows ii).

Now we plug these values into our analysis in Theorem 7.3. Our initial volume $\mathrm{vol}(\mathscr{E}_{init})$ is bounded by the volume of the box with side-lengths $2(n \cdot B)^n$. Thus

$$\mathrm{vol}(\mathscr{E}_{init}) \leqslant (2 \cdot n \cdot B)^{n^2}. \tag{7.19}$$

Above we have shown that

$$L \geqslant 1/(n \cdot B)^{3n^2}. \tag{7.20}$$

Clearly

$$\mathrm{vol}(\mathscr{E}_{init})/L \leqslant (n \cdot B)^{4 \cdot n^2}. \tag{7.21}$$

By Theorem 7.3 the ellipsoid method performs

$$O\left(2 \cdot n \cdot \ln\left((n \cdot B)^{4 \cdot n^2}\right)\right) \tag{7.22}$$

iterations. This is bounded by

$$O(n^3 \cdot \ln(n \cdot B)). \tag{7.23}$$

Now recall that $\log B$ is the number of *bits* which are needed to encode the coefficient with the largest absolute value of the constraint system $Ax \leqslant b$ and that n is the number of variables of this system. Therefore the expression (7.23) is poly-

nomial in the binary input encoding of the system $Ax \leqslant b$. We conclude the following theorem.

Theorem 7.4. *The ellipsoid method (exact version) performs a polynomial number of iterations.*

7.3 The boundedness and full-dimensionality condition

In this section we want to show how the ellipsoid method can be used to solve the following problem.

> Given a matrix $A \in \mathbb{Z}^{m\times n}$ and a vector $b \in \mathbb{Z}^m$, determine a feasible point x^* in the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ or assert that $P = \emptyset$.

7.3.1 Boundedness

We have argued that the matrix $A \in \mathbb{Z}^{m\times n}$ can be assumed to have full column rank. So, if P is not empty, then P does have at least one vertex. The vertices are contained in the box $\{x \in \mathbb{R}^n \mid -n^{n/2}B^n \leqslant x \leqslant n^{n/2}B^n\}$. Therefore, we can append the inequalities $-n^{n/2}B^n \leqslant x \leqslant n^{n/2}B^n$ to $Ax \leqslant b$ without changing the status of $P \neq \emptyset$ or $P = \emptyset$. Notice that the *binary encoding length* of the new inequalities is polynomial in the binary encoding length of the old inequalities.

7.3.2 Full-dimensionality

Exercise 7.5. Let $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ be a polyhedron and $\varepsilon > 0$ be a real number. Show that $P_\varepsilon = \{x \in \mathbb{R}^n \mid Ax \leqslant b + \varepsilon \cdot \mathbf{1}\}$ is full-dimensional if $P \neq \emptyset$.

The above exercise raises the following question. Is there an $\varepsilon > 0$ such that $P_\varepsilon = \emptyset$ if and only if $P = \emptyset$ and furthermore is the binary encoding length of this ε polynomial in the binary encoding length of A and b?

Recall Farkas' Lemma (Theorem 2.9 and Exercise 10 of chapter 2).

Theorem 7.5. *The system $Ax \leqslant b$ does not have a solution if and only if there exists a nonnegative vector $\lambda \in \mathbb{R}^m_{\geqslant 0}$ such that $\lambda^T A = 0$ and $\lambda^T b = -1$.*

Let $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ and let B be the largest absolute value of a coefficient of A and b. If $Ax \leqslant b$ is not feasible, then there exists a $\lambda \geqslant 0$ such that $\lambda^T(A|b) = (\mathbf{0}|-1)$. We want to estimate the largest absolute value of a coefficient of λ with Cramer's rule and the Hadamard inequality. We can choose λ such that the nonzero coefficients of λ are the unique solution of a system of equations

$Cx = d$, where each coefficient has absolute value at most B. By Cramer's rule and the Hadamard inequality we can thus choose λ such that $|\lambda(i)| \leqslant (n \cdot B)^n$. Now let $\varepsilon = 1/((n+1) \cdot (n \cdot B)^n)$. Then $|\lambda^T \mathbf{1} \cdot \varepsilon| < 1$ and thus

$$\lambda^T (b + \varepsilon \cdot \mathbf{1}) < 0. \tag{7.24}$$

Consequently the system $Ax \leqslant b + \varepsilon \mathbf{1}$ is infeasible if and only of $Ax \leqslant b$ is infeasible. Notice again that the encoding length of ε is polynomial in the encoding length of $Ax \leqslant b$ and we conclude with the main theorem of this section.

Theorem 7.6. *The ellipsoid method can be used to decide whether a system of inequalities $Ax \leqslant b$ contains a feasible point, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. The number of iterations is bounded by a polynomial in n and $\log B$, where B is the largest absolute value of a coefficient of A and b.*

7.4 The ellipsoid method for optimization

Suppose that you want to solve a linear program

$$\max\{c^T x \mid x \in \mathbb{R}^n,\ Ax \leqslant b\} \tag{7.25}$$

and recall that if (7.25) is bounded and feasible, then so is its dual and the two objective values are equal. Thus, we can use the ellipsoid method to find a point (x, y) with $c^T x = b^T y$, $Ax \leqslant b$ and $A^T y = c, y \geqslant 0$.

However, we mentioned that the strength of the ellipsoid method lies in the fact that we do not need to write the system $Ax \leqslant b$ down explicitly. The only thing which has to be solvable is the *separation problem*. This is to be exploited in the next exercise.

Exercise 7.6. Show how to solve the optimization problem $\max\{c^T x \mid Ax \leqslant b\}$ with a polynomial number of calls to an algorithm which solves the separation problem for $Ax \leqslant b$. You may assume that A has full column rank and the polynomial bound on the number of calls to the algorithm to solve the separation problem can depend on n and the largest size of a component of A, b and c.

7.5 Numerical issues

We did not discuss the numerical details on how to implement the ellipsoid method such that it runs in polynomial time. One issue is crucial.

> We only want to compute with a precision which is polynomial in the input encoding!

In the formula (7.8) the vector b is defined by taking a square root. The question thus rises on how to round the numbers in the intermediate ellipsoids such that they can be handled on a machine. Also one has to analyze the growth of the numbers in the course of the algorithm. All these issues can be overcome but we do not discuss them in this course. I would like to refer you to the book of Alexander Schrijver [16] for further details. They are not difficult, but a little technical.

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