Situation: Pythagorean Theorem

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Prompt

In a lesson, students were investigating and practicing problems involving the Pythagorean theorem. During a class discussion, a student asked, "Does the Pythagorean theorem only work for right triangles?"

Commentary

The question raised in this prompt is one that is commonly raised among students when exploring the Pythagorean Theorem. This first set of foci aims to examine the geometric concepts behind the theorem, its algebraic representation, as well as the converse of the theorem. The last focus will examine a generalization of the Pythagorean Theorem that is applicable to all triangles.

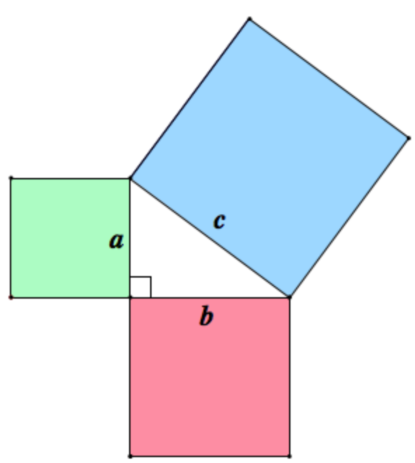
Mathematical Foci

Mathematical Focus 1

The Pythagorean Theorem is a relation in Euclidean geometry among the three sides of a right triangle stating that if a triangle is a right triangle, then the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

Algebraically, the Pythagorean Theorem can be stated that if a triangle is a right triangle where *a* and *b* are the lengths of the legs and *c* is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

The diagram below illustrates the relationship described by the Pythagorean Theorem.



In order to address the question posed in the prompt it seems essential to validate the underlying notion that the Pythagorean Theorem works in the case of any given right triangle. In order to do so, this focus will examine different proofs of the Pythagorean Theorem. While the discovery and proof of the Pythagorean Theorem is attributed to Pythagoras, a Greek mathematician who lived in the 6^{th} century B.C., there is evidence that other ancient cultures, including the Chinese, Egyptians, and Babylonians, also knew about and used this relationship. To date, there are numerous proofs of this theorem, thus only three will be presented in this focus.

Proof One:

Begin by constructing a right triangle with side lengths a, b, and c, where c is the length of the triangle's hypotenuse. Three identical triangles will then be constructed. Using the 4 identical triangles, a square will be formed with side lengths $a + b$, shown as Figure 1 below. Notice the area of the gray square shown in Figure 1 is c^2.

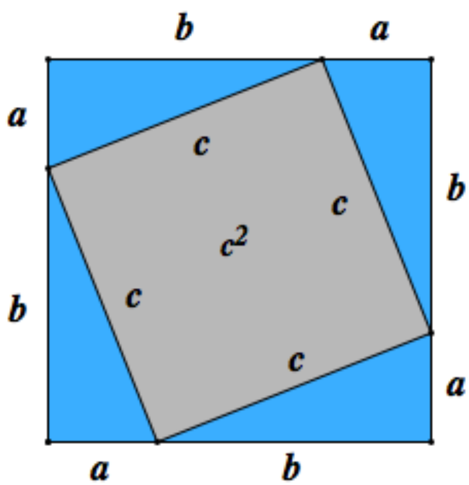


Figure 1

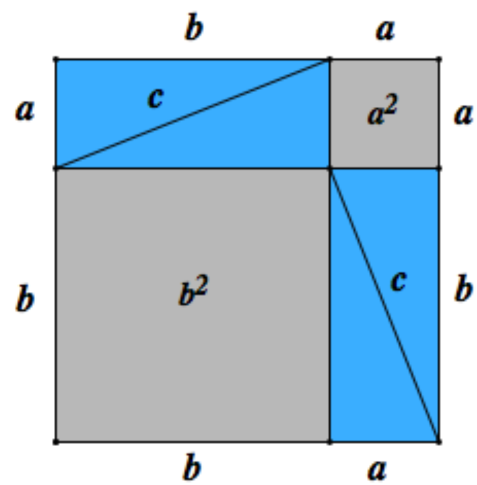


Figure 2

To construct Figure 2, the same 4 identical triangles can be rearranged to result in a square with the same side lengths shown in Figure 1. Thus the area of the gray square shown in Figure 1 is equal to the sum of the areas of the gray squares shown in Figure 2, i.e. $a^2 + b^2 = c^2$.

Proof Two:

For this proof, only Figure 1 above will be utilized. To carry out this proof, we will compute the area of the square with side lengths $a + b$ using two methods. For the first, the formula for the area of a square will be used. For the second, the area of the 4 congruent triangles will be added to the area of the smaller gray square, which has area c^2.

Let A be the area of the square with side lengths $a + b$. Then by the area formula of a square:

$$A = (a + b)(a + b) = a^2 + 2ab + b^2$$

The area of the 4 congruent triangles and the smaller gray square is then:

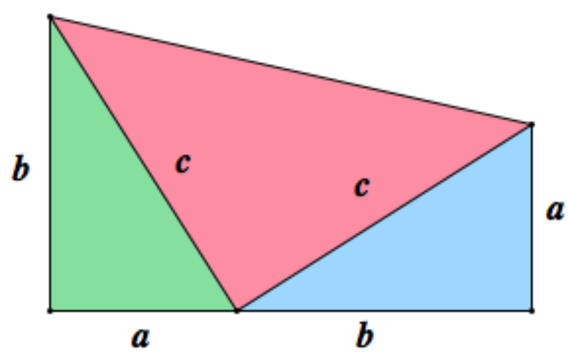
$$A = 4\left(\frac{1}{2}\right)(ab) + c^2 = 2ab + c^2$$

Since the two methods above calculated the area of the same square, then the two area expressions can be equated. Thus equating and simplifying the two area expressions gives:

$$a^2 + 2ab + b^2 = 2ab + c^2 \quad \Rightarrow \quad a^2 + b^2 = c^2$$

Proof Three:

The following proof was one discovered by President James A. Garfield in 1876 while he was a member of the House of Representatives. His proof involved the fact that a trapezoid can be formed from two congruent right triangles and an isosceles right triangle. The proof requires calculating the area of a right trapezoid by two methods, first by using the area formula of a trapezoid and second by summing the area of the three right triangles constructed in the trapezoid. The two expressions for the trapezoid's area are then equated to finish the proof. The trapezoid illustrated below is the one he used to develop this proof, followed by the proof itself.



Let A be the area of the trapezoid shown. Then by the area formula of a trapezoid:

$$A = \frac{(Altitude\ of\ Trapezoid)(Sum\ of\ the\ Base\ Lengths)}{2} = \frac{(a+b)(a+b)}{2} = \frac{a^2 + 2ab + b^2}{2}$$

The area as the sum of the three right triangles is then:

$$A = \left(\frac{1}{2}\right)(ab) + \left(\frac{1}{2}\right)(ab) + \left(\frac{1}{2}\right)(c^2) = \frac{2ab + c^2}{2}$$

Equating the two area expressions and simplifying thus gives:

$$\frac{a^2 + 2ab + b^2}{2} = \frac{2ab + c^2}{2} \quad \Rightarrow \quad a^2 + 2ab + b^2 = 2ab + c^2 \quad \Rightarrow \quad a^2 + b^2 = c^2$$

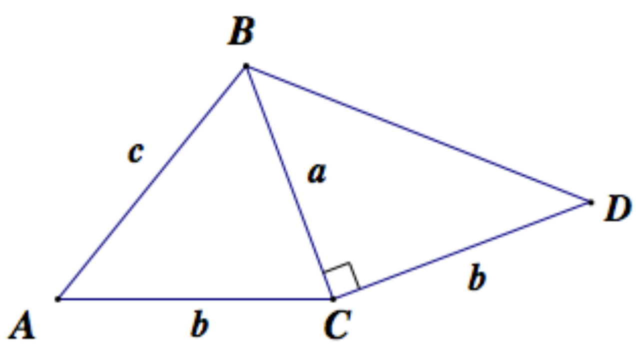
Mathematical Focus 2

If the sum of the squares of the two shorter sides of a triangle is equal to the square of the longest side of a triangle then the triangle is a right triangle.

Algebraically, the Converse of the Pythagorean Theorem can be stated that if $\mathrm{a}^2 + \mathrm{b}^2 = \mathrm{c}^2$, where *a* and *b* are the lengths of the shorter sides of a triangle and *c* is the length of the longest side of a triangle, then the triangle is a right triangle.

Thus the Converse of the Pythagorean Theorem can be used to verify whether a given triangle is a right triangle. Now that the converse has been stated, let us show its proof.

Given: $\triangle ABC$ with $a^2 + b^2 = c^2$.
Prove: $\triangle ABC$ is a right triangle, with right angle at C.

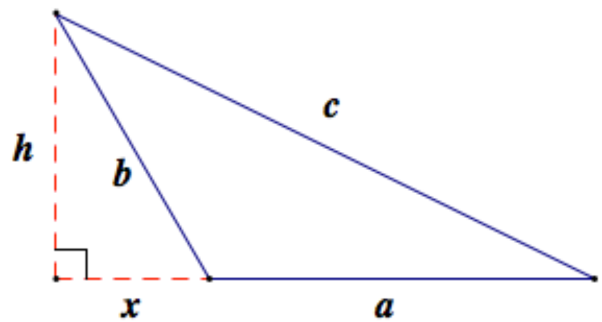


Proof: Construct line segment $\overline{CD} \perp \overline{BC}$ so that $\overline{CD}$ has length b, then construct $\overline{BD}$ (shown above). Now $\overline{AC} \cong \overline{CD}$, and by the Reflexive Property, $\overline{BC} \cong \overline{BC}$. Since $\triangle$BCD is a right triangle then $BC^2 + CD^2 = BD^2$ by the Pythagorean theorem. Further since $BC = a$ and $CD = b$, then $a^2 + b^2 = BD^2$. Given $\triangle ABC$ with $a^2 + b^2 = c^2$, then $BD^2 = c^2$ and thus $BD = c$ since length is never negative. Therefore $\triangle ABC \cong \triangle DBC$ by SSS postulate and it follows that $\triangle ABC$ is a right triangle.

Mathematical Focus 3

Given sides a, b, and c to be the lengths of the three sides of triangle, with length c being the longest and $a + b > c$ by the triangle inequality, then if the triangle is obtuse, then $a^2 + b^2 < c^2$ and if the triangle is acute, then $a^2 + b^2 > c^2$.

If the triangle is obtuse, then $\mathbf{a}^2 + \mathbf{b}^2 < \mathbf{c}^2$:
Consider an obtuse triangle with largest side *c* and consider *a* to be the base. Let *h* denote the altitude to the base *a*, then *h* falls outside of the triangle and intersects the extension of base *a*. Let *x* be the extension of *a* until it meets *h*. This now results in the formation of two right triangles.

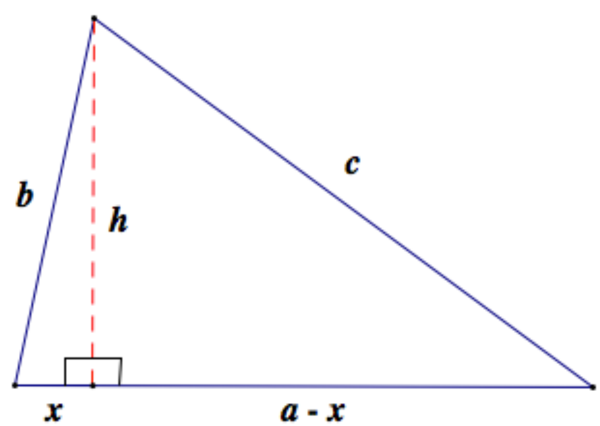


One right triangle that has been formed has h and x as the lengths of the legs and b as the length of the hypotenuse. Thus $b^2 = h^2 + x^2$.

The other right triangle that has been formed has h and $(a + x)$ as the lengths of the legs and c as the length of the hypotenuse. Thus $c^2 = h^2 + (a + x)^2 = h^2 + a^2 + 2ax + x^2$ and substituting b^2 for $h^2 + x^2$ results in $c^2 = a^2 + b^2 + 2ax$.

If the Pythagorean theorem were to hold for an obtuse triangle then it should be the case that $a^2 + b^2 = c^2$, but $a^2 + b^2 + 2ax = c^2$. Thus $a^2 + b^2 < a^2 + b^2 + 2ax$ by a factor of $2ax$. Therefore in any obtuse triangle with side lengths a, b, and c, with side length c being the longest, it always holds that $a^2 + b^2 < c^2$.

If the triangle is acute, then $\mathbf{a^2 + b^2 > c^2}$:
Consider an acute triangle with largest side c and consider a to be the base. Let h denote the altitude to the base a, then h falls onto an internal point of base a and cuts it into two segments. Let x be the segment under side b and $(a - x)$ under side c. This now results in the formation of two right triangles.



One right triangle that has been formed has h and x as the lengths of the legs and b as the length of the hypotenuse. Thus $b^2 = h^2 + x^2$.

The other right triangle that has been formed has h and $(a - x)$ as the lengths of the legs and c as the length of the hypotenuse. Thus $c^2 = h^2 + (a - x)^2 = h^2 + a^2 - 2ax + x^2$ and substituting b^2 for $h^2 + x^2$ results in $c^2 = a^2 + b^2 - 2ax$.

If the Pythagorean theorem were to hold for an acute triangle then it should be the case that $a^2 + b^2 = c^2$, but $a^2 + b^2 - 2ax = c^2$. Thus $a^2 + b^2 > a^2 + b^2 - 2ax$ by a factor of $2ax$. Therefore in any acute triangle with side lengths a, b, and c, with side length c being the longest, it always holds that $a^2 + b^2 > c^2$.

Mathematical Focus 4

The Pythagorean Theorem is a special case of the Law of Cosines which is a more general theorem relating the lengths of sides in any triangle.

The Law of Cosines states that given any triangle with side lengths a, b, and c and opposing angles A, B, and C that

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$b^2 = a^2 + c^2 - 2ac\cos(B)$$
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

The proofs shown below will be for $c^2 = a^2 + b^2 - 2ab\cos(C)$. Now consider three cases :

Case 1: $\triangle ABC$ is a right triangle.

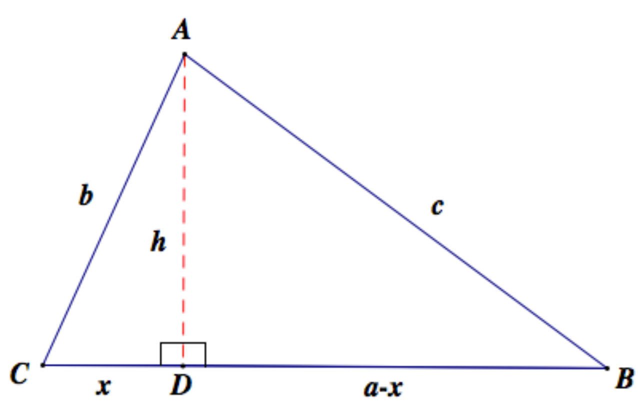
Let's begin with our original equation of $c^2 = a^2 + b^2 - 2ab\cos C$. Since c is the hypotenuse of this triangle, then $\angle C$ is opposite side c and is thus equal to 90°.

If $\angle C = 90°$, then $\cos(C) = \cos(90°) = 0$, which makes $2ab\cos C = 0$.

Therefore $c^2 = a^2 + b^2 - 2ab\cos C$ becomes $c^2 = a^2 + b^2$, the Pythagorean formula.

Case 2: $\triangle ABC$ is an acute triangle.

Consider $\overline{BC}$ to be the base and construct a perpendicular from point A to $\overline{BC}$ to create the point D as shown below.



This now results in right triangle ADC, thus $\cos(C) = \frac{x}{b}$ which is equivalent to $x = b\cos(C)$. By the Pythagorean Theorem, $b^2 = h^2 + x^2$ so that $h^2 = b^2 - x^2$. Since $\triangle ADB$ is also a right triangle, applying the Pythagorean Theorem gives $c^2 = h^2 + (a - x)^2$. Substituting for h^2, simplifying and rearranging the terms then gives:

$$c^2 = b^2 - x^2 + (a - x)^2$$

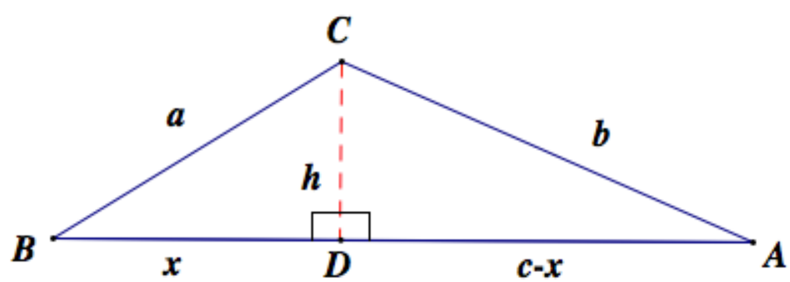
$$c^2 = b^2 - x^2 + a^2 - 2ax + x^2$$

$$c^2 = a^2 + b^2 - 2ax$$

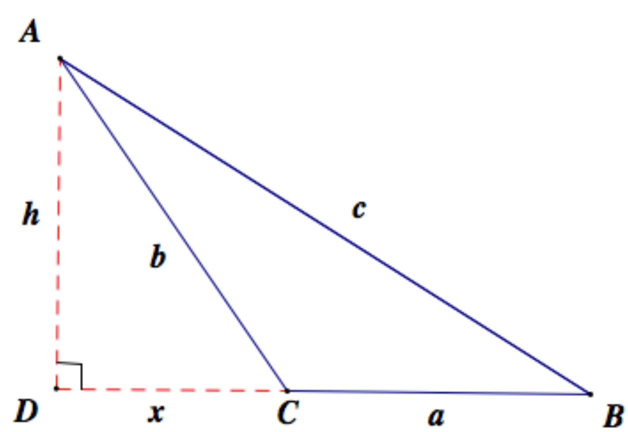
Since $x = b\cos(C)$, then $c^2 = a^2 + b^2 - 2ab\cos(C)$. By applying a similar method as the one done above it can be shown that $a^2 = b^2 + c^2 - 2bc\cos(A)$ and $b^2 = a^2 + c^2 - 2ac\cos(B)$.

Case 3: $\triangle ABC$ is an obtuse triangle.

Consider the triangle to be arranged such that the base is the side opposite the obtuse angle, C. Construct a perpendicular from C to $\overline{AB}$ to create the point D as shown below.



Observing the obtuse triangle in this manner will help in solving for $a^2 = b^2 + c^2 - 2bc\cos(A)$ and $b^2 = a^2 + c^2 - 2ac\cos(B)$ which can be done by using the same technique employed in Case 2. Thus the only case that needs to be checked now is if the triangle is rotated so that $\overline{BC}$ is the base. Construct a perpendicular from A to $\overline{BC}$ to create the point D shown below.



This results in right triangle ADC and since $\angle C$ is obtuse, $\cos(C) = -\cos(180° - C) = -\frac{x}{b}$ which is equivalent to $x = -b\cos(C)$. By the Pythagorean Theorem, $b^2 = h^2 + x^2$ or $h^2 = b^2 - x^2$. Since $\triangle ADB$ is also a right triangle, applying the Pythagorean Theorem gives $c^2 = h^2 + (a + x)^2$. Substituting for h^2, simplifying and rearranging the terms then gives:

$$c^2 = b^2 - x^2 + (a + x)^2$$

$$c^2 = b^2 - x^2 + a^2 + 2ax + x^2$$

$$c^2 = a^2 + b^2 + 2ax$$

Since $x = -b\cos(C)$, then $c^2 = a^2 + b^2 - 2ab\cos(C)$. Thus the Law of Cosines has been established.

References

Bogomolny, A. (n.d.). Pythagorean Theorem and its many proofs. *Interactive Mathematics Miscellany and Puzzles*. Retrieved July 15, 2013, from http://www.cut-the-knot.org/pythagoras/index.shtml