

THE RIEMANN HYPOTHESIS

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0.1 Abstract

The Proof involves Functional Equation of the Riemann Zeta function defined on the whole of Complex plane except for a Pole at $s=1$.

Further we prove that the absolute value of the Riemann Zeta Function is monotonically decreasing and monotonically increasing on specific intervals respectively. By using the above monotonicity property of zeta , work on it's non trivial zeroes

THE RIEMANN HYPOTHESIS (1859): The real part of every non trivial zero of Riemann Zeta Function is $1/2$.

0.2 Proof

The analytic continuation (Ref.-[1]) of the Zeta Function is

$$\zeta^*(s) = s \int_0^\infty ([x] - x)/x^{s+1} dx \quad ; 0 < \text{Re}(s) < 1$$

let, $s = \sigma + i\eta$; $0 < \sigma < 1$.

The Functional equation of the Riemann Zeta function is given as,

$$\zeta^*(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s)$$

$$\zeta^*(\sigma + it) = 2^{\sigma+i\eta} (\pi)^{\sigma-1+i\eta} \sin(\pi(\sigma + i\eta)/2) \Gamma(1-\sigma - i\eta) \zeta(1-\sigma - i\eta)$$

$$| \zeta^*(\sigma + it) | / | \zeta^*(\rho + it) | =$$

$$\frac{(2\pi)^{\sigma-\rho} | \sin(\pi(\sigma + i\eta)/2) | | \Gamma(1-\sigma - i\eta) | | \zeta(1-\sigma - i\eta) |}{| \sin(\pi(\rho + i\eta)/2) | | \Gamma(1-\rho - i\eta) | | \zeta(1-\rho - i\eta) |}$$

0.3 Claim

$|\zeta^*(\sigma + i\eta)|$ is monotonically increasing.

To Prove : $|\zeta^*(\sigma + i\eta)| \leq |\zeta^*(\rho + i\eta)|$.

Idea is to prove each term in Eq.-A less than 1.

let, $\sigma < \rho$.

Consider,

$$\begin{aligned} & |\sin(\pi(\sigma + i\eta)/2)|^2 - |\sin(\pi(\rho + i\eta)/2)|^2 \\ &= |\sin(\pi\sigma/2)\cosh(\pi\eta/2) + i\cos(\pi\sigma/2)\sinh(\pi\eta/2)|^2 - \\ & |\sin(\pi\rho/2)\cosh(\pi\eta/2) + i\cos(\pi\rho/2)\sinh(\pi\eta/2)|^2 \\ &\leq [\sin^2(\pi\sigma/2)\cosh^2(\pi\eta/2) + (\cos^2(\pi\sigma/2)\sinh^2(\pi\eta/2)) - \\ & (\sin^2(\pi\rho/2)\cosh^2(\pi\eta/2) - (\cos^2(\pi\rho/2)\sinh^2(\pi\eta/2))] \end{aligned}$$

Since, $\sin\theta$ is increasing and $\cos\theta$ is decreasing on $[1/2, 1) \subset (0, \pi)$

$$\begin{aligned} &\leq [\sin^2(\pi\rho/2)\cosh^2(\pi\eta/2) + (\cos^2(\pi\sigma/2)\sinh^2(\pi\eta/2)) - \\ & (\sin^2(\pi\rho/2)\cosh^2(\pi\eta/2) - (\cos^2(\pi\rho/2)\sinh^2(\pi\eta/2))] \\ & (\cos^2(\pi\sigma/2)\sinh^2(\pi\eta/2))^2 - \\ & (\cos^2(\pi\rho/2)\sinh^2(\pi\eta/2)) \end{aligned}$$

Since, $1/2 \leq \rho < 1$.

$$\begin{aligned} &\leq (\cos^2(\pi/2) \sinh^2(\pi\eta/2))^2 - \\ &(\cos^2(\pi\rho/2) \sinh^2(\pi\eta/2)) \\ &\leq (\cos^2(\pi/2) \sinh^2(\pi\eta/2))^2 \leq 0. \end{aligned}$$

thus,

$$\begin{aligned} &| \sin(\pi(\sigma + i\eta)/2) |^2 \leq | \sin(\pi(\rho + i\eta)/2) |^2 \text{ hence,} \\ &| \sin(\pi(\sigma + i\eta)/2) | \leq | \sin(\pi(\rho + i\eta)/2) |. \quad [Eq. - 1]. \end{aligned}$$

$$\begin{aligned} &| \Gamma(1 - \sigma - i\eta) | = \left| \int_0^\infty e^{-t} t^{-\sigma - i\eta} dt \right| \\ &\leq \int_0^\infty e^{-t} t^{-\sigma} dt \end{aligned}$$

Now, $1/2 \leq \sigma < 1$,

$$1/t < 1/t^\sigma \leq 1/t^{1/2}.$$

Using the above inequality,

$$\begin{aligned} &| \Gamma(1 - \sigma - i\eta) | \leq \int_0^\infty e^{-t} t^{-1/2} dt \\ &| \Gamma(1 - \sigma - i\eta) | \leq \gamma(1/2) \\ &| \Gamma(1 - \sigma - i\eta) | \leq \pi^{1/2} \end{aligned}$$

Similarly, since ρ also belongs to the same domain $[1/2, 1)$

$$| \Gamma(1 - \rho - i\eta) | \leq \pi^{1/2}$$

$$| \Gamma(1 - \sigma - i\eta) | - | \Gamma(1 - \rho - i\eta) | \leq 0.$$

$$| \Gamma(1 - \sigma - i\eta) | \leq | \Gamma(1 - \rho - i\eta) | \text{ .. - Eq. - 2.}$$

Thus,

$$1/2 \leq \sigma < \rho < 1,$$

Implies,

$$| \zeta^*(\sigma + i\eta) | \leq | \zeta^*(\rho + i\eta) |.$$

So, $\zeta^*(\sigma + it)$ is increasing on $[1/2, 1)$.

$$| \zeta(1 - \sigma - i\eta) | = | \sum_{n=1}^{\infty} 1/n^{1-\sigma-i\eta} |$$

; where] $n^{-in} = e^{-i\eta \ln n}$

$$\leq \sum_{n=1}^{\infty} 1/n^{1-\sigma}$$

$$\sigma < \rho \text{ implies } 1/n^{1-\sigma} < 1/n^{1-\rho}$$

$$\leq \sum_{n=1}^{\infty} 1/n^{1-\rho}$$

$$| \zeta(1 - \sigma - i\eta) | < \sum_{n=1}^{\infty} 1/n^{1-\rho}$$

$$| \zeta(1 - \rho - i\eta) | < \sum_{n=1}^{\infty} 1/n^{1-\rho} \text{ So,}$$

$$| \zeta(1 - \sigma - i\eta) | - | \zeta(1 - \rho - i\eta) | \leq 0$$

$$| \zeta(1 - \sigma - i\eta) | < | \zeta(1 - \rho - i\eta) | - Eq - 3$$

Also, $\sigma < \rho$ implies

$$2\pi^{\sigma-\rho} < 1 \quad Eq - 4$$

So, using Eq. 1, 2, 3 and 4 in Eq A,

When, $\sigma < \rho$; $\sigma, \rho \in [1/2, 1)$

$$| \zeta^*(\sigma + i\eta) | \leq | \zeta^*(\rho + i\eta) |$$

Hence, $| \zeta^*(\sigma + i\eta) |$ is Monotonically Increasing on $[1/2, 1)$,

0.4 Claim

$\zeta^*(\sigma + i\eta)$ is Monotonically Decreasing on $(0, 1/2]$

Let, $\sigma^1 = 1 - \sigma$

$$\rho^1 = 1 - \rho$$

Since, $1/2 \leq \sigma < \rho < 1$

Thus, $0 < \rho^1 < \sigma^1 \leq 1/2$.

$$| \zeta^*(1 - \sigma^1 + i\eta) | \leq | \zeta^*(1 - \rho^1 + i\eta) |.$$

we know $| \zeta(s) | = | \zeta(1 - s) |$

Thus,

when $\rho^1 < \sigma^1$,

$$|\zeta^*(\sigma^1 - i\eta)| \leq |\zeta^*(\rho^1 - i\eta)|.$$

So, $|\zeta^*(\sigma + i\eta)|$ is Monotonically Increasing on $[1/2, 1)$, and

$|\zeta^*(\sigma + i\eta)|$ is Monotonically Decreasing on $(0, 1/2]$

For, $1/2 \leq \sigma < 1$,

$$|\zeta^*(1/2 + it)| \leq |\zeta^*(\sigma + it)| < |\zeta^*(1 + it)|$$

.

For, $0 \leq \sigma \leq 1/2$,

$$|\zeta^*(1/2 + it)| < |\zeta^*(\sigma + it)| \leq |\zeta^*(0 + it)|.$$

Combining above two inequalities, for all $\sigma \in (0, 1/2] \cup [1/2, 1)$,

$$|\zeta^*(1/2 + it)| \leq |\zeta^*(\sigma + it)|.$$

Now $\zeta^*(\sigma + it) = 0$ for, $0 < \sigma < 1$.

$$|\zeta^*(1/2 + it)| \leq 0$$

$$\zeta^*(1/2 + it) = 0$$

0.5 Case

let,

$$\zeta^*(\sigma + i\eta) = 0 \text{ and } \sigma \neq 1/2$$

then,

$$0 \leq |\zeta^*(1/2 + it)| \leq |\zeta^*(\sigma + it)| \leq \zeta_*(1 + it)$$

but,

$$\zeta_*(\sigma + it) = 0, \text{ for } 0 < \sigma < 1 \text{ implies}$$

$$|\zeta_*(1/2 + it)| \leq 0$$

this implies ,

$$|\zeta^*(1/2 + it)| = 0$$

which is a contradiction to our assumption that

$$|\zeta^*(\sigma + i\eta)| = 0$$

if $\sigma \neq 1/2$,

So we have non trivial zeroes of $\zeta^*(s)$ implies $\sigma = 1/2$, on the line,

$$\sigma = 1/2$$

0.6 References:-

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