Learning Linear Dynamical Systems via Spectral Filtering

Elad Hazan Karan Singh Cyril Zhang

Department of Computer Science
Princeton University
Princeton, NJ 08540
{ehazan, karans, cyril.zhang}@cs.princeton.edu

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Abstract

We present an efficient and practical algorithm for the online prediction of discrete-time linear dynamical systems with a symmetric transition matrix. We circumvent the non-convex optimization problem using improper learning: carefully overparameterize the class of LDSs by a polylogarithmic factor, in exchange for convexity of the loss functions. From this arises a polynomial-time algorithm with a near-optimal regret guarantee, with an analogous sample complexity bound for agnostic learning. Our algorithm is based on a novel filtering technique, which may be of independent interest: we convolve the time series with the eigenvectors of a certain Hankel matrix.

1 Introduction

Linear dynamical systems (LDSs) are a class of state space models which accurately model many phenomena in nature and engineering, and are applied ubiquitously in time-series analysis, robotics, econometrics, medicine, and meteorology. In this model, the time evolution of a system is explained by a linear map on a finite-dimensional hidden state, subject to disturbances from input and noise. Recent interest has focused on the effectiveness of recurrent neural networks (RNNs), a nonlinear variant of this idea, for modeling sequences such as audio signals and natural language.

Central to this field of study is the problem of *system identification*: given some sample trajectories, output the parameters for an LDS which generalize to predict unseen future data. Viewed directly, this is a non-convex optimization problem, for which efficient algorithms with theoretical guarantees are very difficult to obtain. A standard heuristic for this problem is expectation-maximization (EM), which can find poor local optima in theory and practice.

We consider a different approach: we formulate system identification as an online learning problem, in which neither the data nor predictions are assumed to arise from an LDS. Furthermore, we slightly overparameterize the class of predictors, yielding an online convex program amenable to efficient regret minimization. This carefully chosen relaxation, which is our main theoretical contribution, expands the dimension of the hypothesis class by only a polylogarithmic factor. This construction relies upon recent work on the spectral theory of Hankel matrices.

The result is a simple and practical algorithm for time-series prediction, which deviates significantly from existing methods. We coin the term *wave-filtering* for our method, in reference to our relaxation's use of convolution by wave-shaped eigenvectors. We present experimental evidence on both toy data and a physical simulation, showing our method to be competitive in terms of predictive performance, more stable, and significantly faster than existing algorithms.

1.1 Our contributions

Consider a discrete-time linear dynamical system with inputs $\{x_t\}$, outputs $\{y_t\}$, and a latent state $\{h_t\}$, which can all be multi-dimensional. With noise vectors $\{\eta_t\}, \{\xi_t\}$, the system's time evolution is governed by the following equations:

$$h_{t+1} = Ah_t + Bx_t + \eta_t$$
$$y_t = Ch_t + Dx_t + \xi_t.$$

If the dynamics A, B, C, D are known, then the Kalman filter [Kal60] is known to estimate the hidden state optimally under Gaussian noise, thereby producing optimal predictions of the system's response to any given input. However, this is rarely the case – indeed, real-world systems are seldom purely linear, and rarely are their evolution matrices known.

We henceforth give a provable, efficient algorithm for the prediction of sequences arising from an unknown dynamical system as above, in which the matrix A is symmetric. Our main theoretical contribution is a regret bound for this algorithm, giving nearly-optimal convergence to the lowest mean squared prediction error (MSE) realizable by a symmetric LDS model:

Theorem 1 (Main regret bound; informal). On an arbitrary sequence $\{(x_t, y_t)\}_{t=1}^T$, Algorithm 1 makes predictions $\{\hat{y}_t\}_{t=1}^T$ which satisfy

$$\mathrm{MSE}(\hat{y}_1, \dots, \hat{y}_T) - \mathrm{MSE}(\hat{y}_1^*, \dots, \hat{y}_T^*) \leq \tilde{O}\left(\frac{\mathrm{poly}(n, m, d, \log T)}{\sqrt{T}}\right),$$

compared to the best predictions $\{y_t^*\}_{t=1}^T$ by a symmetric LDS, while running in polynomial time.

Note that the signal need not be generated by an LDS, and can even be *adversarially* chosen. In the less general batch (statistical) setting, we use the same techniques to obtain an analogous sample complexity bound for agnostic learning:

Theorem 2 (Batch version; informal). For any choice of $\varepsilon > 0$, given access to an arbitrary distribution \mathcal{D} over training sequences $\{(x_t, y_t)\}_{t=1}^T$, Algorithm 2, run on N i.i.d. sample trajectories from \mathcal{D} , outputs a predictor $\hat{\Theta}$ such that

$$\mathbb{E}_{\mathcal{D}}\left[\mathrm{MSE}(\hat{\Theta}) - \mathrm{MSE}(\Theta^*)\right] \leq \varepsilon + \frac{\tilde{O}\left(\mathrm{poly}(n, m, d, \log T, \log 1/\varepsilon)\right)}{\sqrt{N}},$$

compared to the best symmetric LDS predictor Θ^* , while running in polynomial time.

Typical regression-based methods require the LDS to be *strictly* stable, and degrade on ill-conditioned systems; they depend on a spectral radius parameter $\frac{1}{1-||A||}$. Our proposed method of *wave-filtering* provably and empirically works even for the hardest case of ||A|| = 1. Our algorithm attains the first condition number-independent polynomial guarantees in terms of regret (equivalently, sample complexity) and running time for the MIMO setting. Interestingly, our algorithms never need to learn the hidden state, and our guarantees can be sharpened to handle the case when the dimensionality of h_t is infinite.

1.2 Related work

The modern setting for LDS arose in the seminal work of Kalman [Kal60], who introduced the Kalman filter as a recursive least-squares solution for maximum likelihood estimation (MLE) of Gaussian perturbations to the system. The framework and filtering algorithm have proven to be a mainstay in control theory and time-series analysis; indeed, the term *Kalman filter model* is often used interchangeably with LDS. We refer the reader to the classic survey [Lju98], and the extensive overview of recent literature in [HMR16].

Ghahramani and Roweis [RG99] suggest using the EM algorithm to learn the parameters of an LDS. This approach, which directly tackles the non-convex problem, is widely used in practice [Mar10a]. However, it

remains a long-standing challenge to characterize the theoretical guarantees afforded by EM. We find that it is easy to produce cases where EM fails to identify the correct system.

In a recent result of [HMR16], it is shown for the first time that for a restricted class of systems, gradient descent (also widely used in practice, perhaps better known in this setting as backpropagation) guarantees polynomial convergence rates and sample complexity in the batch setting. Their result applies essentially only to the SISO case (vs. multi-dimensional for us), depends polynomially on the spectral gap (as opposed to no dependence for us), and requires the signal to be created by an LDS (vs. arbitrary for us).

2 Preliminaries

2.1 Linear dynamical systems

Many different settings have been considered, in which the definition of an LDS takes on many variants. We are interested in discrete time-invariant MIMO (multiple input, multiple output) systems with a finite-dimensional hidden state.¹ Formally, our model is given as follows:

Definition 2.1. A linear dynamical system (LDS) is a map from a sequence of input vectors $x_1, \ldots, x_T \in \mathbb{R}^n$ to output (response) vectors $y_1, \ldots, y_T \in \mathbb{R}^m$ of the form

$$h_{t+1} = Ah_t + Bx_t + \eta_t \tag{1}$$

$$y_t = Ch_t + Dx_t + \xi_t, \tag{2}$$

where $h_0, \ldots, h_T \in \mathbb{R}^d$ is a sequence of hidden states, A, B, C, D are matrices of appropriate dimension, and $\eta_t \in \mathbb{R}^d, \xi_t \in \mathbb{R}^m$ are (possibly stochastic) noise vectors.

Unrolling this recursive definition gives the *impulse response function*, which uniquely determines the LDS. For notational convenience, for invalid indices $t \leq 0$, we define x_t , η_t , and ξ_t to be the zero vector of appropriate dimension. Then, we have:

$$y_t = \sum_{i=1}^{T-1} CA^i \left(Bx_{t-i} + \eta_{t-i} \right) + CA^t h_0 + Dx_t + \xi_t.$$
 (3)

We will consider the (discrete) time derivative of the impulse response function, given by expanding $y_{t-1} - y_t$ by Equation (3). For the rest of this paper, we focus our attention on systems subject to the following restrictions:

- (i) The LDS is Lyapunov stable: $||A||_2 \le 1$, where $||\cdot||_2$ denotes the operator (a.k.a. spectral) norm.
- (ii) The transition matrix A is symmetric and positive semidefinite.²

The first assumption is standard: when the hidden state is allowed to blow up exponentially, fine-grained prediction is futile. In fact, many algorithms only work when ||A|| is bounded away from 1, so that the effect of any particular x_t on the hidden state (and thus the output) dissipates exponentially. We do not require this stronger assumption.

We take a moment to justify assumption (ii), and why this class of systems is still expressive and useful. First, symmetric LDSs constitute a natural class of linearly-observable, linearly-controllable systems with dissipating hidden states (for example, physical systems with friction or heat diffusion). Second, this constraint has been used successfully for video classification and tactile recognition tasks [HSC $^+$ 16]. Interestingly, though our theorems require symmetric A, our algorithms appear to tolerate some non-symmetric (and even nonlinear) transitions in practice.

¹We assume finite dimension for simplicity of presentation. However, it will be evident that hidden-state dimension has no role in our algorithm, and shows up as $||B||_F$ and $||C||_F$ in the regret bound.

²The psd constraint on A can be removed by augmenting the inputs x_t with extra coordinates $(-1)^t(x_t)$. We omit this for simplicity of presentation.

2.2 Sequence prediction as online regret minimization

A natural formulation of system identification is that of online sequence prediction. At each time step t, an online learner is given an input x_t , and must return a predicted output \hat{y}_t . Then, the true response y_t is observed, and the predictor suffers a squared-norm loss of $||y_t - \hat{y}_t||^2$. Over T rounds, the goal is to predict as accurately as the best LDS in hindsight.

Note that the learner is permitted to access the history of observed responses $\{y_1, \ldots, y_{t-1}\}$. Even in the presence of statistical (non-adversarial) noise, the fixed maximum-likelihood sequence produced by $\Theta = (A, B, C, D, h_0)$ will accumulate error linearly as T. Thus, we measure performance against a more powerful comparator, which fixes LDS parameters Θ , and predicts y_t by the previous response y_{t-1} plus the derivative of the impulse response function of Θ at time t.

We will exhibit an online algorithm that can compete against the best Θ in this setting. Let $\hat{y}_1, \ldots, \hat{y}_T$ be the predictions made by an online learner, and let y_1^*, \ldots, y_T^* be the sequence of predictions, realized by a chosen setting of LDS parameters Θ , which minimize total squared error. Then, we define regret by the difference of total squared-error losses:

Regret
$$(T) \stackrel{\text{def}}{=} \sum_{t=1}^{T} ||y_t - \hat{y}_t||^2 - \sum_{t=1}^{T} ||y_t - y_t^*||^2.$$

This setup fits into the standard setting of online convex optimization (in which a sublinear regret bound implies convergence towards optimal predictions), save for the fact that the loss functions are non-convex in the system parameters. Also, note that a randomized construction (set all $x_t = 0$, and let y_t be i.i.d. Bernoulli random variables) yields a lower bound³ for any online algorithm: $\mathbb{E}[\text{Regret}(T)] \geq \Omega(\sqrt{T})$.

To quantify regret bounds, we must state our scaling assumptions on the (otherwise adversarial) input and output sequences. We assume that the inputs are bounded: $||x_t||_2 \leq R_x$. Also, we assume that the output signal is Lipschitz in time: $||y_t - y_{t-1}||_2 \leq L_y$. The latter assumption exists to preclude pathological inputs where an online learner is forced to incur arbitrarily large regret. For a true noiseless LDS, L_y is not too large; see Lemma F.5 in the appendix.

We note that an optimal $\tilde{O}(\sqrt{T})$ regret bound can be trivially achieved in this setting by algorithms such as Hedge [LW94], using an exponential-sized discretization of all possible LDS parameters; this is the online equivalent of brute-force grid search. Strikingly, our algorithms achieve essentially the same regret bound, but run in polynomial time.

2.3 The power of convex relaxations

Much work in system identification, including the EM method, is concerned with explicitly finding the LDS parameters $\Theta = (A, B, C, D, h_0)$ which best explain the data. However, it is evident from Equation 3 that the CA^iB terms cause the least-squares (or any other) loss to be non-convex in Θ . Many methods used in practice, including EM and subspace identification, heuristically estimate each hidden state h_t , after which estimating the parameters becomes a convex linear regression problem. However, this first step is far from guaranteed to work in theory or practice.

Instead, we follow the paradigm of improper learning: in order to predict sequences as accurately as the best possible LDS $\Theta^* \in \mathcal{H}$, one need not predict strictly from an LDS. The central driver of our algorithms is the construction of a slightly larger hypothesis class $\hat{\mathcal{H}}$, for which the best predictor $\hat{\Theta}^*$ is nearly as good as Θ^* . Furthermore, we construct $\hat{\mathcal{H}}$ so that the loss functions are convex under this new parameterization. From this will follow our efficient online algorithm.

As a warmup example, consider the following overparameterization: pick some time window $\tau \ll T$, and let the predictions \hat{y}_t be linear in the concatenation $[x_t, \ldots, x_{t-\tau}] \in \mathbb{R}^{\tau d}$. When ||A|| is bounded away from 1, this is a sound assumption.⁴ However, in general, this approximation is doomed to either truncate longer-term input-output dependences (short τ), or suffer from overfitting (long τ). Our main theorem uses

³This is a standard construction; see, e.g. Theorem 3.2 in [Haz16].

⁴This assumption is used in *autoregressive models*; see Section 6 of [HMR16] for a theoretical treatment.

an overparameterization whose approximation factor ε is independent of ||A||, and whose sample complexity scales only as $\tilde{O}(\operatorname{polylog}(T, 1/\varepsilon))$.

2.4 Low approximate rank of Hankel matrices

Our analysis relies crucially on the spectrum of a certain *Hankel matrix*, a square matrix whose anti-diagonal stripes have equal entries (i.e. H_{ij} is a function of i+j). An important example is the Hilbert matrix $H_{n,\theta}$, the *n*-by-*n* matrix whose (i,j)-th entry is $\frac{1}{i+j+\theta}$. For example,

$$H_{3,-1} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

This and related matrices have been studied under various lenses for more than a century: see, e.g., [Hil94, Cho83]. A basic fact is that $H_{n,\theta}$ is a positive definite matrix for every $n \ge 1, \theta > -2$. The property we are most interested in is that the spectrum of a positive semidefinite Hankel matrix decays exponentially, a difficult result derived in [BT16] via Zolotarev rational approximations. We state these technical bounds in Appendix E.

3 The wave-filtering algorithm

Our online algorithm (Algorithm 1) runs online projected gradient descent [Zin03] on the squared loss $f_t(M_t) \stackrel{\text{def}}{=} ||y_t - \hat{y}_t(M_t)||^2$. Here, each M_t is a matrix specifying a linear map from featurized inputs \tilde{X}_t to predictions \hat{y}_t . Specifically, after choosing a certain bank of k filters $\{\phi_j\}$, $\tilde{X}_t \in \mathbb{R}^{nk+2n+m}$ consists of convolutions of the input time series with each ϕ_j (scaled by certain constants), along with x_{t-1} , x_t , and y_{t-1} . The number of filters k will turn out to be polylogarithmic in T.

The filters $\{\phi_j\}$ and scaling factors $\{\sigma_j^{1/4}\}$ are given by the top eigenvectors and eigenvalues of the Hankel matrix $Z_T \in \mathbb{R}^{T \times T}$, whose entries are given by

$$Z_{ij} := \frac{2}{(i+j)^3 - (i+j)}.$$

In the language of Section 2.3, one should think of each M_t as arising from an $O(\text{poly}(m, n, d, \log T))$ -dimensional hypothesis class $\hat{\mathcal{H}}$, which replaces the original $O((m+n+d)^2)$ -dimensional class \mathcal{H} of LDS parameters (A, B, C, D, h_0) . Theorem 3 gives the key fact that $\hat{\mathcal{H}}$ approximately contains \mathcal{H} .

Algorithm 1 Online wave-filtering algorithm for LDS sequence prediction

- 1: Input: time horizon T, filter parameter k, learning rate η , radius parameter R_M .
- 2: Compute $\{(\sigma_j, \phi_j)\}_{j=1}^k$, the top k eigenpairs of Z_T .
- 3: Initialize $M_1 \in \mathbb{R}^{m \times k'}$, where $k' \stackrel{\text{def}}{=} nk + 2n + m$.
- 4: **for** t = 1, ..., T **do**
- 5: Compute $\tilde{X} \in \mathbb{R}^{k'}$, with first nk entries $\tilde{X}_{(i,j)} := \sigma_j^{1/4} \sum_{u=1}^{T-1} \phi_j(u) x_{t-u}(i)$, followed by the 2n+m entries of x_{t-1} , x_t , and y_{t-1} .
- 6: Predict $\hat{y}_t := M_t X$.
- 7: Observe y_t . Suffer loss $||y_t \hat{y}_t||^2$.
- 8: Gradient update: $M_{t+1} \leftarrow M_t 2\eta(y_t \hat{y}_t) \otimes \tilde{X}$.
- 9: **if** $||M_{t+1}||_F \ge R_M$ **then**
- 10: Perform Frobenius norm projection: $M_{t+1} \leftarrow \frac{R_M}{\|M_{t+1}\|_F} M_{t+1}$.
- 11: end if
- 12: end for

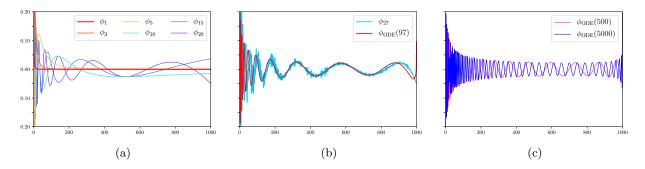


Figure 1: (a) The entries of some typical eigenvectors of Z_{1000} , plotted coordinate-wise. (b) ϕ_{27} of Z_{1000} ($\sigma_{27} \approx 10^{-16}$) computed with finite-precision arithmetic, along with a numerical solution to the ODE in Appendix B.1 with $\lambda = 97$. (c) Some very high-order filters, computed using the ODE, would be difficult to obtain by eigenvector computations.

In Section 4, we provide the precise statement and proof of Theorem 1, the main regret bound for Algorithm 1, with some technical details deferred to the appendix. We also obtain analogous sample complexity results for batch learning; however, on account of some definitional subtleties, we defer all discussion of the offline case, including the statement and proof of Theorem 2, to Appendix A.

We make one final interesting note here, from which the name wave-filtering arises: when plotted coordinate-wise, our filters $\{\phi_j\}$ look like the vibrational modes of an inhomogeneous spring (see Figure 1). We provide some insight on this phenomenon (along with some other implementation concerns) in Appendix B. Succinctly: in the scaling limit, $(Z_T/\|Z_T\|_2)_{T\to\infty}$ commutes with a certain second-order Sturm-Liouville differential operator \mathcal{D} . This allows us to approximate filters with eigenfunctions of \mathcal{D} , using efficient numerical ODE solvers.

4 Analysis

We first state the full form of the regret bound achieved by Algorithm 1:⁵

Theorem 1 (Main). On any sequence $\{(x_t, y_t)\}_{t=1}^T$, Algorithm 1, with a choice of $k = \Theta\left(\log^2 T \log(R_{\Theta}R_xL_yn)\right)$, $R_M = \Theta(R_{\Theta}^2\sqrt{k})$, and $\eta = \Theta((R_x^2L_y\log(R_{\Theta}R_xL_yn)n\sqrt{T}\log^4 T)^{-1})$, achieves regret

$$\operatorname{Regret}(T) \leq O\left(R_{\Theta}^4 R_x^2 L_y \, \log^2(R_{\Theta} R_x L_y n) \cdot n \sqrt{T} \log^6 T\right),\,$$

competing with LDS predictors (A, B, C, D, h_0) with $0 \le A \le I$ and $||B||_F, ||C||_F, ||D||_F, ||h_0|| \le R_{\Theta}$.

Note that the dimensions m,d do not appear explicitly in this bound, though they typically factor into R_{Θ} . In Section 4.1, we state and prove Theorem 3, the convex relaxation guarantee for the filters, which may be of independent interest. This allows us to approximate the optimal LDS in hindsight (the regret comparator) by the loss-minimizing matrix $M_t: \tilde{X} \mapsto \hat{y}_t$. In Section 4.2, we complete the regret analysis using Theorem 3, along with bounds on the diameter and gradient, to conclude Theorem 1.

Since the batch analogue is less general (and uses the same ideas), we defer discussion of Algorithm 2 and Theorem 2 to Appendix A.

4.1 Approximate convex relaxation via wave filters

Assume for now that $h_0 = 0$; we will remove this at the end, and see that the regret bound is asymptotically the same. Recall (from Section 2.2) that we measure regret compared to predictions obtained by adding the

⁵Actually, for a slightly tighter proof, we analyze a restriction of the algorithm which does not learn the portion $M^{(y)}$, instead always choosing the identity matrix for that block.

derivative of the impulse response function of an LDS Θ to y_{t-1} . Our approximation theorem states that for any Θ , there is some $M_{\Theta} \in \hat{\mathcal{H}}$ which produces approximately the same predictions. Formally:

Theorem 3 (Spectral convex relaxation for symmetric LDSs). Let $\{\hat{y}_t\}_{t=1}^T$ be the online predictions made by an LDS $\Theta = (A, B, C, D, h_0 = 0)$. Let $R_{\Theta} = \max\{\|B\|_F, \|C\|_F, \|D\|_F\}$. Then, for any $\varepsilon > 0$, with a choice of $k = \Omega(\log T \log(R_{\Theta}R_xL_ynT/\varepsilon))$, there exists an $M_{\Theta} \in \mathbb{R}^{m \times k'}$ such that

$$\sum_{t=1}^{T} ||M_{\Theta} \tilde{X}_t - y_t||^2 \le \sum_{t=1}^{T} ||\hat{y}_t - y_t||^2 + \varepsilon.$$

Here, k' and \tilde{X}_t are defined as in Algorithm 1 (noting that \tilde{X}_t includes the previous ground truth y_{t-1}).

Proof. We construct this mapping $\Theta \mapsto M_{\Theta}$ explicitly. Write M_{Θ} as the block matrix

$$[M^{(1)} \quad M^{(2)} \quad \cdots \quad M^{(k)} \quad M^{(x')} \quad M^{(x)} \quad M^{(y)}],$$

where the blocks' dimensions are chosen to align with \tilde{X}_t , the concatenated vector

$$\left[\sigma_1^{1/4}(X*\phi_1)_t \quad \sigma_2^{1/4}(X*\phi_2)_t \quad \cdots \quad \sigma_k^{1/4}(X*\phi_k)_t \quad x_{t-1} \quad x_t \quad y_{t-1}\right],$$

so that the prediction is the block matrix-vector product

$$M_{\Theta}\tilde{X}_{t} = \sum_{i=1}^{k} \sigma_{j}^{1/4} M^{(j)} (X * \phi_{j})_{t} + M^{(x')} x_{t-1} + M^{(x)} x_{t} + M^{(y)} y_{t-1}.$$

Without loss of generality, assume that A is diagonal, with entries $\{\alpha_l\}_{l=1}^d$. Let b_l be the l-th row of B, and c_l the l-th column of C. Also, we define a continuous family of vectors $\mu:[0,1]\to\mathbb{R}^T$, with entries $\mu(\alpha)(i)=(\alpha_l-1)\alpha_l^{i-1}$. Then, our construction is as follows:

- $M^{(j)} = \sum_{l=1}^{d} \sigma_j^{-1/4} \langle \phi_j, \mu(\alpha_l) \rangle (c_l \otimes b_l)$, for each $1 \leq j \leq k$.
- $M^{(x')} = -D$, $M^{(x)} = CB + D$, $M^{(y)} = I_{m \times m}$

Below, we give the main ideas for why this M_{Θ} works, leaving the full proof to Appendix C.

Since $M^{(y)}$ is the identity, the online learner's task is to predict the differences $y_t - y_{t-1}$ as well as the derivative Θ , which we write here:

$$\hat{y}_{t} - y_{t-1} = (CB + D)x_{t} - Dx_{t-1} + \sum_{i=1}^{T-1} C(A^{i} - A^{i-1})Bx_{t-i}$$

$$= (CB + D)x_{t} - Dx_{t-1} + \sum_{i=1}^{T-1} C\left(\sum_{l=1}^{d} (\alpha_{l}^{i} - \alpha_{l}^{i-1}) e_{l} \otimes e_{l}\right) Bx_{t-i}$$

$$= (CB + D)x_{t} - Dx_{t-1} + \sum_{l=1}^{d} (c_{l} \otimes b_{l}) \sum_{i=1}^{T-1} \mu(\alpha_{l})(i) x_{t-i}. \tag{4}$$

Notice that the inner sum is an inner product between each coordinate of the past inputs $(x_t, x_{t-1}, \ldots, x_{t-T})$ with $\mu(\alpha_l)$ (or a convolution, viewed across the entire time horizon). The crux of our proof is that one can approximate $\mu(\alpha)$ using a linear combination of the filters $\{\phi_j\}_{j=1}^k$. Writing $Z := Z_T$ for short, notice that

$$Z = \int_0^1 \mu(\alpha) \otimes \mu(\alpha) \, d\alpha,$$

⁶Write the eigendecomposition $A = U\Lambda U^T$. Then, the LDS with parameters $(\hat{A}, \hat{B}, \hat{C}, D, h_0) := (\Lambda, BU, U^TC, D, h_0)$ makes the same predictions as the original, with \hat{A} diagonal.

since the (i, j) entry of the RHS is

$$\int_0^1 (\alpha - 1)^2 \alpha^{i+j-2} d\alpha = \frac{1}{i+j-1} - \frac{2}{i+j} + \frac{1}{i+j+1} = Z_{ij}.$$

What follows is a spectral bound for reconstruction error, relying on the low approximate rank of Z:

Lemma 4.1. Choose any $\alpha \in [0,1]$. Let $\tilde{\mu}(\alpha)$ be the projection of $\mu(\alpha)$ onto the k-dimensional subspace of \mathbb{R}^T spanned by $\{\phi_j\}_{j=1}^k$. Then,

$$\|\mu(\alpha) - \tilde{\mu}(\alpha)\|^2 \le \sqrt{6\sum_{j=k+1}^T \sigma_j} \le O\left(c_0^{-k/\log T} \sqrt{\log T}\right),$$

for an absolute constant $c_0 > 3.4$.

By construction of $M^{(j)}$, $M_{\Theta}\tilde{X}_t$ replaces each $\mu(\alpha_l)$ in Equation (4) with its approximation $\tilde{\mu}(\alpha_l)$. Hence we conclude that

$$M_{\Theta}\tilde{X}_{t} = y_{t-1} + (CB + D)x_{t} - Dx_{t-1} + \sum_{l=1}^{d} (c_{l} \otimes b_{l}) \sum_{i=1}^{T-1} \tilde{\mu}(\alpha_{l})(i) x_{t-i}$$
$$= y_{t-1} + (\hat{y}_{t} - y_{t-1}) + \zeta_{t} = \hat{y}_{t} + \zeta_{t},$$

letting $\{\zeta_t\}$ denote some residual vectors arising from discarding the subspace of dimension T-k. Theorem 3 follows by showing that these residuals are small, using Lemma 4.1: it turns out that $\|\zeta_t\|$ is exponentially small in $k/\log T$, which implies the theorem.

4.2 From approximate relaxation to low regret

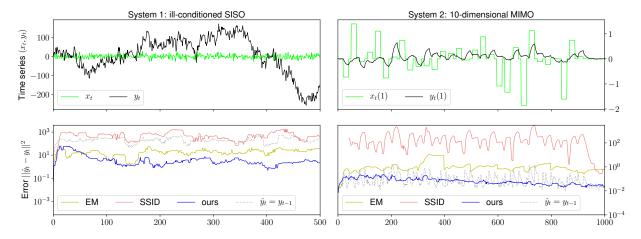
Let $\Theta^* \in \mathcal{H}$ denote the best LDS predictor, and let $M_{\Theta^*} \in \hat{\mathcal{H}}$ be its image under the map from Theorem 3, so that total squared error of predictions $M_{\Theta^*}\tilde{X}_t$ is within ε from that of Θ^* . Notice that the loss functions $f_t(M) \stackrel{\text{def}}{=} \|y_t - M\tilde{X}_t\|^2$ are quadratic in M, and thus convex. Algorithm 1 runs online gradient descent [Zin03] on these loss functions, with decision set $\mathcal{M} \stackrel{\text{def}}{=} \{M \in \mathbb{R}^{m \times k'} \mid \|M\|_F \leq R_M\}$. Let $D_{\max} := \sup_{M,M' \in \mathcal{M}} \|M - M'\|_F$ be the diameter of \mathcal{M} , and $G_{\max} := \sup_{M \in \mathcal{M}, \tilde{X}} \|\nabla f_t(M)\|_F$ be the largest norm of a gradient. We can invoke the classic regret bound:

Lemma 4.2 (e.g. Thm. 3.1 in [Haz16]). Online gradient descent, using learning rate $\frac{D_{\text{max}}}{G_{\text{max}}\sqrt{T}}$, has regret

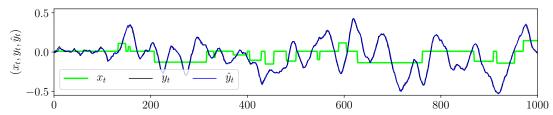
$$\operatorname{Regret}_{\operatorname{OGD}}(T) \stackrel{def}{=} \sum_{t=1}^{T} f_t(M_t) - \min_{M \in \mathcal{M}} \sum_{t=1}^{T} f_t(M) \leq 2G_{\max} D_{\max} \sqrt{T}.$$

To finish, it remains to show that D_{max} and G_{max} are small. In particular, since the gradients contain convolutions of the input by ℓ_2 (not ℓ_1) unit vectors, special care must be taken to ensure that these do not grow too quickly. These bounds are shown in Section D.2, giving the correct regret of Algorithm 1 in comparison with the comparator $M^* \in \hat{\mathcal{H}}$. By Theorem 3, M^* competes arbitrarily closely with the best LDS in hindsight, concluding the theorem.

Finally, we discuss why it is possible to relax the earlier assumption $h_0 = 0$ on the initial hidden state. Intuitively, as more of the ground truth responses $\{y_t\}$ are revealed, the largest possible effect of the initial state decays. Concretely, in Section D.4, we prove that a comparator who chooses a nonzero h_0 can only increase the regret by an additive $\tilde{O}(\log^2 T)$ in the online setting.



(a) Two synthetic systems. For clarity, error plots are smoothed by a median filter. *Left:* Noisy SISO system with a high condition number; EM and SSID finds a bad local optimum. *Right:* High-dimensional MIMO system; other methods fail to learn any reasonable model of the dynamics.



(b) Forced pendulum, a physical simulation our method learns in practice, despite a lack of theory.

Figure 2: Visualizations of Algorithm 1. All plots: blue = ours, yellow = EM, red = SSID, black = true responses, green = inputs, dotted lines = "guess the previous output" baseline. Horizontal axis is time.

5 Experiments

In this section, to highlight the appeal of our provable method, we exhibit two minimalistic cases where traditional methods for system identification fail, while ours successfully learns the system. Finally, we note empirically that our method seems not to degrade in practice on certain well-behaved nonlinear systems. In each case, we use k = 25 filters, and a regularized follow-the-leader variant of Algorithm 1 (see Appendix B.2).

5.1 Synthetic systems: two hard cases for EM and SSID

We construct two difficult systems, on which we run either EM or subspace identification 7 (SSID), followed by Kalman filtering to obtain predictions. Note that our method runs significantly (>1000 times) faster than this traditional pipeline.

In the first example (Figure 2(a), left), we have a SISO system (n = m = 1) and d = 2; all x_t , ξ_t , and η_t are i.i.d. Gaussians, and $B^{\top} = C = [1 \ 1], D = 0$. Most importantly, A = diag([0.999, 0.5]) is ill-conditioned, so that there are long-term dependences between input and output. Observe that although EM and SSID both find reasonable guesses for the system's dynamics, they turns out to be local optima. Our method learns to predict as well as the best possible LDS.

The second example (Figure 2(a), right) is a MIMO system (with n = m = d = 10), also with Gaussian noise. The transition matrix $A = \text{diag}([0, 0.1, 0.2, \dots, 0.9])$ has a diverse spectrum, the observation matrix

 $^{^7\}mathrm{Specifically},$ we use "Deterministic Algorithm 1" from page 52 of [VODM12].

C has i.i.d. Gaussian entries, and $B = I_n, D = 0$. The inputs x_t are random block impulses. This system identification problem is high-dimensional and non-convex; it is thus no surprise that EM and SSID consistently fail to converge.

5.2 The forced pendulum: a nonlinear, non-symmetric system

We remark that although our algorithm has provable regret guarantees only for LDSs with symmetric transition matrices, it appears in experiments to succeed in learning some non-symmetric (even nonlinear) systems in practice, much like the unscented Kalman filter [WVDM00]. In Figure 2(b), we provide a typical learning trajectory for a forced pendulum, under Gaussian noise and random block impulses. Physical systems like this are widely considered in control and robotics, suggesting possible real-world applicability for our method.

6 Conclusion

We have proposed a novel approach for provably and efficiently learning linear dynamical systems. Our online wave-filtering algorithm attains near-optimal regret in theory; and experimentally outperforms traditional system identification in both prediction quality and running time. Furthermore, we have introduced a "spectral filtering" technique for convex relaxation, which uses convolutions by eigenvectors of a Hankel matrix. We hope that this theoretical tool will be useful in tackling more general cases, as well as other non-convex learning problems.

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Guide to the Appendix

- In Appendix A, we present two formulations of the batch learning equivalent of the online algorithm, and derive Theorem 2, a companion sample complexity bound.
- In Appendix B, we discuss some variants of our online algorithm, and offer some tips for implementation. We also provide discussion on the connection of our filters to eigenfunctions of a certain differential operator.
- In Appendix C, we prove the key approximate convex relaxation result (Theorem 3).
- In Appendix D, we complete the details for the proof sketch provided in Section 4.2, concluding the main theorem, the regret bound for the online algorithm. Importantly, we address the subtle issue of deriving upper bounds for the gradient and diameter of the decision set.
- In Appendix E, we derive explicit non-asymptotic bounds for quantities of interest pertaining to the Hankel matrix Z, notably spectral decay. Key results are adapted from [BT16].
- In Appendix F, we verify some easy-to-prove properties of the important vector $\mu(\alpha)$, for sake of completeness.

A Batch variants of the algorithm

The online prediction setting is sensitive to permutation of the time series: that is, the same LDS does not in general map $\{x_{\sigma(1)}, \ldots, x_{\sigma(T)}\}$ to $\{y_{\sigma(1)}, \ldots, y_{\sigma(T)}\}$. As such, one must take care when defining the batch case: the output time series (and thus, loss functions) are correlated, so it is not meaningful to assume that they are i.i.d. samples from a distribution. Thus, our online regret bound, which concerns a single episode, does not translate directly. However, our convex relaxation technique still allows us to do efficient improper learning with least-squares regression, giving interesting and novel statistical guarantees. In this section, we provide two possible formulations of the batch setting, along with accompanying theorems.

In both cases, it is most natural to fix an episode length T, and consider a rollout of the system $\{(x_t, y_t)\}_{t=1}^T$ to be a single example. For short, let $X_i \in \mathbb{R}^{Tn}$ denote the concatenated vector of inputs for a single example, and $Y_i \in \mathbb{R}^{Tm}$ the concatenated responses. The batch formulation is to learn the dynamics of the system using N samples $\{(X_i, Y_i)\}$. Recall that the samples satisfy $||x_t||_2 \leq R_x$ and $||y_t - y_{t-1}||_2 \leq L_y$.

Unlike in the online setting, it will be less confusing in the batch setting to measure the *mean* squared error of predictions, rather than the total squared error. Thus, in this section, $\ell_{X,Y}(h)$ will always refer to mean squared error. As well, to follow statistical learning conventions (for ease of reading), we use h to denote a hypothesis (an LDS) instead of Θ ; this is distinguished from the hidden state h_t .

A.1 Learning the derivative: the direct analogue

Throughout this subsection, assume that $h_0 = 0$.

As noted, the sequential prediction algorithm can be restricted so as to never make updates to the submatrix $M^{(y)}$, keeping it to be the identity matrix. Notice that all other features in \tilde{X} consist of inputs x_t and their convolutions. In other words, we can take the view that the matrix M_t can be used to predict the differences $y_t - y_{t-1}$ between successive responses, as a function of the entire (aligned) input time series $(x_t, x_{t-1}, \ldots, x_{t_T})$.

Thus, we can formulate a direct analogue for the online algorithm: learn the mapping from an input time series $X_i \in \mathbb{R}^{Tn}$ to the differences $Y_i' \in \mathbb{R}^{Tm}$, the concatenation of all $y_t - y_{t-1}$. For this, we can use Theorem 3 (the approximation result) directly, and obtain an improper agnostic learning guarantee.

Specifically, let \mathcal{H} be a subset of the hypothesis class of LDS parameters $\Theta = (A, B, C, D, h_0 = 0)$, subject to $||B||_F$, $||C||_F$, $||D||_F \leq R_{\Theta}$, and choose any approximation tolerance $\varepsilon > 0$. Then, Theorem 3 states that choosing $\hat{\mathcal{H}}$ with $k = \Omega(\log T \log(R_{\Theta}R_xL_ynT/\varepsilon))$ ensures the ε -approximate relaxation property. In the language of the batch setting: for each $h \in \mathcal{H}$ which predicts on the sample (X, Y') with a mean squared error $\ell_X(h)$, there is some $\hat{h} \in \hat{\mathcal{H}}$ so that

$$\ell_{X,Y}(h) \le \ell_{X,Y}(\hat{h}) + \varepsilon.$$

The choice of batch algorithm is clear, in order to mimic Algorithm 1: run least-squares regression on \tilde{X} and Y, where \tilde{X} is the same featurization of the inputs as used in the online algorithm. We describe this procedure fully in Algorithm 2.

Algorithm 2 Offline wave-filtering algorithm for learning the derivative of an LDS

```
1: Input: S = \{(X_i, Y_i')\}, a set of N training samples, each of length T; filter parameter k.
```

- 2: Compute $\{(\sigma_j, \phi_j)\}_{j=1}^k$, the top k eigenpairs of Z_T .
- 3: Initialize matrices $\mathbf{X} \in \mathbb{R}^{(nk+2n)\times NT}, \mathbf{Y}' \in \mathbb{R}^{m\times NT}$
- 4: **for** each sample (X, Y') **do**
- 5: **for** t = 1, ..., T **do**
- 6: Compute $\tilde{X}_t \in \mathbb{R}^{nk+2n}$, with first nk entries $\tilde{X}_{(i,j)} := \sigma_j^{1/4} \sum_{u=1}^{T-1} \phi_j(u) x_{t-u}(i)$, followed by the 2n entries of x_{t-1}, x_t .
- 7: Append (\tilde{X}_t, Y'_t) as new columns to the matrices \mathbf{X}, \mathbf{Y}' .
- 8: end for
- 9: end for
- 10: **return** least-squares solution $(\mathbf{X}\mathbf{X}^{\top})^{\dagger}\mathbf{X}^{\top}\mathbf{Y}'$.

A.1.1 Generalization bound

By definition, Algorithm 2 minimizes the empirical MSE loss on the samples; as such, we can derive a PAC-learning bound for regression. We begin with some definitions and assumptions, so that we can state the theorem.

As in the statement of the online algorithm, as a soft dimensionality restriction, we constrain the comparator class \mathcal{H} to contain LDSs with parameters $\Theta = (A, B, C, D, h_0 = 0)$ such that $0 \leq A \leq I$ and $\|B\|_F, \|C\|_F, \|D\|_F, \|h_0\| \leq R_{\Theta}$. For an empirical sample set S, let $\ell_S(h) = \frac{1}{|S|} \sum_{(X,Y) \in S} \ell_{X,Y}(h)$. Similarly, for a distribution \mathcal{D} , let $\ell_{\mathcal{D}}(h) = \mathbb{E}_{(X,Y) \sim \mathcal{D}}[\ell_{X,Y}(h)]$.

Then, we are able to obtain a sample complexity bound:

Theorem 2 (Generalization of the batch algorithm). Choose any $\varepsilon > 0$. Let $S = \{(X_i, Y_i')\}_{i=1}^N$ be a set of i.i.d. training samples from a distribution \mathcal{D} . Let $\hat{h} \stackrel{def}{=} \operatorname{argmin}_{h \in \hat{\mathcal{H}}} \ell_{\mathcal{D}}(h)$ be the output of Algorithm 2, with a choice of $k = \Theta(\log T \log(R_{\Theta}R_xL_ynT/\varepsilon))$. Let $h^* \stackrel{def}{=} \operatorname{argmin}_{h^* \in \mathcal{H}} \ell_{\mathcal{D}}(h)$ be the true loss minimizer. Then, with probability at least $1 - \delta$, it holds that

$$\ell_{\mathcal{D}}(\hat{h}) - \min_{h \in \mathcal{H}} \ell_{\mathcal{D}}(h) \le \varepsilon + \frac{O\left(R_{\Theta}^4 R_x^2 L_y \log^2(R_{\Theta} R_x L_y n/\varepsilon) n \log^6 T + \sqrt{\log 1/\delta}\right)}{\sqrt{N}}.$$

Proof. Lemma D.1 shows that we can restrict \mathcal{H} by a Frobenius norm bound:

$$||M||_F \le O\left(R_\Theta^2 \sqrt{k}\right).$$

⁸The distinction between measuring total vs. mean squared error is hidden in the constant in front of the $\log T$ when choosing the number of filters k.

Thus, the empirical Rademacher complexity of $\hat{\mathcal{H}}$ on N samples, with this restriction, thus satisfies

$$\mathcal{R}_N(\hat{\mathcal{H}}) \le O\left(\frac{R_{\Theta}^2 R_x \sqrt{k}}{\sqrt{N}}\right).$$

Also, no single prediction error (and thus neither the empirical nor population loss) will exceed the upper bound

 $\ell_{\max} \stackrel{\text{def}}{=} \Theta(R_{\Theta}^4 R_x^2 L_y^2 k).$

Finally, the loss is G_{max} -Lipschitz in the matrix h, where G_{max} is the same upper bound for the gradient as mentioned in Section 4.2. Lemma D.5, states that this is bounded by $O\left(R_{\Theta}^2 R_x^2 L_y \cdot n k^{3/2} \log^2 T\right)$.

With all of these facts in hand, a standard Rademacher complexity-dependent generalization bound holds in the improper hypothesis class $\hat{\mathcal{H}}$ (see, e.g. [BM02]):

Lemma A.1 (Generalization via Rademacher complexity). With probability at least $1 - \delta$, it holds that

$$\ell_{\mathcal{D}}(\hat{h}) - \ell_{\mathcal{D}}(\hat{h}^*) \le G_{\max} \mathcal{R}_N(\hat{\mathcal{H}}) + \ell_{\max} \sqrt{\frac{8 \ln 2/\delta}{N}}$$

With the stated choice of k, an upper bound for the RHS of Lemma A.1 is

$$\frac{O\left(R_{\Theta}^4 R_x^2 L_y \log^2(R_{\Theta} R_x L_y n/\varepsilon) n \log^6 T + \sqrt{\log 1/\delta}\right)}{\sqrt{N}}.$$

Combining this with the approximation result (Theorem 3) yields the theorem.

A.2 The pure batch setting

A natural question is whether there exists a batch learning algorithm that can use X to predict Y directly, as opposed to the differences Y'. This is possible in the regime of low noise: if one has predictions on Y' that are correct up to MSE ε , an easy solution is to integrate and obtain predictions for Y; however, the errors will accumulate to $T\varepsilon$. The same agnostic learning guarantee costs a rather dramatic factor of T^2 in sample complexity.

In the regime of low noise, an analogue of our approximation theorem (Theorem 3) is powerful enough to guarantee low error. For convenience and concreteness, we record this here:

Theorem 3b (Pure-batch approximation). Let Θ be an LDS specified by parameters $(A, B, C, D, h_0 = 0)$, with $0 \leq A \leq I$, and $\|B\|_F, \|C\|_F, \|D\|_F \leq R_{\Theta}$. Suppose Θ takes an input sequence $X = \{x_1, \ldots, x_T\}$, and produces output sequence $Y = \{y_1, \ldots, y_T\}$, assuming all noise vectors ξ_t, η_t are θ . Then, for any $\varepsilon > 0$, with a choice of $k = \Omega(\log T \log(R_{\Theta}R_xL_ynT/\varepsilon))$, there exists an $M_{\Theta} \in \mathbb{R}^{m \times (nk+2n)}$ such that

$$\sum_{t=1}^{T} \left\| \left(\sum_{u=1}^{t} M_{\Theta} \tilde{X}_{u} \right) - y_{t} \right\|^{2} \leq \sum_{t=1}^{T} \|\hat{y}_{t} - y_{t}\|^{2} + \varepsilon,$$

where \tilde{X}_t is defined as in Algorithm 1, without the y_{t-1} entries.

This fact follows from Theorem 3, setting ε/T as the desired precision; the cost of this additional precision is only a constant factor in k. Furthermore, this M_{Θ} is subject to the same Frobenius norm constraint $\|M_{\Theta}\|_F \leq O(R_{\Theta}^2 \sqrt{k})$ as in Lemma D.1.

A.2.1 Filters from the Hilbert matrix

Alternatively, in the realizable case (when the samples from \mathcal{D} are generated by an LDS, possibly with small noise), one can invoke a similar approximate relaxation theorem as Theorem 3. The filters become the eigenvectors of the Hilbert matrix $H_{T,-1}$, the matrix whose (i,j)-th entry is 1/(i+j-1). This matrix exhibits the same spectral decay as Z_T ; see [BT16] for precise statements. the proof follows the sketch from Section 4.1, approximating the powers of α_ℓ by a spectral truncation of a different curve $\mu'(\alpha)(i) = \alpha^{i-1}$, sometimes called the *moment curve* in \mathbb{R}^T . The Hilbert matrix arises from taking the second moment matrix of the uniform distribution on this curve.

However, we find that this approximation guarantee is insufficient to show the strong regret and agnostic learning bounds we exhibit for learning the derivative of the impulse response function. Nonetheless, we find that regression with these filters works well in practice, even interchangeably in the online algorithm; see Section B.1 for some intuition.

A.3 Learning the initial hidden state via hints

In either of the above settings, it is not quite possible to apply the same argument as in the online setting for pretending that the initial hidden state is zero. When this assumption is removed, the quality of the convex relaxation degrades by an additive $\tilde{O}(\frac{\log^2 T}{T})$; see Section D.4. This does not matter much for the regret bound, because it is subsumed by the worst-case regret of online gradient descent.

However, in the batch setting, we take the view of fixed T and increasing N, so the contribution of the initial state is no longer asymptotically negligible. In other words, this additive approximation error hinders us from driving ε arbitrarily close to zero, no matter how many filters are selected. In settings where T is large enough, one may find this acceptable.

We present an augmented learning problem in which we can predict as well as an LDS: the initial hidden state is provided in each sample, up to an arbitrary linear transformation. Thus, each sample takes the form (X, Y, \tilde{h}_0) , and it is guaranteed that $\tilde{h}_0 = Qh_0$ for each sample, for a fixed matrix $Q \in \mathbb{R}^{d' \times d}$. This Q must be well-conditioned for the problem to remain well-posed: our knowledge of h_0 should be in the same dynamic range as the ground truth. Concretely, we should assume that $\sigma_{\max}(Q)/\sigma_{\min}(Q)$ is bounded.

The construction is as follows: append d' "dummy" dimensions to the input, and add an impulse of h_0 in those dimensions at time 0. During the actual episode, these dummy inputs are always zero. Then, replacing B with the augmented block matrix $\begin{bmatrix} B & Q^{-1} \end{bmatrix}$ recovers the behavior of the system. Thus, we can handle this formulation of hidden-state learning in the online or batch setting, incurring no additional asymptotic factors.

A.3.1 Initializations with finite support

We highlight an important special case of the formulation discussed above, which is perhaps the motivating rationale for this altered problem.

Consider a batch system identification setting in which there are only finitely many initial states h_0 in the training and test data, and the experimenter can distinguish between these states. This can be interpreted a set of n_{hidden} known initial "configurations" of the system. Then, it is sufficient to augment the data with a one-hot vector in $\mathbb{R}^{n_{\text{hidden}}}$, corresponding to the known initialization in each sample. An important case is when $n_{\text{hidden}} = 1$: when there is only one distinct initial configuration; this occurs frequently in control problems.

In summary, the stated augmentation takes the original LDS with dimensions (n, m, d, T), and transforms it into one with dimensions $(n + n_{\text{hidden}}, m, d, T + 1)$. The matrix Q^{-1} , as defined above, is the n_{hidden} -by-d matrix whose columns are the possible initial hidden states, which can be in arbitrary dimension. For convenience, we summarize this observation:

Proposition A.2 (Learning an LDS with few, distinguishable hidden states). When there are d' known hidden states, with $d' ||h_0|| \le R_{\Theta}$, Theorems 2, 3, and 3b apply to the modified LDS learning problem, with samples of the form (h_0, X, Y) . The dimension n becomes n + d'.

B Implementation and variants

We discuss the points mentioned in Section 3 at greater length. Unlike the rest of the appendix, this section contains no technical proofs, and is intended as a user-friendly guide for making the wave-filtering method usable in practice.

B.1 Computing the filters via Sturm-Liouville ODEs

We begin by expanding upon the observation, noted in Section 3, that the eigenvectors resemble inhomogeneously-oscillating waves, providing some justification for the heuristic numerical computation of the top eigenvectors of Z_T .

Computing the filters directly from Z_T is difficult. In fact, the Hilbert matrix (its close cousin) is notoriously super-exponentially ill-conditioned; it is probably best known for being a pathological benchmark for finite-precision numerical linear algebra algorithms. One could ignore efficiency issues, and view this as a data-independent preprocessing step: these filters are deterministic. However, this difficult numerical problem poses an issue for using our method in practice.

Fortunately, as briefly noted in Section 3, some recourse is available. In [Grü82], Grünbaum constructs a tridiagonal matrix $T_{n,\theta}$ which commutes with each Hilbert matrix $H_{n,\theta}$, as defined in Section 2.4. In the appropriate scaling limit as $T \to \infty$, this $T_{n,\theta}$ becomes a Sturm-Liouville differential operator \mathcal{D} which does not depend on θ , given by

$$\mathcal{D} = \frac{d}{dx} \left((1 - x^2) x^2 \frac{d}{dx} \right) - 2x^2.$$

Notice that $Z_T = H_{T,-1} - 2H_{T,0} + H_{T,1}$. This suggests that for large T, the entries of the ϕ_j are approximated by solutions to the second-order ODE

$$\mathcal{D}\phi = \lambda\phi. \tag{5}$$

It is difficult to quantify theoretical bounds for this rather convoluted sequence of approximations; however, we find that this observation greatly aids with constructing these filters in practice. In particular, the map between eigenvalues σ_j of Z and λ_j of \mathcal{D} corresponding to the same eigenvector/eigenfunction proves challenging to characterize for finite T. In practice, we find that our method's performance is sensitive to neither the precise eigenvalues nor the ODE boundary conditions.

In summary, aside from the name wave-filtering, this observation yields a numerically stable recipe for computing filters (without a theorem): for each of k hand-selected eigenvalues λ , compute a filter ϕ_{λ} using an efficient numerical solver to Equation 5.

B.2 Alternative low-regret algorithms

We use online gradient descent as our prototypical low-regret learning algorithm due to its simplicity and stability under worst-case noise. However, in practice, particularly when there are additional structural assumptions on the data, we can replace the update step with that of any low-regret algorithm. AdaGrad [DHS11] is a particularly appealing one, as it is likely to find learning rates which are better than those guaranteed theoretically.

Furthermore, if noise levels are relatively low, and it is known a priori that the data are generated from a true LDS, a better approach might be to use follow-the-leader [KV05] or any of its variants. This amounts to replacing the update step with

$$M_{t+1} := \min_{M} \sum_{t'=1}^{t} ||y_{t'} - \hat{y}_{t'}(M)||^{2},$$

a linear regression problem solvable via, e.g. conjugate gradient. For such iterative methods, we further note that it is possible to use the previous predictor M_{t-1} as a warm start.

B.3 Accelerating convolutions

In the batch setting (or in the online setting, when all the inputs x_t are known in advance), it is easy to see that the convolution components of all feature vectors \tilde{X}_t can be computed in a single pass, by pointwise multiplication in the Fourier domain. Using the fast Fourier transform, one can implement all convolutions in time $O(nkT \log T)$, nearly linear in the size of the input. This mitigates what would otherwise be a quadratic dependence on T. Many software libraries provide an FFT-based implementation of convolution.

C Proof of the relaxation theorem

In this section, we follow the proof structure given in Section 4.1, and conclude Theorem 3.

Before proceeding, we note here that the algorithm could have used filters of length T-1 instead of T, obtained from the eigenvectors of Z_{T-1} . However, since carrying this -1 through the statements and analysis degrades clarity significantly, we use a slightly suboptimal matrix throughout this exposition.

C.1 Proof of Lemma 4.1

First, we develop a spectral bound for average reconstruction error of $\mu(\alpha)$, when α is drawn uniformly from the unit interval [0, 1]. This is controlled by the tail eigenvalues of the second moment matrix of $\mu(\alpha)$, just as in PCA:

Lemma C.1. Let $\{(\sigma_j, \phi_j)\}_{j=1}^T$ be the eigenpairs of Z, in decreasing order by eigenvalue. Let Ψ_k be the linear subspace of \mathbb{R}^T spanned by $\{\phi_1, \ldots, \phi_k\}$. Then,

$$\int_0^1 \|\mu(\alpha) - \operatorname{Proj}_{\Psi_k}(\alpha)\|^2 d\alpha \le \sum_{j=k+1}^T \sigma_j.$$

Proof. Let $r(\alpha)$ denote the residual $\mu(\alpha) - \operatorname{Proj}_{\Psi_k}(\alpha)$, and let $U_r \in \mathbb{R}^{T \times r}$ whose columns are ϕ_1, \dots, ϕ_r , so that

$$r(\alpha) = \Pi_r \mu(\alpha) := (I - U_r U_r^{\top}) \mu(\alpha).$$

Write the eigendecomposition $Z_T = U_T \Sigma U_T^{\top}$. Then,

$$\int_0^1 ||r(\alpha)||^2 d\alpha = \int_0^1 \operatorname{Tr}(r(\alpha) \otimes r(\alpha)) d\alpha = \int_0^1 \operatorname{Tr}\left(\Pi_r \mu(\alpha) \mu(\alpha)^\top \Pi_r\right) d\alpha$$
$$= \int_0^1 \operatorname{Tr}\left(\Pi_r Z \Pi_r\right) d\alpha = \int_0^1 \operatorname{Tr}\left(\Pi_r U_T \Sigma U_T^\top \Pi_r\right) d\alpha.$$

Noting that $\Pi_r U_T$ is just U_T with the first r columns set to zero, the integrand becomes $\sum_{j=k+1}^T \Sigma_{jj}$, which is the stated bound.

In fact, this bound in expectation turns into a bound for all α . We show this by noting that $||r(\alpha)||^2$ is Lipschitz in α , so its maximum over $\alpha \in [0,1]$ cannot be too much larger than its mean. We state and prove this here:

Lemma C.2. For all $\alpha \in [0,1]$, it holds that

$$||r(\alpha)||^2 \le \sqrt{6\sum_{j=k+1}^T \sigma_j}.$$

Proof. By part (ii) of Lemma F.4, $\|\mu(\alpha)\|^2$ is 3-Lipschitz; since Π_r is contractive, $\|r(\alpha)\|^2$ is also 3-Lipschitz. Now, let $R := \max_{0 \le \alpha \le 1} \|r(\alpha)\|^2$. Notice that $R \le \max_{0 \le \alpha \le 1} \|\mu(\alpha)\|^2 \le 1$, by part (i) of Lemma F.4. Subject to achieving a maximum at R, the non-negative 3-Lipschitz function $g : [0,1] \to \mathbb{R}$ with the smallest mean is given by the triangle-shaped function

$$\Delta(\alpha) = \max(R - 3\alpha, 0),$$

for which

$$\int_0^1 \Delta(\alpha) \, d\alpha = R^2/6.$$

In other words,

$$R^2/6 \le \int_0^1 ||r(\alpha)||^2 d\alpha.$$

But Lemma C.1 gives a bound on the RHS, so we conclude

$$\max_{\alpha \in [0,1]} \lVert r(\alpha) \rVert^2 \le R \le \sqrt{6 \sum_{j=k+1}^T \sigma_j},$$

as desired. The stated upper bound on this quantity comes a bound of this spectral tail of the Hankel matrix Z_T (see Lemmas E.2 and E.3); this completes the proof of Lemma 4.1.

C.2 Proof of Theorem 3

It remains to apply Lemma 4.1 to the original setting, which will complete the low-rank approximation result of Theorem 3. Indeed, following Section 4.1, we have

$$\zeta_t \stackrel{\text{def}}{=} M_{\Theta} \tilde{X}_t - \hat{y}_t = \sum_{l=1}^d (c_l \otimes b_l) \sum_{i=1}^{T-1} \left[\tilde{\mu}(\alpha_l) - \mu(\alpha_l) \right](i) \cdot x_{t-i}.$$

View each of the n coordinates in the inner summation as an inner product between the length-T sequence $\tilde{\mu}(\alpha_l) - \mu(\alpha_l)$ and coordinates $X(j) := (x_1(j), \dots, x_T(j))$, which are entrywise bounded by R_x . Then, by Hölder's inequality and Lemma 4.1, we know that this inner product has absolute value at most

$$||X(j)||_{\infty} ||\tilde{\mu}(\alpha_l) - \mu(\alpha_l)||_1 \le ||X(j)||_{\infty} \cdot \sqrt{T} ||\tilde{\mu}(\alpha_l) - \mu(\alpha_l)||_2 \le O\left(R_x \sqrt{T} \cdot c_1^{-k/\log T} \log^{1/4} T\right),$$

with $c_1 = \sqrt{c_0}$. There are n such coordinates, so this inner summation is a vector with ℓ_2 norm at most

$$O\left(R_x\sqrt{nT}\cdot c_1^{-k/\log T}\log^{1/4}T\right)$$
.

Thus, in all, we have

$$\|\zeta_t\|_2 \le O\left(\|B\|_F \|C\|_F R_x \sqrt{nT} \cdot c_1^{-k/\log T} \log^{1/4} T\right).$$

In summary, we have shown that for every system Θ from which a predictor for the discrete derivative of the LDS arises, there is some M_{Θ} whose predictions are pointwise $\|\zeta_t\|_2$ -close. This residual bound can be driven down exponentially by increasing the number of filters k.

Finally, to get an inequality on the total squared error, we compute

$$\sum_{t=1}^{T} \|M_{\Theta} \tilde{X}_{t} - y_{t}\|^{2} = \sum_{t=1}^{T} \|\hat{y}_{t} - y_{t} + \zeta_{t}\|^{2} \leq \sum_{t=1}^{T} \left(\|\hat{y}_{t} - y_{t}\|^{2} + \|\zeta_{t}\|^{2} + 2\|\hat{y}_{t} - y_{t}\| \|\zeta_{t}\| \right)
\leq \sum_{t=1}^{T} \|\hat{y}_{t} - y_{t}\|^{2} + O\left(\left(R_{\Theta}^{4} R_{x}^{2} L_{y}^{2} k \right) T^{3/2} n^{1/2} \cdot c_{1}^{-k/\log T} \log^{1/4} T \right), \tag{6}$$

$$\leq \sum_{t=1}^{T} \|\hat{y}_{t} - y_{t}\|^{2} + O\left(R_{\Theta}^{4} R_{x}^{2} L_{y}^{2} T^{5/2} n^{1/2} \cdot c_{1}^{-k/\log T} \log^{1/4} T \right),$$

where inequality (6) invokes Corollary D.2. Thus, in all, it suffices to choose

$$\frac{k}{\log T} \ge \Omega \left(\log \frac{R_{\Theta} R_x L_y \, nT}{\varepsilon} \right)$$

to force the $O(\cdot)$ term to be less than ε , noting that the powers of n and T show up as a constant factor in front of the $\log(\cdot)$. This completes the proof.

D Proof of the main regret bound

In this part of the appendix, we follow the proof structure outlined Section 4.2, to establish Theorem 1. The lemmas involved also appear in the proof of the batch variant (Theorem 2).

D.1 Diameter bound: controlling the comparator matrix

We will show that the M_{Θ} that competes with a system Θ is not too much larger than Θ , justifying the choice of $R_M = \Omega\left(R_{\Theta}^2\sqrt{k}\right)$. Of course, this implies that the diameter term in the regret bound is $D_{\max} = 2R_M$. Concretely:

Lemma D.1. For any LDS parameters $\Theta = (A, B, C, D, h_0 = 0)$ with $0 \leq A \leq I$ and $||B||_F, ||C||_F, ||D||_F, ||h_0|| \leq R_{\Theta}$, the corresponding matrix $M_{\Theta} \in \hat{\mathcal{H}}$ (which realizes the relaxation in Theorem 3) satisfies

$$||M_{\Theta}||_F^2 \le O\left(R_{\Theta}^2\sqrt{k}\right).$$

Proof. Recalling our construction M_{Θ} in the proof of Theorem 3, we have

- $||M^{(j)}||_F \le ||B||_F ||C||_F \cdot \max_{\ell \in [d]} \sigma_j^{-1/4} \langle \phi_j, \mu(\alpha_\ell) \rangle$, for each $1 \le j \le k$.
- $||M^{(x')}||_F = ||D||_F \le O(R_{\Theta}).$
- $||M^{(x)}||_F \le ||B||_F ||C||_F + ||D||_F \le O(R_{\Theta}^2).$

Recall that we do not consider $M^{(y)}$ as part of the online learning algorithm; it is always the identity matrix. Thus, for the purposes of this analysis, it does not factor into regret bounds.

In Lemma E.4, we show that the reconstruction coefficients $\sigma_j^{-1/4} \langle \phi_j, \mu(\alpha_l) \rangle$ are bounded by an absolute constant; thus, those matrices each have Frobenius norm at most $O(R_{\Theta}^2)$. These terms dominate the Frobenius norm of the entire matrix, concluding the lemma.

This has a very useful consequence:

Corollary D.2. The predictions $\hat{y}_t = M\tilde{X}_t$ made by choosing M such that $||M||_F \leq O(R_{\Theta}^2\sqrt{k})$ satisfy

$$\|\hat{y}_t - y_t\|^2 \le O(R_{\Theta}^4 R_x^2 L_y^2 k).$$

D.2 Gradient bound and final details

A subtle issue remains: the gradients may be large, as they depend on \tilde{X}_t , defined by convolutions of the entire input time series by some filters ϕ_j . Note that these filters do *not* preserve mass: they are ℓ_2 unit vectors, which may cause the norm of the part of \tilde{X}_t corresponding to each filter to be as large as \sqrt{T} .

Fortunately, this is not the case. Indeed, we have:

Lemma D.3. Let $\{(\sigma_j, \phi_j)\}_{j=1}^T$ be the eigenpairs of Z, in decreasing order by eigenvalue. Then, for each $1 \leq j, t \leq T$, it holds that

$$\|\sigma^{1/4}(\phi_j * X)_t\|_{\infty} \le O\left(R_x \log T\right).$$

Proof. Each coordinate of $(\sigma^{1/4}\phi_j * X)_t$ is the inner product between ϕ_j and a sequence of T real numbers, entrywise bounded by $\sigma_j^{1/4}R_x$. Corollary E.6 shows that this is at most $O(\log T)$, a somewhat delicate result which uses matrix perturbation.

Thus, \tilde{X}_t has nk entries with absolute value bounded by $O(R_x \log T)$, concatenated with x_t and x_{t-1} . So, we have:

Corollary D.4. Let \tilde{X}_t be defined as in Algorithm 1, without the y_{t-1} portion. Then,

$$\|\tilde{X}_t\|_2 \le O\left(R_x \log T\sqrt{nk}\right).$$

Our bound on the gradient follows:

Lemma D.5. Suppose \mathcal{M} is chosen with diameter $O(R_{\Theta}^2)$. Then, the gradients satisfy

$$G_{\max} \stackrel{def}{=} \max_{\substack{M \in \mathcal{M}, \\ 1 < t < T}} \|\nabla f_t(M)\|_F \le O\left(R_{\Theta}^2 R_x^2 L_y \cdot nk^{3/2} \log^2 T\right).$$

Proof. We compute the gradient, and apply Lemma D.3:

$$\nabla f_t(M) = \nabla \left(\|y_t - M\tilde{X}_t\|^2 \right) = 2(M\tilde{X}_t - y) \otimes \tilde{X}_t,$$

so that

$$\begin{split} \|\nabla f_t(M)\|_F &= 2\|M\tilde{X}_t - y_t\|_2 \cdot \|\tilde{X}_t\|_2 \\ &\leq 2\left(\|M\|_F \|\tilde{X}_t\|_2 + L_y\right) \|\tilde{X}_t\|_2 \\ &\leq 2\left(\left(R_{\Theta}^2 \sqrt{k}\right) \left(R_x \log T \sqrt{nk}\right) + L_y\right) \left(R_x \log T \sqrt{nk}\right) \\ &\leq O\left(R_{\Theta}^2 R_x^2 L_y \cdot nk^{3/2} \log^2 T\right), \end{split}$$

as desired.

D.3 Assembling the regret bound

Using Lemma 4.2, collecting all terms from Lemmas D.1 and D.5, we have in summary

$$D_{\max}G_{\max} = O\left(R_{\Theta}^2 \sqrt{k}\right) \cdot O\left(R_{\Theta}^2 R_x^2 L_y \cdot nk^{3/2} \log^2 T\right)$$
$$= O\left(R_{\Theta}^4 R_x^2 L_y nk^2 \log^2 T\right).$$

To compete with systems with parameters bounded by R_{Θ} , in light of Theorem 3, k should be chosen to be $\Theta\left(\log^2T\log(R_xL_yR_{\Theta}n)\right)$. It suffices to set the relaxation approximation error ε to be a constant; in the online case, this is not the bottleneck of the regret bound. In all, the regret bound from online gradient descent is

$$\operatorname{Regret}(T) \leq O\left(R_{\Theta}^4\,R_x^2\,L_y\,\log^2(R_{\Theta}R_xL_yn)\cdot n\sqrt{T}\log^6T\right),$$

as claimed. \square

D.4 Diminishing effect of the initial hidden state

Finally, we show that h_0 is not significant in this online setting, thereby proving a slightly more general result. Throughout the rest of the analysis, we considered the comparator Θ^* , which forces the initial hidden state to be the zero vector. We will show that this does not make much worse predictions than Θ^{**} , which is allowed to set $||h_0||_2 \leq R_{\Theta}$. We quantify this below:

Lemma D.6. Relaxing the condition $h_0 = 0$ for the comparator in Theorem 1 increases the regret (additively) by at most

$$O\left(R_{\Theta}^4 R_x L_y \log(R_{\Theta} R_x L_y n) \log^2 T\right)$$

Proof. First, an intuitive sketch: Lemma F.1 states that for any α , there is an "envelope" bound $\mu(\alpha)(t) \leq \frac{1}{t+1}$. This means that the influence of h_0 on the derivative of the impulse response function decays as 1/t; thus, we can expect the total "loss of expressiveness" caused by forcing $h_0 = 0$ to be only logarithmic in T. Indeed, with a nonzero initial hidden state, we have

$$\hat{y}_t - y_{t-1} = (CB + D)x_t - Dx_{t-1} + \sum_{i=1}^{T-1} C(A^i - A^{i-1})Bx_{t-i} + C(A^t - A^{t-1})h_0.$$

Let $\hat{y}_1, \ldots, \hat{y}_T$ denote the predictions made by an LDS $\Theta^{**} = (A, B, C, D, h_0)$ whose; $\hat{y}_1^{\emptyset}, \ldots, \hat{y}_T^{\emptyset}$ denote the predictions made by the LDS with the same (A, B, C, D), but h_0 set to 0. Then, we have

$$\|\hat{y}_t - \hat{y}_t^{\emptyset}\| = \|C(A^t - A^{t-1})h_0\| = \|\sum_{l=1}^d C[\mu(\alpha_l)(t) \cdot e_l \otimes e_l] h_0\|$$

$$\leq \frac{\|C\|_F \|h_0\|\sqrt{n}}{t} \leq \frac{R_{\Theta}^2 \sqrt{n}}{t}.$$

Thus we have, for vectors u_t satisfying $||u_t|| \leq R_{\Theta}^2/t$:

$$\sum_{t=1}^{T} \|\hat{y}_{t}^{\emptyset} - y_{t}\|^{2} = \sum_{t=1}^{T} \|\hat{y}_{t} + u_{t} - y_{t}\|^{2} \leq \sum_{t=1}^{T} \|\hat{y}_{t} - y_{t}\|^{2} + \|u_{t}\|^{2} + 2\|\hat{y}_{t} - y_{t}\| \|u_{t}\|$$

$$\leq \sum_{t=1}^{T} \|\hat{y}_{t} - y_{t}\|^{2} + O\left(R_{\Theta}^{4} n\right) + O\left(\left(R_{\Theta}^{2} R_{x} L_{y} \sqrt{k}\right) \cdot R_{\Theta}^{2} \sqrt{n} \log T\right)$$

$$\leq \sum_{t=1}^{T} \|\hat{y}_{t} - y_{t}\|^{2} + O\left(R_{\Theta}^{4} R_{x} L_{y} \log(R_{\Theta} R_{x} L_{y} n) n \log^{2} T\right),$$

where the inequalities respectively come from Cauchy-Schwarz, Lemma F.1, and Lemma D.2. This completes the proof. \Box

Thus, strengthening the comparator by allowing a nonzero h_0 does not improve the asymptotic regret bound from Theorem 1.

E Properties of the Hankel matrix Z_T

In this section, we show some technical lemmas about the family of Hankel matrices Z_T , whose entries are given by

$$Z_{ij} = \frac{2}{(i+j)^3 - (i+j)}.$$

E.1 Spectral tail bounds

We use the following low-approximate rank property of positive semidefinite Hankel matrices, from [BT16]:

Lemma E.1 (Cor. 5.4 in [BT16]). Let H_n be a psd Hankel matrix of dimension n. Then,

$$\sigma_{j+2k}(H_n) \le 16 \left[\exp\left(\frac{\pi^2}{4\log(8|n/2|/\pi)}\right) \right]^{-2k+2} \sigma_j(H_n).$$

Note that the Hankel matrix Z_T is indeed positive semidefinite, because we constructed it as

$$Z = \int_0^1 \mu(\alpha) \otimes \mu(\alpha) \, d\alpha,$$

for a certain $\mu(\alpha) \in \mathbb{R}^T$. Also, note that at no point do we rely upon Z_T being positive definite or having all distinct eigenvalues, although both seem to be true.

The first result we need is an exponential decay of the tail spectrum of Z.

Lemma E.2. Let σ_i be the j-th top singular value of $Z := Z_T$. Then, for all $T \geq 10$, we have

$$\sigma_j \le \min\left(\frac{3}{4}, K \cdot c^{-j/\log T}\right),$$

where $c = e^{\pi^2/4} \approx 11.79$, and $K < 10^6$ is an absolute constant.

Proof. We begin by noting that for all j, $\sigma_j \leq \text{Tr}(Z) = \sum_{i=1}^T \frac{1}{(2i)^3 - 2i} < \sum_{i=1}^\infty \frac{1}{4i^3} < \frac{3}{4}$. Now, since $T \geq 10$ implies $8\lfloor T/2 \rfloor / \pi > T$, we have

$$\sigma_{2+2k} \le \sigma_{1+2k} < 12 \cdot \left[\exp\left(\frac{\pi^2}{2\log T}\right) \right]^{-k+1}$$

$$< 1680 \cdot c^{-2k/\log T}.$$

Thus, we have that for all j,

$$\sigma_j < 1680 \cdot c^{-(j-2)/\log T} < 235200 \cdot c^{-j/\log T},$$

completing the proof.

We also need a slightly stronger claim: that all spectral gaps are large. Lemma E.2 does not preclude that there are closely clustered eigenvalues under the exponential tail bound. In fact, this cannot be the case:

Lemma E.3. Let σ_i be the j-th top singular value of $Z := Z_T$. Then, if $T \geq 60$, we have

$$\sum_{j'>j} \sigma_{j'} < 400 \log T \cdot \sigma_j.$$

Proof. For convenience, define $\sigma_j := 0$ when $j \geq T$. Picking k = 4 and using Lemma E.1, we have that

$$\beta_j := \sum_{q=1}^T \sigma_{j+4q} < 16\sigma_j \sum_{q=1}^\infty \left[\exp\left(\frac{-\pi^4}{4\log T}\right) \right]^q = 16\sigma_j \cdot \frac{1}{1 - \exp\left(\frac{-\pi^4}{4\log T}\right)} < 100\log T \cdot \sigma_j,$$

where the last inequality follows from the fact that

$$\frac{1}{1-e^{-x}} < \frac{6}{x}$$

whenever x < 6, and setting $x := \frac{-\pi^4}{4 \log T} \le \frac{-\pi^4}{4 \log 60} < 6$.

Thus, we have

$$\sum_{j'>j} \sigma_{j'} = \beta_j + \beta_{j+1} + \beta_{j+2} + \beta_{j+3} < 4\beta_j < 400 \log T \cdot \sigma_j,$$

as desired.

E.2 Decaying reconstruction coefficients

To show a bound on the entries of M_{Θ} , we need the following property of Z_T :

Lemma E.4. For any $0 \le \alpha \le 1$ and $1 \le j \le T$, we have

$$|\langle \phi_j, \mu(\alpha) \rangle| \le 6^{1/4} \, \sigma_j^{1/4}.$$

Proof. We have

$$\int_0^1 \langle \phi_j, \mu(\alpha) \rangle^2 d\alpha = \int_0^1 \phi_j^T (\mu(\alpha) \otimes \mu(\alpha)) \phi_j$$
$$= \phi_j^T Z_T \phi_j = \sigma_j.$$

Thus, we have a bound on the expectation of the squared coefficient, when α is drawn uniformly from [0,1]. We proceed with the same argument as was used to prove Lemma C.2: since $\|\mu(\alpha)\|^2$ is 3-Lipschitz in α , so is $\langle \phi_j, \mu(\alpha) \rangle^2$ (since projection onto the one-dimensional subspace spanned by ϕ_j is contractive). Thus it holds that

$$\max_{\alpha \in [0,1]} \langle \phi_j, \mu(\alpha) \rangle^2 \le \sqrt{6\sigma_j},$$

from which the claim follows.

E.3 Controlling the ℓ_1 norms of filters

To bound the size of the convolutions, we need to control the ℓ_1 norm of the eigenvectors ϕ_j with a tighter bound than \sqrt{T} . Actually, we prove a more general result, bounding the $\ell_2 \to \ell_1$ subordinate norm of $Z^{1/4}$:

Lemma E.5. Let $Z := Z_T$. Then, for every T > 0, and $v \in \mathbb{R}^n$ with $||v||_2 = 1$, we have

$$||Z^{1/4}v||_1 \le 2 + 2\log_2 T.$$

Proof. We take the following steps:

- (i) Start with a constant T_0 ; the subordinate norm of Z_{T_0} is clearly bounded by a constant.
- (ii) Argue that doubling the size of the matrix $(T \mapsto 2T)$ comprises only a small perturbation, which will only affect the eigenvalues of the matrix by a small amount. This will show up in the subordinate norm as an additive constant.
- (iii) Iterate the doubling argument $O(\log T)$ times to reach Z_T from Z_{T_0} , to conclude the lemma.

The only nontrivial step is (ii), which we prove first. Consider the doubling step from T to 2T. Let Z denote the 2T-by-2T matrix which has Z_T as its upper left T-by-T submatrix, and zero everywhere else. Let Z' denote Z_{2T} , and call E = Z' - Z, which we interpret as the matrix perturbation associated with doubling the size of the Hankel matrix.

Notice that when $T \geq 2$, E is entrywise bounded by $\frac{2}{(T+2)^3-(T+2)} \leq \frac{2}{T^3}$, which we call e_{max} for short. Then, $||E||_{\text{op}}$ is at most $Te_{\text{max}} \leq \frac{2}{T^2}$.

Hence, by the generalized Mirsky inequality of [Aud14] (setting $f(x) = x^{1/4}$), we have a bound on how much E perturbs the fourth root of Z:

$$||Z^{1/4} - Z'^{1/4}||_2 \le ||E||_2^{1/4} \le \left(\frac{2}{T^2}\right)^{1/4} < \frac{2}{\sqrt{T}}.$$

Thus we have

$$\begin{split} \|Z'^{1/4}\|_{2\to 1} &\leq \|Z^{1/4}\|_{2\to 1} + \|Z^{1/4} - Z'^{1/4}\|_{2\to 1} \\ &\leq \|Z^{1/4}\|_{2\to 1} + \sqrt{T} \cdot \|Z^{1/4} - Z'^{1/4}\|_{2} \\ &\leq \|Z^{1/4}\|_{2\to 1} + \sqrt{T} \cdot \frac{2}{\sqrt{T}} \\ &= \|Z^{1/4}\|_{2\to 1} + 2. \end{split}$$

Thus, doubling the dimension increases the subordinate norm by at most a constant. We finish the argument: start at $T_0 = 2$, for which it clearly holds that

$$||Z_2^{1/4}||_{2\to 1} < \sqrt{2}||Z_2^{1/4}||_F < \sqrt{2}||Z_4||_F < 2.$$

Noting that the norm is clearly monotonic in T, we repeat the doubling argument $\lfloor \log_2 T \rfloor$ times, so that

$$\|Z_T^{1/4}\|_{2 \to 1} \leq \|Z_{2,2\lfloor \log_2 T \rfloor}^{1/4}\|_{2 \to 1} < \|Z_2^{1/4}\|_{2 \to 1} + 2\lfloor \log_2 T \rfloor < 2 + 2\log_2 T,$$

as claimed. \Box

We give an alternate form here:

Corollary E.6. Let (σ_j, ϕ_j) be the j-th largest eigenvalue-eigenvector pair of Z. Then,

$$\|\phi_j\|_1 \le O\left(\frac{\log T}{\sigma_j^{1/4}}\right).$$

F Properties of $\mu(\alpha)$

Throughout this section, fix some $T \geq 1$; then, recall that $\mu(\alpha) \in \mathbb{R}^T$ is defined as the vector whose *i*-th entry is $(1-\alpha)\alpha^{i-1}$. At various points, we will require some elementary properties of $\mu(\alpha)$, which we verify here.

Lemma F.1 (1/t envelope of μ). For any $t \geq 0$ and $0 \leq \alpha \leq 1$, it holds that

$$(1-\alpha)\alpha^t \le \frac{1}{t+1}.$$

Proof. Setting the derivative to zero, the global maximum occurs at $\alpha^* = \frac{t}{t+1}$. Thus,

$$(1 - \alpha^*)(\alpha^*)^t = \frac{1}{t+1} \left(1 - \frac{1}{t+1}\right)^t \le \frac{1}{t+1},$$

as claimed. \Box

Corollary F.2. Let $T \geq 1$. For t = 1, ..., T, let $\alpha_t \in [0, 1]$ be different in general. Then,

$$\sum_{t=1}^{T} (1 - \alpha_t) \alpha_t^{t-1} \le H_n = O(\log T),$$

where H_n denotes the n-th harmonic number.

Lemma F.3 (ℓ_1 -norm is small). For all $T \geq 1$ and $0 \leq \alpha \leq 1$, we have

$$\|\mu(\alpha)\|_1 \le 1.$$

Proof. We have

$$\|\mu(\alpha)\|_1 = (1-\alpha)\sum_{t=1}^T \alpha^{t-1} \le (1-\alpha)\sum_{t=1}^\infty \alpha^{t-1} = 1,$$

proving the claim.

Lemma F.4 (ℓ_2 -norm is small and Lipschitz). For all $T \geq 1$ and $0 \leq \alpha \leq 1$, we have

- (i) $\|\mu(\alpha)\|^2 \le 1$.
- (ii) $\left| \frac{d}{d\alpha} \| \mu(\alpha) \|^2 \right| \leq 3$.

Proof. For the first inequality, compute

$$\|\mu(\alpha)\|^2 = \sum_{i=1}^T \left((\alpha - 1)\alpha^{i-1} \right)^2 = \sum_{i=1}^T \alpha^{2i} - 2\alpha^{2i-1} + \alpha^{2i-2}$$
$$= \frac{(\alpha^2 - 2\alpha + 1)(1 - \alpha^{2T})}{1 - \alpha^2} = \frac{(1 - \alpha)(1 - \alpha^{2T})}{1 + \alpha} \le 1.$$

For the second, differentiate the closed form to obtain

$$\left| \frac{d}{d\alpha} \|\mu(\alpha)\|^2 \right| = \left| \frac{2(\alpha^T - 1) + T\alpha^{T-1}(\alpha^2 - 1)}{(1 + \alpha)^2} \right| \le \frac{2(1 - \alpha^T) + T\alpha^{T-1}(1 - \alpha^2)}{(1 + \alpha)^2}$$
$$= \frac{2 - \alpha^T}{(1 + \alpha)^2} + \frac{T\alpha^{T-1}(1 - \alpha)}{1 + \alpha} \le 2 + T\alpha^{T-1}(1 - \alpha) \le 3,$$

where the final inequality uses Lemma F.1.

F.1 The Lipschitzness of a true LDS

We claim in Section 2.2 that L_y , the Lipschitz constant of a true LDS, is bounded by $||B||_F ||C||_F R_x$. We now prove this fact, which is a consequence of the above facts.

Lemma F.5. Let $\Theta = (A, B, C, D, h_0)$ be a true LDS, which produces outputs y_1, \ldots, y_T from inputs x_1, \ldots, x_T by the definition in the recurrence, without noise. Let $0 \le A \le I$, and $||B||_F, ||C||_F, ||D||_F, ||h_0|| \le R_{\Theta}$. Then, we have that for all t,

$$||y_t - y_{t-1}|| \le O(R_{\Theta}^2 R_x).$$

Proof. We have that for all $1 \le t \le T$,

$$||y_t - y_{t-1}|| = \left| |(CB + D)x_t - Dx_{t-1} + \sum_{i=1}^{T-1} C(A^i - A^{i-1})Bx_{t-i} + C(A^t - A^{t-1})h_0 \right|$$

$$\leq (||B||_F ||C||_F + 2||D||_F)R_x + ||B||_F ||C||_F R_x + \frac{||C||_F ||h_0||}{t},$$

where the inequality on the second term arises from Lemma F.3 and the inequality on the third from Lemma F.2. This implies the lemma. \Box