Two-Dimensional Imaging

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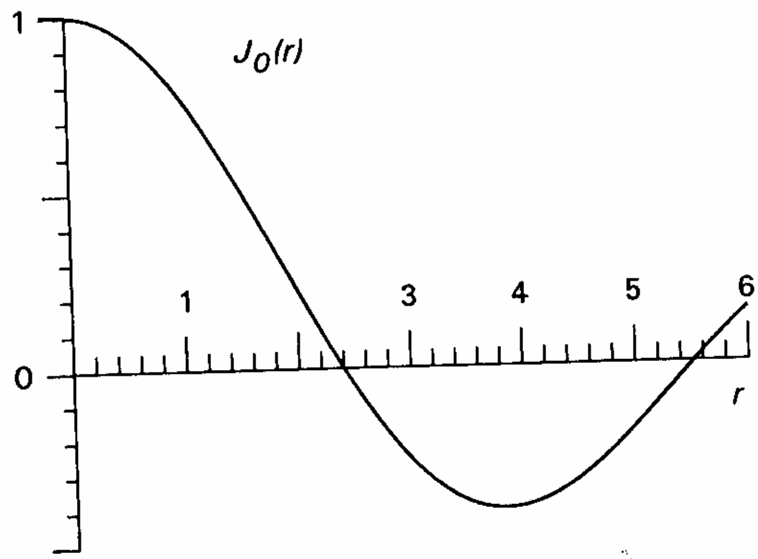

Figure 9-6 The zero-order Bessel function $J_0(r)$.

of electrical and mechanical systems that are both linear and time invariant, so the Bessel function will be found in higher dimensionalities where linearity and space invariance exist and where circular boundary conditions are imposed.

THE HANKEL TRANSFORM

To obtain the Fourier transform of a function that is constant over a central circle in the (x, y)-plane and zero elsewhere, or which, while not being constant, is a function of $r = (x^2 + y^2)^{1/2}$ only, one may of course use the standard transform definition in cartesian coordinates. Let the function $f(x, y)$ be $\mathbf{f}(r)$. Then

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{f}(r) e^{-i2\pi(ux+vy)}\, dx\, dy.$$

Some comments should be made about the limits of integration. They may certainly be written as above, provided it is understood that $\mathbf{f}(r)$ covers the whole plane $0 < r < \infty$. In the example mentioned, where the function is a constant K over a central circular region C of radius a and zero elsewhere, there would be a simplified alternative

$$F(u, v) = K \int \int_C e^{-i2\pi(ux+vy)}\, dx\, dy.$$

Of course, the boundary of a circle does not lend itself to the simplest kind of representation in rectangular coordinates. Nevertheless, it is possible to do so, in which case we could write

$$F(u,v) = K\int_{-a}^{a} dy \int_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} e^{-i2\pi(ux+vy)}\,dx.$$

Sometimes this kind of integration over elaborate boundaries works out painlessly; let us try to proceed in this case.

$$\begin{aligned}
F(u,v) &= K\int_{-a}^{a} e^{-i2\pi vy}\,dy \int_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} e^{-i2\pi ux}\,dx \\
&= K\int_{-a}^{a} e^{-i2\pi vy}\left[\frac{e^{-i2\pi ux}}{-i2\pi u}\right]_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} dy \\
&= K\int_{-a}^{a} e^{-i2\pi vy}\left[\frac{e^{-i2\pi u(a^2-y^2)^{1/2}} - e^{i2\pi u(a^2-y^2)^{1/2}}}{-i2\pi u}\right] dy \\
&= \frac{K}{\pi u}\int_{-a}^{a} e^{-i2\pi vy}\sin\left[2\pi u(a^2-y^2)^{1/2}\right] dy \\
&= 2\frac{K}{\pi u}\int_{0}^{a} \cos(2\pi vy)\sin\left[2\pi u(a^2-y^2)^{1/2}\right] dy.
\end{aligned}$$

This is the maximum simplification that we can hope to make before turning to lists of integrals for help. We find the integral on p. 399 as entry 3.711 in GR (1965). The answer is

$$F(u,v) = Ka(u^2+v^2)^{-1/2}J_1\left[2\pi a(u^2+v^2)^{1/2}\right] = Ka\frac{J_1(2\pi aq)}{q},$$

where J_1 is the first-order Bessel function of the first kind and $q = \sqrt{u^2+v^2}$.

As an alternative approach, consider making use of the circular symmetry of the exercise rather than forcing rectangular coordinates upon it. Let (r,θ) be the polar coordinates of (x,y). Then

$$F(u,v) = \int\int_C \mathbf{f}(r)e^{-i2\pi(ux+vy)}\,dx\,dy = \int_0^{\infty}\int_0^{2\pi} \mathbf{f}(r)e^{-i2\pi qr\cos(\theta-\phi)}r\,dr\,d\theta.$$

We are now integrating from 0 to ∞ radially and through 0 to 2π in azimuth, and, because of the independence of azimuth, the latter integral should drop out. The new symbols q and ϕ are polar coordinates in the (u,v)-plane. Thus

$$q^2 = u^2+v^2 \quad \text{and} \quad \tan\phi = v/u.$$

Then the new kernel $\exp[-i2\pi qr\cos(\theta-\phi)]$ arises from recognizing $ux+vy$ as the scalar product of two two-dimensional vectors. Thus

$$ux+vy = \Re[(x+iy)(u-iv)] = \Re[re^{i\theta}qe^{-i\phi}] = qr\cos(\theta-\phi).$$

Removing the integration with respect to θ,

$$F(u, v) = \int_0^\infty \mathbf{f}(r)\left[\int_0^{2\pi} e^{-i2\pi qr\cos(\theta-\phi)}d\theta\right] r\,dr$$
$$= \int_0^\infty \mathbf{f}(r)\left[\int_0^{2\pi} \cos(2\pi qr\cos\theta)d\theta\right] r\,dr.$$

Since u and v, and therefore q and ϕ, are fixed during the integration over the (x, y)-plane, we may drop ϕ, because it merely represents an initial angle in an integration that will run over one full rotation from 0 to 2π. Therefore, the result of the integration will be the same regardless of the value of ϕ; take it to be zero. Again, the sine component of the imaginary exponential may be dropped, because it will integrate to zero; to see this, make sketches of cos (cos θ) and sin (cos θ) for $0 < \theta < 2\pi$.

We now make use of the fundamental integral representation for the zero-order Bessel function $J_0(z)$, namely,

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\cos\theta)d\theta.$$

This basic relation will be returned to later. Meanwhile, incorporating it into the development, we have finally

$$F(u, v) = 2\pi\int_0^\infty \mathbf{f}(r)J_0(2\pi qr)r\,dr.$$

Here we have the statement of the integral transform that takes the place of the two-dimensional Fourier transform when $f(x, y)$ possesses circular symmetry and is representable by $\mathbf{f}(r)$. It follows that $F(u, v)$ also possesses circular symmetry (the rotation theorem is an expression of this) and may be written $\mathbf{F}(q)$, depending only on the radial coordinate $q = (u^2 + v^2)^{1/2}$ in the (u, v)-plane and not on azimuth ϕ. The transform

Hankel transform.

$$\mathbf{F}(q) = 2\pi\int_0^\infty \mathbf{f}(r)J_0(2\pi qr)r\,dr$$

is known as the Hankel transform. It is a one-dimensional transform. The functions, $\mathbf{f}$ and $\mathbf{F}$ are functions of one variable. They are not a Fourier transform pair, but a Hankel transform pair. As functions of *one* variable they may be used to represent two-dimensional functions, which will be two-dimensional Fourier transform pairs.

Hankel Transform of a Disk

Our first example of a Hankel transform pair, obtained directly by integration, was

$$\text{rect}\left(\frac{r}{2a}\right) \quad \text{has Hankel transform} \quad \frac{aJ_1(2\pi aq)}{q}.$$

This is such an important pair that we adopt the special name jinc q for the Hankel transform of rect r. Thus

$$\text{rect}\, r \quad \text{has Hankel transform} \quad \text{jinc}\, q \equiv \frac{J_1(\pi q)}{2q}.$$

From thc integral transform formulation it follows that

$$\text{jinc}\, q = 2\pi \int_0^\infty \text{rect}\, r\, J_0(2\pi q r) r\, dr.$$

Likewise, from the reversibility of the two-dimensional Fourier transform, it follows that

$$\text{rect}\, r = 2\pi \int_0^\infty \text{jinc}\, q\ J_0(2\pi q r)\ q\, dq.$$

Frequently needed properties of the jinc function are collected below.

Hankel Transform of the Ring Impulse

As an example of the Hankel transform we used the rectangle function of radius 0.5 and found by direct integration in two dimensions that the Hankel transform was jinc q. Now we make use of the Hankel transform formula to obtain another important transform pair. Let

$$\mathbf{f}(r) = \delta(r - a)$$

which describes a unit-strength ring impulse. Its Hankel transform $\mathbf{F}(q)$ is given by

$$\begin{aligned} \mathbf{F}(q) &= 2\pi \int_0^\infty \mathbf{f}(r) J_0(2\pi q r) r\, dr \\ &= 2\pi \int_0^\infty \delta(r - a) J_0(2\pi q r) r\, dr. \end{aligned}$$

Apply the sifting property to obtain immediately

$$\mathbf{F}(q) = 2\pi a J_0(2\pi a q).$$

From the reciprocal property of the Hankel transform it also follows that

$$\delta(r - a) = 2\pi \int_0^\infty 2\pi a J_0(2\pi a q) J_0(2\pi r q) q\, dq$$

a relationship that can be recognized as expressing an orthogonality relationship between zero-order Bessel functions of different "frequencies." Unless the two Bessel functions have the same "frequency," the infinite integral of their product is zero, just as with sines and cosines. Although $J_0(\omega t)$ is not a monochromatic waveform, nevertheless as t elapses, the waveform decays away rather slowly in amplitude and settles down more and more closely to a definite angular frequency ω and a fixed phase, as may be seen from the asymptotic expression

$$J_0(\omega t) \sim \sqrt{\frac{2}{\pi \omega t}} \cos(\omega t - \tfrac{1}{4}\pi).$$

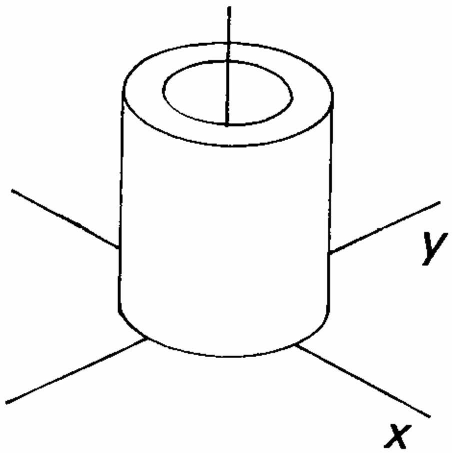

Figure 9-7 A circularly symmetrical function suitable for representing the distribution of light over a uniformly illuminated annular slit. By an extraordinary visual illusion connected with the evolution of vision, the height and outside diameter appear unequal.

Just as $\operatorname{jinc} r$ can be described as a circularly symmetrical two-dimensional function that contains all spatial frequencies up to a certain cutoff, in uniform amount for all frequencies and orientations, so also $J_0(r)$ can be seen as a circularly symmetrical two-dimensional function that contains only one spatial frequency but equally in all orientations. The jinc function is thus like the sinc function, and J_0 is like the cosine function.

Annular Slit

We have established the following two Hankel transform pairs:

$$\operatorname{rect} r \supset \operatorname{jinc} q$$
$$\delta(r-a) \supset 2\pi a J_0(2\pi a q),$$

and in what follows we need to recall the similarity theorem in its form applicable to circular symmetry:

$$\text{If } \mathbf{f}(r) \supset \mathbf{F}(q) \quad \text{then} \quad \mathbf{f}(r/a) \supset a^2\mathbf{F}(aq).$$

Note that the sign $\supset$ may be read "has Hankel transform" if you picture **f** and **F** as one dimensional, but may alternatively be read "has two-dimensional Fourier transform" if you prefer to think of **f** and **F** as functions of radius representing two-dimensional entities.

Both the unit circular patch represented by $\operatorname{rect} r$ and the ring impulse $\delta(r-a)$ are constantly needed. A third important circularly symmetrical function (Fig. 9-7) is unity over an annulus. For concreteness of description it may be referred to as an *annular slit*, but of course the function is of wider significance.

A narrow circular slit could be cut in an opaque sheet and, if uniformly illuminated from behind, would be reasonably represented by a ring impulse. If the mean radius of the annulus were a, the slit width w, and the amplitude of illumination A, then the distribution of light would be expressible as

$$A \operatorname{rect}\left(\frac{r}{2a+w}\right) - A \operatorname{rect}\left(\frac{r}{2a-w}\right).$$

The "quantity" of light would be $2\pi awA$, and the quantity per unit arc length would be wA. Therefore, since the ring impulse $\delta(r-a)$ has unit weight per unit arc length, the appropriate impulse representation would be $wA\delta(r-a)$.

We know the Hankel transforms of both the annulus and the ring delta, and the transforms will approach equality as the slit width w approaches zero, provided at the same time the amplitude of illumination A is increased so as to maintain constant the integrated amplitude over the slit. We are saying that as the slit width w goes to zero while wA remains constant, then

$$A(2a+w)^2\,\mathrm{jinc}[(2a+w)q] - A(2a-w)^2\,\mathrm{jinc}[(2a-w)q] \to (wA)2\pi a J_0(2\pi aq).$$

It follows that

$$\lim_{w\to 0} w^{-1}\Big[(2a+w)^2\,\mathrm{jinc}[(2a+w)q] - (2a-w)^2\,\mathrm{jinc}[(2a-w)q]\Big] = 2\pi a J_0(2\pi aq).$$

The left-hand side is recognizable as a derivative, and therefore the conclusion implies the identity

$$\frac{\partial}{\partial a}(4a^2\,\mathrm{jinc}\,2qa) = 2\pi a J_0(2\pi aq),$$

a result that can be deduced independently from properties of Bessel functions.

For computing purposes we may sometimes wish to represent a ring impulse by an annulus of small but nonzero width, and we may also wish to do the reverse for purposes of theory—namely, to represent an annular slit by a ring impulse. A slit width equal to 10 percent of the mean radius may seem a rather crude example to take, but with $a=1$ and $w=0.1$ let us compare $10[(2.1)^2\,\mathrm{jinc}(2.1q) - (1.9)^2\,\mathrm{jinc}(1.9q)]$ with $2\pi J_0(2\pi q)$.

We quickly find from a few test points that the agreement is good.

q	0	0.38277	−0.5	1.0
LHS	6.28318	−0.004	−1.9061	1.365
RHS	6.28318	0	−1.9116	1.384

Thus, as far as the transform is concerned, a 10 percent slit width, which seems far from a slit of zero width, gives results within 1 percent or so.

It is worthwhile doing numerical calculations of this sort from time to time to develop a sense of how crude an approximation may be and still be useful. Over the whole range $0<q<1$ the discrepancy ranges between limits of 0.0192 and −0.0197 or just under 2 percent of the central value. For less crude approximations the results would, of course, be even more accurate. An approximate solution to an urgent problem is most welcome, provided you have the experience to feel confidence in the quality of the approximation. You gain this feeling for magnitudes by making a habit of comparing rough approximations with correct solutions. Reference lists of Hankel transforms can be found in FTA (1986), in Erd (1954), and others occur in GR (1965).

Table 9-1 Table of Hankel transforms.

$\mathbf{f}(r)$	$\mathbf{F}(q) = 2\pi \int_0^\infty \mathbf{f}(r) J_0(2\pi q r) r dr$
$\mathbf{f}(ar)$	$a^{-2}\mathbf{f}(q/a)$
$\mathbf{f} ** \mathbf{g}$	$\mathbf{FG}$
$r^2\mathbf{f}(r)$	$-\nabla^2\mathbf{F}$
$\operatorname{rect} r$	$\operatorname{jinc} q$
$\delta(r-a)$	$2\pi a J_0(2\pi a q)$
$e^{-\pi r^2}$	$e^{-\pi q^2}$
$r^2 e^{-\pi r^2}$	$\pi^{-1}(\pi^{-1} - q^2)e^{-\pi q^2}$
$(1+r^2)^{-1/2}$	$q^{-1}e^{-2\pi q}$
$(1+r^2)^{-3/2}$	$2\pi e^{-2\pi q}$
$(1-4r^2)\operatorname{rect} r$	$J_2(\pi q)/\pi q^2$
$(1-4r^2)^\nu \operatorname{rect} r$	$2^{\nu-1}\nu! J_{\nu+1}(\pi q)/\pi^\nu q^{\nu+1}$
r^{-1}	q^{-1}
e^{-r}	$2\pi(4\pi^2 q^2 + 1)^{-3/2}$
$r^{-1}e^{-r}$	$2\pi(4\pi^2 q^2 + 1)^{-1/2}$
${}^2\delta(x,y)$	1

Theorems for the Hankel Transform

Theorems for the Hankel transform are deducible from those for the two-dimensional Fourier transform, with appropriate change of notation. For example, the similarity theorem $f(ax, by) \overset{2}{\supset} |ab|^{-1}F(u/a, v/b)$ will apply, provided $a = b$, a condition that is necessary to preserve circular symmetry. Thus $\mathbf{f}(ar)$ has Hankel transform $a^{-2}\mathbf{F}(q/a)$. The shift theorem does not have any meaning for the Hankel transform, since shift of origin destroys circular symmetry. The convolution theorem, $f ** g \overset{2}{\supset} FG$, retains meaning for the Hankel transform on the understanding that $\mathbf{f}(r)$ and its Hankel transform $\mathbf{F}(q)$ are both taken as representing two-dimensional functions on the (x, y)-plane. Then $\mathbf{f} ** \mathbf{g}$ has Hankel transform $\mathbf{FG}$. Some theorems have been incorporated in Table 9-1.

Computing the Hankel Transform

It is perfectly feasible to compute the Hankel transform from the integral definition. The infinite upper limit causes no trouble in practice when the given function either cuts off or dies away rapidly. To evaluate the Bessel functions needed for all the q values one uses the series approximation given above for arguments less than 3 and an asymptotic expansion otherwise. In the following sample program the given function $\mathbf{f}(r)$ is defined to be $\exp[-\pi(r/7)^2]$, which falls to 3×10^{-6} at $r = 14$, and is integrated from 0 to 14. The Bessel function appears in the inner loop with three explicit multiplies, but at least ten more occur in the function definition for $J_0(x)$. Consequently this program is not fast.

A faster method starts by taking the Abel transform of $\mathbf{f}(r)$ (see below) followed by a standard fast Fourier transform; or, since only the real part of the complex output will be utilized, some may prefer to call a standard fast Hartley transform, which will give exactly the same result faster.

HANKEL TRANSFORM

```
DEF FNf(r)=EXP(-PI*(r/7)^2        Function definition
dr=0.1                            Step in r
FOR q=0 TO 0.25 STEP 0.05
   k=2*PI*q
   s=0
   FOR r=dr/2 TO 14 STEP dr
      s=s+FNf(r)*FNJ0(k*r)* r
   NEXT r
   PRINT q;2*PI*s*dr
NEXT q
END
```

THE JINC FUNCTION

Just as in one dimension there is a sinc function which contains all frequencies equally up to a cutoff, and no higher frequencies, so in two dimensions there is a jinc function (Figs. 9-1, 9-8 and Table 9-1) that has already been referred to. The following material, which is collected in one place for reference, mentions the Abel transform, and the Struve function of order unity, which are discussed later. A table of the jinc function is given as Table 9-4 at the end of the chapter.

Properties of the jinc Function

Definition.

$$\operatorname{jinc} x = \frac{J_1(\pi x)}{2x}.$$

Series Expansion.

$$\begin{aligned}\operatorname{jinc} x &= \frac{\pi}{4} - \frac{\pi^3}{2^5}x^2 + \frac{\pi^5}{2^8.3}x^4 - \frac{\pi^7}{2^{12}.3}x^6 + \frac{\pi^9}{2^{16}.3^2.5}x^8 - \cdots \\ &= .785398 - .968946x^2 + .398463x^4 - .245792x^6 + .010108x^8 + \cdots.\end{aligned}$$

Asymptotic Expression, x > 3.

$$\operatorname{jinc} x \sim \frac{\cos[\pi(x - 3/4)]}{\sqrt{2\pi^2 x^3}}.$$

Asymptotic Behavior. The slow decay of $J_0(r)$ with r is connected with the fact that its Hankel transform is impulsive, while the relatively rapid decay of jinc r to small values

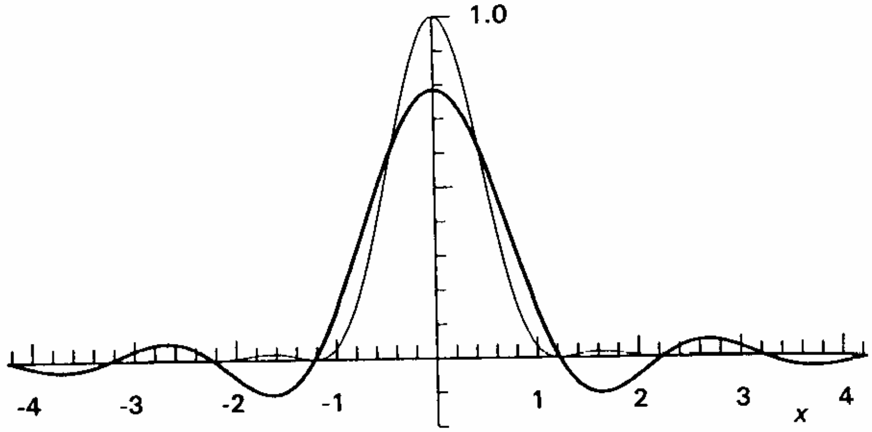

Figure 9-8 The jinc function, Hankel transform of the unit rectangle function (heavy line). The first null is at 1.22, the constant that is familiar in the expression $1.22\lambda/D$ for the angular resolution of a telescope of diameter D. See Table 9-4 for tabulated values. The jinc^2 function normalized to unity at the origin (light line) describes the intensity in the Airy disc, the diffraction pattern of a circular aperture.

occurs because its transform has only a finite discontinuity. The even greater compactness of $\text{jinc}^2(r)$ is associated with the even smoother form of its transform, the chat function. Thus $J_0(r) \sim r^{-1/2}$, $\text{jinc}\, r \sim r^{-3/2}$, $\text{jinc}^2 r \sim r^{-3}$. It appears that, if n derivatives of $\mathbf{F}(q)$ have to be taken to make the result impulsive, then $\mathbf{f}(r) \sim r^{-(n+1/2)}$. The similar theorem for the Fourier transform is that if the nth derivative of $f(x)$ is impulsive, then $F(s) \sim s^{-n}$. However, in the presence of circular symmetry a qualification is necessary, because there cannot be a finite discontinuity at the origin, and a discontinuity in slope at the origin, such as chat r exhibits, counts for less.

Zeros. $\text{jinc}\, x_n = 0$

n	1	2	3	4	5	6	...	n
x_n	1.2197	2.2331	3.2383	4.2411	5.2428	6.2439	...	$\sim n + 1/4$

Derivative.

$$\text{jinc}'\, x = \frac{\pi}{2x} J_0(\pi x) - \frac{1}{x^2} J_1(\pi x) = -\frac{\pi}{2x} J_2(\pi x).$$

Maxima and Minima.

Location	1.6347	2.6793	3.6987	4.7097	5.7168	6.7217
Value	−0.1039	0.0506	−0.0314	0.0219	−0.016	0.013

Integral. The jinc function has unit area under it:

$$\int_{-\infty}^{\infty} \operatorname{jinc} x \, dx = 1.$$

Half Peak and 3 dB Point.

$$\operatorname{jinc}(0.70576) = 0.5 \operatorname{jinc} 0 = \pi/8 = 0.39270.$$

$$\operatorname{jinc} \theta_{3\mathrm{dB}} = \frac{1}{\sqrt{2}} \operatorname{jinc} 0 = 0.55536,$$

where $\theta_{3\mathrm{dB}} = 0.51456$.

Fourier Transform. The one-dimensional Fourier transform of the jinc function is semi-elliptical with unit height and unit base.

$$\int_{-\infty}^{\infty} \operatorname{jinc} x \, e^{-i2\pi sx} \, dx = \sqrt{1-(2s)^2} \operatorname{rect} s.$$

Hankel Transform. The Hankel transform of the jinc function is the unit rectangle function

$$\int_{0}^{\infty} \operatorname{jinc} r \, J_0(2\pi qr) \, 2\pi r \, dr = \operatorname{rect} q.$$

Abel Transform. The Abel transform (line integral) of the jinc function is the sinc function

$$2\int_{x}^{\infty} \frac{\operatorname{jinc} r \, r \, dr}{\sqrt{r^2 - x^2}} = \operatorname{sinc} x.$$

Two-dimensional Aspect. Regarded as a function of two variables x and y, jinc r (where $r^2 = x^2 + y^2$) describes a circularly symmetrical hump surrounded by null circles separating positive and negative annuli.

The Null Circles. Nulls occur at radii 1.220, 2.233, 3.239, etc. As the radii approach values of 0.25 + integer, their spacing approaches unity.

Two-dimensional Integral. The volume under jinc r is unity:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{jinc} \sqrt{x^2+y^2} \, dx \, dy = \int_{0}^{\infty} \operatorname{jinc} r 2\pi r \, dr = 1.$$

Two-dimensional Fourier Transform. A disc function of unit height and diameter:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{jinc} \sqrt{x^2+y^2} \, e^{-i2\pi(ux+vy)} dx \, dy = \operatorname{rect}(\sqrt{u^2+v^2}).$$

Two-dimensional Autocorrelation Function of the jinc function is the jinc function

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{jinc} \sqrt{\xi^2+\eta^2} \; \operatorname{jinc} \sqrt{(\xi+x)^2+(\eta+y)^2} \, d\xi \, d\eta = \operatorname{jinc} \sqrt{x^2+y^2}.$$

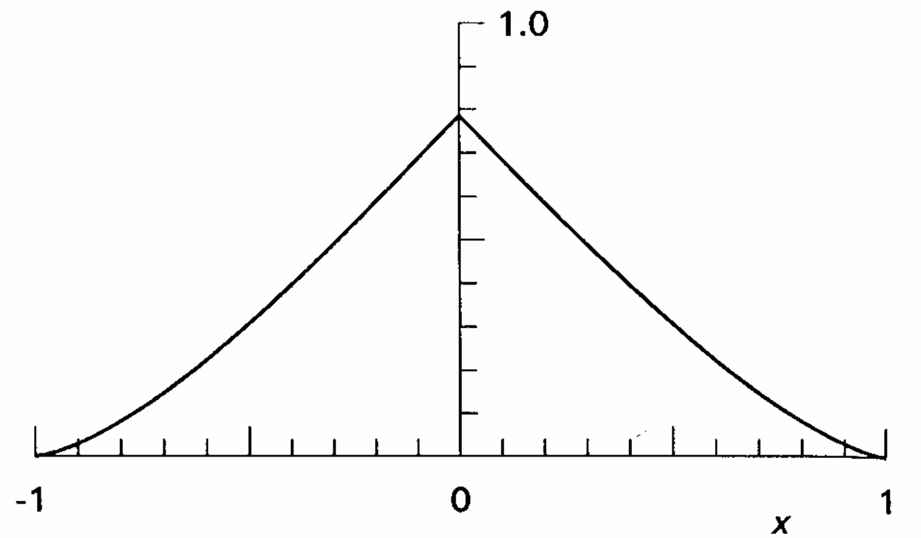

Figure 9-9 The Chinese hat function, autocorrelation function of rect r.

The Chinese Hat Function (Fig. 9-9) is the Hankel transform of $\text{jinc}^2 r$.

$$\text{chat}\, q \equiv \tfrac{1}{2}\left(\cos^{-1}|q| - |q|\sqrt{1-q^2}\right) \text{rect}\, \tfrac{1}{2}q = \int_0^\infty \text{jinc}^2 r\, J_0(2\pi q r)\, 2\pi r\, dr.$$

q	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
chat q	0.7854	0.6856	0.5867	0.4900	0.3963	0.3071	0.2236	0.1477	0.0817	0.0294	0.0

Autocorrelation of the Unit Disk Function is the Chinese hat function of radius 1.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{rect}(\sqrt{\alpha^2+\beta^2})\, \text{rect}(\sqrt{(\alpha+u)^2+(\beta+v)^2})\, d\alpha\, d\beta = \text{chat}\sqrt{u^2+v^2}.$$

Two-dimensional Fourier Transform of the jinc^2 function is the Chinese hat function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{jinc}^2\sqrt{x^2+y^2}\, e^{-i2\pi(ux+vy)} dx\, dy = \text{chat}\sqrt{u^2+v^2}.$$

See Fig. 9-9.

Abel Transform of jinc^2 is (See Fig. 9-10.)

$$2\int_x^\infty \frac{\text{jinc}^2 r\, r\, dr}{\sqrt{r^2-x^2}} = \frac{\mathbf{H}_1(2\pi x)}{4\pi x^2}.$$

Fourier Transform of Chat.

$$\int_{-\infty}^{\infty} \text{chat}\, x e^{-i2\pi s x}\, dx = \frac{\mathbf{H}_1(2\pi s)}{4\pi s^2}.$$

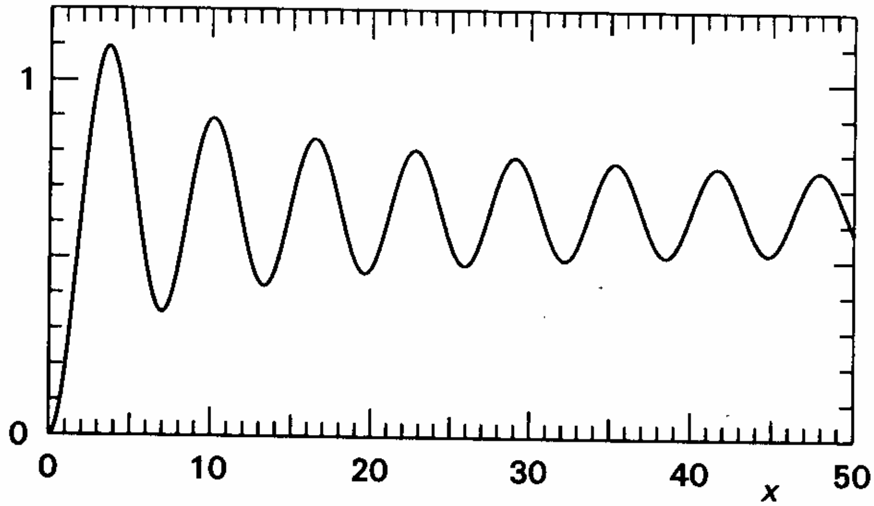

Figure 9-10 The Struve function $\mathbf{H}_1(x)$.

Abel Transform of Chat.

$$2\int_x^\infty \frac{\text{chat}\, r\, r\, dr}{\sqrt{r^2 - x^2}} = \text{See Fig. 9-12.}$$

Fourier Transform of jinc^2 **is**

$$\int_{-\infty}^{\infty} \text{jinc}^2 x\, e^{-i2\pi sx}\, dx =$$

See Fig. 9-12.

Integrals and Central Values.

$$\int_0^\infty \text{jinc}^2 r\, 2\pi r\, dr = \text{chat}\, 0 = \pi/4,$$

$$\int_0^\infty \text{chat}\, r\, 2\pi r\, dr = \text{jinc}^2\, 0 = \pi^2/16.$$

To summarize, we arrange the various functions in groups of four to display their relationships according to the pattern shown in Fig. 9-11. Each algebraic quartet in the table can also be illustrated graphically. We have been accustomed to arranging functions on the left and transforms on the right. A certain convenience accrues from the adoption of conventions of this sort, which provide a constant framework within which different cases may be considered. In the graphical version of the new organization proposed, the

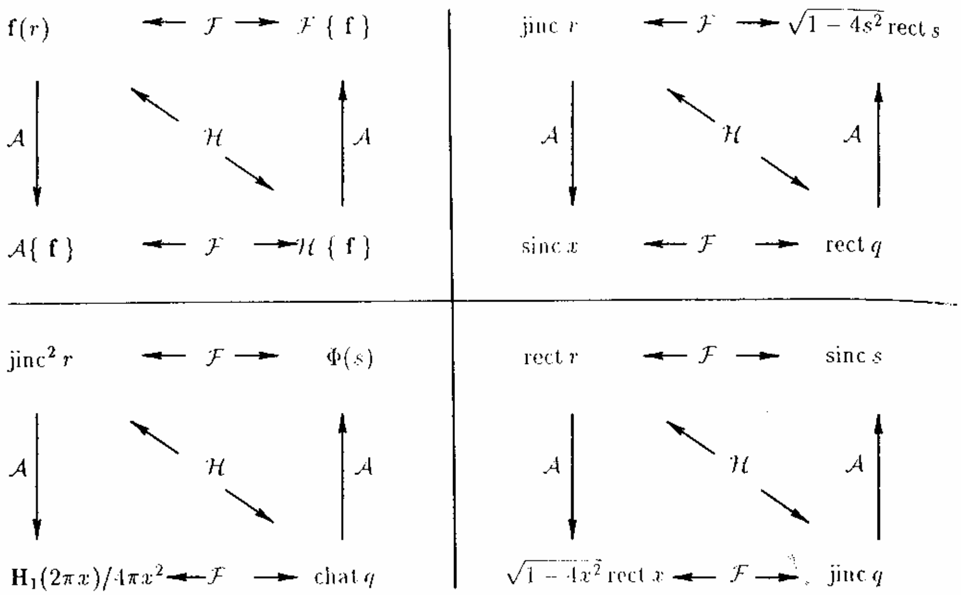

Figure 9-11 The jinc function and its relatives arranged in quartets obeying the relationships specified in the upper left. The function $\Phi(s)$ is both the Abel transform of the chat function and the Fourier transform of the jinc^2 function.

function $f(x, y)$ (or $\mathbf{f}(r)$ in the case of circular symmetry) goes in the northwest and its two-dimensional Fourier transform $F(u, v)$ [or $\mathbf{F}(q)$] goes in the southeast, giving a diagonal arrangement on the page, because in this way the left-right juxtaposition of one-dimensional Fourier transforms can be preserved. In the example where jinc r is in the top left-hand corner and rect q in the bottom right-hand corner, the cross section of each two-dimensional function along the east-west axis can be shown rabatted into the plane. These one-dimensional functions of x and u, respectively, constitute a Hankel transform pair. Where circular symmetry happens to exist and $f(x, y) \;{}^2\!\supset\; F(u, v)$, then $f(x, 0)$ has Hankel transform $F(u, 0)$. The general situation of no symmetry has more to do with data than with properties of instruments, which can often be designed with cylindrical symmetry, and is taken up later in connection with the projection-slice theorem.

The cross section of $f(x, y)$ along a line $x = \text{const}$ has an area which is the ordinate of the Abel transform of $f(x, y)$, viz., sinc x. We see that sinc x and its Fourier transform rect q are arranged left-to-right as planned.

The whole story can now be repeated, since the Hankel transform is reciprocal, starting in the bottom right-hand corner. Thus the cross section of rect q has an area equal to the ordinate of $(1 - 4u^2)^{1/2}$ rect u, which in turn is the one-dimensional Fourier transform of jinc r, the function we began with.

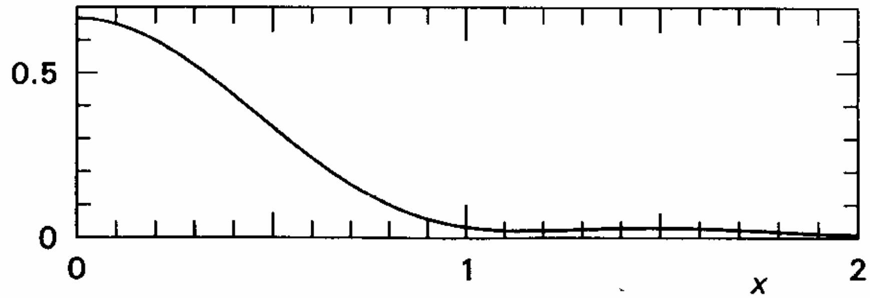

Figure 9-12 The Abel transform of $\text{jinc}^2 r$. This function is also the one-dimensional Fourier transform of the chat function.

THE STRUVE FUNCTION

The Fourier transform of the chat function, which is the same as the Abel transform of the Airy diffraction pattern, or jinc^2 function, both arise naturally in optical systems and may be expressed in terms of the Struve function $\mathbf{H}_1(x)$ as $\mathbf{H}_1(2\pi x)/4\pi x^2$. For values of x up to about 5 the Struve function can be calculated from the Taylor series $(2/\pi)(1 + x^2/3 - x^4/3^2.5 + x^6/3^2 \cdot 5^2 \cdot 7 - \cdots)$. For larger values of x use the asymptotic expansion 12.1.31 given in A&S (1964). A graph is shown in Fig. 9-10; the function oscillates with a period close to 2π about a limiting value of $2/\pi$. The oscillations decay rather slowly in amplitude, inversely as the square root of x; the only null is the one at $x = 0$.

THE ABEL TRANSFORM

A two-dimensional function $\mathbf{f}(r)$ that has circular symmetry possesses a line integral, or projection, that is the same in all directions. Call this function $f_A(x)$. The subscript A refers to Abel and the variable x can be thought of as being the abscissa in the (x, y)-coordinate system to which the radial coordinate r belongs. Thus $f_A(x)$ is the projection in the y-direction, or the line integral in the y-direction. As an example, if $\mathbf{f}(r) = \text{rect}(r)$, then $f_A(x) = (1 - 4x^2)^{1/2} \text{rect}\, x$ (Fig. 9-13). This is because a disk function of unit height and unit diameter has a cross-section area $(1 - 4x^2)^{1/2}$ on the line $x = \text{const}$, provided $|x| < \frac{1}{2}$. Where $|x| \geq \frac{1}{2}$, the cross-section area is zero, a fact that the factor $\text{rect}\, x$ reminds us of. The shape of the Abel transform in this example is semi-elliptical, which is connected with the fact that the given outline was circular. If you wanted to know what function of r has a semicircular Abel transform, the answer would be $\frac{1}{2}\,\text{rect}\, r$.

In lieu of these explanatory remarks it would be sufficient simply to introduce the Abel transform $f_A(\ )$ of a function $\mathbf{f}(\ )$ by this definition:

$$f_A(x) \triangleq \int_{-\infty}^{\infty} \mathbf{f}(\sqrt{x^2 + y^2})\, dy.$$

Then if the question arose as to the Abel transform of rect( ), we would evaluate it as follows:

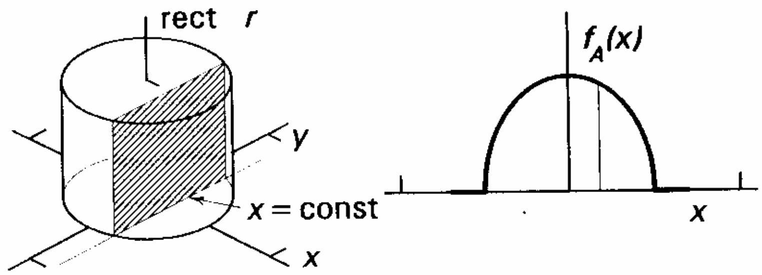

Figure 9-13 The area of the shaded cross section is the Abel transform of the function of r for the particular value of x chosen. In the case of rect r the Abel transform $f_A(x)$ is the semi-ellipse $\sqrt{1-4x^2}$, $|x| < \frac{1}{2}$.

$$\begin{aligned} f_A(x) &= \int_{-\infty}^{\infty} \operatorname{rect}(\sqrt{x^2+y^2})\,dy \\ &= \int_{-(1/4-x^2)^{1/2}}^{(1/4-x^2)^{1/2}} dy \ \operatorname{rect} x \\ &= 2\sqrt{\tfrac{1}{4}-x^2}\ \operatorname{rect} x \\ &= \sqrt{1-4x^2}\ \operatorname{rect} x. \end{aligned}$$

The limits of integration were arrived at by noting that points on a line parallel to the y-axis at abscissa x must lie within $y = \pm(\frac{1}{4} - x^2)^{1/2}$ in order for $\operatorname{rect}(\sqrt{x^2+y^2})$ to be unity rather than zero. The integration involved is not difficult, but it is obvious that some geometrical reasoning based on the explanatory introduction is helpful in arriving at the limits of integration.

An alternative form of the definition can be given in terms of r, which is the natural variable to think of as underlying a circularly symmetrical function $\mathbf{f}(r) = f(x, y)$. Thus

Abel transform definition.

$$f_A(x) \triangleq 2\int_x^{\infty} \frac{\mathbf{f}(r)r\,dr}{\sqrt{r^2-x^2}}.$$

To convert from dy to dr write $dr/dy = \sin\theta = \sqrt{r^2-x^2}/r$. Thus $dy = r\,dr/\sqrt{r^2-x^2}$. To relate this definition to the previous one, note that the minimum value of r is the given value of x. Thus $\int_{-\infty}^{\infty}\ldots dy$ may be replaced by $2\int_x^{\infty}\ldots \frac{dy}{dr}\,dr$, the factor 2 arising from the equal contributions from above and below the x-axis. From $y^2 = r^2 - x^2$ we deduce that $dy/dr = r/y$ at constant x. Alternatively, from Fig. 9-14, we can see from similar triangles that $dy/dr = r/y$.

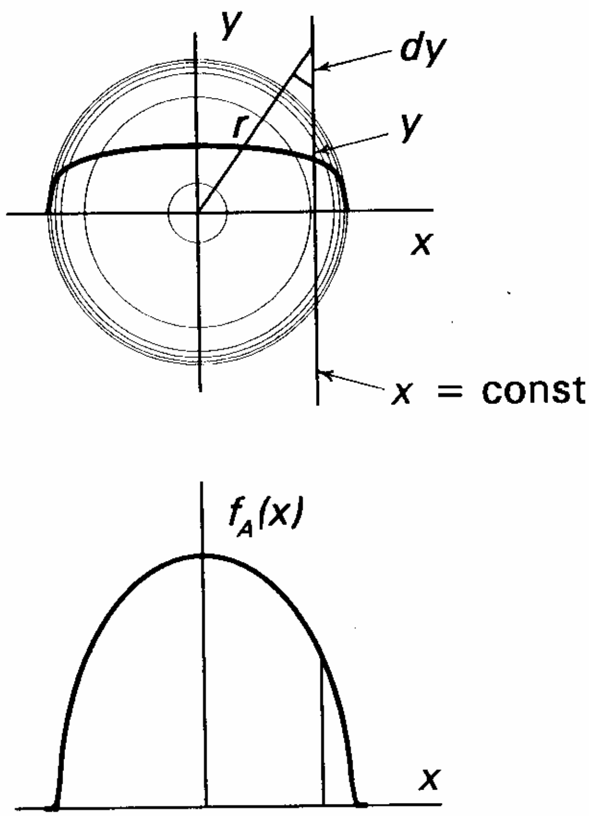

Figure 9-14 A contour map of the plane of $\mathbf{f}(r)$ (above) with a diametral cross section and a graph (below) of the line integral $\int \mathbf{f}(r)dy$ along the line $x =$ const.

Some Theorems for the Abel Transform

Many theorems for the Fourier transform do not have a counterpart when circular symmetry is imposed, but a small number of interesting theorems for the Abel theorem can be mentioned.

Similarity Theorem. If $\mathbf{f}(r)$ is contracted by a factor a to $\mathbf{f}(ar)$, then clearly $f_A(x)$ will be contracted in the same proportion and, in addition, the values of $f_A()$ will be reduced in magnitude by the same factor. Thus

$$\mathbf{f}(ar) \quad \text{has Abel transform} \quad a^{-1} f_A(ax).$$

This result is verifiable immediately by substituting ar for r in the definition.

Linear Superposition. If $\mathbf{f}(r)$ has Abel transform $f_A(ax)$, and $\mathbf{g}(r)$ has Abel transform $g_A(ax)$, then

$$\mathbf{f}(r) + \mathbf{g}(r) \quad \text{has Abel transform} \quad f_A(x) + g_A(x),$$

for any choice of **f** and **g**.

Convolution Theorem. If $\mathbf{f}(r)$ and $\mathbf{g}(r)$ are convolved in two dimensions, then the Abel transform of the result can be obtained as follows. From each of the Abel transforms $f_A(x)$ and $g_A(x)$ construct a circularly symmetrical function with the same radial section. These two functions are correctly written $f_A(r)$ and $g_A(r)$. After f_A and g_A are convolved two-dimensionally, the radial section in any direction θ is the desired Abel transform. The theorem is:

$$\mathbf{f}(r) ** \mathbf{g}(r) \quad \text{has Abel transform} \quad \Big[f_A(r) ** g_A(r)\Big]_{\theta=\text{const}}.$$

The derivation of this theorem can be written down starting from the definition integrals, but such a simple result must have a simple explanation, and it is given in Chapter 14.

Conservation Theorem. As the Abel transform is a simple projection of a two-dimensional function, the area integral of $\mathbf{f}(r)$ equals the integral of $f_A(x)$:

$$2\pi \int_0^\infty \mathbf{f}(r) r\,dr = \int_{-\infty}^\infty f_A(x)\,dx.$$

Central Value Theorem. Putting $x = 0$ in the defining integral, we see that

$$f_A(0) = 2\int_0^\infty \mathbf{f}(r)dr,$$

a relation that is useful for normalizing at the end of a computation in which unnecessary multiplications by constants are dropped.

Table 9-2 lists a variety of Abel transforms for ready reference.

Inverting the Abel Transform

Inversion of the Abel transform is performed by

$$\mathbf{f}(r) = -\frac{1}{\pi}\int_r^\infty \frac{f_A'(x)\,dx}{\sqrt{x^2 - r^2}}.$$

An important special case is where $f_A(x)$ is zero for x greater than some cutoff value r_0. Then

$$\mathbf{f}(r) = -\frac{1}{\pi}\int_r^{r_0} \frac{f_A'(x)\,dx}{\sqrt{x^2 - r^2}} + \frac{f_A(r_0-)}{\pi\sqrt{r_0^2 - r^2}}.$$

The final term, which might be overlooked, arises from the possibility of $f_A(x)$ being discontinuous at $x = r_0$. Numerical inversion of the Abel transform is important, because

Table 9-2 Table of Abel transforms. For compactness rect x is written $\Pi(x)$.

$f(r)$		$f_A(x) = 2\int_x^{\infty}(r^2 - x^2)^{-1/2} f(r) r\,dr$	
$\Pi(r/2a)$	Disk	$2(a^2 - x^2)^{1/2}\Pi(x/2a)$	Semiellipse
$(a^2 - r^2)^{-1/2}\Pi(r/2a)$		$\pi\Pi(x/2a)$	Rectangle
$(a^2 - r^2)^{1/2}\Pi(r/2a)$	Hemisphere	$\frac{1}{2}\pi(a^2 - x^2)\Pi(x/2a)$	Parabola
$(a^2 - r^2)\Pi(r/2a)$	Paraboloid	$\frac{4}{3}(a^2 - x^2)^{3/2}\Pi(x/2a)$	
$(a^2 - r^2)^{3/2}\Pi(r/2a)$		$\frac{3\pi}{8}(a^2 - x^2)^2\Pi(x/2a)$	
$(1 - \lvert r\rvert)\Pi(r/2)$	Cone	$[(a^2 - x^2)^{1/2} - (x^2/a)\cosh^{-1}(a/x)]\Pi(x/2a)$	
$\cosh^{-1}(a/r)\Pi(r/2a)$		$\pi a(1 - \lvert r/a\rvert)\Pi(r/2a)$	Triangle
$\delta(r - a)$	Ring impulse	$2a(a^2 - x^2)^{-1/2}\Pi(x/2a)$	
$\exp(-\pi r^2)$	Gaussian	$W\exp(-\pi x^2/W^2)$	Gaussian
$\exp(-r^2/2\sigma^2)$	Normal	$\sqrt{2\pi}\sigma\exp(-x^2/2\sigma^2)$	Normal
$r^2\exp(-r^2/2\sigma^2)$		$\sqrt{2\pi}\sigma(x^2 + \sigma^2)\exp(-x^2/2\sigma^2)$	
$(r^2 - \sigma^2)\exp(-r^2/2\sigma^2)$		$\sqrt{2\pi}\sigma x^2\exp(-x^2/2\sigma^2)$	
r^{-2}		π/x	
$(a^2 + r^2)^{-1}$		$\pi(a^2 + x^2)^{-1/2}$	
$J_0(2\pi ar)$	Bessel	$(\pi a)^{-1}\cos 2\pi ax$	Cosine
$2\pi[r^{-3}\int_0^r J_0(r)dr - r^{-2}J_0(r)]$		$\operatorname{sinc}^2 x$	
$\delta(r)/\pi\lvert r\rvert$		$\delta(x)$	Impulse
$2a\operatorname{sinc}(2ar)$		$J_0(2\pi ax)$	Bessel
$\frac{1}{2}r^{-1}J_1(2\pi ar)$		$\operatorname{sinc} 2ax$	
$\operatorname{jinc} r$		$\operatorname{sinc} x$	

line-integrated data can often be obtained in situations where values of $f(r)$ itself are inaccessible. In such circumstances a formula containing a derivative looks unattractive if the derivative $f_A'(x)$ must be formed by differencing, because measurement error is unfavorable. A numerical inversion procedure is described in FTA (1986) which avoids the derivative. A different method of inversion is to take advantage of the Fourier-Abel-Hankel cycle: take the one-dimensional Fourier transform and then take the Hankel transform.

One can take the Hankel transform without the need to invoke Bessel functions, by first taking the Abel transform and then taking the one-dimensional Fourier transform, as displayed in the following diagram.

$$\begin{array}{ccc} f(r) & \overset{\mathcal{F}}{\longleftrightarrow} & F(\cdot) \\ \mathcal{A}\big\downarrow & & \big\uparrow\mathcal{A} \\ f_A(x) & \overset{\mathcal{F}}{\longleftrightarrow} & H(q) \end{array}$$

It follows that $f(r)$ can be recovered from $f_A(x)$ as indicated by

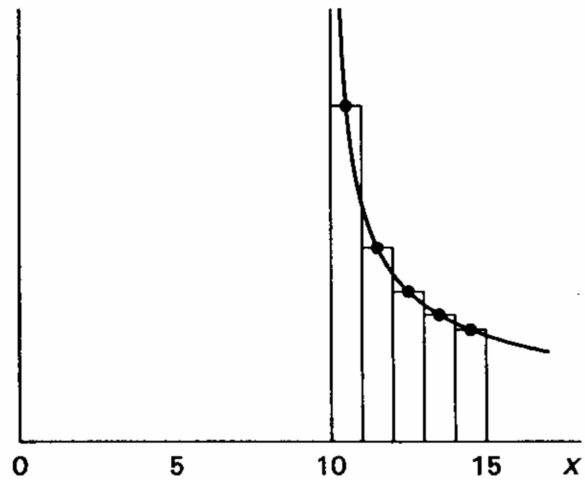

Figure 9-15 Staggering the samples of an integrand to avoid the infinite sample at the discontinuity.

$$f = \mathfrak{F} A \mathfrak{F} f_A.$$

Implementation of this sequence of operations numerically is straightforward.

Computing the Abel Transform

A rather interesting item of numerical analysis arises when one is called on to evaluate

$$\int_r^{r_0} \frac{dx}{\sqrt{x^2 - r^2}}$$

since the integrand is infinite at the left edge. For example, if we wanted $I = \int_{10}^{15}(x^2 - 10^2)^{-1/2}\,dx$, there would be trouble at $x = 10$, even though the full integral is finite. A way around this would be to stagger the samples (Fig. 9-15). Let $(x^2 - 10^2)^{-1/2} = \phi(x)$; then if one computes $\sum_{j=10.5}^{14.5} \phi(j)$, where $j = 10.5, 11.5, \ldots 14.5$, the result is 0.827, but the exact integral is $I = 0.962$. Obviously the approximation is crude, and fine subdivision of the interval might be needed to achieve desired accuracy. The correct approach is to note that $\int_{10}^{10+w} \phi(x)\,dx = C_1\phi(10 + 0.5w)w$ and that $\int_{10+w}^{10+2w} = C_2\phi(10 + 1.5w)w$, where C_1 and C_2 are coefficients and w is the sampling interval. As $w \to 0$, C_1 and C_2 assume definite values 1.414 and 1.015 that may be used for general-purpose integration in cases such as this, where the integrand diverges inversely as the square root of distance from the pole. Thus

$$\int_r \frac{\mathbf{f}(r)\,dx}{\sqrt{x^2 - r^2}} \approx \left[1.414 f(r + 0.5w) + 1.015 f(r + 1.5w) + \sum_{j=2\frac{1}{2}} f(r + jw)\right] w.$$

With $w = 1$, which is rather coarse, the approximation to the correct value 0.962 is 0.959, which is already better than 1 percent. A similar approach works with integrands that go infinite as the inverse three-halves power.

With this useful background, the reader may enjoy the following complete program for the Abel transform in which I avoid the pole at $x = r$. This application assumes that the function of radius is expressible in algebraic form and that the abscissa is scaled so that the function is zero where $r > 1$. The example applies to $f(r) = 1 - r$, whose Abel transform is known to be $\sqrt{1-x^2} - x^2 \cosh^{-1}(1/x)$. The program can readily be modified for *data* given at equal intervals.

ABEL TRANSFORM

```
DEF FNf(r)=1-r                    Define given function as cone
d=0.1                             Step in x
dy=0.01                           Step in y
FOR x=d/2 TO 1 STEP d
   s=0.5*FNf(x)
   FOR y=dy TO SQR(1-x²) STEP dy
      s=s+FNf(SQR(x²+y²))
   NEXT y
   PRINT x;2*s*dy
NEXT x
```

The results are as follows:

r	.05	.15	.25	.35	.45	.55	.65	.75	.85	.95
$f(r)$	0.98952	0.93053	0.83928	0.72718	0.60210	0.47067	0.33909	0.21404	0.10363	0.02071

Comparison with the theoretical expression shows that the largest error is one digit in the fifth decimal place. Whether the value of dy is too coarse can be checked empirically.

SPIN AVERAGING

A function $f(x)$ may be spin averaged to obtain a new function $f_S(r)$ defined by

$$f_S(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r\cos\alpha)\, d\alpha.$$

If $f(x)$ is an even function, as in all that follows here, we may evaluate $f_S(r)$ from

$$f_S(r) = \frac{2}{\pi} \int_0^{\pi/2} f(r\cos\alpha)\, d\alpha.$$

Two ways of viewing spin averaging will now be described.

Imagine a function defined on the (x, y)-plane so that at any point (x, y) the value is $f(x)$, i.e., independent of y. If the function value represented the height of a surface above the (x, y)-plane, the surface would be a cylindrical ridge running in the y-direction. Then if we traveled on the surface of this ridge so that our track projected onto the (x, y)-plane was a circle of radius r, then our average height would be $f_S(r)$, as given by either of the