The Nine Dots Puzzle Extended to *n*x*n*x…x*n* Points

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Abstract. The classic thinking problem, the "Nine Dots Puzzle", is widely used in courses on creativity and appears in a lot of games magazines. One of the earliest appearances is in "Cyclopedia of Puzzles" by Sam Loyd in 1914. Here is a review of the generic solution of the problem of the 9 points spread to n^2 points. Basing it on a specific pattern, we show that any *n*x*n* (for $n \geq 5$) points puzzle can also be solved 'Inside the Box', using only $2 \cdot n - 2$ straight lines (connected at their end-points), through the square spiral method. The same pattern is also useful to "bound above" the minimal number of straight lines we need to connect n^k points in a *k*-dimensional space, while to "bound below" the solution of the *n*x*n*x…x*n* puzzle we start from a very basic consideration.

Keywords: dots, straight line, inside the box, outside the box, plane, upper bound, lower bound, graph theory, segment, points.

MSC2010: Primary 91A43; Secondary 05E30, 91A46.

§1. Introduction

The classic thinking problem, the *nine points puzzle*, reads: "Since the 9 points as shown in **Fig. 1**, we must join with straight line and continuous stroke, without this overlap more than once, using the smallest number of lines possible" [6]. For the solution to this problem, we must make some exceptions, and one of them is that a line must be attached to at least two points, such that the least number of lines that can be used in this 3x3 grid is 4. That is obvious, since it would be meaningless to do a line for each point, although there is nothing to prevent it.

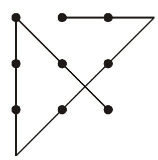

Fig. 1. The nine points connected by four lines.

The interesting thing about this problem is not the solution, but rather, the procedure in reaching it. This problem requires lateral thinking for its solution [7]. The problem appears in a lot of places, for example, in the book "*The art of creative thinking, how to be innovative and develop great ideas*" [1].

Thinking outside the box (sometimes erroneously called "thinking out of the box" or "thinking outside the square") is to think differently, unconventionally or from a new perspective. This phrase often refers to novel, creative and smart thinking [3].

The phrase means something like "think creatively" or "be original" and its origin is generally attributed to consultants in the 1970s and 1980s who tried to make clients feel inadequate by drawing nine dots on a piece of paper and asking those clients to connect the dots without lifting their pen, using only four lines [5].

§2. *n*x*n* points problem in a bi-dimensional space

From the 3x3 grid, there has grown the problem of extending it to a grid of *n*x*n* points, and to find a solution under the same conditions as the original problem. **Fig. 2** shows a grid of 4x4 points.

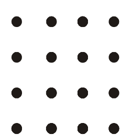

Fig. 2. 4x4 grid points.

Fig. 3 shows some of the possible solutions for a grid of 4x4. Given the grid symmetry, it is enough to exhibit some solutions, because the remaining cases are obtained by rotating the grid. Therefore, it is possible to solve the 4x4 version of the puzzle using 6 lines starting from any point of the grid. In addition, each starting point, in any of the solutions, may well be the point of arrival. These solutions are using the least number of lines.

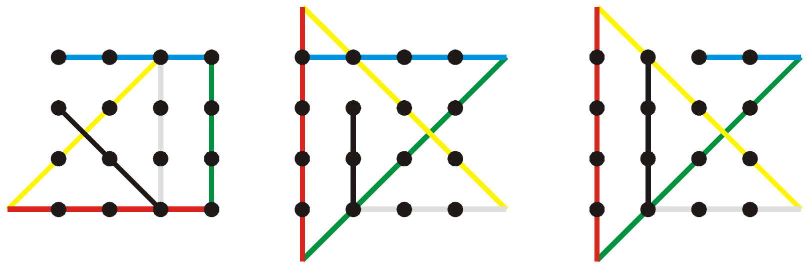

Fig. 3. 4x4 grid points and some solutions.

Another curiosity that arises is that for n greater than 4, it is possible to construct solutions "Inside the Box" and "Outside the Box". **Fig. 4** illustrates the 5x5 case.

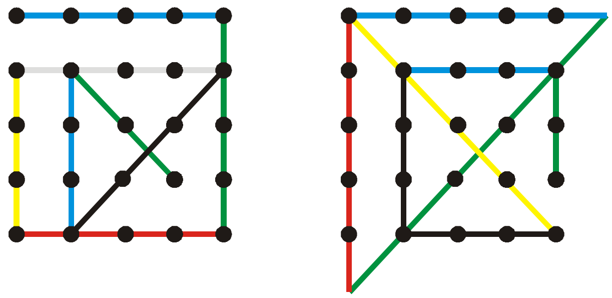

Fig. 4. 5x5 grid points solutions inside / outside the box.

Fig. 5 shows the solution for a grid with n equal to 3, 4, 5 and 6 respectively, using a pattern with a spiral shape. In figure **c,** the solution is given by a pattern "Inside the Box" and compared with figure **b,** it has two lines more. In turn, comparing **b** with **a**, we can also see two additional lines. It's the same with **d** and **c**. Likewise, when n is increased by one unit of the number of lines, the solution to the problem is increased by two. This occurs for any pattern solution to the problem, whether or not it is the spiral type. In fact, we can draw a square spiral around the pattern in figure **c** (or considering a different solution), so it is trivial that we add two straight lines more for any further row / column we have. In the mentioned figure, we show the spiral shape of the solution (a square spiral frame for $n \geq 5$).

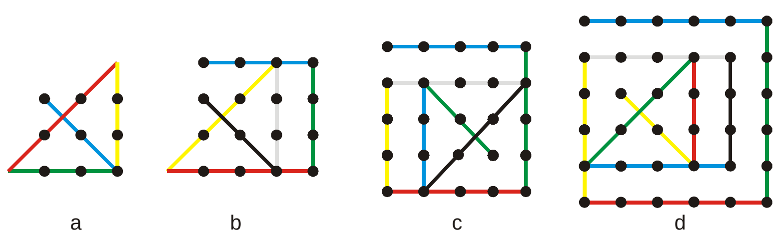

Fig. 5. Some solutions for $n = 3$, $n = 4$, $n = 5$ and $n = 6$, which show the square spiral frame starting from $n = 5$.

Stated another way, the **Eq. 1** gives the minimum number of lines required [2]. Where h represents the number of straight lines to connect all the points and n is the number of rows or columns of the grid. It should be mentioned that this result is independent of the grid pattern solution for any value of n, excepting for 1 and 2.

$$h = 2 \cdot (n-1) \quad \forall\, n \in \mathbb{N} - \{0, 1, 2\} \qquad \textbf{(1)}$$

A special case is represented by a mono-dimensional space, we have n points in a row. In this case, $\forall$ $n \geq 2$, $h = 1$, and this puzzle can be solved *inside the box* or *outside the box*.

§3. Problem generalization: *n*x*n*x…x*n* points corresponding to a *k*-dimensional space

After showing the general solution for the case of nxn points on a plane, a new problem arises: extending the same puzzle to nxnx...xn points in a k-dimensional space, where k is equal to the number of occurrences of n (n^k total points, indeed).

First we show the problem and the solution to a three-dimensional space, afterwards, the general problem and the solution to a k-dimensional space.

We distinguish two types of solutions: first, called "Upper Bound", considering the spiral solution method, and second, called "Lower Bound" [4], based on the consideration that we cannot connect more than n points with the first line and the maximum of $n-1$ points for any additional line (i.e., it is possible to connect $n-1$ points with the first line, n points with the second line and $n-1$ points using any further line, but this clarification does not change the previous result).

Let, h_u be the number of lines from the Upper Bound and h_l the constraint based on the previous assumption; the minimum number of lines, h, we need to connect the nxnx...xn points, is $h_l \leq h \leq h_u$.

Table 1 shows the number of lines for Upper and Lower Bound cases, in two and three dimensions (based on the square spiral method applying to the pattern shown in figure **c**, when n ranges from 1 to 20. Moreover, the Gap column shows the difference in the number of lines between the Upper and Lower Bound. The last column shows the increase in the number of lines for the case in three-dimensions, Upper Bound, when incrementing the value of n.

Table 1: Upper / Lower bounds in 2 and 3 dimensions.

	Two Dimensions			Three Dimensions			
n	Lower Bound	Upper Bound	Gap (Upper-Lower)	Lower Bound	Upper Bound	Gap (Upper-Lower)	Upper B. Increments [$n \to n+1$]
1	/	/	/	/	/	/	/
2	3	3	0	7	7	0	6
3	4	4	0	13	14	1	7
4	5	6	1	21	26	5	12
5	6	8	2	31	43	12	17
6	7	10	3	43	64	21	21
7	8	12	4	57	89	32	25
8	9	14	5	73	118	45	29
9	10	16	6	91	151	60	33
10	11	18	7	111	188	77	37
11	12	20	8	133	229	96	41
12	13	22	9	157	274	117	45

13	14	24	10	183	323	140	49
14	15	26	11	211	376	165	53
15	16	28	12	241	433	192	57
16	17	30	13	273	494	221	61
17	18	32	14	307	559	252	65
18	19	34	15	343	628	285	69
19	20	36	16	381	701	320	73
20	21	38	17	421	778	357	77

In the three-dimensional space case, we used a "plane by plane" solution, from the pattern of the *n*x*n* puzzle and linking each plane by a line.

The Upper Bound column of **Table 1** shows that h, the number of lines needed, as we increase n by a unit, is given by $h_{n+1} = h_n + 4\cdot(n-1) + 5$, for $n \geq 3$.

Fig. 6 shows an Upper Bound solution when $n = 5$ ($h = 43$).

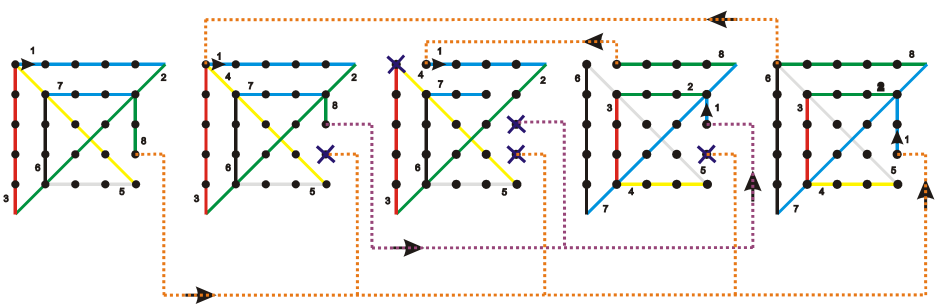

Fig. 6. 5x5x5 points, 43 straight lines.

Using the **Eq. 1** and by an extension of this to a three-dimensional space, we multiply this solution by the number of planes given by the n value and add the $n-1$ necessary lines to connect each plane. This gives the number of lines needed to connect all the points. Thus, the Upper Bound for an arbitrary large number of dimensions, k, where $k \geq 2$, is given by the **Eq. 2**, and h is the number of lines.

$$h = 2\cdot(n-1)\cdot n^{k-2} + n^{k-2} - 1 = (2\cdot n - 1)\cdot n^{k-2} - 1 \qquad \textbf{(2)}$$

Extending the Lower Bound constraint we have previously explained to k dimensions, where $k \geq 2$, we obtain the **Eq. 3**. It indicates the number of needed lines to connect n^k points in a k-dimensional space.

$$n^k = n + (h-1)\cdot(n-1) \qquad \text{Thus} \qquad \frac{n^k - n}{n-1} = h - 1 \;\rightarrow\; h = \frac{n^k - n}{n-1} + 1$$

It follows that

$$h = \frac{n^k - 1}{n-1} \qquad \textbf{(3)}$$

For the "Lower Bound" on the three-dimensional case considering "plane by plane solutions only", joining the *n*x*n* solutions with a line, the result is given by the **Eq. 4**.

$$h = (2\cdot n-2)\cdot 2 + (2\cdot n-3)\cdot 2 + (2\cdot n-4)\cdot 4 + (2\cdot n-5)\cdot 4 + (2\cdot n-6)\cdot 6 + (2\cdot n-7)\cdot 6 + \ldots + n - 1$$

Then

$$h = n - 1 + \sum_{i=1}^{i_{max}} 2 \cdot (2 \cdot n - i - 1) \cdot \left\lceil \frac{i}{2} \right\rceil + (2 \cdot n - i_{max} - 2) \cdot \left(n - \sum_{i=1}^{i_{max}} 2 \cdot \left\lceil \frac{i}{2} \right\rceil \right)$$

Where i_{max} is the maximum (integer) value of "i" inside the summation (the maximum value $\tilde{\imath}$ such that $n \geq \sum_{i=1}^{\tilde{\imath}} 2 \cdot \left\lceil \frac{i}{2} \right\rceil \rightarrow n \geq \left\lceil \frac{1-\tilde{\imath}}{2} \right\rceil^2 - 3 \cdot \left\lceil \frac{1-\tilde{\imath}}{2} \right\rceil + \left\lfloor \frac{\tilde{\imath}}{2} \right\rfloor^2 + \left\lfloor \frac{\tilde{\imath}}{2} \right\rfloor + 2$).

It follows that

$$h = \begin{cases} \frac{4}{3} \cdot {i_{max}}^3 + 7 \cdot {i_{max}}^2 + \left(\frac{35}{3} - 2 \cdot n\right) \cdot i_{max} + 2 \cdot n^2 - 3 \cdot n + 5 \\ \qquad \text{if} \quad n \leq 2 \cdot (i_{max} + 2)^2 \\ \\ \frac{4}{3} \cdot {i_{max}}^3 + 9 \cdot {i_{max}}^2 + \left(\frac{59}{3} - 2 \cdot n\right) \cdot i_{max} + 2 \cdot n^2 - 4 \cdot n + 13 \\ \qquad \text{if} \quad n > 2 \cdot (i_{max} + 2)^2 \end{cases} \quad \textbf{(4)}$$

Where $i_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{2 \cdot n + 1} - 3\right) \right\rfloor$.

Table 2 shows the number of needed lines using a "plane to plane" solution for *n*x*n*x*n* points. The Gap column is the difference between "Upper Bound" and "Lower Bound".

Table 2: Upper / Lower Bounds in 3 dimensions [9].

n	Lower Bound	Upper Bound	Gap Upper-Lower	Upper B. Increments [*n*→*n*+1]	Guessing the Plane Bound
1	/	/	/	/	/
2	7	7	0	6	7
3	13	14	1	7	14
4	21	26	5	12	26
5	31	43	12	17	40
6	43	64	21	21	59
7	57	89	32	25	82
8	73	117	44	28	109
9	91	148	57	31	139
10	111	183	72	35	173
11	133	222	89	39	211
12	157	265	108	43	253
13	183	311	128	46	298
14	211	361	150	50	347
15	241	415	174	54	400
16	273	473	200	58	457
17	307	535	228	62	518
18	343	601	258	66	583
19	381	670	289	69	651
20	421	743	322	73	723

Fig. 7 shows points of connection without crossing the line and without additional constraint intersections. We called this the "pure" square spiral pattern. The square spiral is not only a frame connected to another internal pattern; it is solving the problem inside the box, connecting points without crossing a line and visiting any dot just once.

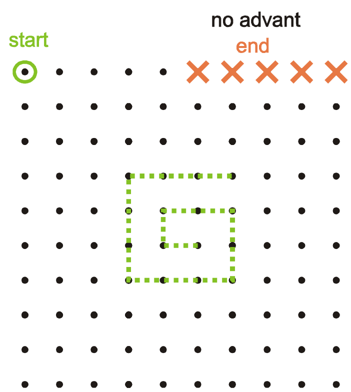

Fig. 7. The "pure" square spiral pattern in three dimensions.

$$h = (2\cdot n-1)\cdot 2 + (2\cdot n-2)\cdot 2 + (2\cdot n-3)\cdot 4 + (2\cdot n-4)\cdot 4 + (2\cdot n-5)\cdot 6 + (2\cdot n-6)\cdot 6 + (2\cdot n-7)\cdot 8 + \ldots + n-1$$

So,

$$h = n-1+\sum_{i=1}^{i_{\max}} 2\cdot(2\cdot n-i)\cdot\left\lceil\frac{i}{2}\right\rceil + (2\cdot n-i_{\max}-1)\cdot\left(n-\sum_{i=1}^{i_{\max}}2\cdot\left\lceil\frac{i}{2}\right\rceil\right)$$

Thus (for $n \geq 4$)

$$h = \begin{cases} \frac{4}{3}\cdot \mathrm{i_{max}}^3 + 7\cdot \mathrm{i_{max}}^2 + \left(\frac{35}{3}-2\cdot \mathrm{n}\right)\cdot \mathrm{i_{max}} + 2\cdot \mathrm{n}^2 - 2\cdot \mathrm{n} + 5 \\ \qquad \text{if} \quad \mathrm{n} \leq 2\cdot(\mathrm{i_{max}}+2)^2 \\ \\ \frac{4}{3}\cdot \mathrm{i_{max}}^3 + 9\cdot \mathrm{i_{max}}^2 + \left(\frac{59}{3}-2\cdot \mathrm{n}\right)\cdot \mathrm{i_{max}} + 2\cdot \mathrm{n}^2 - 3\cdot \mathrm{n} + 13 \\ \qquad \text{if} \quad \mathrm{n} > 2\cdot(\mathrm{i_{max}}+2)^2 \end{cases} \tag{5}$$

Where $i_{max} = \left\lfloor \frac{1}{2}\cdot\left(\sqrt{2\cdot n+1}-3\right)\right\rfloor$.

A method to reduce the gap between the Upper and the Lower Bound in three dimensions is combining the pattern [10] on **Fig. 8** with the square spiral one.

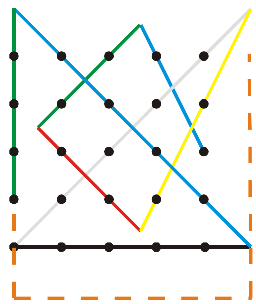

Fig. 8. 5x5 points, 8 lines basic pattern.

This is not the best Upper Bound that defines under the “plane by plane” additional constraint. In fact, there are other patterns which enhance the solution. As per **Fig. 9**, **Fig. 10** and **Fig. 11**. The pattern in **Fig. 11** is valid for any even value of n, for $n \geq 6$, while it improves the “standard” Upper Bound in **Fig. 8** for $n = 6, 8, 10, 12$ and 14.

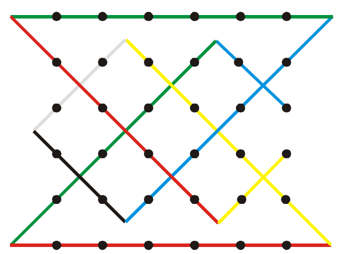

Fig. 9. 6x6x6 points, 62 straight lines.

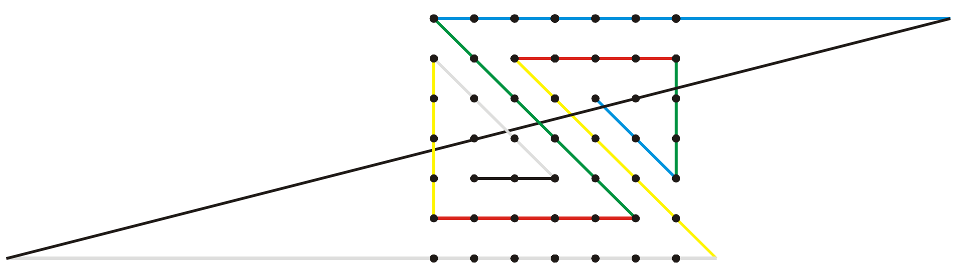

Fig. 10. 7x7x7 points, 85 straight lines.

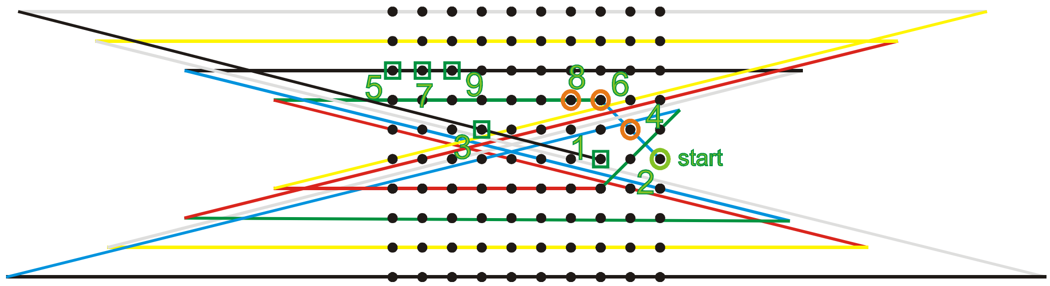

Fig. 11. 10x10x10 points, 178 straight lines.

Analyzing the different patterns, the best "Upper Bound", for $n \geq 15$, is the one derived from the pattern by **Fig. 8**. **Table 3**, and shows the three-dimensional "Upper Bound", based on the standard solution of **Fig. 8**.

Table 3: *n*x*n*x*n* points puzzle Upper Bounds considering the pattern by **Fig. 8** only.

n	Upper Bound (*n*x*n*x*n*)	*n*	Upper Bound (*n*x*n*x*n*)	*n*	Upper Bound (*n*x*n*x*n*)	*n*	Upper Bound (*n*x*n*x*n*)
1	/	16	471	31	1799	46	4003
2	7	17	532	32	1919	47	4181
3	14	18	597	33	2043	48	4363
4	26	19	666	34	2171	49	4549
5	43	20	739	35	2302	50	4739
6	63	21	816	36	2437	51	4932
7	87	22	897	37	2576	52	5129
8	115	23	982	38	2719	53	5330
9	146	24	1071	39	2866	54	5535
10	181	25	1163	40	3017	55	5744
11	220	26	1259	41	3172	56	5957
12	263	27	1359	42	3331	57	6174
13	309	28	1463	43	3493	58	6395
14	359	29	1571	44	3659	59	6620
15	413	30	1683	45	3829	60	6849

Table 4 shows the three-dimensional problem Upper Bounds, based on the square spiral pattern. This is the best Upper Bound we have currently found for an arbitrary large value of *n* (i.e., $n \geq 51$).

Table 4: *n*x*n*x*n* points puzzle Upper Bounds following the "pure" square spiral pattern and the one in **Fig. 8**: if $n \geq 42$, we get the same result.

n	Square Spiral	Best Upper Bound Currently Discovered	Gap	n	Square Spiral	Best Upper Bound Currently Discovered	Gap	n	Square Spiral	Best Upper Bound Currently Discovered	Gap
1	/	/	/	18	601	**597**	4	35	2304	**2302**	2
2	**7**	**7**	0	19	670	**666**	4	36	2439	**2437**	2
3	16	**14**	2	20	743	**739**	4	37	2578	**2576**	2
4	29	**26**	3	21	820	**816**	4	38	2721	**2719**	2
5	45	**43**	2	22	901	**897**	4	39	2868	**2866**	2
6	65	63→**62**	2→3	23	986	**982**	4	40	3019	**3017**	2
7	89	87→**85**	2→4	24	1075	**1071**	4	41	3173	**3172**	1
8	117	115→**112**	2→5	25	1167	**1163**	4	42	**3331**	**3331**	0
9	148	**146**	2	26	1263	**1259**	4	43	**3493**	**3493**	0
10	183	181→**178**	2→5	27	1363	**1359**	4	44	**3659**	**3659**	0
11	222	**220**	2	28	1467	**1463**	4	45	**3829**	**3829**	0
12	265	263→**260**	2→5	29	1575	**1571**	4	46	**4003**	**4003**	0
13	311	**309**	2	30	1687	**1683**	4	47	**4181**	**4181**	0
14	361	359→**358**	2→3	31	1803	**1799**	4	48	**4363**	**4363**	0
15	415	**413**	2	32	1923	**1919**	4	49	**4549**	**4549**	0
16	473	**471**	2	33	2046	**2043**	3	50	**4739**	**4739**	0
17	535	**532**	3	34	2173	**2171**	2	51	**4932**	**4932**	0

As already stated, for n = 6, 8, 10, 12 or 14, the best “plane by plane” to “Upper Bound” is given by $h = 2 \cdot (n-1) \cdot n + n - 1 - (1 + 2 \cdot (n-5)) = 2 \cdot n^2 - 3 \cdot n + 8$, following the pattern of Roger Phillips [8].

For any $n \geq 42$, the number of lines is given by the **(5)**.

§4. Conclusion

When n becomes very large (i.e. $n \geq 42$), the spiral pattern is the best three-dimensional model “plane by plane”, allowing a good solution. It is as good as the one deriving from the pattern of **Fig. 8** for any $n \geq 42$ (for $n \geq 51$, considering a generic pattern of 5x5, the last / external parts of the two patterns overlap – it is a square spiral frame). In addition, the spiral pattern allows a solution “Inside the Box”, without crossing any line and passing through each point more than once. It is also the best pattern available without crossing lines, for dimensions from 1 to k.

Let us call t the least “Upper Bound” found for the case of three dimensions, see **Table 3**, ∀ $n \geq 42$, we obtain the **Eq. (6)**.

$$t = \begin{cases} \frac{4}{3} \cdot {i_{max}}^3 + 7 \cdot {i_{max}}^2 + \left(\frac{35}{3} - 2 \cdot n\right) \cdot i_{max} + 2 \cdot n^2 - 2 \cdot n + 5 \\ \qquad \boldsymbol{if} \quad n \leq 2 \cdot (i_{max} + 2)^2 \\ \\ \frac{4}{3} \cdot {i_{max}}^3 + 9 \cdot {i_{max}}^2 + \left(\frac{59}{3} - 2 \cdot n\right) \cdot i_{max} + 2 \cdot n^2 - 3 \cdot n + 13 \\ \qquad \boldsymbol{if} \quad n > 2 \cdot (i_{max} + 2)^2 \end{cases} \qquad \textbf{(6)}$$

Where $i_{max} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{2 \cdot n + 1} - 3\right) \right\rfloor$.

Thus h, the "Upper Bound" for the k-dimensions problem, can be further lowered as:
∀ $n \in \mathbb{N}-\{0\}$, let us define t as the lowest "Upper Bound" we have previously proven for the standard nxnxn points problem (see **Eq. (6)** and **Table 3** - e.g., $n = 6 \rightarrow t = 62$),

$$h = t \cdot n^{k-3} + n^{k-3} - 1 \quad \rightarrow \quad h = (t+1) \cdot n^{k-3} - 1 \qquad \textbf{(7)}$$

Let l be the minimum amount of straight lines needed to solve the nxnx…xn=n^k points problem ($k, n \in \mathbb{N}-\{0, 1, 2\}$), we have just proven that:

$$\frac{n^k - 1}{n - 1} \leq l \leq (2 \cdot n - 1) \cdot n^{k-2} - 1 \qquad \textbf{(8)}$$

The **Eq. (8)** can be further improved, by the **Eq. (6)** and **Table 3**, as:

$$\frac{n^k - 1}{n - 1} \leq l \leq (t + 1) \cdot n^{k-3} - 1 \qquad \textbf{(9)}$$

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