SCHMIDT ANALYSIS FOR STIRLING ENGINES

This Analysis was written by David Berchowitz and Israel Urieli and published in their book "Stirling Cycle Engine Analysis". The analysis was written for engines but applies also to coolers. The difference between the two analysis is that the Rejector (k) in the engine is the lower temperature reject sink and in the cooler is the higher temperature reject sink. The Acceptor is the heat accept in both engine and cooler but corresponds to the lower temperature in the cooler and to the higher temperature in the engine.

Nomenclature		**Subscripts**	
T	Temperature	c	Compression Space
P	Pressure	e	Expansion Space
M	Mass	r	Regenerator
R	Gas Constant	k	Rejector
V	Volume	h	Acceptor
W	Work	sw	swept

A.1 The Schmidt analysis

A.1.1 Background

The apparent conceptual simplicity of the Stirling engine belies its intractability to mathematical analysis. The difficulty of describing even idealized models of the engine in terms of simple closed-form equations is one of the primary reasons for the widespread skepticism and lack of understanding which exists even today.

In Chapter 2 we derived the basic set of equations which describe the Ideal Isothermal model (table 2.1). Gustav Schmidt of the German Polytechnic Institute of Prague published an analysis in 1871 in which he obtained closed-form solutions of these equations for the special case of sinusoidal volume variations of the working spaces (Schmidt 1871). This analysis is still used today as the classic Stirling cycle analysis. It was derived in order to describe the highly successful Lehmann engine shown in figure A. 1. 1. The paper includes a detailed description of the engine and displays a clear insight and appreciation of it. From figure A.1.1 we see that a very long horizontal cylinder *ABC* was used, in which a concentric displacer L and power piston D reciprocate in accordance with a rather complex driving mechanism. A full size complete Lehmann engine is on permanent display at the Deutscher Museum in Munich, and a clear, modern description of the engine operation has been recently presented by Kolin (1972).

The driving mechanism does not produce sinusoidal motion. Schmidt derived the equivalent mechanism 'which by means of an imagined, infinitely long thrust-rod, would attain the true movement' (sinusoidal motion) of the working and displacer pistons.

The Lehmann engine piston seal was ingeniously constructed similarly to a bicycle pump, allowing limited pressurized operation. 'The working piston is isolated by means of a leather sleeve turned towards the inside. So long as the air inside the machine has a higher pressure than the outside atmosphere, this sleeve effectively prevents the escape of air towards the outside. However as soon as the pressure inside sinks below the ordinary atmospheric pressure, it permits the entrance of external air into the machine.

'The Lehmann arrangement has the advantage that the working piston is in constant contact only with the cooler air, thereby preventing the inward turned leather sleeve from burning!

Schmidt was acutely aware of the advantages of operating the cycle at a higher pressure, and further states. 'Undoubtedly this is the only system which holds any promise for the future, because with high pressure one can use lower temperatures and therefore produce a durable machine.'

Significantly, there is no mention throughout the paper of the importance of the regenerator, even though the use of regenerators in the earlier Ericsson machines was described. In figure A.1.1 we notice that about a third of the cylinder is enclosed by the furnace and the rest of the cylinder by a water jacket. 'The displacer leaves between itself and the wall of cylinder *A,* the intervening piece *B* and the heating pot C exactly so much room that the cross section of the circular intervening space is large enough to allow minimal resistance to the passage of the air, and small enough to produce a thin layer of air in order that heating and cooling may be achieved as rapidly as possible.' The Lehmann machine apparently was not fitted with a regenerator. Now, since over the cycle under cyclic steady conditions the net heat transferred to the regenerator is zero, it is conceivable that Schmidt did not appreciate the importance of the regenerator. He refers to the textbook by Zeuner as containing a 'complete, simple and clear theory' of air engines, but in the same textbook Zeuner decries the use of regenerators for air engines (Finkelstein 1959).

A.1.2 The analysis

The approach taken by Schmidt for the analysis follows the Isothermal Analysis used in Chapter 2 quite closely, up to the derivation of the pressure relation given by equation (2.5), reproduced as follows:

$$p = MR\left(\frac{V_c}{T_k} + \frac{V_k}{T_k} + \frac{V_r \ln(T_h/T_k)}{(T_h - T_k)} + \frac{V_h}{T_h} + \frac{V_e}{T_h}\right)^{-1} \quad \text{(A. 1. 1)}$$

The sinusoidal volume variations are given as in equations (2.15) and (2-16) as follows:

$$V_c = V_{clc} + V_{swc}(1 + \cos\theta)/2$$
$$V_e = V_{cle} + V_{swe}[1 + \cos(\theta + \alpha)]/2. \quad \text{(A.1.2) (A.1.3)}$$

Substituting (A.1.2) and (A.1.3) in (A.1.1) and simplifying we obtain

$$p = MR\left[s + \left(\frac{V_{swe}\cos\alpha}{2T_h} + \frac{V_{swc}}{2T_k}\right)\cos\theta - \left(\frac{V_{swe}}{2T_h}\sin\alpha\right)\sin\theta\right]^{-1} \quad \text{(A.1.4)}$$

where

$$s = \left[\frac{V_{swc}}{2T_k} + \frac{V_{clc}}{T_k} + \frac{V_k}{T_k} + \frac{V_r \ln(T_h/T_k)}{(T_h - T_k)} + \frac{V_h}{T_h} + \frac{V_{swe}}{2T_h} + \frac{V_{cle}}{T_h}\right].$$

Referring to figure *A.1.2* we consider the following trigonometric substitutions:

$$c\sin\beta = \frac{V_{swe}\sin\alpha}{2T_h} \quad \text{(A.1.5)}$$

$$c\cos\beta = \frac{V_{swe}\cos\alpha}{2T_h} + \frac{V_{swc}}{2T_k} \quad \text{(A.1.6)}$$

where

$$\beta = \tan^{-1}\left(\frac{V_{swe}\sin\alpha/T_h}{V_{swe}\cos\alpha/T_h + V_{swc}/T_k}\right) \quad \text{(A.1.7)}$$

and

$$c = \frac{1}{2}\left[\left(\frac{V_{swe}}{T_h}\right)^2 + 2\frac{V_{swe}}{T_h}\frac{V_{swc}}{T_k}\cos\alpha + \left(\frac{V_{swc}}{T_k}\right)^2\right]^{1/2} \quad \text{(A.1.8)}$$

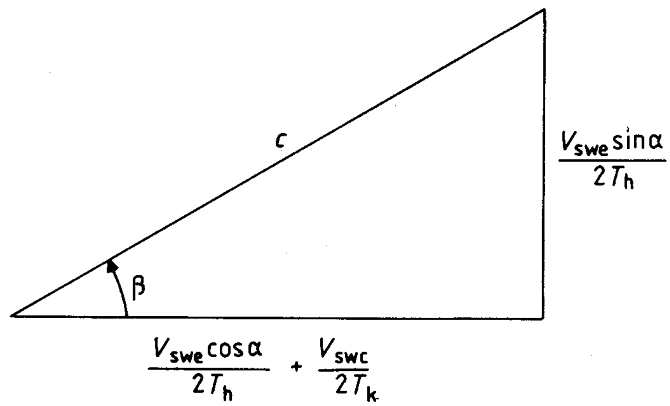

Figure A.1.2 Trigonometric substitutions.

Substituting equations *(A.1.5)* and *(A.1.6)* into equation *(A.1.4)* and simplifying, we obtain

$$p = \frac{MR}{s(1 + b\cos\phi)} \qquad \text{(A.1.9)}$$

where

$$\phi = \theta + \beta \qquad b = c/s.$$

Equation (A. 1.9) is the 'equation of the caloric line' and has essentially the same form as that derived by Schmidt. The maximum and minimum values of pressure are easily evaluated for the extreme values of cos ∅:

$$p_{\text{max}} = \frac{MR}{s(1-b)} \qquad \text{(A.1.10)}$$

$$p_{\text{min}} = \frac{MR}{s(1+b)}. \qquad \text{(A.1.11)}$$

The average pressure over the cycle is given by

$$p_{\text{mean}} = \frac{1}{2\pi}\int_0^{2\pi} p\,\mathrm{d}\phi$$

$$= \frac{MR}{2\pi s}\int_0^{2\pi} \frac{1}{(1 + b\cos\phi)}\mathrm{d}\phi. \qquad \text{(A.1.12)}$$

From tables of integrals (Dwight 1961), equation (A.1.12) reduces to

$$p_{\text{mean}} = MR/(s\sqrt{1-b^2}). \qquad \text{(A.1.13)}$$

Equation (A. 1.13) is the most convenient method of relating the total mass of working gas to the more conveniently specified mean operating pressure and is used for this purpose throughout this book.

Work is done by the engine on the surroundings by virtue of the varying volumes of the working spaces Vc and Ve The total work done by the engine is therefore the algebraic sum of the work done by the compression and expansion spaces. Over a complete cycle we have

$$W_c = \oint p\,\mathrm{d}V_c = \int_0^{2\pi} p\frac{\mathrm{d}V_c}{\mathrm{d}\theta}\mathrm{d}\theta \qquad \text{(A.1.14)}$$

$$W_e = \oint p\,\mathrm{d}V_e = \int_0^{2\pi} p\frac{\mathrm{d}V_e}{\mathrm{d}\theta}\mathrm{d}\theta \qquad \text{(A.1.15)}$$

$$W = W_c + W_e. \qquad \text{(A.1.16)}$$

Differentiating equations (A.1.2) and (A.1.3), the volume derivatives are

$$\frac{\mathrm{d}V_c}{\mathrm{d}\theta} = -\tfrac{1}{2}V_{swc}\sin\theta \qquad \text{(A.1.17)}$$

$$\frac{\mathrm{d}V_e}{\mathrm{d}\theta} = -\tfrac{1}{2}V_{swe}\sin(\theta+\alpha). \qquad \text{(A.1.18)}$$

Substituting equations (A.1.17), (A.1.18) and (A.1.9) into equations (A.1.14) and (A.1.15), we obtain

$$W_c = -\frac{V_{swc}MR}{2s}\int_0^{2\pi}\frac{\sin\theta}{1+b\cos(\beta+\theta)}\mathrm{d}\theta \qquad \text{(A.1.19)}$$

$$W_e = -\frac{V_{swe}MR}{2s}\int_0^{2\pi}\frac{\sin(\theta+\alpha)}{1+b\cos(\beta+\theta)}\mathrm{d}\theta. \qquad \text{(A.1.20)}$$

The following approach to the solution of integrals (A. 1.19) and (A. 1.20) is somewhat different from that due to Schmidt. However, it is considered by the authors to be more easily comprehended.

The Fourier series expansion of the pressure function is first considered. It is shown that only one of the terms of this expansion will return a non-zero integral. This integral is then evaluated, giving the exact solution.

The Fourier series expansion of p (0) in equation (A. 1.9) is given as follows (Arfken 1970):

$$p(\phi) = p_0 + \sum_{i=1}^{\infty}[p_{ci}\cos(i\phi) + p_{si}\sin(i\phi)] \qquad \text{(A.1.21)}$$

where

$$p_0 = \frac{1}{2\pi}\int_0^{2\pi}p(\phi)\,\mathrm{d}\phi$$

$$p_{ci} = \frac{1}{\pi}\int_0^{2\pi}p(\phi)\cos(i\phi)\,\mathrm{d}\phi$$

$$p_{si} = \frac{1}{\pi}\int_0^{2\pi}p(\phi)\sin(i\phi)\,\mathrm{d}\phi.$$

Now, referring to the graph of equation (A.1.9) for a typical value of b (figure A. 1.3) we observe that p (∅) is an even function of ∅ and can thus be represented exclusively by the cosine terms. Equation (A. 1.2 1) thus reduces to

$$p(\phi) = p_0 + \sum_{i=1}^{\infty}p_{ci}\cos(i\phi). \qquad \text{(A.1.22)}$$

Substituting equations (A.1.22) and (A.1.17) into (A.1.14) we obtain

$$W_c = -\frac{V_{swc}}{2}\int_0^{2\pi}\left(p_0 + \sum_{i=1}^{\infty}p_{ci}\cos(i\phi)\right)\sin\theta\,\mathrm{d}\theta. \qquad \text{(A.1.23)}$$

Expanding equation (A.1.23)

$$W_c = -\frac{V_{swc}p_0}{2}\int_0^{2\pi}\sin\theta\,\mathrm{d}\theta - \frac{V_{swc}}{2}\sum_{i=2}^{\infty}p_{ci}\int_0^{2\pi}\cos[i(\theta+\beta)]\sin\theta\,\mathrm{d}\theta -$$
$$\frac{V_{swc}p_{c1}}{2}\int_0^{2\pi}\cos(\theta+\beta)\sin\theta\,\mathrm{d}\theta. \qquad \text{(A.1.24)}$$

It can easily be shown that the first two terms on the right-hand side of equation (A.1.24) are zero, resulting in

$$W_c = -\frac{V_{swc}p_{c1}}{2}\int_0^{2\pi}\cos(\theta+\beta)\sin\theta\,\mathrm{d}\theta. \qquad \text{(A.1.25)}$$

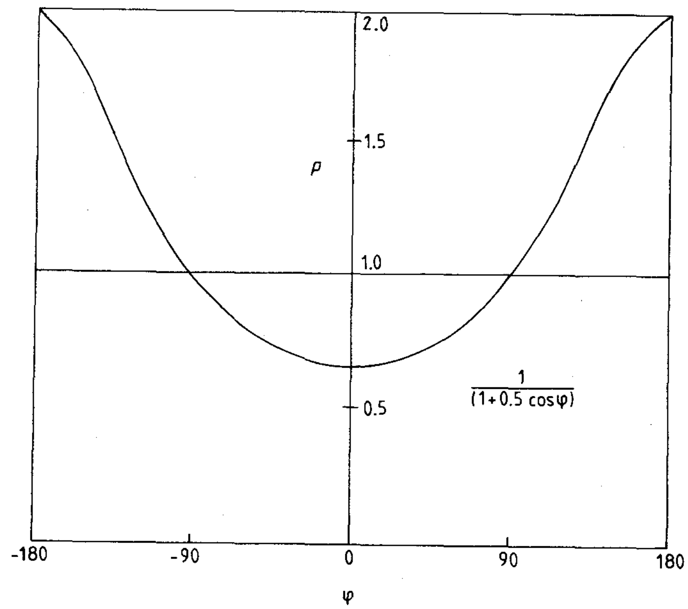

Figure A.1.3 Normalized pressure p versus composite angle 0.

Evaluating the integral of equation (A.1.25)

$$W_c = \tfrac{1}{2}\pi V_{swc} p_{c1} \sin\beta. \qquad \text{(A.1.26)}$$

Similarly, for the expansion space we find

$$W_e = \tfrac{1}{2}\pi V_{swe} p_{c1} \sin(\beta - \alpha). \qquad \text{(A.1.27)}$$

Now, from equations (A.1.21) and (A.1.9)

$$p_{c1} = \frac{MR}{\pi s}\int_0^{2\pi} \frac{\cos\phi}{(1 + b\cos\phi)} d\phi. \qquad \text{(A.1.28)}$$

Equation (A.1.28) can be evaluated in two steps using tables of integrals, as follows (Dwight 1961):

$$\begin{aligned} p_{c1} &= \frac{MR}{\pi s}\left(\frac{2\pi}{b} - \frac{1}{b}\int_0^{2\pi} \frac{1}{(1 + b\cos\phi)} d\phi\right) \\ &= \frac{MR}{\pi s}\left(\frac{2\pi}{b} - \frac{2\pi}{b\sqrt{1-b^2}}\right) \\ &= \frac{2MR}{sb}\left(1 - \frac{1}{\sqrt{1-b^2}}\right). \end{aligned} \qquad \text{(A.1.29)}$$

Substituting equations (A.1.29) and (A.1.13) into equations (A.1.26) and (A.1.27) we finally obtain

$$W_c = \pi V_{swc} p_{mean} \sin\beta (\sqrt{1-b^2} - 1)/b \qquad \text{(A.1.30)}$$

$$W_e = \pi V_{swe} p_{mean} \sin(\beta - \alpha)(\sqrt{1-b^2} - 1)/b. \qquad \text{(A.1.31)}$$

Equations (A.1.30) and (A.1.31) are essentially the same results as those obtained by Schmidt, and constitute the major analytical results of the analysis.

Now, since the Schmidt analysis is based on the Ideal Isothermal model, the thermal efficiency should reduce to the Carnot efficiency. The thermal efficiency is defined by the ratio of the work done by the engine to the heat supplied externally to the engine. In Chapter 2 we showed that the heat supplied externally is equal to the work done by the expansion space (equations (2.12) and (2.13)) thus:

$$\eta = W/W_e = (W_c + W_e)/W_e. \qquad \text{(A.1.32)}$$

Substituting equations (A.1.30) and (A.1.31) into equation (A.1.32)

$$\eta = 1 + \frac{V_{swc}\sin\beta}{V_{swe}\sin(\beta-\alpha)}. \qquad \text{(A.1.33)}$$

Expanding equation (A.1.33) and simplifying

$$\eta = 1 - \frac{V_{swc}}{V_{swe}}\left(\frac{\tan\beta}{\sin\alpha - \tan\beta\cos\alpha}\right). \qquad \text{(A.1.34)}$$

Substituting equation (A.1.7) into equation (A.1.34)and simplifying, we obtain

$$\eta = 1 - T_k/T_h \qquad \text{(A.1.35)}$$

which is the Carnot efficiency.

A.2 Regenerator mean effective temperature

In order to evaluate the total mass of gas in the regenerator void space correctly, the lengthwise distribution of the gas temperature must be known. It has been shown that for real regenerators, the temperature profile is very nearly linear (Urieli 1980), and thus we assume that the ideal regenerator has a linear temperature profile between the cold temperature T, and the hot temperature *Th,* as in figure A.2.1 (Creswick 1965).

From figure A.2.1 we observe

$$T(x) = (T_h - T_k)x/L_r + T_k \qquad \text{(A.2.1)}$$

where *L,* is the regenerator length.

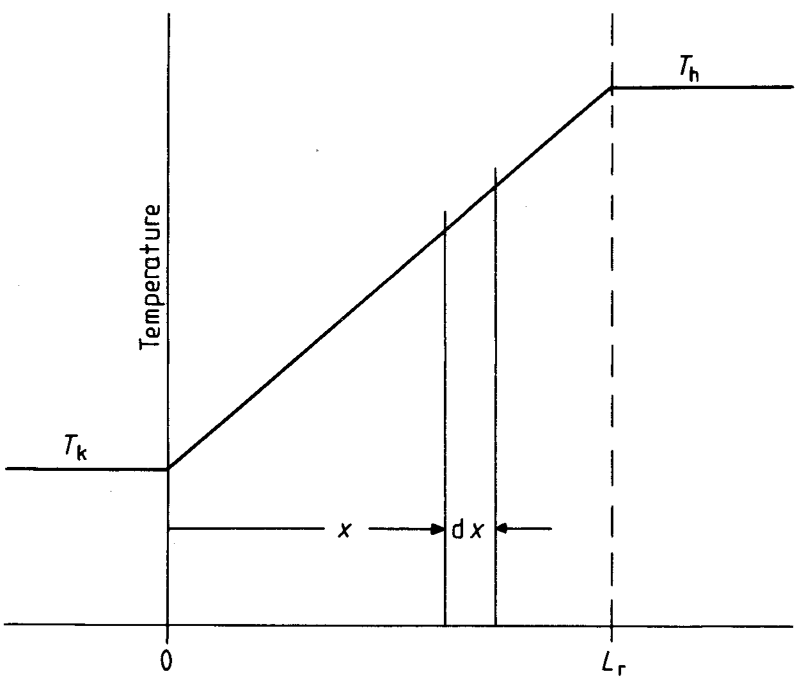

Figure A.2.1 Regenerator linear temperature profile.

The total mass of gas m, in the regenerator void volume is given by

$$m_r = \int_0^{V_r} \rho\, dV_r \qquad \text{(A.2.2)}$$

where p is the density, d V, = A, dx is the lengthwise volume derivative for constant free-flow area *A,* and *Vr* = *ArL,.*

Substituting for the ideal gas law p = pRT in equation (A.2.2) and simplifying

$$m_r = \frac{V_r p}{R} \int_0^{L_r} \frac{1}{[(T_h - T_k)x + T_k L_r]} dx. \quad (A.2.3)$$

Integrating the right-hand side of equation (A.2.3) and simplifying

$$m_r = \frac{V_r p}{R} \frac{\ln (T_h/T_k)}{(T_h - T_k)}. \quad (A.2.4)$$

We define the mean effective regenerator temperature T, in terms of the ideal gas equation of state:

$$m_r = V_r p/(RT_r). \quad (A.2.5)$$

Comparing equations (A.2.4) and (A.2.5) we obtain

$$T_r = (T_h - T_k)/\ln(T_h/T_k). \quad (A.2.6)$$

Equation (A.2.6) gives the mean effective regenerator temperature. Tr as a function of Tk and Th, as required.

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