Trigonometric Identities $(Revision : 1.4)$

1 Trigonometric Identities you must remember

The "big three" trigonometric identities are

$$\sin^2 t + \cos^2 t = 1 \tag{1}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{2}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \tag{3}$$

Using these we can derive many other identities. Even if we commit the other useful identities to memory, these three will help be sure that our signs are correct, etc.

2 Two more easy identities

From equation (1) we can generate two more identities. First, divide each term in (1) by $\cos^2 t$ (assuming it is not zero) to obtain

$$\tan^2 t + 1 = \sec^2 t. \tag{4}$$

When we divide by $\sin^2 t$ (again assuming it is not zero) we get

$$1 + \cot^2 t = \csc^2 t. \tag{5}$$

3 Identities involving the difference of two angles

From equations (2) and (3) we can get several useful identities. First, recall that

$$\cos(-t) = \cos t, \qquad \sin(-t) = -\sin t. \tag{6}$$

From (2) we see that

$$\begin{aligned} \sin(A - B) &= \sin(A + (-B)) \\ &= \sin A \cos(-B) + \cos A \sin(-B) \end{aligned}$$

which, using the relationships in (6), reduces to

$$\sin(A - B) = \sin A \cos B - \cos A \sin B. \tag{7}$$

In a similar way, we can use equation (3) to find

$$\begin{aligned} \cos(A - B) &= \cos(A + (-B)) \\ &= \cos A \cos(-B) - \sin A \sin(-B) \end{aligned}$$

which simplifies to

$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \tag{8}$$

Notice that by remembering the identities (2) and (3) you can easily work out the signs in these last two identities.

4 Identities involving products of sines and cosines

If we now add equation (2) to equation (7)

$$\begin{aligned} \sin(A-B) &= \sin A\cos B - \cos A\sin B \\ +(\sin(A+B) &= \sin A\cos B + \cos A\sin B) \end{aligned}$$

we find

$$\sin(A-B) + \sin(A+B) = 2\sin A\cos B$$

and dividing both sides by 2 we obtain the identity

$$\sin A\cos B = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B). \tag{9}$$

In the same way we can add equations (3) and (8)

$$\begin{aligned} \cos(A-B) &= \cos A\cos B + \sin A\sin B \\ +(\cos(A+B) &= \cos A\cos B - \sin A\sin B) \end{aligned}$$

to get

$$\cos(A-B) + \cos(A+B) = 2\cos A\cos B$$

which can be rearranged to yield the identity

$$\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B). \tag{10}$$

Suppose we wanted an identity involving $\sin A\sin B$. We can find one by slightly modifying the last thing we did. Rather than adding equations (3) and (8), all we need to do is subtract equation (3) from equation (8):

$$\begin{aligned} \cos(A-B) &= \cos A\cos B + \sin A\sin B \\ -(\cos(A+B) &= \cos A\cos B - \sin A\sin B) \end{aligned}$$

This gives

$$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$$

or, in the form we prefer,

$$\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B). \tag{11}$$

5 Double angle identities

Now a couple of easy ones. If we let $A = B$ in equations (2) and (3) we get the two identities

$$\sin 2A = 2\sin A\cos A, \tag{12}$$

$$\cos 2A = \cos^2 A - \sin^2 A. \tag{13}$$

6 Identities for sine squared and cosine squared

If we have $A = B$ in equation (10) then we find

$$\begin{aligned}\cos A \cos B &= \frac{1}{2}\cos(A - A) + \frac{1}{2}\cos(A + A) \\ \cos^2 A &= \frac{1}{2}\cos 0 + \frac{1}{2}\cos 2A.\end{aligned}$$

Simplifying this and doing the same with equation (11) we find the two identities

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A), \tag{14}$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A). \tag{15}$$

7 Identities involving tangent

Finally, from equations (2) and (3) we can obtain an identity for $\tan(A + B)$:

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Now divide numerator and denominator by $\cos A \cos B$ to obtain the identity we wanted:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \tag{16}$$

We can get the identity for $\tan(A - B)$ by replacing B in (16) by $-B$ and noting that tangent is an odd function:

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \tag{17}$$

8 Summary

There are many other identities that can be generated this way. In fact, the derivations above are not unique — many trigonometric identities can be obtained many different ways. The idea here is to be very familiar with a small number of identities so that you are comfortable manipulating and combining them to obtain whatever identity you need to.