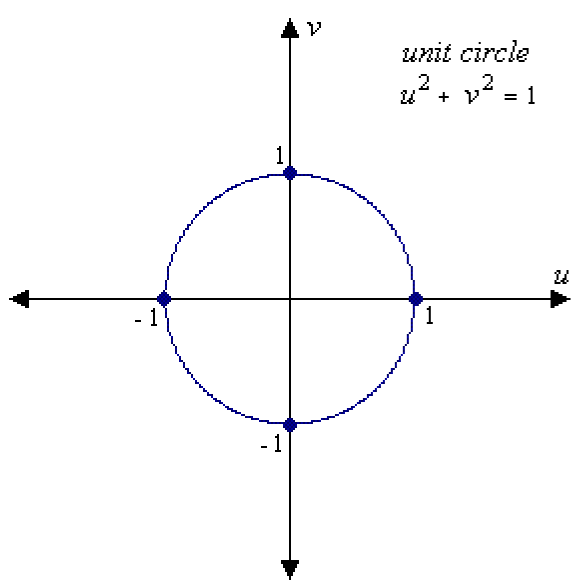
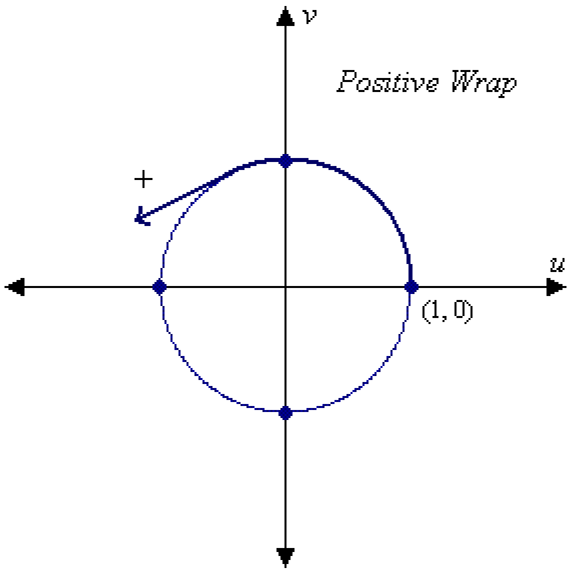
Trigonometric Functions: The Unit Circle

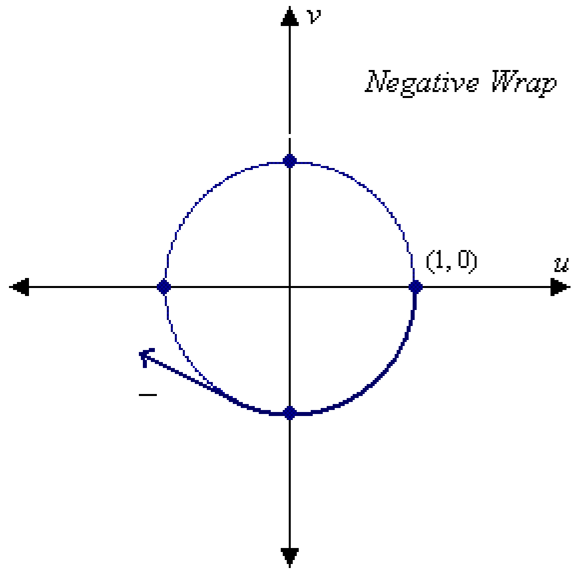
This chapter deals with the subject of trigonometry, which likely had its origins in the study of distances and angles by the ancient Greeks. The word trigonometry is based on the Greek words for *triangle measurement.* Today, trigonometry has much broader applications than that of the measurement of angles and distances, now including the study of sound, light, and electrical waves.

This section introduces trigonometry in terms of what is called the ***wrapping function***, which takes the real number line and wraps it around the ***unit circle***. The unit circle is a circle having a radius value of 1 and its center at the origin of a rectangular coordinate system. For example, if u and v are the variables in a rectangular coordinate system, the equation of the unit circle would be given by $u^2 + v^2 = 1$, and the graph of the circle would be as shown below.



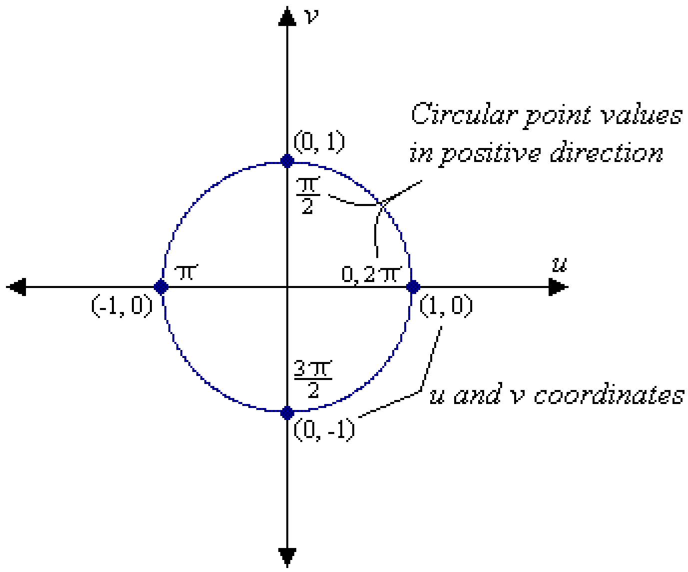
The origin or zero value of the real number line in the wrapping function is at the point (1,0) on the unit circle. Positive values of the line move around the circumference of the circle in a counterclockwise direction and negative values in a clockwise direction, as shown below.



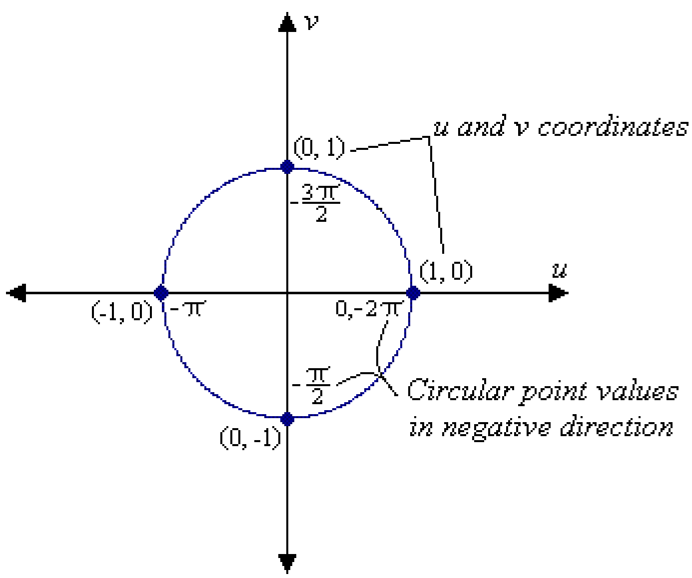


We would like to associate any point on the real number line with a corresponding point on the unit circle, called a ***circular point***. We know that the circumference of a circle is equal to the diameter of the circle times π, or $2r\pi$, where r is the radius of the circle. For the unit circle, $r = 1$, so the circumference of the circle is equal to 2π.

To find specific values on the unit circle, we can divide the circle into quadrants and then further subdivide the quadrants into lesser segments. For example, in the figure below, look at the first quadrant going in a positive (counterclockwise) direction from the origin of the circle. The point (0,1) is one quarter of the distance around the circle, so its value on the associated number line is equal to 2π divided by 4, or $\pi/2$. Moving quadrant by quadrant around the circle in a positive direction, we can see that the value of the line at point (-1,0) is equal to π and at point (0,-1) is equal to $3\pi/2$. If we continue on in a positive direction back around to the origin, point (1,0), the line's value is now 2π. We could continue going around the unit circle in a positive direction any number of times, after each quadrant adding $\pi/2$ to the value of the number line.



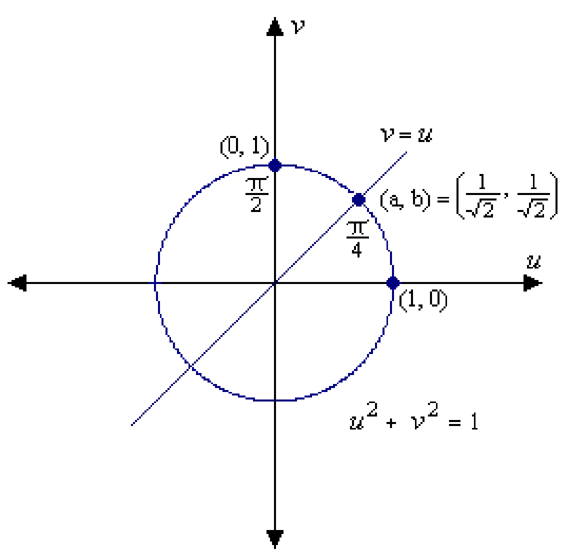
Moving in a negative (clockwise) direction around the circle from the origin, the process is simply reversed, as shown below. The values on the associated number line are all negative, and as each quadrant is passed, the value becomes more negative by an amount equal to $\pi/2$. For example, moving in a negative direction, the value of the number line at the point (0,1) would be equal to $-3\pi/2$.



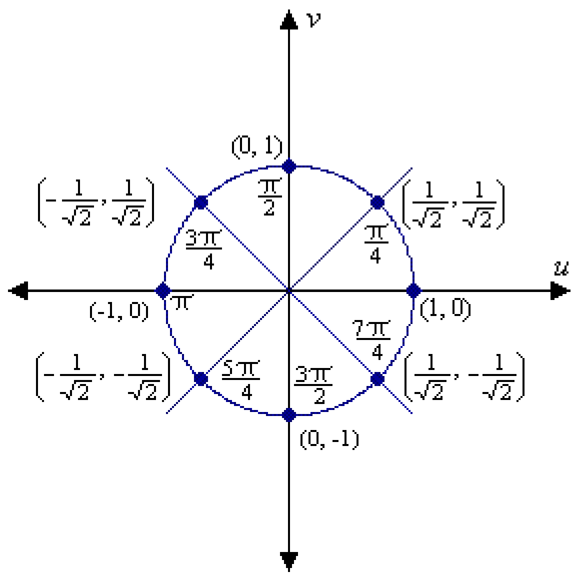
Now concentrating on just the first quadrant in a positive direction from the origin of the unit circle, there are several specific points of interest. If we go halfway around the first quadrant to point (a,b) as shown below, the value of the number line at this point would be $\pi/2$ divided by 2, or $\pi/4$. The coordinates of this point on the unit circle are not obvious, but we can derive them fairly easily. First, we know the equation of the unit circle is $u^2 + v^2 = 1$. We also can see that the equation of the straight line that passes through this point and the center of the circle is $v = u$, since the line is symmetric with respect to the axes. The point (a,b) lies on both the straight line and the circle, so it must satisfy both equations. Since $v = u$ at this point, we can substitute u for v in the unit circle equation and obtain

$$
\begin{aligned}
u^2 + v^2 &= 1 \\
u^2 + u^2 &= 1 \\
2u^2 &= 1 \\
u^2 &= \frac{1}{2} \\
u &= \pm\sqrt{\frac{1}{2}} \\
u &= \pm\frac{1}{\sqrt{2}}
\end{aligned}
$$

Since we are in the first quadrant of the circle, both u and v must be positive and $v = u$, so $u = \dfrac{1}{\sqrt{2}}$ and $v = \dfrac{1}{\sqrt{2}}$ at point (a,b) .



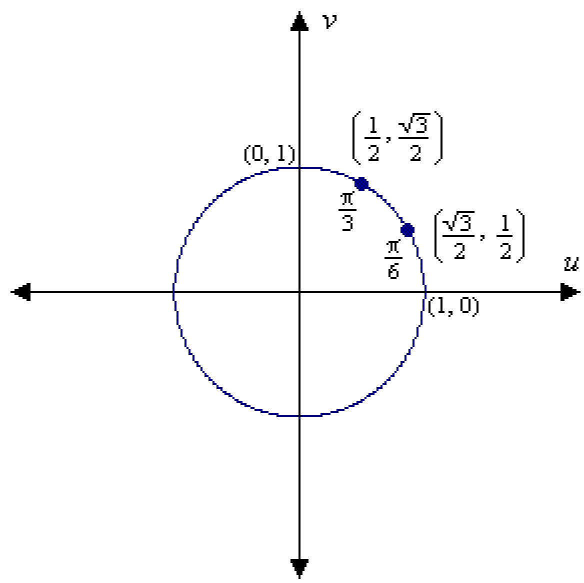
Using symmetry with respect to the axes and the origin, we can easily find the coordinates of any point located halfway around each quadrant, as shown below.



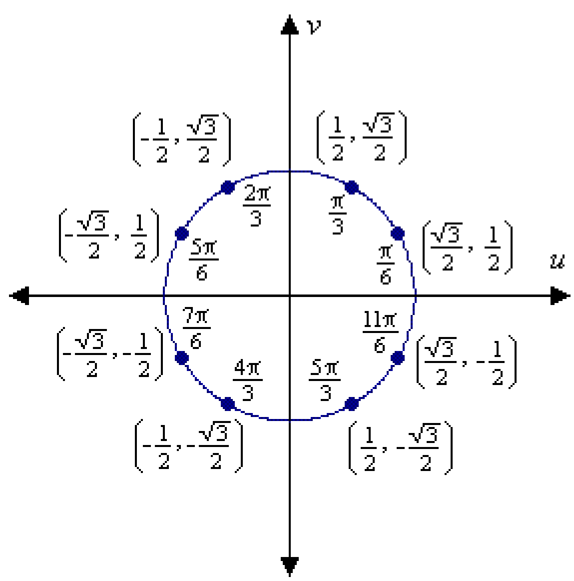
Looking at the first quadrant again, besides the point located halfway around, we can derive the coordinates of points located one-third and two-thirds of the way around the quadrant. These points have values on the number line of $\pi/6$ and $\pi/3$, respectively, as shown below. Using the Pythagorean Theorem for right triangles and symmetry with respect to the axes, it is possible to derive the coordinates for these points. After the calculation, the results are:

For the point (a,b) having the value $\frac{\pi}{6}$, $u = \frac{\sqrt{3}}{2}$ and $v = \frac{1}{2}$.

For the point (a,b) having the value $\frac{\pi}{3}$, $u = \frac{1}{2}$ and $v = \frac{\sqrt{3}}{2}$.



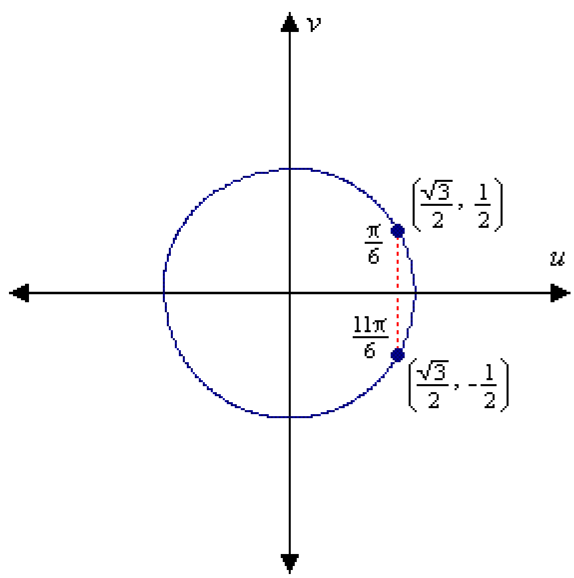
It is useful to memorize the coordinates u and v at the points on the first quadrant of the unit circle having the values $\pi/6$, $\pi/4$, and $\pi/3$. Using symmetry, it is then easy to find the coordinates of corresponding points on any of the quadrants, as shown below.



Example 1: Find the coordinates of the circular point W having a value of $\frac{11\pi}{6}$.

Solution:

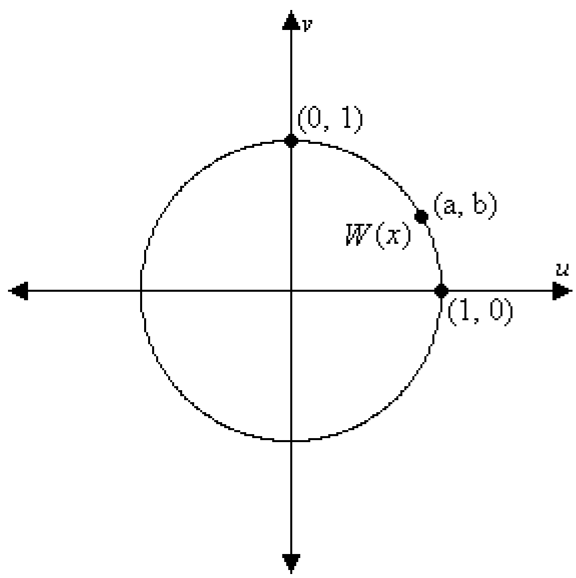
The point $W\left(\frac{11\pi}{6}\right)$ lies in the fourth quadrant since its value is between $\frac{3\pi}{2}$ and 2π. Therefore, the u coordinate of this point will be positive and the v coordinate will be negative. The point in the first quadrant symmetrical to this point with respect to the horizontal axis is $\frac{\pi}{6}$, and as already discussed, this point has coordinates of $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$.



We need only to change the v coordinate to a negative number. The coordinates of the circular point $W\left(\frac{11\pi}{6}\right)$ are thus $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$.

We will now continue with the concept of the wrapping function $W(x)$, in which a real number x is associated with a pair of coordinates on the unit circle, and introduce the six primary trigonometric functions, which are also called **circular functions**. These six functions and their abbreviations are **sine** (sin), **cosine** (cos), **tangent** (tan), **secant** (sec), **cosecant** (csc), and **cotangent** (cot). The values of these functions for any real number x are denoted by $\sin x$, $\cos x$, etc.

Consider the unit circle as shown below, having a radius equal to 1, and with the center of the circle at the origin of a rectangular coordinate system. The circular point $W(x)$ is associated with a pair of coordinates (a,b) on the unit circle.



The definitions of the six trigonometric (circular) functions then are:

$$\sin x = b \qquad \cos x = a$$

$$\tan x = \frac{b}{a}; \quad a \neq 0 \qquad \cot x = \frac{a}{b}; \quad b \neq 0$$

$$\sec x = \frac{1}{a}; \quad a \neq 0 \qquad \csc x = \frac{1}{b}; \quad b \neq 0$$

It is important to note that the coordinates *a* and *b* on the unit circle will always have absolute values which are less than or equal to 1, since they can never exceed the radius of the circle which is equal to 1. The coordinates may have positive or negative values, depending on the quadrant in which they lie.

We can now make several statements about the range of values the circular functions can assume:

> sin *x* and cos *x* each have ranges between -1 and +1, inclusive.
>
> tan *x* and cot *x*, since they are quotients of two numbers each of which can vary in value, have ranges including all real numbers, as long as the denominator is not equal to 0, in which case the function is undefined.
>
> sec *x* and csc *x* have absolute values which can never be less than 1, since the quotient of 1 divided by a number less than or equal to 1 is always greater than or equal to 1. Again, the denominator can not be equal to 0. The range of values then is all real numbers excluding the interval between -1 and +1.

Looking again at the six basic circular functions, we can easily derive several identities that relate the functions together:

$$\sec x = \frac{1}{a} = \frac{1}{\cos x} \qquad \cos x = a = \frac{1}{\sec x}$$

$$\csc x = \frac{1}{b} = \frac{1}{\sin x} \qquad \sin x = b = \frac{1}{\csc x}$$

$$\tan x = \frac{b}{a} = \frac{\sin x}{\cos x} = \frac{1}{\cot x} \qquad \cot x = \frac{a}{b} = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$$

If we move in a clockwise (negative) direction around the unit circle from the origin, we can easily determine several additional identities. Looking at the first and fourth quadrants, as shown below, the value of a is positive whether we move a distance of x (counterclockwise) or $-x$ (clockwise). Since $\cos x = a$,

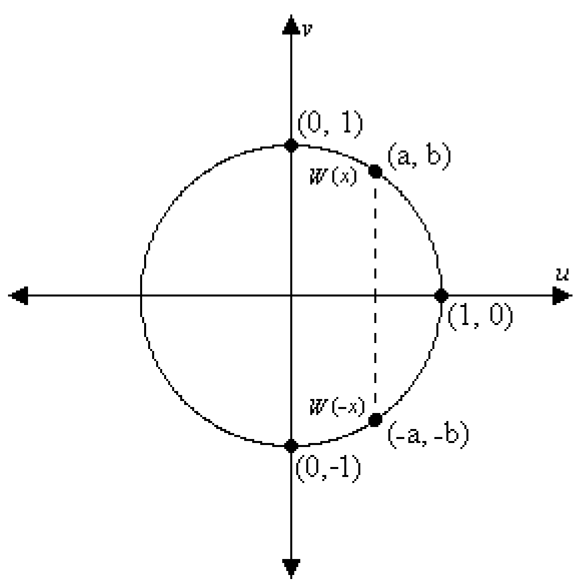
$$\cos(-x) = \cos x$$

Also, for x in the first quadrant, the coordinate b is positive, while for $-x$ in the fourth quadrant, the b coordinate is negative. Since $\sin x = b$ and $\sin(-x) = -b$

$$\sin(-x) = -\sin x$$

As previously defined, $\tan x = \frac{b}{a} = \frac{\sin x}{\cos x}$. If x is negative, then

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x$$



Since the point (a,b) lies on the unit circle $u^2 + v^2 = 1$, and $\cos x = a$ and $\sin x = b$,

$$(\cos x)^2 + (\sin x)^2 = 1$$

$(\cos x)^2$ and $(\sin x)^2$ are usually written as $\cos^2 x$ and $\sin^2 x$, respectively, so the above identity can be written as

$$\sin^2 x + \cos^2 x = 1$$

All of the above identities are known as the basic identities. It is useful to memorize these identities along with the definitions of the circular functions, since they are all frequently used in the study of trigonometry.

It is useful to memorize the exact values of the sine and cosine functions when x is equal to 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. Since $\cos x = a$ and $\sin x = b$ on the unit circle, the values of the a and b coordinates derived with the wrapping function can be restated here for the sine and cosine functions.

$$\sin 0 = 0 \qquad \cos 0 = 1$$

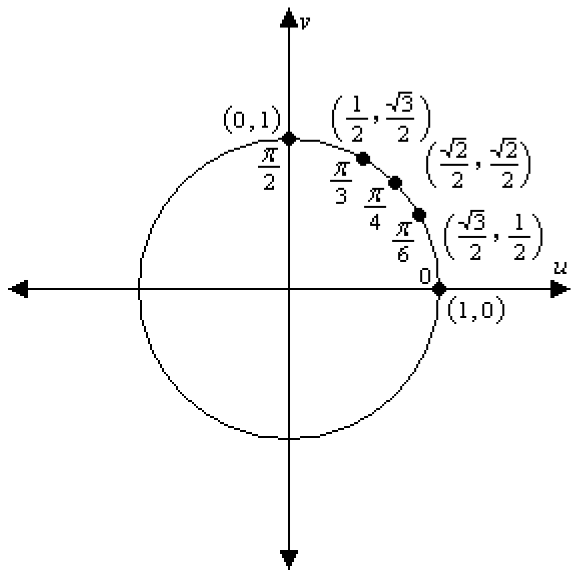
$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

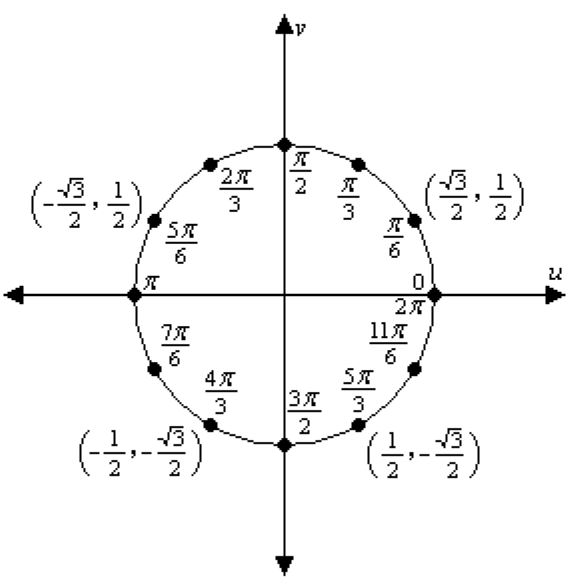
$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sin\frac{\pi}{2} = 1 \qquad \cos\frac{\pi}{2} = 0$$

The points on the first quadrant of the unit circle having these x values are shown below. The a coordinate is equal to the cosine function, and the b coordinate is equal to the sine function.



It is possible to easily derive the sine and cosine functions for multiples of these specific x values using symmetry with respect to the axes of the rectangular coordinate system, as shown below. Again, the a coordinate is equal to the cosine function, and the b coordinate is equal to the sine function.



Example 2: Evaluate exactly $\tan \frac{7\pi}{6}$.

Solution:

First, determine the quadrant in which the x value $\frac{7\pi}{6}$ lies.

This x value lies in the third quadrant, since it is greater than π and less than $\frac{3\pi}{2}$.

Second, determine the value of the sine and cosine functions of this point.

$$\sin\frac{7\pi}{6} = -\frac{1}{2} \qquad\qquad \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

Now, determine the value of $\tan \frac{7\pi}{6}$.

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan\frac{7\pi}{6} = \frac{\sin\frac{7\pi}{6}}{\cos\frac{7\pi}{6}}$$

$$= \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{3}}$$