<u>Unit 5 – Analytical Trigonometry – Classwork</u>

A) Verifying Trig Identities: Definitions to know:

Equality: a statement that is always true. example: $2 = 2$, $3 + 4 = 7$, $6^2 = 36$, $2(3+5) = 6+10$.
Equation: a statement that is conditionally true, depending on the value of a variable. example: $2x+3=11$, $(x-1)^2 = 25$, $x^3 - 2x^2 + 5x - 12 - 0$, $2\sin\theta = 1$.
Identity: a statement that is always true no matter the value of the variable. example: $2x + 3x = 5x$, $4(x-3) = 4x - 12$, $(x-1)^2 = x^2 - 2x + 1$, $\dfrac{1}{x-1} - \dfrac{1}{x+1} = \dfrac{2x}{x^2-1}$. In the last example, it could be argued that this is not an identity, because it is not true for all values of the variable (x cannot be 1 or -1). However, when such statements are written, we assume the domain is taken into consideration although we don't always write it. So a better definition of an identity is: a statement that is always true for all values of the variable within its domain.

The 8 Fundamental Trigonometric Identities:

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Trig Identities proofs (assuming θ in standard position)

$$\frac{1}{\sin\theta} = \frac{1}{\frac{y}{r}} = \frac{r}{y} = \csc\theta$$
$$\frac{1}{\cos\theta} = \frac{1}{\frac{x}{r}} = \frac{r}{x} = \sec\theta$$
$$\frac{1}{\tan\theta} = \frac{1}{\frac{y}{x}} = \frac{x}{y} = \cot\theta$$
$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta$$
$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{y} = \cot\theta$$

$x^2 + y^2 = r^2$	$x^2 + y^2 = r^2$	$x^2 + y^2 = r^2$
$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$	$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$	$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$
$\cos^2\theta + \sin^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$

Corollaries: a statement that is true because another statement is true:

Examples (you write the others):

Reciprocal identities: $\sin\theta\csc\theta = 1$ $\quad \sin\theta = \dfrac{1}{\csc\theta}$ $\quad \boxed{\sin\theta\cos\theta = 1}$

Quotient identities: $\tan\theta\cos\theta = \sin\theta$ $\quad \boxed{\cos\theta = \dfrac{\sin\theta}{\tan\theta}}$

Pythagorean identities: $\sin^2\theta = 1 - \cos^2\theta$ $\quad \cos^2\theta = 1 - \sin^2\theta$ $\quad \boxed{\sin\theta = \pm\sqrt{1-\cos^2\theta} \qquad \cos\theta = \pm\sqrt{1-\sin^2\theta}}$

In this section, you will be given a number of trigonometric identities. Remember – they are true. Your job will be proving that they are true. Your tools will be your knowledge of algebra, the 8 trig identities, and your ingenuity. Some are easy like example 1 and others are more difficult like example 2.

Example 1) $\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$

$\sin\theta\csc\theta - \sin^2\theta$

$1 - \sin^2\theta = \cos^2\theta$

Example 2) $\dfrac{\sec^2 x}{\tan x} = \sec x \csc$

$\sec x\left(\dfrac{\sec x}{\tan x}\right) = \sec x\left(\dfrac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}\right) = \sec x\left(\dfrac{1}{\sin x}\right) = \sec x \csc x$

Guidelines for verifying trigonometric identities:

1) Your job is to prove one side of an identity is equal to the other so you will only work on one side of the identity, so…
2) Always work on the most complicated side and try to transform it to the simpler side. More complicated can mean the side that is "longer" or has more complicated expressions. Additions (or subtractions) are generally more complicated than multiplications.
3) If an expression can be multiplied out, do so.
4) If an expression can be factored, do so.
5) If you have a polynomial over a single term, you can "split it" into several fractions.
6) If you have an expression, that involves adding fractions, do so finding a lowest common denominator.
7) When in doubt, convert everything to sines and cosines.
8) Don't be afraid to create complex fractions. Once you do that, many problems are a step away from solution.
9) Always try something! You don't have to see the solution before you actually do the problem. Sometimes when you try something, the solution just evolves.

3) $\sin x(\csc x + \sin x \sec^2 x) = \sec^2 x$

$\sin x \csc x + \sin^2 x \sec^2 x$

$1 + \dfrac{\sin^2 x}{\cos^2 x}$

$1 + \tan^2 x$

$\sec^2 x$

4) $2\cos^2 x + \sin^2 x = \cos^2 x + 1$

$\cos^2 x + \cos^2 x + \sin^2 x$

$\cos^2 x + 1$

5) $2\cos^2 x - 1 = 1 - 2\sin^2 x$

$2(1 - \sin^2 x) - 1$

$2 - 2\sin^2 x - 1$

$1 - 2\sin^2 x$

6) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

$\sin^2 x + 2\sin x\cos x + \cos^2 x + \sin^2 x - 2\sin x\cos x + \cos^2 x$

$\sin^2 x + \cos^2 x + \sin^2 x + \cos^2 x$

2

7) $\dfrac{\cot x}{\csc x} = \cos x$

$\dfrac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}}$

$\cos x$

8) $\tan x + \cot x = \sec x \csc x$

$\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$

$\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}$

$\dfrac{1}{\sin x} \cdot \dfrac{1}{\cos x}$

$\sec x \csc x$

9) $\sec x - \cos x = \sin x \tan x$

$$\frac{1}{\cos x} - \cos x$$
$$\frac{1-\cos^2 x}{\cos x}$$
$$\frac{\sin^2 x}{\cos x} = \sin x \tan x$$

10) $\sin x + \cos x \cot x = \csc x$

$$\sin x + \frac{\cos^2 x}{\sin x}$$
$$\frac{\sin^2 x + \cos^2 x}{\sin x}$$
$$\frac{1}{\sin x} = \csc x$$

11) $\dfrac{\cot x + 1}{\cot x - 1} = \dfrac{1+\tan x}{1-\tan x}$

$$\left(\frac{\frac{1}{\tan x}+1}{\frac{1}{\tan x}-1}\right)\left(\frac{\tan x}{\tan x}\right)$$
$$\frac{1+\tan x}{1-\tan x}$$

12) $\dfrac{\sec^2 x - 1}{\sec^2 x} = \sin^2 x$

$$1 - \frac{1}{\sec^2 x}$$
$$1 - \cos^2 x$$
$$\sin^2 x$$

13) $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = 2\sec^2 x$

$$\frac{1+\sin x + 1 - \sin x}{(1-\sin x)(1+\sin x)}$$
$$\frac{2}{1-\sin^2 x}$$
$$\frac{2}{\cos^2 x} = 2\sec^2 x$$

14) $\dfrac{\csc x + \cot x}{\tan x + \sin x} = \cot x \csc x$

$$\left(\frac{\frac{1}{\sin x}+\frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x}+\sin x}\right)\left(\frac{\sin x \cos x}{\sin x \cos x}\right)$$
$$\frac{\cos x + \cos^2 x}{\sin^2 x + \sin^2 x \cos x} = \frac{\cos x(1+\cos x)}{\sin^2 x(1+\cos x)}$$
$$\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x} = \cot x \csc x$$

B) Sum and difference Formulas

Determine whether the sine function is distributive: that is $\sin(A+B) = \sin A + \sin B$. Let's try it with different values of A and B. Check out whether $\sin(30° + 60°) = \sin 30° + \sin 60°$. $1 \neq \frac{1}{2} + \frac{\sqrt{3}}{2}$

There are geometric proofs to determine the sum and difference formulas for trig functions:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad \cos(A+B) = \cos A \cos B - \sin A \sin B \qquad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B \qquad \cos(A-B) = \cos A \cos B + \sin A \sin B \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example 1) Find the exact value of $\sin 75°$

$$\sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°$$
$$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Example 2) Find the exact value of $\cos 75°$

$$\cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45°$$
$$\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Example 3) Find the exact value of $\tan 75°$ in two ways.

$$\tan 75° = \frac{\sin 75°}{\cos 75°} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{3+\sqrt{3}}{3-\sqrt{3}}$$

Example 4) Find the exact value of $\tan 15°$

$$\tan 15° = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \frac{3-\sqrt{3}}{3+\sqrt{3}}$$

Example 5) Given $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, both A and B in quadrant I, find

a. $\sin(A+B)$ b. $\cos(A+B)$ c. $\tan(A+B)$ d. quadrant of $(A+B)$

$\frac{4}{5}\left(\frac{5}{13}\right)+\frac{3}{5}\left(\frac{12}{13}\right)=\frac{56}{65}$

$\frac{3}{5}\left(\frac{5}{13}\right)-\frac{4}{5}\left(\frac{12}{13}\right)=\frac{-33}{65}$

$\frac{-56}{33}$

quadrant II

Example 6) Given $\cos A = \frac{1}{3}$, A in quadrant IV and $\cos B = \frac{-\sqrt{7}}{4}$, B in quadrant II, find

a. $\sin(A-B)$ b. $\cos(A-B)$ c. $\tan(A-B)$ d. quadrant of $(A-B)$

$\frac{-2\sqrt{2}}{3}\left(\frac{-\sqrt{7}}{4}\right)-\frac{1}{3}\left(\frac{3}{4}\right)$

$\frac{2\sqrt{14}-3}{12}$

$\frac{1}{3}\left(\frac{-\sqrt{7}}{4}\right)+\left(\frac{-2\sqrt{2}}{3}\right)\left(\frac{3}{4}\right)$

$\frac{-\sqrt{7}-6\sqrt{2}}{12}$

$\frac{2\sqrt{14}-3}{-\sqrt{7}-6\sqrt{2}}$

quadrant II

Example 7) Verify that $\sin(x+90°)=\cos x$

$\sin\cos 90° - \cos x \sin 90°$

$\cos x$

Example 8) Verify that $\tan(x+180°)=\tan x$

$\frac{\tan x - \tan 180°}{1-\tan x 180°}$

$\tan x$

C) Double Angle formulas

Recall that $\sin(A+B)=\sin A\cos B+\cos A\sin B$. If $A = B$, we get $\sin(A+A)=\sin A\cos A+\cos A\sin A$
So $\sin 2A = 2\sin A\cos A$. This works for the other trig functions as well getting the double angle formulas.

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A \text{ or } 2\cos^2 A - 1 \text{ or } 1-2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Example 1) Using trig functions of 30°, find the values of:

a) $\sin 60°$

$2\sin 30°\cos 30°$

$2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{2}$

b) $\cos 60°$

$\cos^2 30° - \sin^2 30°$

$\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$

c) $\tan 60°$

$\frac{2\tan 30°}{1-\tan^2 30}$

$\frac{2\frac{\sqrt{3}}{3}}{1-\frac{1}{3}}=\frac{2\sqrt{3}}{3-1}=\sqrt{3}$

Example 2) Given $\sin A = \frac{4}{5}$, A in quadrant I find

a. $\sin 2A$ b. $\cos 2A$ c. $\tan 2A$ d. quadrant of 2A

$2\sin A\cos A$

$2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)=\frac{24}{25}$

$\cos^2 A - \sin^2 A$

$\frac{9}{25}-\frac{16}{25}=\frac{-7}{25}$

$\frac{-24}{7}$

quadrant II

Example 3) Given $\tan A = \frac{-2}{3}$, A in quadrant II find

a. $\sin 2A$ b. $\cos 2A$ c. $\tan 2A$ d. quadrant of 2A

a. $2\sin A\cos A$

$2\left(\frac{-2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right) = \frac{-12}{13}$

b. $\cos^2 A - \sin^2 A$

$\frac{9}{13} - \frac{4}{13} = \frac{5}{13}$

c. $\frac{-12}{5}$

d. quadrant IV

Example 4) Express $\sin 4x$ in terms of the angle x.

$2\sin 2x\cos 2x = 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)$

$4\sin x\cos^3 x - 2\sin^3 x\cos x$

Example 5) Verify the following identities:

a) $\frac{\sin x}{\sin 2x} = \frac{1}{2}\sec x$

$\frac{\sin x}{2\sin x\cos x} = \frac{1}{2\cos x} = \frac{1}{2}\sec x$

b) $(\sin x - \cos x)^2 = 1 - \sin 2x$

$\sin^2 x - 2\sin x\cos x + \cos^2 x$

$1 - 2\sin x\cos x$

$1 - \sin 2x$

D) Half-angle formulas: These formulas are more obscure and are not used that much. Still, you should know that they exist and be able to use them.

$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$ $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$ $\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$ or $\frac{\sin A}{1+\cos A}$

The signs of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ depend on the quadrant in which $\frac{A}{2}$ lies.

Example 1) Find the exact values of the following using half-angle formulas.

a) $\sin 15°$

$\sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$

$\sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$

b. $\cos 15°$

$\sqrt{\frac{1+\cos 30°}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$

$\sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$

c) $\tan 15°$

$\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$2-\sqrt{3}$

Example 2) Given $\sin A = -\frac{4}{5}$, A in quadrant III find

a. $\sin\frac{A}{2}$ b. $\cos\frac{A}{2}$ c. $\tan\frac{A}{2}$ d. quadrant of $\frac{A}{2}$

a. $\sqrt{\frac{1-\frac{-3}{5}}{2}}$

$\sqrt{\frac{8}{10}} = \sqrt{\frac{4}{5}}$

b. $-\sqrt{\frac{1+\frac{-3}{5}}{2}}$

$-\sqrt{\frac{2}{10}} = -\sqrt{\frac{1}{5}}$

c. $\sqrt{\frac{4}{5}} \Big/ -\sqrt{\frac{1}{5}} = -2$

d. quadrant II

E) Solving trigonometric equations

Just as we solved equations for a value of x that satisfied the equation, we do the same for trig equations – in this case finding the value of an angle that satisfies the equation.

Example 1) $\sin x = 1$. We can do this by inspection – we know from our knowledge of graphing and quadrant angles that the angle that satisfies this equation is $90° \left(\text{or } \frac{\pi}{2}\right)$. However, there are other angles that satisfy this equation like $450°, 810°, 1170°, ...$ So we usually solve the equation on a certain domain. Usually we will solve it on $0 \le x < 360°$ or $0 \le x < 2\pi$. We can also verify our solutions by graphing the equation.

Example 2) $2\sin x + 1 = 0$

$$\begin{aligned} &2\sin x = -1 \\ &\sin x = \frac{-1}{2} \\ &x = 210°, 330° \end{aligned}$$

Example 3) $\sin^2 x = \sin x$

$$\begin{aligned} &\sin^2 x - \sin x = 0 \\ &\sin x(\sin x - 1) \\ &x = 0°, 180°, 90° \end{aligned}$$

Example 4) $\sin x = \cos x$

$$\begin{aligned} &\frac{\sin x}{\cos x} = 1 \\ &\tan x = 1 \\ &x = 45°, 225° \end{aligned}$$

Example 5) $2\sin^2 x = 1$

$$\begin{aligned} &\sin^2 x = \frac{1}{2} \\ &\sin x = \pm\frac{\sqrt{2}}{2} \\ &x = 45°, 135°, 225°, 315° \end{aligned}$$

Example 6) $2\sin^2 x - 5\sin x + 2 = 0$

$$\begin{aligned} &(2\sin x - 1)(\sin x - 2) = 0 \\ &2\sin x = 1 \qquad \sin x = 2 \\ &\sin x = \frac{1}{2} \qquad \text{No solution} \\ &x = 30°, 150° \end{aligned}$$

Example 7) $2\cos^2 x - \cos x = 1$

$$\begin{aligned} &2\cos^2 x - \cos x - 1 = 0 \\ &(2\cos x + 1)(\cos x - 1) = 0 \\ &2\cos x = -1 \qquad \cos x = 1 \\ &\cos x = -\frac{1}{2} \\ &x = 120°, 240°, 0 \end{aligned}$$

Example 8) $\sin x - 1 = \cos x$

$$\begin{aligned} &(\sin x - 1)^2 = \cos^2 x \\ &\sin^2 x - 2\sin x + 1 = \cos^2 x \\ &\sin^2 x - 2\sin x + 1 = 1 - \sin^2 x \\ &2\sin^2 x - 2\sin x = 0 \\ &2\sin x(\sin x - 1) = 0 \\ &x = \emptyset°, 180°, 90° \end{aligned}$$

Example 9) $\sin 2x = 0$

$$\begin{aligned} &2\sin x\cos x = 0 \\ &\sin x = 0 \qquad \cos x = 0 \\ &x = 0°, 180°, 90°, 270° \end{aligned}$$

Unit 5 – Analytical Trigonometry – Homework

1. Verify the following identities: There are additional problems in your book.

a) $\csc^2 x\left(1-\cos^2 x\right)=1$

$\frac{1}{\sin^2 x}\left(\sin^2 x\right)$
1

b) $(\sin x+\cos x)^2-(\sin x-\cos x)^2=4\sin x\cos x$

$\sin^2 x+2\sin x\cos x+\cos^2 x-\left(\sin^2 x-2\sin x\cos x+\cos^2 x\right)$
$1+2\sin x\cos x-1+2\sin x\cos x$
$4\sin x\cos x$

c) $\sin x\left(\csc x+\sin x\sec^2 x\right)=\sec^2 x$

$\sin x\left(\frac{1}{\sin x}\right)+\frac{\sin^2 x}{\cos^2 x}$
$1+\tan^2 x$
$\sec^2 x$

d) $\cot^2 x+5=\csc^2 x+4$

$\csc^2 x-1+5$
$\csc^2 x+4$

e) $\cos^4 x-\cos^2 x=\sin^4 x-\sin^2 x$

$\cos^2 x\left(\cos^2 x-1\right)$
$\left(1-\sin^2 x\right)\left(-\sin^2 x\right)$
$\sin^4 x-\sin^2 x$

f) $\sin x\tan x+\cos x=\sec x$

$\sin x\left(\frac{\sin x}{\cos x}\right)+\cos x$
$\frac{\sin^2 x+\cos^2 x}{\cos x}$
$\frac{1}{\cos x}=\sec x$

g) $\sin x-\csc x=\frac{-\cos^2 x}{\sin x}$

$\sin x-\frac{1}{\sin x}$
$\frac{\sin^2 x-1}{\sin x}$
$\frac{-\cos^2 x}{\sin x}$

h) $\frac{1}{\sec x}-\frac{1}{\cos x}=\cos x-\sec x$

$\cos x-\sec x$

i) $\sin x+\cos x\cot x=\csc x$

$\sin x+\cos x\left(\frac{\cos x}{\sin x}\right)$
$\frac{\sin^2 x+\cos^2 x}{\sin x}$
$\frac{1}{\sin x}=\csc x$

j) $\frac{(\sin x-\cos x)^2}{\cos x}=\sec x-2\sin x$

$\frac{\sin^2 x-2\sin x\cos x+\cos^2 x}{\cos x}$
$\frac{1-2\sin x\cos x}{\cos x}$
$\sec x-2\sin x$

k) $\frac{1}{1+\cos x}+\frac{1}{1-\cos x}=2\csc^2 x$

$$\frac{1-\cos x+1+\cos x}{(1+\cos x)(1-\cos x)}$$
$$\frac{2}{1-\cos^2 x}$$
$$\frac{2}{\sin^2 x}=2\csc^2 x$$

l) $\frac{\cos}{1-\sin x}-\frac{\cos x}{1+\sin x}=2\tan x$

$$\frac{\cos x(1+\sin x)-\cos x(1-\sin x)}{(1+\sin x)(1-\sin x)}$$
$$\frac{\cos x+\sin x\cos x-\cos x+\sin x\cos x}{1-\sin^2 x}$$
$$\frac{2\sin x\cos x}{\cos^2}=\frac{2\sin x\cos x}{\cos x\cos x}=2\tan x$$

m) $\frac{1}{\sin x+1}+\frac{1}{\csc x+1}=1$

$$\frac{1}{\sin x+1}+\left(\frac{1}{\frac{1}{\sin x}+1}\right)\cdot\frac{\sin x}{\sin x}$$
$$\frac{1}{\sin x+1}+\frac{\sin x}{1+\sin x}$$
$$\frac{1+\sin x}{\sin x+1}=1$$

n) $\sin^4 x-\cos^4 x=2\sin^2 x-1$

$$(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)$$
$$\left[\sin^2 x-(1-\sin^2 x)\right]1$$
$$\sin^2 x-1+\sin^2 x$$
$$2\sin^2 x-1$$

o) $(\tan x+\sin x)(1-\cos x)=\sin^2 x\tan x$

$$\tan x-\tan x\cos x+\sin x-\sin x\cos x$$
$$\frac{\sin x}{\cos x}-\frac{\sin x\cos x}{\cos x}+\sin x-\sin x\cos x$$
$$\frac{\sin x}{\cos x}-\sin x+\sin x-\sin x\cos x$$
$$\frac{\sin x}{\cos x}-\sin x\cos x\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{\sin x(1-\cos^2 x)}{\cos x}=\tan x\sin^2 x$$

p) $\frac{\tan x}{\sec x-1}=\frac{\sec x+1}{\tan x}$

$$\left(\frac{\tan x}{\sec x-1}\right)\left(\frac{\sec x+1}{\sec x+1}\right)$$
$$\frac{\tan x(\sec x+1)}{\sec^2 x-1}$$
$$\frac{\tan x(\sec x+1)}{\tan^2 x}$$
$$\frac{\sec x+1}{\tan x}$$

q) $\cos x(1+\tan x)^2=\sec x+2\sin x$

$$\cos x(1+2\tan x+\tan^2 x)$$
$$\cos x(\sec^2 x+2\tan x)$$
$$\cos x\left(\frac{1}{\cos^2 x}+2\frac{\sin x}{\cos x}\right)=\sec x+2\sin x$$

r) $\frac{\sec x}{\tan x+\cot x}=\sin x$

$$\left(\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}\right)\left(\frac{\sin x\cos x}{\sin x\cos x}\right)$$
$$\frac{\sin x}{\sin^2 x+\cos^2 x}=\sin x$$

s) $\cos^4 x-\sin^4 x=2\cos^2 x-1$

$$(\cos^2 x-\sin^2 x)(\cos^2 x+\sin^2 x)$$
$$\cos^2 x-(1-\cos^2 x)$$
$$2\cos^2 x-1$$

t) $\frac{\cos x}{1-\sin x}=\frac{1+\sin x}{\cos x}$

$$\left(\frac{\cos x}{1-\sin x}\right)\left(\frac{1+\sin x}{1+\sin x}\right)=\frac{\cos x(1+\sin x)}{1-\sin^2 x}$$
$$\frac{\cos x(1+\sin x)}{\cos^2 x}=\frac{1+\sin x}{\cos x}$$

2. Find the exact values of the following expressions. Make appropriate pictures.

a. $\sin 105°, \cos 105°, \tan 105°$

$105° = (60° + 45°)$

By formulas:

$\sin 105° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$\cos 105° = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

$\tan 105° = \dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{2}-\sqrt{6}}$

b. $\sin 255°, \cos 255°, \tan 255°$

$255° = (225° + 30°)$ - there are other combinations

By formulas:

$\sin 255° = \dfrac{-\sqrt{6}-\sqrt{2}}{4}$

$\cos 255° = \dfrac{-\sqrt{6}+\sqrt{2}}{4}$

$\tan 255° = \dfrac{-\sqrt{6}-\sqrt{2}}{-\sqrt{6}+\sqrt{2}}$

3. Given $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{8}{17}$, both A and B in quadrant I, find

a. $\sin(A+B)$ b. $\cos(A+B)$ c. $\tan(A+B)$ d. quadrant of $(A+B)$

a. $\sin A\cos B + \cos A\sin B$

$\left(\dfrac{3}{5}\right)\left(\dfrac{8}{17}\right)+\left(\dfrac{4}{5}\right)\left(\dfrac{15}{17}\right)$

$\dfrac{84}{85}$

b. $\cos A\cos B - \sin A\sin B$

$\left(\dfrac{4}{5}\right)\left(\dfrac{8}{17}\right)-\left(\dfrac{3}{5}\right)\left(\dfrac{15}{17}\right)$

$\dfrac{-13}{85}$

c. $-\dfrac{84}{13}$

d. Quadrant 2

4. Given $\cos A = \dfrac{-2}{5}$, A in quadrant III and $\cos B = \dfrac{1}{4}$, B in quadrant I, find

a. $\sin(A+B)$ b. $\cos(A+B)$ c. $\tan(A+B)$ d. quadrant of $(A+B)$

a. $\sin A\cos B + \cos A\sin B$

$\left(\dfrac{-\sqrt{21}}{5}\right)\left(\dfrac{1}{4}\right)+\left(\dfrac{-2}{5}\right)\left(\dfrac{\sqrt{15}}{4}\right)$

$\dfrac{-\sqrt{21}-2\sqrt{15}}{20}$

b. $\cos A\cos B - \sin A\sin B$

$\left(\dfrac{-2}{5}\right)\left(\dfrac{1}{4}\right)-\left(\dfrac{-\sqrt{21}}{5}\right)\left(\dfrac{\sqrt{15}}{4}\right)$

$\dfrac{-2+\sqrt{315}}{20}$

c. $\dfrac{-\sqrt{21}-2\sqrt{15}}{-2+\sqrt{315}}$

d. Quadrant 4

5. Given $\tan A = 5$, A in quadrant III and $\sin B = \dfrac{2}{3}$, B in quadrant II, find

a. $\sin(A-B)$ b. $\cos(A-B)$ c. $\tan(A-B)$ d. quadrant of $(A-B)$

a. $\sin A\cos B - \cos A\sin B$

$\left(\dfrac{-5}{\sqrt{26}}\right)\left(\dfrac{-\sqrt{5}}{3}\right)-\left(\dfrac{-1}{\sqrt{26}}\right)\left(\dfrac{2}{3}\right)$

$\dfrac{5\sqrt{5}+2}{3\sqrt{26}}$

b. $\cos A\cos B + \sin A\sin B$

$\left(\dfrac{-1}{\sqrt{26}}\right)\left(\dfrac{-\sqrt{5}}{3}\right)+\left(\dfrac{-5}{\sqrt{26}}\right)\left(\dfrac{2}{3}\right)$

$\dfrac{\sqrt{5}-10}{3\sqrt{26}}$

c. $\dfrac{5\sqrt{5}+2}{\sqrt{5}-10}$

d. Quadrant 2

6. Verify the following identities:

a) $\cos(270° - x) = -\sin x$

$$\begin{array}{|l|}\hline \cos 270° \cos x + \sin 270° \sin x \\ 0(\cos x) - 1\sin x \\ -\sin x \\ \hline\end{array}$$

b) $\sin(x - 30°) = \frac{1}{2}\left(\sqrt{3}\sin x - \cos x\right)$

$$\begin{array}{|l|}\hline \sin x \cos 30° - \cos x \sin 30° \\ \sin x\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\cos x \\ \frac{1}{2}\left(\sqrt{3}\sin x - \cos x\right) \\ \hline\end{array}$$

c) $\sin(x - \pi) - \cos\left(x + \frac{\pi}{2}\right) = 0$

$$\begin{array}{|l|}\hline \sin x \cos\pi - \cos x \sin\pi - \left(\cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2}\right) \\ -\sin x - 0 - 0 + \sin x = 0 \\ \hline\end{array}$$

d) $\tan(x + 60°) = \dfrac{\sqrt{3}\tan x + 3}{\sqrt{3}\tan x - 3\tan x}$

$$\begin{array}{|l|}\hline \dfrac{\tan x + \tan 60°}{1 - \tan x \tan 60°} \\ \left(\dfrac{\tan x + \sqrt{3}}{1 - \tan x\sqrt{3}}\right)\left(\dfrac{\sqrt{3}}{\sqrt{3}}\right) = \dfrac{\sqrt{3}\tan x + 3}{\sqrt{3}\tan x - 3\tan x} \\ \hline\end{array}$$

e) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

$$\begin{array}{|l|}\hline (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B) \\ \sin^2 A\cos^2 B - \cos^2 A\sin^2 B \\ \sin^2 A\left(1 - \sin^2 B\right) - \left(1 - \sin^2 A\right)\sin^2 B \\ \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B = \sin^2 A - \sin^2 B \\ \hline\end{array}$$

f) $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$

$$\begin{array}{|l|}\hline \cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B \\ 2\cos A\cos B \\ \hline\end{array}$$

7. Using trig functions of 60°, find the values of

a) $\sin 120°$

$$\begin{array}{|l|}\hline \sin 120° = \sin(2\cdot 60°) \\ 2\sin 60°\cos 60° \\ 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2} \\ \hline\end{array}$$

b) $\cos 120°$

$$\begin{array}{|l|}\hline \cos 120° = \cos(2\cdot 60°) \\ \cos^2 60° - \sin^2 60° \\ \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} \\ \hline\end{array}$$

c) $\tan 120°$

$$\begin{array}{|l|}\hline \tan 120° = \dfrac{\sin 120°}{\cos 120°} \\ -\sqrt{3} \\ \hline\end{array}$$

8. Given $\sin A = \frac{7}{25}$, A in quadrant II find

a. $\sin 2A$

$$\begin{array}{|l|}\hline 2\sin A\cos A \\ 2\left(\frac{7}{25}\right)\left(\frac{-24}{25}\right) = \frac{-336}{625} \\ \hline\end{array}$$

b. $\cos 2A$

$$\begin{array}{|l|}\hline \cos^2 A - \sin^2 A \\ \left(\frac{-24}{25}\right)^2 - \left(\frac{7}{25}\right)^2 = \frac{527}{625} \\ \hline\end{array}$$

c. $\tan 2A$

$$\begin{array}{|l|}\hline \sin 2A/\cos 2A \\ \dfrac{\sin 2A}{\cos 2A} = \dfrac{-336}{527} \\ \hline\end{array}$$

d. quadrant of 2A

$$\begin{array}{|l|}\hline \sin 2A = -,\ \cos 2A = + \\ \text{Quadrant IV} \\ \hline\end{array}$$

9. Given $\cos A = \frac{1}{10}$, A in quadrant I find

a. $\sin 2A$

$$\begin{array}{|l|}\hline 2\sin A\cos A \\ 2\left(\frac{\sqrt{99}}{10}\right)\left(\frac{1}{10}\right) = \frac{\sqrt{99}}{50} \\ \hline\end{array}$$

b. $\cos 2A$

$$\begin{array}{|l|}\hline \cos^2 A - \sin^2 A \\ \left(\frac{1}{10}\right)^2 - \left(\frac{\sqrt{99}}{10}\right)^2 = \frac{-49}{50} \\ \hline\end{array}$$

c. $\tan 2A$

$$\begin{array}{|l|}\hline \sin 2A/\cos 2A \\ \dfrac{\sin 2A}{\cos 2A} = -\dfrac{\sqrt{99}}{49} \\ \hline\end{array}$$

d. quadrant of 2A

$$\begin{array}{|l|}\hline \sin 2A = +,\ \cos 2A = - \\ \text{Quadrant II} \\ \hline\end{array}$$

10. Given $\tan A = 2$, A in quadrant III find

a. $\sin 2A$ — $2\sin A\cos A$; $2\left(\frac{-2}{\sqrt{5}}\right)\left(\frac{-1}{\sqrt{5}}\right) = \frac{4}{5}$

b. $\cos 2A$ — $\cos^2 A - \sin^2 A$; $\left(\frac{-1}{\sqrt{5}}\right)^2 - \left(\frac{-2}{\sqrt{5}}\right)^2 = \frac{-3}{5}$

c. $\tan 2A$ — $\sin 2A/\cos 2A$; $\frac{\sin 2A}{\cos 2A} = -\frac{4}{3}$

d. quadrant of 2A — $\sin 2A = +,\ \cos 2A = -$; Quadrant II

11. Verify the following identities:

a. $\cos^4 x - \sin^4 x = \cos 2x$

$$\left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right)$$
$$\left(\cos^2 x - \sin^2 x\right)(1) = \cos 2x$$

b. $\csc 2x = \frac{\sec x \csc x}{2}$

$$\frac{1}{\sin 2x} = \frac{1}{2\sin x\cos x} = \frac{\csc x \sec x}{2}$$

c. $\sin 3x = 3\sin x - 4\sin^3 x$

$$\sin(2x + x)$$
$$\sin 2x\cos x + \cos 2x \sin x$$
$$2\sin x\cos x\cos x + \left(1 - 2\sin^2 x\right)\sin x$$
$$2\sin x\cos^2 x + \sin x - 2\sin^3 x$$
$$2\sin x\left(1 - \sin^2 x\right) + \sin x - 2\sin^3 x$$
$$2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$
$$3\sin x - 4\sin^3 x$$

d. $\cos 3x = 4\cos^3 x - 3\cos x$

$$\cos 3x = 4\cos^3 x - 3\cos x$$
$$\cos 2x\cos x - \sin 2x \sin x$$
$$\left(2\cos^2 x - 1\right)\cos x - \left(2\sin x\cos x\right)\sin x$$
$$2\cos^3 x - 2\cos x - 2\sin^2 x\cos x$$
$$2\cos^3 x - 2\cos x - 2\left(1 - \cos^2 x\right)\cos x$$
$$2\cos^3 x - 2\cos x - 2\cos x + 2\cos^3 x$$
$$4\cos^3 x - 3\cos x$$

12. Find the exact values of the following using half-angle formulas.

a) $\sin 22.5°$ — $\sqrt{\frac{1 - \cos 45°}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{2}}{4}}$

b. $\cos 22.5°$ — $\sqrt{\frac{1 + \cos 45°}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{2}}{4}}$

c) $\tan 22.5°$ — $\sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}}$

13. Given $\sin A = \frac{8}{17}$, A in quadrant II find

a. $\sin\frac{A}{2}$ — $\sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{1 - \frac{-15}{17}}{2}}$; $\sqrt{\frac{17 + 15}{34}} = \sqrt{\frac{16}{17}}$

b. $\cos\frac{A}{2}$ — $\sqrt{\frac{1 + \cos A}{2}} = \sqrt{\frac{1 + \frac{-15}{17}}{2}}$; $\sqrt{\frac{17 - 15}{34}} = \sqrt{\frac{1}{17}}$

c. $\tan\frac{A}{2}$ — $\sqrt{\frac{16}{1}} = 4$

d. quadrant of $\frac{A}{2}$ — Quadrant I

14. Given $\cos A = \frac{1}{3}$, A in quadrant IV find

a. $\sin\frac{A}{2}$ — $\sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{1 - \frac{1}{3}}{2}}$; $\sqrt{\frac{3 - 1}{6}} = \sqrt{\frac{1}{3}}$

b. $\cos\frac{A}{2}$ — $-\sqrt{\frac{1 + \cos A}{2}} = -\sqrt{\frac{1 + \frac{1}{3}}{2}}$; $-\sqrt{\frac{3 + 1}{6}} = -\sqrt{\frac{2}{3}}$

c. $\tan\frac{A}{2}$ — $-\frac{1}{\sqrt{2}}$

d. quadrant of $\frac{A}{2}$ — Quadrant 2

15. Solve the following equations on $\left[0, 360^\circ\right)$

a) $2\cos x = 1$

$$\cos x = \frac{1}{2}$$
$$x = 60^\circ, 300^\circ$$

b) $\cos^2 x - \cos x = 0$

$$\cos x(\cos x - 1) = 0$$
$$\cos x = 0 \qquad \cos x = 1$$
$$x = 90^\circ, 270^\circ \qquad x = 0^\circ$$

c) $4\sin^2 x = 3$

$$\sin^2 x = \frac{3}{4}$$
$$\sin x = \pm\frac{\sqrt{3}}{2}$$
$$x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$$

d) $2\cos^2 x = 1$

$$\cos^2 x = \frac{1}{2}$$
$$\cos x = \pm\frac{\sqrt{2}}{2}$$
$$x = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

e) $2\cos^2 x = \cos x$

$$\cos x(2\cos x - 1) = 0$$
$$\cos x = 0 \qquad \cos x = \frac{1}{2}$$
$$x = 90^\circ, 270^\circ \qquad x = 60^\circ, 300^\circ$$

f) $\tan^2 x = 1$

$$\tan x = 1 \qquad \tan x = -1$$
$$x = 45^\circ, 225^\circ \qquad x = 135^\circ, 315^\circ$$

f) $2\cos^2 x + 7\cos x + 3 = 0$

$$(2\cos x + 1)(\cos x + 3) = 0$$
$$\cos x = \frac{-1}{2} \qquad \cos x = -3$$
$$x = 120^\circ, 240^\circ \qquad \text{No solution}$$

h) $\sin^3 x = \sin x$

$$\sin x(\sin x - 1)(\sin x + 1) = 0$$
$$\sin x = 0 \qquad \sin x = 1 \qquad \sin x = -1$$
$$x = 0^\circ, 180^\circ, 90^\circ, 270^\circ$$

i) $\cos^2 x = 3\sin^2 x$

$$1 - \sin^2 x = 3\sin^2 x$$
$$\sin x = \pm\frac{1}{2}$$
$$x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

j) $\cos x = 1 - \sin x$

$$\cos^2 x = 1 - 2\sin x + \sin^2 x$$
$$1 - \sin^2 x = 1 - 2\sin x + \sin^2 x$$
$$2\sin x(\sin x - 1) = 0$$
$$x = 0^\circ, \cancel{180^\circ}, 90^\circ$$

k) $\sin 2x = \cos x$

$$\cos x(2\sin x - 1) = 0$$
$$\cos x = 0 \qquad \sin x = \frac{1}{2}$$
$$x = 90^\circ, 270^\circ, 30^\circ, 150^\circ$$

l) $\cos 2x = \cos x$

$$2\cos^2 x - 1 = \cos x$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$\cos x = \frac{-1}{2} \qquad \cos x = 1$$
$$x = 120^\circ, 240^\circ, 0^\circ$$