Section 5.3
Double Angle, Power-Reducing, and Half-Angle Formulas

How Far Can You Throw That?

When you throw an object, the distance that the object will travel before hitting the ground depends on the initial speed of the object as well as the angle at which the object leaves your hand. This distance can be modeled by a formula that involves both sines and cosines.

In the Exercise Set, you will see how a trigonometric identity introduced in this section can be used to rewrite a formula involving two trigonometric functions as a formula involving only one such function. The simpler form can be used to determine an angle that maximizes throwing distance.

Objective #1: Use the double-angle formulas.

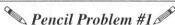| ✔ *Solved Problem #1*	✏ *Pencil Problem #1*✏

1a. If $\sin\theta = \dfrac{4}{5}$ and θ lies in quadrant II, find the exact value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

The formulas $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ require that we know both $\sin\theta$ and $\cos\theta$. We are given that $\sin\theta = \dfrac{4}{5}$. We need to find $\cos\theta$. Since θ lies in quadrant II, the value of $\cos\theta$ is negative. Use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$, choosing the negative value.

$$\left(\frac{4}{5}\right)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \left(\frac{4}{5}\right)^2 = \frac{25}{25} - \frac{16}{25} = \frac{9}{25}$$

$$\cos\theta = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

1a. If $\sin\theta = \dfrac{15}{17}$ and θ lies in quadrant II, find the exact value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

Now substitute the values into the formulas.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

The formula $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ requires that we know $\tan\theta$. Since we know $\sin\theta = \dfrac{4}{5}$ and $\cos\theta = -\dfrac{3}{5}$, we can use a quotient identity to find $\tan\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\dfrac{4}{5}}{-\dfrac{3}{5}} = \frac{4}{5}\cdot\left(-\frac{5}{3}\right) = -\frac{4}{3}$$

Now we can use this value in the formula.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\dfrac{4}{3}\right)}{1-\left(-\dfrac{4}{3}\right)^2}$$

$$= \frac{-\dfrac{8}{3}}{1-\dfrac{16}{9}} = \frac{-\dfrac{8}{3}}{-\dfrac{7}{9}} = -\frac{8}{3}\cdot\left(-\frac{9}{7}\right) = \frac{24}{7}$$

Note that we can also find $\tan 2\theta$ using the values of $\sin 2\theta$ and $\cos 2\theta$ and a quotient identity.

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\dfrac{24}{25}}{-\dfrac{7}{25}} = \frac{24}{7}$$

1b. Find the exact value of $\cos^2 15° - \sin^2 15°$.

The given expression looks like the right side of the double-angle formula $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ with $\theta = 15°$. We use the formula to rewrite the expression and then evaluate.

$$\cos^2 15° - \sin^2 15° = \cos(2 \cdot 15°) = \cos 30° = \frac{\sqrt{3}}{2}$$

1b. Find the exact value of $2\sin 15° \cos 15°$.

1c. Verify the identity: $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$.

Notice that the sine on the left side has 3θ as its argument while the ones on the right have just θ, so we need to use one or more identities that allow us to change the argument. The right side may look more complicated, but we will start with the left side because it contains the more complicated argument and we see more opportunities to apply identities.

We first rewrite $\sin 3\theta$ as $\sin(2\theta + \theta)$ and use the sum formula for sine. Then we use double-angle formulas to simplify the occurrences of $\sin 2\theta$ and $\cos 2\theta$ that result from the first step. Since the right side contains only sine, we choose to use the form of the double-angle formula for cosine that expresses $\cos 2\theta$ in terms of sine only. Still keeping in mind that the right side contains only sine, we replace an occurrence of $\cos^2 \theta$ with $1 - \sin^2 \theta$, using a form of the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$.

$$\begin{aligned}
\sin 3\theta &= \sin(2\theta + \theta) \\
&= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta \\
&= (2\sin \theta \cos \theta)\cos \theta + (1 - 2\sin^2 \theta)\sin \theta \\
&= 2\sin \theta \cos^2 \theta + \sin \theta - 2\sin^3 \theta \\
&= 2\sin \theta(1 - \sin^2 \theta) + \sin \theta - 2\sin^3 \theta \\
&= 2\sin \theta - 2\sin^3 \theta + \sin \theta - 2\sin^3 \theta \\
&= 3\sin \theta - 4\sin^3 \theta
\end{aligned}$$

1c. Verify the identity:
$$\sin 4t = 4\sin t \cos^3 t - 4\sin^3 t \cos t.$$

Objective #2: Use the power-reducing formulas.

✔ Solved Problem #2

2. Write an equivalent expression for $\sin^4 x$ that does not contain powers of trigonometric functions greater than 1.

    We begin by writing $\sin^4 x$ as $(\sin^2 x)^2$ and applying the power-reducing formula for sine. After some simplification, we will use the power-reducing formula for cosine.

    $\sin^4 x = (\sin^2 x)^2$

    $\qquad = \left(\dfrac{1 - \cos 2x}{2}\right)^2$

    $\qquad = \dfrac{1 - 2\cos 2x + \cos^2 2x}{4}$

    $\qquad = \dfrac{1}{4} - \dfrac{1}{2}\cos 2x + \dfrac{1}{4}\cos^2 2x$

    $\qquad = \dfrac{1}{4} - \dfrac{1}{2}\cos 2x + \dfrac{1}{4}\left[\dfrac{1 + \cos 2(2x)}{2}\right]$

    $\qquad = \dfrac{1}{4} - \dfrac{1}{2}\cos 2x + \dfrac{1}{8} + \dfrac{1}{8}\cos 4x$

    $\qquad = \dfrac{3}{8} - \dfrac{1}{2}\cos 2x + \dfrac{1}{8}\cos 4x$

✎ Pencil Problem #2 ✎

2. Write an equivalent expression for $\sin^2 x \cos^2 x$ that does not contain powers of trigonometric functions greater than 1.

Objective #3: Use the half-angle formulas.	
✔ *Solved Problem #3*	✎ *Pencil Problem #3*✎

3. Use $\cos 210° = -\dfrac{\sqrt{3}}{2}$ and a half-angle formula to find the exact value of $\cos 105°$.

Since an angle that measures $105°$ lies in quadrant II and cosine is negative in quadrant II, we choose the negative sign in the half-angle formula for cosine.

$$\cos 105° = \cos \frac{210°}{2}$$

$$= -\sqrt{\frac{1 + \cos 210°}{2}}$$

$$= -\sqrt{\frac{1 + \left(-\dfrac{\sqrt{3}}{2}\right)}{2}}$$

$$= -\sqrt{\frac{2 - \sqrt{3}}{4}}$$

$$= -\frac{\sqrt{2 - \sqrt{3}}}{2}$$

Note that to simplify the fraction $\dfrac{1 + \left(-\dfrac{\sqrt{3}}{2}\right)}{2}$ we multiplied the numerator and denominator both by

2: $\dfrac{\left[1 + \left(-\dfrac{\sqrt{3}}{2}\right)\right] \cdot 2}{2 \cdot 2} = \dfrac{1 \cdot 2 + \left(-\dfrac{\sqrt{3}}{2}\right) \cdot 2}{4} = \dfrac{2 - \sqrt{3}}{4}$.

3. Use $\cos 315° = \dfrac{\sqrt{2}}{2}$ and a half-angle formula to find the exact value of $\cos 157.5°$.

Answers for Pencil Problems (*Textbook Exercise references in parentheses*):

1a. $\sin 2\theta = -\dfrac{240}{289}$; $\cos 2\theta = -\dfrac{161}{289}$; $\tan 2\theta = \dfrac{240}{161}$ *(5.3 #7)*

1b. $\dfrac{1}{2}$ *(5.3 #15)*

1c. $\sin 4t = \sin(2 \cdot 2t) = 2\sin 2t \cos 2t = 2(2\sin t \cos t)(\cos^2 t - \sin^2 t) = 4\sin t \cos^3 t - 4\sin^3 t \cos t$ *(5.3 #33)*

2. $\dfrac{1}{8} - \dfrac{1}{8}\cos 4x$ *(5.3 #37)*

3. $-\dfrac{\sqrt{2 + \sqrt{2}}}{2}$ *(5.3 #41)*