Mathematics

A Geometric Perspective on the Riemann Zeta Function's Partial Sums

Carl Erickson

The Riemann Zeta Function, $\zeta(s)$, is an important complex function whose behavior has implications for the distribution of the prime numbers among the natural numbers. Most notably, the still unsolved Riemann Hypothesis, which states that all non-trivial zeros of the zeta function have real part one-half, would imply the most regular distribution of primes possible in the context of current theory. The Riemann Zeta Function is the simplest of the Dirichlet series and is represented in its Dirichlet series form as $\zeta(s) = \sum_{n+1}^{\infty} n^{-s} = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$. This series only converges when the real part of s, $\Re(s)$, is greater than 1, outside the area of the complex plane relevant to the distribution of the primes. This area is called the critical strip: $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. The result of our investigation of the geometric distribution will be to draw connections between the partial sums of the Dirichlet series and the value of $\zeta(s)$ with s in the critical strip despite the series' divergence. This article will illustrate connections between existing theory of the Riemann Zeta Function and geometric analysis of the partial sums through visual representations. From the connections between the visually accessible geometry and this theory, we illuminate and explore potentially provable improvements of the theory based on symmetry among the partial sums.

1. The Importance of the Riemann Zeta Function

Very complex mathematical ideas often spring from the investigation of questions that are simple to understand. The subject of this article - the behavior of the Riemann Zeta Function - is one such complex mathematical object. However, the study of the function's theory sprung from an investigation of the prime numbers' distribution. The prime numbers will seem very removed from most of the discussion in this article. 2, 3, 5, 7, 11, 13... they appear so simple at first. And the question of their distribution? Simply finding the next known prime might appear to be sufficient, but this is not the case. Mathematicians are unsatisfied with knowing simply a large number of primes and observing their distribution, for there are an infinite number of primes[1] [5]; instead, they seek a general rule that will dictate the distribution of the primes of any magnitude. This search gradually led mathematicians like Bernhard Riemann to utilize the theory of complex functions to describe the distribution of the primes [7].

Before discussing Riemann's work more, preliminaries about complex numbers and complex function must be laid out. Also, the experienced math reader should keep in mind that a rigorous exposition and careful attention to detail is beyond the scope of this journal; the focus is to illustrate a connection between geometry and analysis.

Complex Numbers

Complex numbers are less familiar than prime numbers. Yet most of the numbers discussed, like the argument of the function s, the value of the function $\zeta(s)$, and each partial sum, are complex numbers. Thus, if complex numbers are unfamiliar, read this section before moving on or see [1] or more briefly [11].

A complex number is a number $a + bi$ where a and b are real numbers ($a, b \in \mathbb{R}$) and $i = \sqrt{-1}$. By high school we learn that -1 has no square root, which is correct considering only the real numbers, for no real number squared is negative. Complex numbers are critical objects exactly because they solve this problem: given any complex polynomial equation, all of its solutions are complex.[2] So, for example, the polynomial equation $x^2 + 1 = 0$ has solutions i and $-i$, which are complex numbers.

Geometric Interpretation of Complex Numbers

Whether or not complex numbers are new to the reader, it will be convenient to simply visualize of the complex numbers as a plane of numbers of the form $z = (a, b)$ where a is the

[1] Euclid proved that there are an infinite number of primes around 300 B.C.

[2] This property is called algebraic closure.

real part of z ($a = \Re(z)$), its component along the real axis, and b is the imaginary part of z ($b = \Im(z)$), its component along the imaginary axis. This plane is called the complex plane, and is shown in figure 1.

The "polar form" of complex numbers is important to their visual, geometric interpretation: any complex number can be represented by its non-negative distance from the origin r and its angle from the positive real axis θ as in figure 2. It is then equal to $re^{i\theta}$; an important geometric corollary to this fact is that multiplication by $e^{i\theta}$ is equivalent to a rotation by θ around the origin.

A final important complex operation is that of conjugation; its geometric meaning is also important. The conjugate of s is denoted by $\overline{s}$ and the conjugate of $a + bi$ is $a - bi$; the sign of the imaginary part is changed, which is equivalent to a flip over the real axis. For example, $1 + i$ is flipped to $1 - i$. A critical fact about conjugation is that for any complex function,

$$f(\overline{s}) = \overline{f(s)}. \tag{1.1}$$

Conventions Regarding the Zeta Function

For the remainder of this text, let s represent the argument of the zeta function and let $\sigma = \Re(s)$ and $t = \Im(s)$ so that $s = \sigma + it$. Assume t to be positive, which does not result in any loss of generality because $\zeta(\sigma + it) = \overline{\zeta(\sigma - it)}$. The function $\zeta(s)$ accepts a point on the plane, $s = (\sigma, t)$, and gives an output that is a point on the plane. Likewise, a partial sum is the sum of the two components of each summand. Also, let $P_s(n)$ represent the n^{th} partial sum of the series with argument s. Keep in mind that addition of complex numbers follows the pattern of vector addition:

$$\begin{aligned} P_s(n) &= \sum_{k=1}^{n} k^{-s} = \sum_{k=1}^{n} \big(\Re(k^{-s}), \Im(k^{-s})\big) \\ &= \Big(\sum_{k=1}^{n} \Re(k^{-s}), \sum_{k=1}^{n} \Im(k^{-s})\Big). \end{aligned} \tag{1.2}$$

For example, figure 3 shows the the first five[3] partial sums of the zeta function when $s = 1/2 + 500i$. Each partial sum n differs from partial sum $n-1$ by the n^{-s}, or in vector form by $\big(\Re(n^{-s}), \Im(n^{-s})\big)$. This vector forms the n^{th} branch in the "connect the dots" pattern of the partial sums.

Analytic Continuation

A final preliminary issue with repect to complex numbers to resolve is the fact that $\zeta(s)$ is well defined for s for which the Dirichlet series $\sum_{k=1}^{\infty} k^{-s}$ does not converge. Many analytic complex functions like $\zeta(s)$ have an analytic continuation that extends a primary definition to additional areas of the complex plane. In the case of $\zeta(s)$, its Dirichlet series is defined for $\Re(s) > 1$, but additionally, $\zeta(s)$ has a unique value for all values of s by continuation. For more infomation see [1] or [9].

[3] The 0^{th} partial sum is 0.

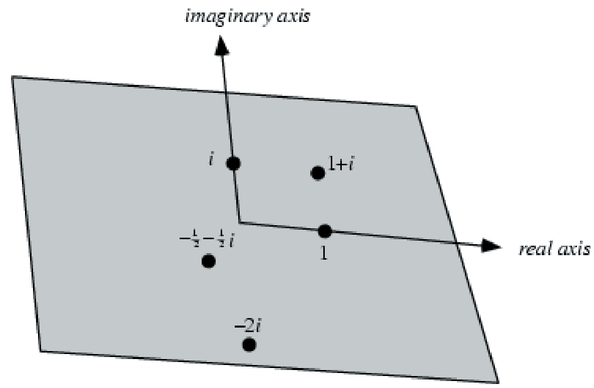

Figure 1: The complex plane. From [12].

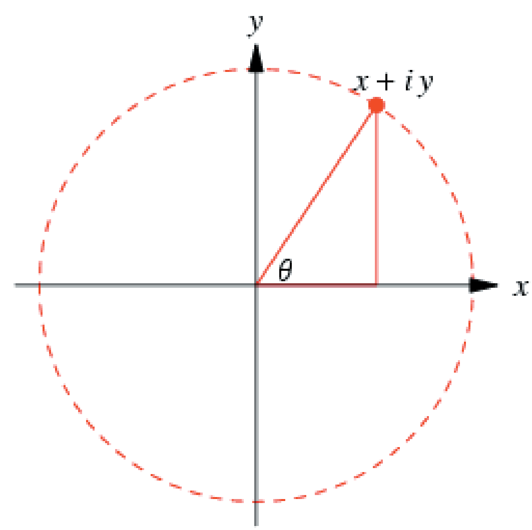

Figure 2: A complex number x+iy expressed in polar form, $re^{i\theta}$ where r is the circle's radius. From [11].

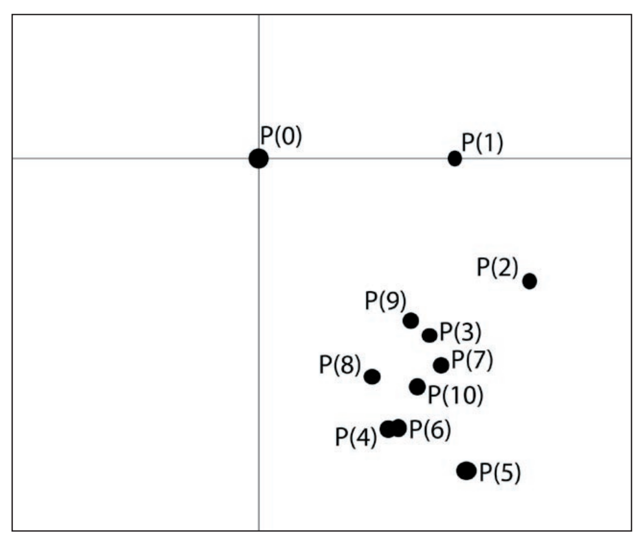

Figure 3: The first ten partial sums, Ps(n), with s = 1/2 + 500i.

Riemann's Work

In 1859, Riemann proved a connection between the non-trivial[4] zeros of $\zeta(s) = \sum_{k\geq 1} k^{-s}$ and the density of the prime numbers among the natural numbers. In 1896, De la Vallée Poussin and Hadamard proved independently that the non-trivial zeros of $\zeta(s)$ lie in the critical strip (see figure 4), $0 < \sigma < 1$ (recalling that σ represents the real part of s), a fact from which the prime number theorem follows [8]. The prime number theorem is a rule for the density of the primes: it states that

$$\pi(x) \approx \int_2^x \frac{1}{\ln x} dx \tag{1.3}$$

where $\pi(x)$ is the number of prime numbers less than than x [5].

Additionally, in Riemann's 1859 paper he stated a functional equation that implies symmetry across the line $\sigma = 1/2$: $\zeta(1-s) = \gamma(s)\zeta(s)$ where the gamma factor will be described later. Also, he formulated the famous Riemann Hypothesis, which conjectures that all non-trivial zeros lie on the critical line, $\sigma = 1/2$ [18]. The hypothesis, which states that the zeros of the zeta function lies on the critical line, implies that to the prime number theorem's estimate of the distribution of the prime numbers being as accurate as possible [8], [5]. Figure 5 shows the critical line in the complex plane along with trivial and non-trivial zeros.

The Behavior of the Zeta Function in the Critial Strip

To this day, the Riemann Hypothesis remains neither disproven nor proven and the behavior of $\zeta(s)$ in the critical strip remains mysterious; ten trillion zeros have been calculated, and all lie on the critical line [20]. See [3] for a detailed description of this mystery. The work I describe in this article may be the start of a new perspective from which to demystify the zeta function's behavior. Certainly, I do not claim any sort of significant progress toward the Riemann Hypothesis or important questions in analytic number theory; however, these methods may inform the analysis of the zeta function. Any conclusions that can be drawn about $\zeta(s)$ in the critical strip are significant, and these methods are first steps toward such conclusions.

2. Observing Geometric Patterns in the Partial Sums

The Partial Sums and $\zeta(s)$

Figure 6 shows the the first 13000 partial sums of the Riemann Zeta Function $\zeta(s)$ with $s \approx 1/2 + 33704.56i$. The swirling shapes of the partial sums stand out in the figure. In fact, that shape is called a Cornu spiral (see figure 7).

While these spiraling shapes are significant, the first interesting pattern is not inherent in the picture itself: If a dot were put down for each partial sum after the 13000$^{\text{th}}$ one, the dots would not end at the final spiral, labeled C_0. They only get close to that point. But continuing onward, they fill up the picture spiraling outward in a circle. This final outward spiraling is a visual representation of the series' divergence: there is no point at which the series will end. This divergence follows from the fact that $1/2 + 33704.56i$ is in the critical strip, and there the Dirichlet series of $\zeta(s)$ does not converge.

The first key observation that began this research is that this final spiral appears to diverge in a regular, outward spiraling pattern around $\zeta(s)$ for all values of s in the critical strip. For example, C_0 is around 0 in figure 6, and $\zeta(1/2 + 33704.56) = 0$. Note that because the Dirichlet series does not converge for this point, $\zeta(s)$ is derived by analytic continuation and is not given by its series representation. Suspecting that this circling was no coincidence, I investigated a method for finding the complex number at C_0 with the hypothesis that any formula for it would be the value of the zeta function.

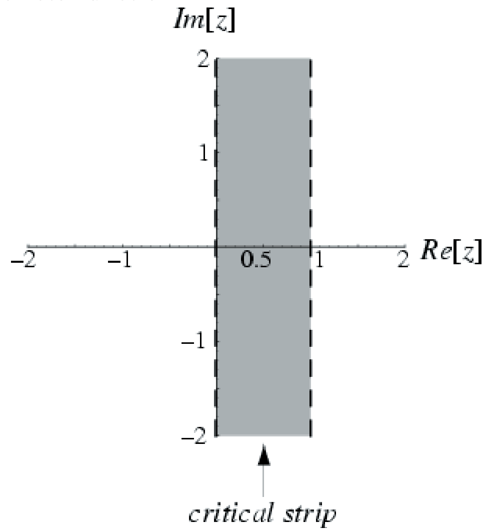

Figure 4: The critical strip: all of those complex number with real part greater than 0 and less than 1. From [15].

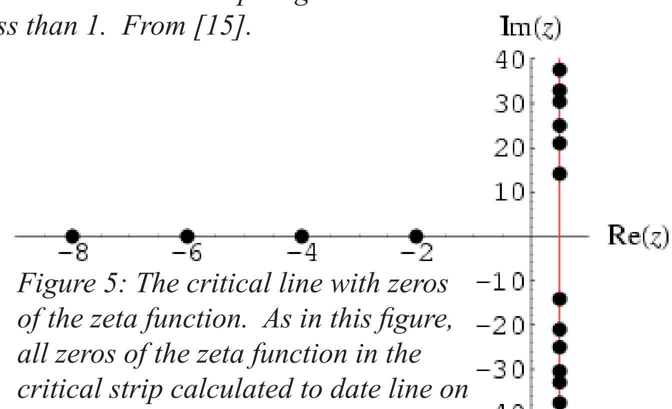

Figure 5: The critical line with zeros of the zeta function. As in this figure, all zeros of the zeta function in the critical strip calculated to date line on the critical line. From [14].

[4]The Riemann Zeta Function has trivial zeros at all negative even integers [7].

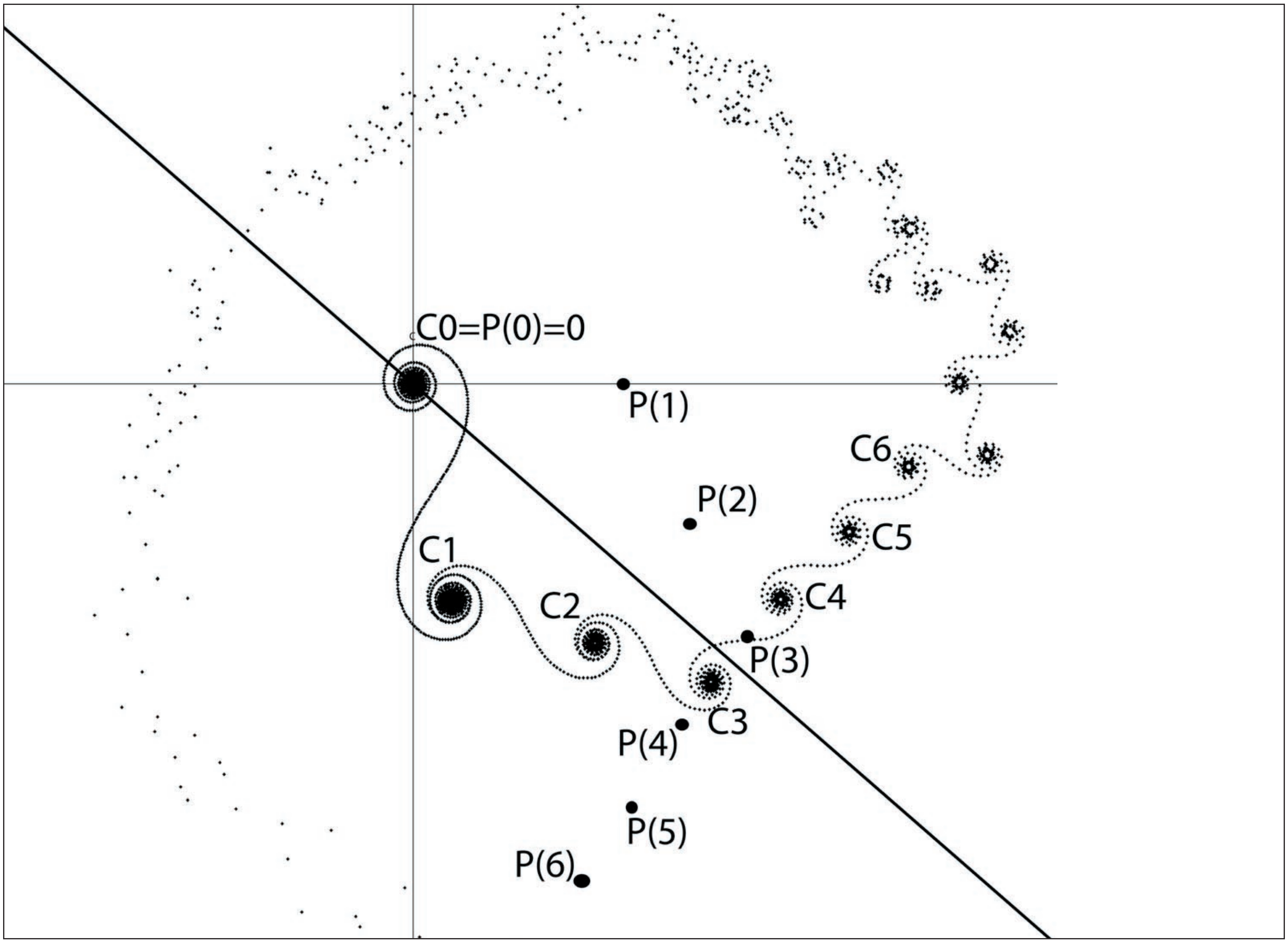

Figure 6: The first 13000 partial sums with s = 1/2 + 33704.56. This is a zero of the zeta function, evidenced by C_0, which is the same as ζ(s), being at the origin. Note the mirror symmetry between partial sums and the centers of the spirals across the line of symmetry.

Geometrically Finding the Center of the Spiral

The method to find this center consists of applying properties of smooth functions to three consecutive partial sums with the goal of finding a function to "correct" the outward spiraling back to its center. Properties of smooth functions can be applied to three consecutive partial sums because, in the critical strip, the distance between consecutive partial sums vanishes as the partial sum index increases. Visually, the partial sums become increasingly close together as their index increases, and begin to look like a smooth curve. Mathematically,

$$\lim_{n\to\infty} |n^{-s}| = 0.$$

Then, differential properties of curves can be expressed in terms of the partial sum index after being derived from the geometry of the partial sums. The experienced mathematical reader will notice that I discuss approximations without giving bounds on error. Unfortunately, this critical issue is beyond the scope of this article, but error bounds in central results will be stated in "big-O" form: $\mathcal{O}(f(x))$ denotes an error term bounded by a constant times $f(x)$ as $x \to \infty$ [10].

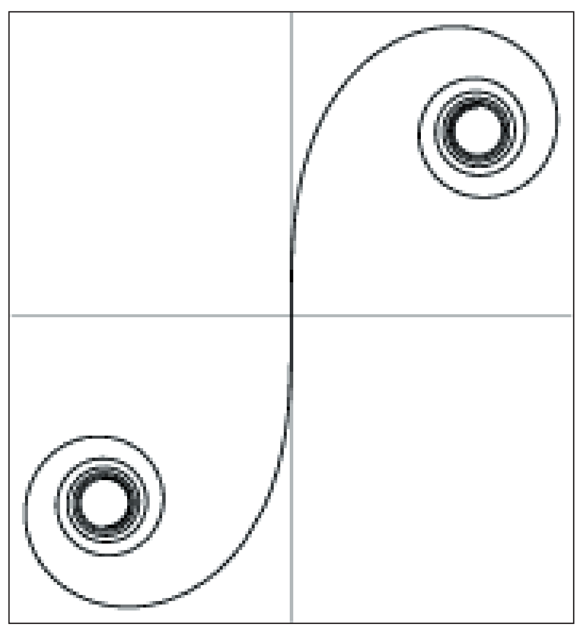

Figure 7: A cornu spiral. In a cornu spiral, the curvature is proportional to the distance along the curve from the origin. From [13].

Observe figure 8, showing nine partial sums from index $n+1$ to $n+4$ and their spiraling pattern. Because the angle change between the vector from $P_s(n-1)$ to $P_s(n)$ and the vector from $P_s(n)$ to $P_s(n+1)$ is approximately t/n and the length of each vector is $n^{-\sigma}$, the radius r_n of the uniquely defined circle going through partial sums $P_s(n-1)$, $P_s(n)$, and $P_s(n+1)$ (displayed in figure 8) is approximately

$$r_n = \frac{n^{1-\sigma}}{2t}. \tag{2.1}$$

Recall that $s = \sigma + it$. Also, the length that the smooth curve covers as the index n increases by 1 is $n^{-\sigma}$ and the total length δ is approximately

$$\delta = \frac{n^{1-\sigma}}{1-\sigma}.$$

Combining these two facts, the following expression for curvature[5], κ, in terms of arc length δ follows:

$$\kappa = \frac{1}{r_n} = \frac{t}{\delta(1-\sigma)}. \tag{2.2}$$

An equation expressing the curvature of a smooth curve in terms of arc length is called an "intrinsic equation" and determines the curve up to the factor of scale [2], [16]. This intrinsic equation is that of a logarithmic spiral (see figure 9). A logarithmic spiral is a polar function of the form $r(\theta) = ae^{b\theta}$. Solving this equation given the intrinsic equation, the b-constant is

$$b = \frac{1-\sigma}{t}. \tag{2.3}$$

Then it remains to state the angle θ in terms of the partial sum index n and find the a-constant. A convenient fact is that the logarithmic spiral is also equiangular; that is, it intersects a given axis at the same angle each time it passes around. Therefore the tangent angle of the curve and the central angle θ differ by a constant, which is

$$\arctan\left(\frac{t}{1-\sigma}\right). \tag{2.4}$$

The tangent angle in terms of the partial sum index n is

$$-t \ln n. \tag{2.5}$$

With θ and b set, the scaling constant a is

$$a = \frac{1}{\sqrt{(1-\sigma)^2 + t^2}}. \tag{2.6}$$

Combining all of these results, the function that shares its differential behavior with the partial sums is

$$\frac{n^{1-\sigma}}{\sqrt{(1-\sigma)^2+t^2}} e^{-it\ln n + i\arctan\frac{t}{1-\sigma}}. \tag{2.7}$$

Amazingly, recalling that $s = \sigma + it$, this function of n simplifies via complex arithmetic to

$$\frac{n^{1-s}}{1-s}. \tag{2.8}$$

To check the equivalence between 2.7 and 2.8, apply the complex arithmetic rules from the introduction section.

Thus the center of this final spiral in the progression of the partial sums, which I label C_0, is given by

$$C_0 := \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^{-s} - \frac{n^{-s}}{1-s}\right). \tag{2.9}$$

It is necessary to take n to infinity because of the issues with error mentioned earlier. This error vanishes for large n.

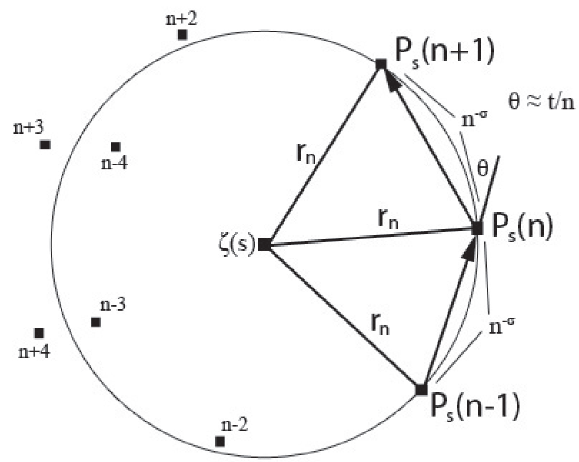

Figure 8: A demonstration of the geometric process for determining C_0. Given the lengths and angles, the curvature of the spiral and its center can be calculated. Keep in mind that this portion of the spiraling occurs as the partial sums spiral regularly away from C_0 after the point that they are displayed in Figure 6.

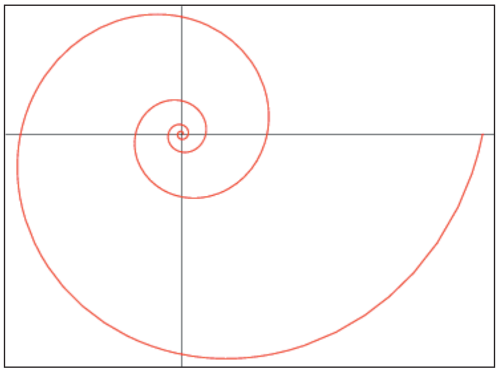

Figure 9: The logarithmic spiral. This is the family of curve that the partial sums approximate as they diverge in this spiraling pattern with C_0 in the center. From [17].

[5]The curvature of a smooth curve is the reciprocal of the radius of a tangent circle. Thus a curve with tangent circle of radius 2 has curvature of 1/2 and a straight line has curvature 0.

The Significance of this Result

Via geometric operations, we have found a center to the spiraling of the partial sums. By itself, the center has no meaning with respect to the zeta function. However, in 1921 the mathematicians Hardy and Littlewood proved by completely different means that this same formula is the value of $\zeta(s)$ [8], [4]. We may then conclude that

$$\boxed{\zeta(s) = C_0} \tag{2.10}$$

This conclusion is the first key component of my research; however, the logical implictions are somewhat subtle. Note that nothing new has been proven about the actual value of the zeta function per se. Rather, we now know that the geometry of the partial sums is connected to the value $\zeta(s)$ in the pattern of divergence of the partial sums. Or in other words, the geometry of the partial sums has a connection to the value of $\zeta(s)$, giving an even finer characterization of $\zeta(s)$ than the Hardy-Littlewood identity alone implies. The patterns in the partial sums have been investigated by Hugh Montgomery and others (see [6]), but a direct geometric derivation of the connection between geometric properties of the partial sums (most importantly C_0) and $\zeta(s)$ is evidently new to this work. This new technique for approaching the zeta function may be useful, for any conclusions that can be drawn regarding the geometry of the partial sums are important; they may be used as a foundation for conclusions about C_0, and thereby $\zeta(s)$, even when s is in the critical strip.

3. Symmetry in the Partial Sums

The Approximate Functional Equation

We are now ready to discuss the main result of this article: the correspondence between geometric patterns and the approximate function equation for the Riemann Zeta Function. To do this, we will examine symmetries among partial sums for $\Re(s) = \frac{1}{2}$ that allow us to draw conclusions about C_0 that parallel the approximate functional equation. Then because we know from the previous section that $C_0 = \zeta(s)$, any fact about C_0 applies to $\zeta(s)$.

More precisely, the approximate functional equation is stated as follows.

Theorem 3.1 (Approximate Functional Equation). *Given $s \in \mathbb{C}$ in the critical strip $(0 < \Re(s) < 1)$ and real parameters $X, Y \geq 1$ such that $2\pi XY = \Im(s)$, then*

(3.1)
$$\zeta(s) = \sum_{k \leq X} k^{-s} + \gamma(1-s) \sum_{k \leq Y} k^{s-1} + \mathcal{O}(X^{1/2-\Re(s)}(X^{-1/2} + Y^{-1/2}) \log XY).$$

Here $\gamma(s)$ denotes the arithmetic factor in the functional equation for the Riemann zeta function. It can be written as

$$\gamma(s) = \pi^{1/2-s} \frac{\Gamma(s/2)}{\Gamma((1-s)/2)}.$$

This is an analog to the previously mentioned functional equation of Riemann, replacing the infinite series of the zeta function with finite sums and accounting for the error. Without a context, the consequences of this theorem are rather opaque; the geometric picture illuminates the statement and the structure that it describes. Consider figure 6, a example of the structure of the approximate functional equation. The figure shows the first partial sums of the Dirichlet series of the zeta function for $s \approx 1/2 + 33704.56i$. The part of the approximate function equation equal to these partial sums is the leftmost term, $\sum_{k \leq X} k^{-s}$. Each dot is a representation of this sum for successive X's - for example, $P(2)$ is this sum with $X = 2$. Now we will explain the rest of the approximate functional equation geometrically, referring to figure 6.

Understanding the Approximate Functional Equation Geometrically

Let us consider the simplest case for the value of Y: X and Y must be at least 1 and the second summand is taken over all integers less than or equal to Y, so let us consider $1 \leq Y < 2$. Then the sum $\sum_{k \leq Y} k^{-s} = 1^{-s} = 1$ and the value of the second term in the approximate functional equation is simply $\gamma(1-s)$.

To understand what values X takes on as Y varies from 1 to 2, the relation $2\pi XY = t$ is key; as Y varies from 1 to 2, X varies from $\frac{t}{2\pi}$ to $\frac{t}{4\pi}$ respectively. The approximate functional equation implies that all of these partial sums should approximate $\zeta(s) - \gamma(1-s)$, for the sum up to X is the only part of the equation that varies.

How can all of these partial sums of index in this range of X give approximations to $\zeta(s)$? The answer to this predicament lies in the Cornu spiral shape of the partial sums mentioned previously. We will investigate how the partial sum index X corresponds to the partial sum's position on the progression of spirals. Since we know that for very large X the partial sums spiral out centered at $\zeta(s)$, we can start by examining the index of the partial sums as they get closest to $\zeta(s)$. At this point, consecutive partial sums remain close to $\zeta(s)$ because consecutive differences between partial sums cancel each other out. This happens when the vector connecting $P_s(X-1)$ to $P_s(X)$ and the vector connecting $P_s(X)$ to $P_s(X+1)$ is near π; for then each difference between partial sums doubles back to where the last one started. Stated in terms of "connect the dots," if one drew an arrow from dot $X-1$ to dot X, and then from dot X to dot $X+1$ as in figure 8, the arrows would point opposite directions for partial sums near one of the C_n in figure 6; we call these places C_n "centers."

We have seen that centers occur when consecutive partial sums double back on themselves. Now let's characterize this mathematically: th change in angle between these consecutive vectors in terms of the index X is

$$\frac{t}{X}, \tag{3.2}$$

which is an approximation for the difference of two consecutive terms of equation 2.5. Now we can turn this equation on its head to find at what partial sums these centers occur. Solving for X,

$$\frac{t}{X} \equiv \pi \mod 2\pi,$$

or, equivalently,

$$X = \frac{t}{\pi(2j-1)}$$

for some positive integer j. The mod 2π equivalence reflects the fact that the quantities are angles and all functions of angles are periodic with period 2π. We have now determined a family of X's for which $P_X(s)$ is near a center.

As labeled in figure 6, C_0 is the final center in the ordered progression of partial sums; therefore the index X corresponding to C_0 is thus the greatest value for which $t/X \equiv \pi \mod 2\pi$. This value is $X = \frac{t}{\pi}$. The index $X = \frac{t}{3\pi}$ is the second largest X for which the change in angle will be π, so the partial sums around this index must be at the second to last swirl - this swirl is labelled C_1 in figure 6. Each of the consecutive swirls obey this property as well, and each is labelled C_n leading up to C_0. Thus, in general, the index X of the partials sums around C_n is

$$X = \frac{t}{\pi(2n+1)}. \tag{3.3}$$

Likewise, solving for the change in angle between consecutive partial sums congruent to $0 \mod 2\pi$, we may determine the indices X for which partial sums are halfway between swirls. By applying equation 3.2 to 0 instead of π, we find that the index of the partial sums halfway between C_n and C_{n-1} is

$$X = \frac{t}{2n\pi}. \tag{3.4}$$

Parallels between Geometry and the Approximate Functional Equation

Now that we've found at what indices X the partial sums approximate the centers C_0, C_1, etc., let's use this knowledge to explain the relationship between X and Y in the approximate functional equation. Again, we will refer to figure 6.

As shown in the previous section, as Y varies from 1 to 2, X varies from $\frac{t}{4\pi}$ to $\frac{t}{2\pi}$. We now know that the partial sum with index $X = \frac{t}{3\pi}$ is located at the center C_1. According to the identity $2\pi XY = t$, Y must be $3/2$ for $X = \frac{t}{3\pi}$. Therefore, the approximate functional equation claims, with appropriate error bounds, that

$$\zeta(s) \approx P_s(\frac{t}{3\pi}) + \gamma(1-s). \tag{3.5}$$

This statement can be rephrased in terms of geometrically meaningful quantities:

$$C_0 \approx C_1 + \gamma(1-s). \tag{3.6}$$

That is, the difference between the locations C_0 and C_1 is $\gamma(1-s)$!

To state other C_n in terms of $\gamma(1-s)$, let's consider other choices of Y and X in the approximate functional equation such that $Y < X$. When $2 \leq Y < 3$, then X ranges from $\frac{t}{4\pi}$ to $\frac{t}{6\pi}$ respectively, a of partial sums range on either side of the swirl C_2 around index $X = \frac{t}{5\pi}$. The resulting equation stated in terms of geometrical quantities is

$$\zeta(s) = C_0 \approx C_2 + \gamma(1-s)\big(1^{s-1} + 2^{s-1}\big). \tag{3.7}$$

Extending this analysis in general for $Y = \frac{2m+1}{2}$ with m a positive integer, then $X = \frac{t}{m\pi}$ is the index of partial sums around C_m, and according to the approximate functional equation

$$C_0 \approx C_m + \gamma(1-s)\sum_{k=1}^{\frac{2m+1}{2}} k^{s-1} = C_m + \gamma(1-s)P_{1-s}(m). \tag{3.8}$$

Note that the sum with Y has argument $1-s$ instead of s as usual, and that summing up to $\frac{2m+1}{2}$ is the same as summing up to m.

This conclusion gives an approximate formula for the C_n:

$$C_n = \zeta(s) - \gamma(1-s)P_{1-s}(n). \tag{3.9}$$

Now not only do we have simple approximation for C_n, but this equation suggests a correspondence among the partial sums of the Riemann Zeta Function. However, this correspondence is not direct; the factors that remain to be resolved are:

- The γ-factor.
- The relationship between $P_s(n)$ and $P_{1-s}(n)$.

Symmetry among the Partial Sums with s on the Critical Line

Both of these factors behave simply and imply a symmetry among the partial sums exactly when s is on the critical line; that is, the real part of s is one half: $\sigma = \frac{1}{2}$. For the remainder of this section, assume that $\Re(s) = \frac{1}{2}$.

The $\gamma(1-s)$ variable comes from the functional equation for $\zeta(s)$, which closely relates the value of $\zeta(s)$ and $\zeta(1-s)$. An expression for $\gamma(1-s)$ was given in the statement of the approximate functional equation. The absolute value of $\gamma(1-s)$ is one if and only if s is on the critical line:

(3.10)
$$|\gamma(1-s)| = 1 \Longleftrightarrow 1-\sigma = \frac{1}{2} \Longleftrightarrow \sigma = \frac{1}{2} \text{ (see [8]).}$$

Likewise, there is no simple relationship between $P_s(n)$ and $P_{1-s}(n)$ except when $\sigma = \frac{1}{2}$. When $\sigma = \frac{1}{2}$, then s and $1-s$ are conjugate, for recalling the definition of conjugation we get that

$$\overline{s} = \overline{\sigma + it} = \frac{1}{2} - it = 1 - \big(\frac{1}{2} + it\big) = 1 - s. \tag{3.11}$$

Any complex analytic function f satisfies $f(\overline{s}) = \overline{f(s)}$. Since a given partial sum is an analytic function of s, this powerful property ensures that

$$P_{1-s}(n) = \overline{P_s(n)} \text{ when } \sigma = \frac{1}{2}. \tag{3.12}$$

These two facts together, $|\gamma(1-s)| = 1$ and $P_{1-s}(n) = \overline{P_s(n)}$, interpreted geometrically, imply a symmetry among the partial sums. First, note that because $|\gamma(1-s)| = 1$, $\gamma(1-s) = e^{i\theta}$ for some θ, using the polar form of complex numbers. Recall that multiplication by $e^{i\theta}$ is equivalent to rotation by θ and that conjugation is a flip over the real axis as we saw in the geometric interpretation of complex numbers.

Applying the new identity $\overline{s} = s-1$ and keeping in mind the geometric interpretation of conjugation and multiplication by $\gamma(1-s)$, then $P_{1-s}(n) = P_{\overline{s}}(n) = \overline{P_s(n)}$; with equation 3.9,

(3.13)
$$\boxed{C_n = \zeta(s) + \gamma(1-s)P_{1-s}(n) = \zeta(s) + e^{i\theta}\overline{P_s(n)}.}$$

The geomtric interpretation of this equation yields an algorithm to calculate C_n from $P_s(n)$:

- $\overline{P_s(n)}$: Flip the point $P_s(n)$ over the real axis
- $e^{i\theta}\overline{P_s(n)}$: Rotate the result by θ
- $\zeta(s) + e^{i\theta}\overline{P_s(n)}$: Translate the result by $\zeta(s)$

One may check that a composition of a reflection with translations and rotations is simply a reflection over a different line. Thus the approximate functional equation predicts a mirror symmetry over a certain line which is given by the factors $\gamma(s)$ and $\zeta(s)$. This symmetry is denoted by the line in figure 6.

The Implications of this Symmetry

The reader should keep in mind that this symmetry only holds for $\Re(s) = \frac{1}{2}$. Certainly, the conjugation functions and multiplication by $\gamma(s)$ have meaning when s is off the critical line, but recall that $|\gamma(s)| \neq 1$ for $\Re(s) \neq \frac{1}{2}$, so the geometric correspondence between C_n and $P_s(n)$ involves a scaling factor, and hence destroys the symmetry. Perhaps the most important symmetry to keep in mind is that between C_0 and $P_s(0)$[6].

Now reach back to the beginning and recall the motivation for this entire discussion. Any sort of conclusion about the behavior of the Riemann Zeta Function in the critical strip is valuable, especially one that has possible relations to the Riemann Hypothesis, which conjectures that

$$\zeta(s) = 0 \Longrightarrow \Re(s) = \frac{1}{2}. \tag{3.14}$$

Ideally, a symmetry result on C_0, which is identical to $\zeta(s)$, would show that the zeta function can be zero only when there is the prescribed symmetry among the partial sums. Since $P_s(0) = 0$ for any s, and there is only symmetry when $\sigma = \frac{1}{2}$, it may seem intuitively true that when $\zeta(s) = 0$, then there must be symmetry since $C_0 = P_s(0)$ and one point has mirror symmetry with itself.

Clearly this is not possible because huge approximations were taken along the way, and to boot, we defined the line of symmetry in terms of $\zeta(s)$, the quantity in question. There is a long and perhaps impassible path from any of these conclusions to any important conclusions. My work has a great deal to do with seeking well-defined notions of the C_n other than C_0.

However, this geometric approach is not without substance. The approximate functional equation only predicts an approximate symmetry between approximate locations of partial sums. It predicts nothing about the Cornu spiral shape! Of course, the proof of the approximate functional equation depends on this underlying truth that we can observe visually, but it does not capture the whole reality. This shortcoming suggests that the geometric interpretation of the zeta function's partial sums may have a surprisingly wide vocabulary to address the value of $\zeta(s)$ in the critical strip.

4. Applications to the Behavior of $\zeta(s)$ in the Critical Strip

Chronologically, my research progressed in the opposite direction of this exposition. After completing the work on finding that $\zeta(s) = C_0$, I visually observed the symmetry among the partial sums described in the previous section. It then took a good deal of investigation to find that this symmetry was predicted by existing theory, namely the approximate functional equation.

However, this fact does not mean that all geometric efforts are doomed to be fruitless because of their duplication of analytic results like the approximate functional equation. Though it goes beyond the scope of this journal, my work

[6] The 0^{th} partial sum is defined to be 0, the sum of the first 0 terms

now consists in reproving the approximate functional equation using geometric techniques. This is surprisingly difficult given how "obvious" the approximate functional equation seems given the geometry. In other words, we can see visually that the approximate functional equation holds, so the geometric perspective on the partial sums and $\zeta(s)$ seems stronger than that of analysis. With this highly descriptive perspective, where we can "*see* the Riemann Hypothesis hold true," it would seem natural for a result like the less descriptive approximate functional equation would follow easily from the geometric perspective.

Though there is certainly no ease, there is some progress. Starting with the connection between the analytically meaningful $\zeta(s)$ and the geometrically meaningful C_0, my current work seeks to prove the approximate functional equation by proving equation 3.13, which is an expression for C_n, by induction, using a base case of $C_0 = \zeta(s)$ and then attempting to show by geometrically analyzing the partial sums between C_n and C_{n+1} that the difference between the two is appropriate for equation 3.13. As of press time I have derived sufficient conditions in error bounds for results and am now working on those areas.

Ultimately, complete proofs having to do with the Riemann Zeta Function must use analysis, for geometry quickly falls into approximation. The usefulness of the geometric perspective on this problem lies in its applications to analysis. Research into smooth representations of the progression of the partial sums as in figure 6 suggest via the geometric perspective that variations on the methods of Fourier analysis, which unfortunately exceed the bounds of this article, may be sufficient to define the C_n clearly and investigate an exact symmetry. This method would provide a powerful tool for analyzing the Riemann Zeta Function in the critical strip.

5. Acknowledgements

I would like to thank Professor Ben Brubaker for his invaluable mentorship in this research and Undergraduate Research Programs for a Small Grant for the purchase of texts related to this work.

References

[1] Ahlfors, Lars V. Complex Analysis, 3rd ed. New York: McGraw-Hill, 1979.
[2] Do Carmo, Manfredo P. Differential Geometry of Curves and Surfaces. Upper Saddle River, New Jersey: Prentice-Hall, 1976.
[3] Conrey, J. Brian. The Riemann Hypothesis. Notices of the American Mathematical Society 2003; 50, 341-353, http://www.ams.org/notices/200303/fea-conrey-web.pdf
[4] Hardy, G. H. and J. E. Littlewood. The Zeros of Riemann's Zeta Function on the Critical Line. Mathematische Zeitschrift 1921.
[5] Ingham, A. E. The Distribution of Prime Numbers. New York: Cambridge University Press, 1932.
[6] Montgomery, H. L. Ten lectures on the interface between analytic number theory and harmonic analysis. CBMS Regional Conference Series in Mathematics, Number 84. Providence, Rhode Island: American Mathematical Society, 1994.
[7] Patterson, S. J. An Introduction to the Theory of the Riemann Zeta-function. New York: Cambridge University Press, 1988.
[8] Titchmarsh, E. C. The Theory of the Riemann Zeta Function, 2nd ed. revised by D. R. Heath Brown. Oxford: Oxford Science Publications, 1986.
[9] Weisstein, Eric W. "Analytic Continuation." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/AnalyticContinuation.html.
[10] Weisstein, Eric W. "Asymptotic Notation." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/AsymptoticNotation.html.
[11] Weisstein, Eric W. "Complex Number." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/ComplexNumber.html.
[12] Weisstein, Eric W. "Complex Plane." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/ComplexPlane.html.
[13] Weisstein, Eric W. "Cornu Spiral." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CornuSpiral.html.
[14] Weisstein, Eric W. "Critical Line." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CriticalLine.html.
[15] Weisstein, Eric W. "Critical Strip." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CriticalStrip.html.
[16] Weisstein, Eric W. "Intrinsic Equation." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/IntrinsicEquation.html.
[17] Weisstein, Eric W. "Logarithmic Spiral." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LogarithmicSpiral.html.
[18] Weisstein, Eric W. "Riemann Hypothesis." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/RiemannHypothesis.html.
[19] Weisstein, Eric W. "The Riemann Zeta Function." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/RiemannZetaFunction.html.
[20] Weisstein, Eric W. "Riemann Zeta Function Zeros." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/RiemannZetaFunctionZeros.html.

Carl Erickson

Carl Erickson is a sophomore from Milwaukee, Wisconsin. He is majoring in Math with his principal interest in number theory, and also dabbles in Computer Science and Religious Studies. He has interned with Lucent Technologies and will soon be researching algebraic number theory at Williams College and studying math abroad in Hungary. At Stanford, he works as a tutor and grader and has fun through racquet sports and musical pursuits. Carl would like to thank his advisor, Ben Brubaker, for his enthusiasm and support, and Professor Brent Sockness for his guidance in writing.