



## Lesson 4: Addition and Subtraction Formulas

### Student Outcomes

- Students use the addition formulas to derive the double-angle formulas for sine, cosine, and tangent and evaluate trigonometric expressions for specific values of  $\theta$ .
- Students use the double-angle formulas to derive the half-angle formulas for sine, cosine, and tangent and evaluate trigonometric expressions for specific values of  $\theta$ .

### Lesson Notes

In the previous lesson, students applied an analytic method to prove the subtraction formula for cosine. They then derived the remaining addition and subtraction formulas using their understanding of the periodicity and symmetry of the sine, cosine, and tangent functions. In this lesson, students continue to apply their understanding of the properties of the trigonometric functions to determine double and half-angle formulas for sine, cosine, and tangent, which they apply to evaluate trigonometric functions for specific input values.

### Classwork

#### Opening (5 minutes)

- Let's return one more time to our carousel model of the unit circle. We know that a rider's position after a rotation  $\theta$  is  $(x_\theta, y_\theta)$ , where  $x_\theta = \cos(\theta)$  and  $y_\theta = \sin(\theta)$ . In the last lesson, we developed the sum and difference formulas for sine and cosine; that is, if we know  $\sin(\alpha)$ ,  $\sin(\beta)$ ,  $\cos(\alpha)$ , and  $\cos(\beta)$ , then we can find the values of  $\sin(\alpha + \beta)$ ,  $\sin(\alpha - \beta)$ ,  $\cos(\alpha + \beta)$ , and  $\cos(\alpha - \beta)$ .

Students should reflect on the following prompts. After a minute, they should share their reflections with a partner. Select several students to share their ideas; ask them to explain a strategy or display a procedure on the board.

- We know the position of the rider after rotating by  $\theta = \frac{\pi}{3}$ . How can we use the addition formulas to find her position after rotating by  $\frac{2\pi}{3}$ ?
  - Answers will vary but might include using the formulas we have derived in previous lessons; for examples, we could rewrite  $\sin(2\theta)$  as  $\sin(\theta + \theta)$  and apply the formula for the sine of a sum to find the position of a rider after a rotation of  $2\theta$ .
- We know the position of the rider after rotating by  $\theta = \pi/4$ . How can we determine his position after rotating by  $\frac{\pi}{8}$ ?
  - Answers will vary but might include applying the angle sum formulas in reverse after writing  $\sin(2\theta)$  as  $\sin(\theta + \theta)$ , using this particular angle measure,  $\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{8} + \frac{\pi}{8}\right)$ .

MP.3  
&  
MP.7

**Exercise 1 (7 minutes)**

In this exercise, students derive the double-angle formula for sine using the addition formula. Allow students to struggle with this task before explaining that they apply the sum formulas with  $2\theta = \theta + \theta$ . This provides an opportunity for them to further develop their mathematical reasoning skills and to practice looking for structure. Students later apply the double-angle formulas to determine the half-angle formulas, which allows them to evaluate trigonometric functions for a larger number of input values.

*Scaffolding:*

Encourage advanced students to think about ways to write these formulas in terms of sine or cosine only.

The exercise should be completed in pairs and then discussed in a whole-class setting after a few minutes. Be sure to identify these formulas as the double-angle formulas for sine and cosine.

**Exercises**

1. Derive formulas for the following:

a.  $\sin(2\theta)$

$$\begin{aligned} \sin(2\theta) &= \sin(\theta + \theta) \\ &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) \\ &= 2\sin(\theta)\cos(\theta) \end{aligned}$$

b.  $\cos(2\theta)$

$$\begin{aligned} \cos(2\theta) &= \cos(\theta + \theta) \\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta) \end{aligned}$$

MP.7

- Why was it helpful to rewrite  $2\theta$  as  $(\theta + \theta)$  when deriving the double-angle formula?
  - While we don't know how to find the value of  $\sin(2\theta)$  directly, we do know how to evaluate the sine of the sum  $(\theta + \theta)$  using the formula  $\sin(\theta + \theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) = 2\sin(\theta)\cos(\theta)$ .
- Our formula for calculating  $\cos(2\theta)$  is written in terms of  $\cos^2(\theta)$  and  $\sin^2(\theta)$ . Is there a way to write this formula in terms of the cosine function only? Explain.
  - Yes. Since  $\cos^2(\theta) + \sin^2(\theta) = 1$ , we can substitute  $\sin^2(\theta) = 1 - \cos^2(\theta)$ . Then  $\cos^2(\theta) - \sin^2(\theta) = \cos^2(\theta) - (1 - \cos^2(\theta)) = 2\cos^2(\theta) - 1$ .
- Can we write  $\cos(2\theta)$  in terms of the sine function only? Explain.
  - Yes. Since  $\cos^2(\theta) + \sin^2(\theta) = 1$ , we can substitute  $\cos^2(\theta) = 1 - \sin^2(\theta)$ . Then  $\cos^2(\theta) - \sin^2(\theta) = (1 - \sin^2(\theta)) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$ .

**Exercises 2–3 (10 minutes)**

Students should complete the exercises in pairs or small groups. Each pair or small group should verify one of the identities in Exercise 1. After a few minutes, volunteers could display their solutions. Other students should be allowed to offer alternative approaches or critiques. Exercise 2 could be completed in pairs or as part of a teacher-led discussion. The discussion of Exercise 2 could include a verification, using the unit circle, of the signs on the coordinates of the rider's position.

MP.3

2. Use the double-angle formulas for sine and cosine to verify these identities:

a.  $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

$$\begin{aligned} \tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)} \\ &= \left( \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)} \right) \left( \frac{\frac{1}{\cos^2(\theta)}}{\frac{1}{\cos^2(\theta)}} \right) \\ &= \frac{\frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)}}{\frac{\cos^2(\theta) - \sin^2(\theta)}{\cos^2(\theta)}} \\ &= \frac{\frac{2\sin(\theta)\cos(\theta)}{\cos(\theta)\cos(\theta)}}{\frac{\cos^2(\theta) - \sin^2(\theta)}{\cos^2(\theta)}} \\ &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)} \end{aligned}$$

b.  $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$

$$\frac{1 - \cos(2\theta)}{2} = \frac{1 - (1 - 2\sin^2(\theta))}{2} = \frac{2\sin^2(\theta)}{2} = \sin^2(\theta)$$

c.  $\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta)$

$$\begin{aligned} \sin(3\theta) &= \sin(2\theta + \theta) \\ &= \sin(2\theta)\cos(\theta) + \cos(2\theta)\sin(\theta) \\ &= 2\sin(\theta)(\cos(\theta))(\cos(\theta)) + (1 - 2\sin^2(\theta))(\sin(\theta)) \\ &= 2\sin(\theta)\cos^2(\theta) + \sin(\theta) - 2\sin^3(\theta) \\ &= 2\sin(\theta)(1 - \sin^2(\theta)) + \sin(\theta) - 2\sin^3(\theta) \\ &= 2\sin(\theta) - 2\sin^3(\theta) + \sin(\theta) - 2\sin^3(\theta) \\ &= -4\sin^3(\theta) + 3\sin(\theta) \end{aligned}$$

3. Suppose that the position of a rider on the unit circle carousel is  $(0.8, -0.6)$  for a rotation  $\theta$ . What is the position of the rider after rotation by  $2\theta$ ?

$$x_{2\theta} = \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 0.8^2 - (-0.6)^2 = 0.64 - 0.36 = 0.28$$

$$y_{2\theta} = \sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2(-0.6)(0.8) = -0.96$$

The rider's position is  $(0.28, -0.96)$ .

*Scaffolding:*

- Point students to a visual representation of the definition of the tangent as  $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ .
- Cue students to look at the structure of the right side of the identity in Exercise 2 part (a) to help them simplify the expression on the left side. For instance, ask the students, "What term could we divide our expression by so the first term in the denominator reduces to 1?"
- Cue students to use a procedure similar to that used in Exercise 1 to verify the identity in Exercise 2 part (c).

MP.2

**Discussion (5 minutes)**

In this example, students use the double-angle formula for cosine to derive the half-angle formula for cosine. The half-angle formula provides students with an efficient means of evaluating trigonometric functions for a wider variety of inputs. The example should be completed as part of a teacher-led discussion.

- What are some of the limitations of the formulas we have derived so far in evaluating trigonometric expressions for specific inputs?
  - *Since we only determined the exact values of trigonometric functions for multiples of  $\frac{\pi}{2}$ ,  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ , and  $\frac{\pi}{3}$ , we are limited in the values of  $\theta$  for which we can evaluate trigonometric expressions. In other words, we can only evaluate  $\sin(\theta)$ ,  $\cos(\theta)$ , and  $\tan(\theta)$  for values of  $\theta$  that can be found by adding or subtracting multiples of  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ ,  $\frac{\pi}{3}$ , and  $\frac{\pi}{2}$ .*

- In the Opening, we thought about how we could find  $\sin\left(\frac{\pi}{8}\right)$  and  $\cos\left(\frac{\pi}{8}\right)$  since we know the values of sine and cosine at  $\frac{\pi}{4}$ . Do we have a way to evaluate sine and cosine at  $\frac{\pi}{8}$ ?
  - *No, we do not yet have a way to evaluate sine and cosine at  $\frac{\pi}{8}$ .*

- Why might it be helpful to derive half-angle formulas for sine, cosine, and tangent?
  - *Answers will vary but should address that we could evaluate the trigonometric functions for more values of  $\theta$ .*
- Recall that the double-angle formula for cosine is  $\cos(2\theta) = 2\cos^2(\theta) - 1$ . How can we use this formula to find a value for  $\cos\left(\frac{\pi}{8}\right)$ ?
  - *Allow students to work with a partner and struggle with this question for a minute or two before revealing the answer.*

*We can use the double-angle formula with  $\cos\left(\frac{\pi}{4}\right) = \cos\left(2 \cdot \frac{\pi}{8}\right)$ , which gives*

$$\cos\left(\frac{\pi}{4}\right) = \cos\left(2 \cdot \frac{\pi}{8}\right) = 2\cos^2\left(\frac{\pi}{8}\right) - 1. \text{ Then we can solve } \frac{\sqrt{2}}{2} = 2\cos^2\left(\frac{\pi}{8}\right) - 1 \text{ for } \cos\left(\frac{\pi}{8}\right).$$

- This gives  $\cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + 1\right)$ . How do we know whether or not we need the positive or negative square root when we solve for  $\cos\left(\frac{\pi}{8}\right)$ ?
  - *It will depend on the quadrant in which the initial ray lands after rotating by  $\frac{\pi}{8}$ . Cosine is positive in Quadrants I and IV and negative in Quadrants II and III. Since rotation by  $\frac{\pi}{8}$  terminates in Quadrant I, the value of  $\cos\left(\frac{\pi}{8}\right)$  is positive.*

**Exercises 4–7 (10 minutes)**

Students should complete the exercises in pairs, first working on the problems independently and then verifying their solutions with a partner. At an appropriate time, selected students should explain their solutions, including how they determined the signs for the position coordinates in Exercise 6 and the signs for the evaluated trigonometric functions in Exercise 7. As time permits, additional students should be encouraged to share alternative approaches to solving the problems; for example, there could be multiple ways to evaluate  $\cos\left(\frac{\pi}{12}\right)$  in Exercise 7 part (b). Note: if calculators are available, students could use them to verify their solutions and approximate the values in Exercises 6 and 7. If they are not available, answers could be left in their exact form.

4. Use the double-angle formula for cosine to establish the identity  $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta)+1}{2}}$ .

Since  $\theta = 2\left(\frac{\theta}{2}\right)$ , the double-angle formula gives  $\cos\left(2\left(\frac{\theta}{2}\right)\right) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$ . Then we have

$$\begin{aligned}\cos(\theta) &= 2\cos^2\left(\frac{\theta}{2}\right) - 1 \\ 1 + \cos(\theta) &= 2\cos^2\left(\frac{\theta}{2}\right) \\ \frac{1 + \cos(\theta)}{2} &= \cos^2\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{\cos(\theta) + 1}{2}}\end{aligned}$$

5. Use the double-angle formulas to verify these identities:

a.  $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$

Since  $\theta = 2\left(\frac{\theta}{2}\right)$ , the double-angle formulas give  $\cos\left(2\left(\frac{\theta}{2}\right)\right) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$ .

Then we have

$$\begin{aligned}\cos(\theta) &= 1 - 2\sin^2\left(\frac{\theta}{2}\right) \\ 1 - \cos(\theta) &= 2\sin^2\left(\frac{\theta}{2}\right) \\ \frac{1 - \cos(\theta)}{2} &= \sin^2\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 - \cos(\theta)}{2}}\end{aligned}$$

b.  $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{\pm\sqrt{\frac{1-\cos(\theta)}{2}}}{\pm\sqrt{\frac{\cos(\theta)+1}{2}}} = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$$

*Scaffolding:*

- Have advanced students derive a formula for the one-third angle for sine instead of verifying the identities in Exercise 5.
- Cue students in Exercise 7 by asking them, “Is there a way to use the addition or subtraction formulas to evaluate  $\cos\left(\frac{\pi}{12}\right)$ ?”

MP.2

6. The position of a rider on the unit circle carousel is  $(0.8, -0.6)$  after a rotation by  $\theta$  where  $0 \leq \theta < 2\pi$ . What is the position of the rider after rotation by  $\frac{\theta}{2}$ ?

*Given that  $\cos(\theta)$  is positive and  $\sin(\theta)$  is negative, the rider is located in Quadrant IV after rotation by  $\theta$ , so  $\frac{3\pi}{2} < \theta < 2\pi$ . This means that  $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$ , which is in Quadrant II, so  $\cos\left(\frac{\theta}{2}\right)$  is negative and  $\sin\left(\frac{\theta}{2}\right)$  is positive.*

$$x_{\frac{\theta}{2}} = \cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{\cos(\theta) + 1}{2}} = -\sqrt{\frac{0.8 + 1}{2}} \approx -0.95$$

$$y_{\frac{\theta}{2}} = \sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{2}} = \sqrt{\frac{1 - 0.8}{2}} \approx 0.32$$

*The rider's position is approximately  $(-0.95, 0.32)$  after rotation by  $\frac{\theta}{2}$ .*

7. Evaluate the following trigonometric expressions.

a.  $\sin\left(\frac{3\pi}{8}\right)$

$$\sin\left(\frac{3\pi}{8}\right) = \sin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \sqrt{\frac{1 - \cos\left(\frac{3\pi}{4}\right)}{2}} = \sqrt{\frac{1 - \left(-\cos\left(\frac{\pi}{4}\right)\right)}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \approx 0.92$$

b.  $\tan\left(\frac{\pi}{24}\right)$

$$\tan\left(\frac{\pi}{24}\right) = \tan\left(\frac{\pi}{12} - \frac{\pi}{24}\right) = \sqrt{\frac{1 - \cos\left(\frac{\pi}{12}\right)}{1 + \cos\left(\frac{\pi}{12}\right)}}$$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\tan\left(\frac{\pi}{24}\right) = \sqrt{\frac{1 - \frac{\sqrt{6} + \sqrt{2}}{4}}{1 + \frac{\sqrt{6} + \sqrt{2}}{4}}} = \sqrt{\frac{4 - \sqrt{6} - \sqrt{2}}{4 + \sqrt{6} + \sqrt{2}}} \approx 0.13$$

### Closing (3 minutes)

Have students respond in writing to one or more of the prompts below:

- Write the double-angle formulas that we studied in this lesson for  $\sin(2\theta)$ ,  $\cos(2\theta)$ , and  $\tan(2\theta)$ .
- Write the half-angle formulas that we studied in this lesson for  $\sin\left(\frac{\theta}{2}\right)$ ,  $\cos\left(\frac{\theta}{2}\right)$ , and  $\tan\left(\frac{\theta}{2}\right)$ .
- How can our understanding of the trigonometric functions help us determine the position coordinates for a carousel rider on our unit circle model regardless of the size of the rotation  $\theta$ ? Share your thoughts with a partner.

Answers will vary. Possible acceptable responses could include the following:

- We have found the exact position coordinates for rotations  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ , and  $\frac{\pi}{3}$ .
- Our understanding of the unit circle allows us to determine the position coordinates for rotations that are multiples of  $\frac{\pi}{2}$ .
- The double- and triple-angle formulas enable us to find position coordinates for rotations that are multiples of  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ , and  $\frac{\pi}{3}$ .
- The half-angle and addition and subtraction formulas enable us to find position coordinates for many other rotational values.

#### Lesson Summary

The double-angle and half-angle formulas for sine, cosine, and tangent are summarized below.

For all real numbers  $\theta$  for which the expressions are defined,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta)\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{\cos(\theta) + 1}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$$

#### Exit Ticket (5 minutes)



Exit Ticket Sample Solutions

1. Show that  $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ .

$$\begin{aligned} \cos(3\theta) &= \cos(2\theta + \theta) = \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\ &= (2\cos^2(\theta) - 1)(\cos(\theta)) - 2\sin(\theta)\cos(\theta)(\sin(\theta)) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2(1 - \cos^2(\theta))(\cos(\theta)) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta) \\ &= 4\cos^3(\theta) - 3\cos(\theta) \end{aligned}$$

2. Evaluate  $\cos\left(\frac{7\pi}{12}\right)$  using the half-angle formula, and then verify your solution using a different formula.

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{7\pi}{6}\right) = -\sqrt{\frac{\cos\left(\frac{7\pi}{6}\right) + 1}{2}} = -\sqrt{\frac{-\frac{\sqrt{3}}{2} + 1}{2}} \approx -0.26$$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{2} - \sqrt{6}}{4} \approx -0.26$$

Problem Set Sample Solutions

1. Evaluate the following trigonometric expressions.

a.  $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

b.  $\frac{1}{2}\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$

$$\frac{1}{4}\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{1}{8}$$

c.  $4\sin\left(-\frac{5\pi}{12}\right)\cos\left(-\frac{5\pi}{12}\right)$

$$2\sin\left(-\frac{5\pi}{6}\right) = -2\sin\left(\frac{\pi}{6}\right) = -2\left(\frac{1}{2}\right) = -1$$

d.  $\cos^2\left(\frac{3\pi}{8}\right) - \sin^2\left(\frac{3\pi}{8}\right)$

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

e.  $2\cos^2\left(\frac{\pi}{12}\right) - 1$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

f.  $1 - 2\sin^2\left(-\frac{\pi}{8}\right)$

$$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

g.  $\cos^2\left(-\frac{11\pi}{12}\right) - 2$

$$\frac{1}{2}\left(2\cos^2\left(-\frac{11\pi}{12}\right) - 1\right) - \frac{3}{2} = \frac{1}{2}\left(\cos\left(-\frac{11\pi}{6}\right)\right) - \frac{3}{2} = \frac{1}{2}\left(\cos\left(\frac{\pi}{6}\right)\right) - \frac{3}{2} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{3}{2} = \frac{\sqrt{3}}{4} - \frac{3}{2}$$

h.  $\frac{2\tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)}$

$$\tan\left(\frac{\pi}{4}\right) = 1$$

i.  $\frac{2\tan\left(-\frac{5\pi}{12}\right)}{1 - \tan^2\left(-\frac{5\pi}{12}\right)}$

$$\tan\left(-\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

j.  $\cos^2\left(\frac{\pi}{8}\right)$

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{1 + \cos\left(\frac{\pi}{4}\right)}{2} = \frac{1 + \frac{\sqrt{2}}{2}}{2} = \frac{1}{2} + \frac{\sqrt{2}}{4}$$

k.  $\cos\left(\frac{\pi}{8}\right)$

*Rotation by  $\theta = \frac{\pi}{8}$  terminates in Quadrant I; therefore,  $\cos\left(\frac{\pi}{8}\right)$  has a positive value.*

$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

l.  $\cos\left(-\frac{9\pi}{8}\right)$

*Rotation by  $\theta = -\frac{9\pi}{8}$  terminates in Quadrant II; therefore,  $\cos\left(-\frac{9\pi}{8}\right)$  has a negative value.*

$$\cos\left(-\frac{9\pi}{8}\right) = -\sqrt{\frac{1 + \cos\left(-\frac{9\pi}{4}\right)}{2}} = -\sqrt{\frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}} = -\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = -\frac{\sqrt{2 + \sqrt{2}}}{2}$$

m.  $\sin^2\left(\frac{\pi}{12}\right)$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1 - \cos\left(\frac{\pi}{6}\right)}{2} = \frac{1 - \frac{\sqrt{3}}{2}}{2} = \frac{1}{2} - \frac{\sqrt{3}}{4}$$

n.  $\sin\left(\frac{\pi}{12}\right)$

Rotation by  $\theta = \frac{\pi}{12}$  terminates in Quadrant I; therefore,  $\sin\left(\frac{\pi}{12}\right)$  has a positive value.

$$\sin\left(\frac{\pi}{12}\right) = \sqrt{\frac{1 - \cos\left(\frac{\pi}{6}\right)}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

o.  $\sin\left(-\frac{5\pi}{12}\right)$

Rotation by  $\theta = -\frac{5\pi}{12}$  terminates in Quadrant IV; therefore,  $\sin\left(-\frac{5\pi}{12}\right)$  has a negative value.

$$\sin\left(-\frac{5\pi}{12}\right) = -\sqrt{\frac{1 - \cos\left(-\frac{5\pi}{6}\right)}{2}} = -\sqrt{\frac{1 + \cos\left(\frac{\pi}{6}\right)}{2}} = -\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = -\frac{\sqrt{2 + \sqrt{3}}}{2}$$

p.  $\tan\left(\frac{\pi}{8}\right)$

Rotation by  $\theta = \frac{\pi}{8}$  terminates in Quadrant I; therefore,  $\tan\left(\frac{\pi}{8}\right)$  has a positive value.

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{\frac{1 - \cos\left(\frac{\pi}{4}\right)}{1 + \cos\left(\frac{\pi}{4}\right)}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}} = \sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}} = \sqrt{3 - 2\sqrt{2}} \approx 0.414$$

$$\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{1 + \cos\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2 + \sqrt{2}} \approx 0.414$$

$$\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{\pi}{4}\right) = \frac{1 - \cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{1 - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1 \approx 0.414$$

q.  $\tan\left(\frac{\pi}{12}\right)$

$$\tan\left(\frac{\pi}{12}\right) = \sqrt{\frac{1 - \cos\left(\frac{\pi}{6}\right)}{1 + \cos\left(\frac{\pi}{6}\right)}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}}} = \sqrt{\frac{2 - \sqrt{3}}{2 + \sqrt{3}}} = \sqrt{7 - 4\sqrt{3}} \approx 0.268$$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{1 + \cos\left(\frac{\pi}{6}\right)} = \frac{\frac{1}{2}}{1 + \frac{\sqrt{3}}{2}} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3} \approx 0.268$$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1 - \cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3} \approx 0.268$$

r.  $\tan\left(-\frac{3\pi}{8}\right)$

Rotation by  $\theta = -\frac{3\pi}{8}$  terminates in Quadrant IV; therefore,  $\tan\left(-\frac{3\pi}{8}\right)$  has a negative value.

$$\tan\left(-\frac{3\pi}{8}\right) = -\frac{\sqrt{1 - \cos\left(-\frac{3\pi}{4}\right)}}{\sqrt{1 + \cos\left(-\frac{3\pi}{4}\right)}} = -\frac{\sqrt{1 + \cos\left(\frac{\pi}{4}\right)}}{\sqrt{1 - \cos\left(\frac{\pi}{4}\right)}} = -\frac{\sqrt{1 + \frac{\sqrt{2}}{2}}}{\sqrt{1 - \frac{\sqrt{2}}{2}}} = -\frac{\sqrt{2 + \sqrt{2}}}{\sqrt{2 - \sqrt{2}}} = -\sqrt{3 + 2\sqrt{2}} \approx -2.414$$

$$\tan\left(-\frac{3\pi}{8}\right) = \tan\left(\frac{-\frac{3\pi}{4}}{2}\right) = \frac{\sin\left(-\frac{3\pi}{4}\right)}{1 + \cos\left(-\frac{3\pi}{4}\right)} = \frac{-\frac{\sqrt{2}}{2}}{1 - \frac{\sqrt{2}}{2}} = \frac{-\sqrt{2}}{2 - \sqrt{2}} \approx -2.414$$

$$\tan\left(-\frac{3\pi}{8}\right) = \tan\left(\frac{-\frac{3\pi}{4}}{2}\right) = \frac{1 - \cos\left(-\frac{3\pi}{4}\right)}{\sin\left(-\frac{3\pi}{4}\right)} = \frac{1 + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{2 + \sqrt{2}}{-\sqrt{2}} = -\sqrt{2} - 1 \approx -2.414$$

2. Show that  $\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x)$ . (Hint: Use  $\sin(2x) = 2\sin(x)\cos(x)$  and the sine sum formula.)

$$\begin{aligned} \sin(3x) &= \sin(x + (2x)) \\ &= \sin(x)\cos(2x) + \cos(x)\sin(2x) \\ &= \sin(x)[\cos^2(x) - \sin^2(x)] + \cos(x)[2\sin(x)\cos(x)] \\ &= \sin(x)\cos^2(x) - \sin^3(x) + 2\sin(x)\cos^2(x) \\ &= 3\sin(x)\cos^2(x) - \sin^3(x) \end{aligned}$$

3. Show that  $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$ . (Hint: Use  $\cos(2x) = \cos^2(x) - \sin^2(x)$  and the cosine sum formula.)

$$\begin{aligned} \cos(3x) &= \cos(x + (2x)) \\ &= \cos(x)\cos(2x) - \sin(x)\sin(2x) \\ &= \cos(x)[\cos^2(x) - \sin^2(x)] - \sin(x)[2\sin(x)\cos(x)] \\ &= \cos^3(x) - \cos(x)\sin^2(x) - 2\cos(x)\sin^2(x) \\ &= \cos^3(x) - 3\cos(x)\sin^2(x) \end{aligned}$$

4. Use  $\cos(2x) = \cos^2(x) - \sin^2(x)$  to establish the following formulas.

a.  $\cos^2(x) = \frac{1 + \cos(2x)}{2}$

$$\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= \cos^2(x) - (1 - \cos^2(x)) \\ &= 2\cos^2(x) - 1 \end{aligned}$$

Therefore,  $\cos^2(x) = \frac{1 + \cos(2x)}{2}$ .

b.  $\sin^2(x) = \frac{1 - \cos(2x)}{2}$

$$\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= (1 - \sin^2(x)) - \sin^2(x) \\ &= 1 - 2\sin^2(x) \end{aligned}$$

Therefore,  $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ .

5. Jamia says that because sine is an odd function,  $\sin\left(\frac{\theta}{2}\right)$  is always negative if  $\theta$  is negative. That is, she says that for negative values of  $\sin\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1 - \cos(\theta)}{2}}$ . Is she correct? Explain how you know.

*Jamia is not correct. Consider  $\theta = -\frac{7\pi}{3}$ . In this case,  $\frac{\theta}{2} = -\frac{7\pi}{6}$ , and rotation by  $-\frac{7\pi}{6}$  terminates in Quadrant II. Thus,  $\sin\left(-\frac{7\pi}{6}\right)$  is positive.*

6. Ginger says that the only way to calculate  $\sin\left(\frac{\pi}{12}\right)$  is using the difference formula for sine since  $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ . Fred says that there is another way to calculate  $\sin\left(\frac{\pi}{12}\right)$ . Who is correct and why?

*Fred is correct. We can use the half-angle formula with  $\theta = \frac{\pi}{6}$  to calculate  $\sin\left(\frac{\pi}{12}\right)$ .*

7. Henry says that by repeatedly applying the half-angle formula for sine we can create a formula for  $\sin\left(\frac{\theta}{n}\right)$  for any positive integer  $n$ . Is he correct? Explain how you know.

*Henry is not correct. Repeating this process will only give us formulas for  $\sin\left(\frac{\theta}{2^k}\right)$  for positive integers  $k$ . There is no way to derive a formula for quantities such as  $\sin\left(\frac{\theta}{5}\right)$  using this method.*