Archimedes' Determination of Circular Area

presentation by

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MA 330 - History of Mathematics

Archimedes of Syracuse

- Born 287 B.C. at Syracuse
- Father was an astronomer
- Developed life-long interest in the heavens
- Most of what we know about Archimedes comes from Plutarch's <u>Life of Marcellus</u> and Archimedes writings to the scholars at Alexandria

Archimedes (Cont'd)

- Spent time in Egypt as boy
- Studied at the great Library of Alexandria
- Trained in the Euclidean tradition
- Created "Archimedean screw"

Archimedes (Cont’d)

- Archimedes wrote a treatise entitled *On Floating Bodies,* which discussed the principles of hydrostatics
- Archimedes invented a water pump, known as the *Archimedean Screw,* which is still used today
- Archimedes was instrumental in the development of pulleys, levers, and optics

Marcellus' Siege

- Romans invaded Syracuse, led by the General Marcellus
- In defense of his homeland, Archimedes developed weapons to ward off the Romans

Marcellus' (Cont'd)

- Archimedes prepared the city for attacks during both day and night
- The Syracusans dropped their guard during feast to Diana
- Opportunistic Romans seized their chance and invaded the city
- The death of Archimedes brought great sorrow to Marcellus

The Death of Archimedes

- …as fate would have it, intent upon working out some problem by a diagram, and having fixed his mind alike and his eyes upon the subject of his speculation, [Archimedes] never noticed the incursion of the Romans, nor that the city was taken. In this transport of study and contemplation, a soldier, unexpectedly coming up to him, commanded him to follow to Marcellus; which he declining to do before he had worked out his problem to a demonstration, the soldier, enraged, drew his sword and ran him through.

Plutarch

Ancients Knowledge of Circular Area Pre Archimedes

- The ancients knew the ratio of C over D was equal to the value π
- Proposition 12.2 of Euclid states the ratio of circular area to D^2 is constant
- The Area of a regular polygon is ½hQ
- They knew that an inscribed polygon's area would be less than that of a circle regardless of how many sides it had

Measurement of a Circle

- Proposition I
  - The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle

Measurement (cont'd)

- Proof
  - Archimedes began with two figures:
    - a circle with a center O, radius r, and circumference C
    - a right triangle with base of length C and height of r
  - the area of the Circle being equal to A
  - the area of the Triangle equal to T
  - To prove A=T, Archimedes used a double *reductio ad absurdum*

Measurement (cont'd)

- Case 1
  - Suppose A>T then A-T is positive
  - Archimedes knew by inscribing a square within his circle and repeatedly bisecting it he could create a regular polygon with area Z with A-Z < A-T
  - Adding Z+T-A to both sides of A-Z < A-T
    - T < Z

Case 1 (cont'd)

- But this is an inscribed polygon therefore the perimeter of the polygon Q is less than C (the circumference of the circle) and it's apothem is less than the circle's radius
- Hence $Z = \frac{1}{2}hQ < \frac{1}{2}rC = T$ (contradiction)

Measurement (cont’d)

- Case 2
  - Suppose A < T then T-A > 0
  - Circumscribe about the circle a regular polygon who’s area exceeds the circle by no more than T-A
  - With area of circumscribed polygon = Z
    - Z-A < T-A
  - Adding A to both sides results in Z < T

Case 2 (cont'd)

- But the apothem of the polygon equals r while the polygon's perimeter Q exceeds the circle's circumference
- Thus $Z = \frac{1}{2}hQ > \frac{1}{2}rC = T$ (contradiction)
- Therefore A must equal T
- **Q.E.D.**

Results of Proposition I

- Archimedes had related a circles area not to that of another circle as Euclid did, but to its own circumference and radius
- Remembering that $C = \pi D = 2\pi r$, we rephrase his theorem as
  - $A = \frac{1}{2}rC = \frac{1}{2}r(2\pi r) = \pi r^2$

Archimedes' Other Works

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Arhimedes' Method of Discovery

- Presented his method of discovery before each proof
- The "Method" is a treatise of the results of his methods
- Discovered in 1899 unexpectedly
  - Parts of it had been washed out
- Sold for two million

π rap

Measurement of a Circle

- Proposition 3
  - The ratio of the circumference of any to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.

Proposition 3 (cont'd)

- Used inscribed and circumscribed polygons starting with a hexagon
- Archimedes bisected sides of polygons until had 96-gon
- At each stage he had to approximate square roots
  - Very difficult to do back then

A Chronology of π

- Ca. 240 B.C. - Archimedes
- Ca. A.D. 150 - Ptolemy
- Ca. 480 - Tsu Ch'ung-chih
- Ca. 530 - Aryabhata
- Ca. 1150 - Bhaskara
- 1429 - Al-Kashi

π (cont'd)

- 1579 - Francois Viete
- 1585 - Adriaen Anthoniszoon
- 1593 - Adriaen von Roomen
- 1610 - Ludolph van Ceulen
- 1621 - Willebrord Snell
- 1630 - Grienberger

π (cont'd)

- 1650 - John Wallis
- 1671 - James Gregory
- 1699 - Abraham Sharp
- 1706 - John Machin
- 1719 - De Lagny
- 1737 - William Oughtred, Isaac Barrow, David Gregory

π (cont'd)

- 1754 - Jean Etienne Montucla
- 1755 - French Academy of Sciences
- 1767 - Johann Heinrich
- 1777 - Comte de Buffon
- 1794 - Adrien-Marie Legendre
- 1841 - William Rutherford

π (cont'd)

- 1844 - Zacharias Dase
- 1853 - Rutherford
- 1873 - William Shanks
- 1882 - F. Lindemann
- 1906 - A. C. Orr
- 1948 - D. F. Ferguson

π (cont'd)

- 1949 - ENIAC
- 1959 - Francois Genuys
- 1961 - Wrench and Daniel Shanks
- 1965 - ENIAC
- 1966 - M. Jean Guilloud
- 1973 - Guilloud again

π (cont'd)

- 1981 - Kazunori Miyoshi and
  Kazuhika Nakayama
- 1986 - D. H. Bailey

On the Sphere and the Cylinder

- Considered Archimedes masterpiece
- Achieved for 3D bodies what *Measurement of a Circle* had done for 2D bodies

On the Sphere and the Cylinder

- Proposition 13
  - The surface of any right circular cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side of the cylinder and the diameter of the base.

Proposition 13 (cont'd)

- In modern terms lateral surface of cylinder (height h, radius r) is equal to area of a circle w/ radius x

$$\frac{h}{x} = \frac{x}{2r}$$

On the Sphere and the Cylinder

- Proposition 33
  - The surface of any sphere is equal to four times the greatest circle in it

Proposition 33 (cont'd)

- Archimedes proved using double *reductio ad absurdam*
- Can restate proof as *the surface area of a sphere is equal to* $4\pi r^2$
- There's nothing intuitive regarding about this result
- Archimedes states he was only lucky enough to glimpse at these internal truths

On the Sphere and the Cylinder

- Proposition 34
  - Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.

Proposition 34 (cont'd)

- Archimedes expresses volume of sphere in terms of a simpler solid
- Restating formula gives
  - Volume (cone) = $1/3\pi r^2(r) = 1/3\pi r^3$
  - Volume (sphere) = $4(1/3\pi r^3) = 4/3\pi r^3$

Archimedes' Personal Best

- Archimedes restated Proposition 33 & 34 with a cylinder circumscribed about a sphere
  - *The volume and the surface area of the cylinder is half again as large as the sphere's.*
- Archimedes' was so proud of this that he requested the result adorn his tomb once he passed