CHAPTER 9
MULTILINEAR ALGEBRA

In this chapter we study multilinear algebra, functions of several variables that are linear in each variable separately. Multilinear algebra is a generalization of linear algebra since a linear function is also multilinear in one variable. If $V_1, V_2, \cdots, V_k$ and W are vector spaces, then we wish to understand what are all the multilinear maps $g : V_1 \times V_2 \times \cdots \times V_k \to W$ and notation to systematically express them. This may seem like a difficult and involved problem. After all the reader has probably taken considerable effort to learn linear algebra and multilinear algebra must be more complicated. The method employed is to convert g into a linear map $\tilde{g}$ on a different vector space, a vector space called the tensor product of $V_1, V_2, \cdots, V_k$. Since $\tilde{g}$ is a linear map on a vector space, we are now in the realm of linear algebra again. The benefit is that we know about linear maps and how to represent all of them. The cost is that the new space is a complicated space.

Definition 9.1*.** *Suppose $V_1, V_2, \cdots, V_k$ and W are vector spaces. A function $f : V_1 \times V_2 \times \cdots \times V_k \to W$ is called multilinear if it is linear in each of its variables, i.e.,*

$$\begin{aligned} &f(v_1, \cdots, v_{i-1}, av_i + bv_i', v_{i+1}, \cdots, v_k) \\ &\qquad\qquad = af(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k) + bf(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_k) \end{aligned}$$

for all $a, b \in \mathbf{R}$, $v_j \in V_j$ for $j = 1, \cdots, k$ and $v_i' \in V_i$ for $i = 1, \cdots, k$.

Our objective is to reduce the study of multilinear maps to the study of linear maps. We use $F(V_1, V_2, \cdots, V_k)$ to denote the vector space having its basis $\{(v_1, \cdots, v_k) \in V_1 \times \cdots \times V_k\} = V_1 \times \cdots \times V_k$. Each element of $V_1 \times \cdots \times V_k$ is a basis element of $F(V_1, V_2, \cdots, V_k)$. For example, if $V = \mathbf{R}$, then $F(V)$ is an infinite dimesional vector space in which each $r \in \mathbf{R}$ is a basis element. This vector space is enormous, but it is just an intermediate stage. It has the following important property:

Lemma 9.2*.** *If $V_1, V_2, \cdots, V_k$ and W are vector spaces, then linear maps from $F(V_1, V_2, \cdots, V_k)$ to W are in one to one correspondence with set maps from $V_1 \times \cdots \times V_k$ to W.*

Proof. This property follows since a linear map is exactly given by specifying where a basis should map, and $V_1 \times \cdots \times V_k$ is a basis of $F(V_1, V_2, \cdots, V_k)$. Given any set map $g : V_1 \times \cdots \times V_k \to W$ we obtain a linear map $\tilde{\tilde{g}} : F(V_1, V_2, \cdots, V_k) \to W$. $\square$

We next improve upon the construction of F by forming a quotient of F to make a smaller space. We can do this improvement since we are not interested in set maps from

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$V_1 \times \cdots \times V_k$ to W but only in multilinear maps. Let $R \subset F$ be the vector subspace of F spanned by the following vectors

$$(1^{***}) \qquad \begin{aligned}&(v_1, \cdots, v_{i-1}, av_i + bv_i', v_{i+1}, \cdots, v_k)\\ &\qquad - a(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k) - b(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_k)\end{aligned}$$

for each $a, b \in \mathbf{R}$, $v_j \in V_j$ for $j = 1, \cdots, k$ and $v_i' \in V_i$ for $i = 1, \cdots, k$. The vector given in $(1)^{***}$ is a single vector expressed as a sum of three basis elements, and each basis element is an n-tuple in $V_1 \times \cdots \times V_k$. The subspace R has the following important property

Lemma 9.3*.** *If $V_1, V_2, \cdots, V_k$ and W are vector spaces, then linear maps from $F(V_1, V_2, \cdots, V_k)$ to W which vanish on R are in one to one correspondence with multilinear maps from $V_1 \times \cdots \times V_k$ to W.*

Proof. The correspondence is the same correspondence as is given in Lemma 9.2***. Using the same notation as in the proof of Lemma 9.2***, we must show that g is multilinear if and only if $\tilde{\tilde{g}}$ vanishes on R.

Suppose $g : V_1 \times \cdots \times V_k \to W$ is a multilinear map. Then

$$\begin{aligned}&g(v_1, \cdots, v_{i-1}, av_i + bv_i', v_{i+1}, \cdots, v_k)\\ &\qquad = ag(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k) + bg(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_k)\end{aligned}$$

which is true if and only if

$$\begin{aligned}&\tilde{\tilde{g}}(v_1, \cdots, v_{i-1}, av_i + bv_i', v_{i+1}, \cdots, v_k)\\ &\qquad = a\tilde{\tilde{g}}(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k) + b\tilde{\tilde{g}}(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_k)\\ &\qquad = \tilde{\tilde{g}}(a(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k)) + \tilde{\tilde{g}}(b(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_k))\end{aligned}$$

which is true if and only if

$$\begin{aligned}&\tilde{\tilde{g}}((v_1, \cdots, v_{i-1}, av_i + bv_i', v_{i+1}, \cdots, v_k)\\ &\qquad - a(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k) - b(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_k)) = 0.\end{aligned}$$

In the computation above, $\tilde{\tilde{g}}$ is a linear map and each n-tuple is a basis vector in the vector space. The last line states that $\tilde{\tilde{g}}$ vanishes on R and the first line states that g is multilinear. Hence g is multilinear if and only if $\tilde{\tilde{g}}$ vanishes on R. □

We are now ready to define the vector space discussed in the beginning of the chapter.

Definition 9.4*.** *Suppose that $V_1, V_2, \cdots, V_k$ are vector spaces. Then the vector space $F(V_1, \cdots, V_k)/R$ along with the map $\phi : V_1 \times \cdots \times V_k \to F(V_1, \cdots, V_k)/R$ is call the tensor product of $V_1, V_2, \cdots, V_k$. The vector space F/R is denoted $V_1 \otimes \cdots \otimes V_k$. The image $\phi((v_1, \cdots, v_k))$ is denoted $v_1 \otimes \cdots \otimes v_k$. Usually the map ϕ is supressed, but it is understood to be present.*

Usually the map ϕ is suppressed, but it is understood to be present. Although the vector space F is infinite dimensional, we will soon show that $V_1 \otimes \cdots \otimes V_k$ is finite dimensional (Proposition 9.8***). We first show that

Proposition 9.5*.** *The map ϕ in the definition of the tensor product is a multilinear map.*

Proof. We must show that

$$\begin{aligned}\phi(v_1, \cdots, v_{i-1}, &av_i + bv_i', v_{i+1}, \cdots v_k)\\ &= a\phi(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots v_k) + b\phi(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots v_k)\end{aligned}$$

or, using the notation of Definition 9.4***,

$$\begin{aligned}(v_1 \otimes \cdots \otimes v_{i-1} \otimes &av_i + bv_i' \otimes v_{i+1} \otimes \cdots \otimes v_k)\\ &- a(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \cdots \otimes v_k)\\ &- b(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i' \otimes v_{i+1} \otimes \cdots \otimes v_k) = 0\end{aligned}$$

This equation is equivalent to the following statement in F,

$$\begin{aligned}(v_1, \cdots, v_{i-1}, &av_i + bv_i', v_{i+1}, \cdots v_k)\\ &- a(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots v_k) - b(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots v_k) \in R\end{aligned}$$

The vector on the left is in R since it is the element in expression (1)***. □

The main property of the tensor product is the universal mapping property for multilinear maps. It is stated in the following theorem.

Proposition 9.6*.** *Suppose $V_1, V_2, \cdots, V_k$ are vector spaces. The tensor product $\phi : V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$ satisfies the following property, the universal mapping property for multilinear maps:*

If W is a vector space and $g : V_1 \times \cdots \times V_k \to W$ is a multilinear map, then there is a unique linear map $\tilde{g} : V_1 \otimes \cdots \otimes V_k \to W$ such that $\tilde{g} \circ \phi = g$.

Proof. Given the multilinear map g, there is a unique linear map $\tilde{\tilde{g}} : F(V_1, \cdots, V_k) \to W$ by Lemma 9.2***. Since g is multilinear, the map $\tilde{\tilde{g}}$ vanishes on R by Lemma 9.3***. Hence there is a unique well-defined map induce by $\tilde{\tilde{g}}$, call it $\tilde{g} : F/R \to W$. □

The ability of the tensor product to convert multilinear maps into linear maps is an immediate consequence of Proposition 9.6***.

Theorem 9.7*.** *Suppose $V_1, V_2, \cdots, V_k$ and W are vector spaces. Linear maps $\tilde{g} : V_1 \otimes \cdots \otimes V_k \to W$ are in one to one correspondence with multilinear maps $g : V_1 \times \cdots \times V_k \to W$.*

Proof. Given a multilinear map g, Proposition 9.6*** produces the unique linear map $\tilde{g}$. Given a linear map $\tilde{g}$ let $g = \tilde{g} \circ \phi$. The map g is a composition of a linear map and a multilinear map, Proposition 9.5***. The composition of a linear map and a multilinear map is a multilinear linear map. The reader should check this fact. □

Theorem 9.8*.** *Suppose $V_1, V_2, \cdots, V_k$ are vector spaces and $\dim V_i = n_i$. Let $\{e^i_j \mid j = 1, \cdots, n_i\}$ be a basis for V_i. Then $\dim V_1 \otimes \cdots \otimes V_k = n_1 n_2 \cdots n_k$ and $\{e^1_{j_1} \otimes e^2_{j_2} \otimes \cdots \otimes e^k_{j_k} \mid j_i = 1, \cdots, n_i, i = 1, \cdots, k\}$ is a basis for the tensor product $V_1 \otimes \cdots \otimes V_k$.*

Proof. We first show that $\dim V_1 \otimes \cdots \otimes V_k \geq n_1 n_2 \cdots n_k$. Let W be the vector space of dimension $n_1 n_2 \cdots n_k$ and label a basis $E_{j_1, \cdots, j_k}$ for $j_i = 1, \cdots, n_k$. Define $L : V_1 \times \cdots \times V_k \to W$ by

$$L(\sum_{j_1=1}^{n_1} a_{1j_1} e^1_{j_1}, \cdots, \sum_{j_k=1}^{n_k} a_{kj_k} e^k_{j_k}) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} a_{1j_1} \cdots a_{kj_k} E_{j_1, \cdots, j_k}$$

The map L maps onto a basis of W since $L(e^1_{j_1}, \cdots, e^k_{j_k}) = E_{j_1, \cdots, j_k}$.

We next observe that L is multilinear. Let $v_r = \sum_{j_r=1}^{n_r} a_{rj_r} e^r_{j_r}$ for $r = 1, \cdots, k$ and $v'_i = \sum_{j_i=1}^{n_i} a'_{ij_i} e^i_{j_i}$ so that $av_i + bv'_i = \sum_{j_i=1}^{n_i} (aa_{ij_i} + ba'_{ij_i}) e^i_{j_i}$ and

$$\begin{aligned}
&L(v_1, \cdots, v_{i-1}, av_i + bv'_i, v_{i+1} \cdots, v_k) \\
&\qquad = \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} a_{ij_1} \cdots a_{i-1 j_{i-1}} (aa_{ij_i} + ba'_{ij_i}) a_{i+1 j_{i+1}} \cdots a_{kj_k} E_{j_1, \cdots, j_k} \\
&\qquad = a \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} a_{ij_1} \cdots a_{i-1 j_{i-1}} (a_{ij_i}) a_{i+1 j_{i+1}} \cdots a_{kj_k} E_{j_1, \cdots, j_k} \\
&\qquad \quad + b \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} a_{ij_1} \cdots a_{i-1 j_{i-1}} (a'_{ij_i}) a_{i+1 j_{i+1}} \cdots a_{kj_k} E_{j_1, \cdots, j_k} \\
&\qquad = aL(v_1, \cdots, v_{i-1}, v_i, v_{i+1} \cdots, v_k) + bL(v_1, \cdots, v_{i-1}, v'_i, v_{i+1} \cdots, v_k)
\end{aligned}$$

By Proposition 9.6***, there is an induced linear map $\tilde{L} : V_1 \otimes \cdots \otimes V_k \to W$. This map hits a basis since L maps onto a basis. Since $\tilde{L}$ is linear, it is onto. Therefore $\dim V_1 \otimes \cdots \otimes V_k > \dim W = n_1 n_2 \cdots n_k$.

We show that $\{e^i_j \mid j = 1, \cdots, n_i\}$ is a spanning set. The set $V_1 \times \cdots V_k$ is a basis of $F(V_1, \cdots, V_k)$ and $F(V_1, \cdots, V_k)$ maps onto the tensor product $V_1 \otimes \cdots \otimes V_k = F/R$. Therefore the elements of the form $v_1 \otimes \cdots \otimes v_k$ span the tensor product. Let $v_i = \sum_{j_i=1}^{n_i} a_{ij_i} e^i_{j_i}$ for each $i = 1, \cdots, k$. Then

$$\begin{aligned}
v_1 \otimes \cdots \otimes v_k &= \sum_{j_1=1}^{n_1} a_{1j_1} e^1_{j_1} \otimes \cdots \otimes \sum_{j_k=1}^{n_k} a_{kj_i} e^k_{j_k} \\
&= \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} a_{1j_1} \cdots a_{kj_i} e^1_{j_1} \otimes \cdots \otimes e^k_{j_k}
\end{aligned}$$

Since $\{e^i_j \mid j = 1, \cdots, n_i\}$ is a spanning set and its cardinality is $n_1 n_2 \cdots n_k$, it is basis. $\square$

Example 9.9*.** *A multilinear map $g : \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$.*

Suppose that e_1, e_2, e_3 is the standard basis for $\mathbf{R}^3$. The vector space $\mathbf{R}^3 \otimes \mathbf{R}^3 \otimes \mathbf{R}^3$ is 27 dimensional and has as a basis $\{e_i \otimes e_j \otimes e_k \mid i, j, k = 1, 2, 3\}$ This basis is usually represented as a triple index, indexed by the basis elements of $\mathbf{R}^3$. The linear map $\tilde{g} : \mathbf{R}^3 \otimes \mathbf{R}^3 \otimes \mathbf{R}^3 \to \mathbf{R}$ can be represented as a 1×27 matrix, but this is not the usual way to represent tensors. Suppose that $\tilde{g}(e_i \otimes e_j \otimes e_k) = a_{ijk}$ using the triple index to write the basis elements of the tensor product. Then, using the e_i basis for $\mathbf{R}^3$, we have

$$\tilde{g}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ijk} x_i y_j z_k$$

Example 9.10*.** *$\mathbf{R} \otimes \cdots \otimes \mathbf{R}$ is isomorphic to $\mathbf{R}$, $\mathbf{R} \otimes W$ is isomorphic to W.*

A basis for $\mathbf{R} \otimes \cdots \otimes \mathbf{R}$ is $1 \otimes \cdots \otimes 1$ and $a_1 \otimes \cdots \otimes a_k = (a_1 \cdots a_k) 1 \otimes \cdots \otimes 1$. The isomorphism is to multiply the entries.

Similarly, $\mathbf{R} \otimes W \to W$ by $r \times w \mapsto rw$ induces the second isomorphism.

These isomorphisms are natural and standard. They are natural because $\mathbf{R}$ is not an abstract 1-dimensional vector space, but $\mathbf{R}$ has a distinguished multiplicative unit 1. These isomorphisms are used as an identification in these notes and in physics literature.

The following theorem defines and gives properites of induced maps between tensor products.

Theorem 9.11*.** *Suppose $V_1, \cdots, V_k$ and $W_1, \cdots, W_k$ are vector spaces. Further suppose that $f_i : V_i \to W_i$ is a linear map for each $i = 1, \cdots, k$. Then there is an induced linear map $f_1 \otimes \cdots \otimes f_k : V_1 \otimes \cdots \otimes V_k \to W_1 \otimes \cdots \otimes W_k$ defined by $f_1 \otimes \cdots \otimes f_k(v_1 \otimes \cdots \otimes v_k) = f_1(v_1) \otimes \cdots \otimes f_k(v_k)$. These induced maps satisfy the following two properties*

(1) *If $Q_1, \cdots, Q_k$ are another collection of vector spaces and $g_i : W_i \to Q_i$ is a linear map for each $i - 1, \cdots, k$, then $g_1 \otimes \cdots \otimes g_k \circ f_1 \otimes \cdots \otimes f_k = (g_1 \circ f_1) \otimes \cdots \otimes (g_k \circ f_k)$*

(2) *If I_X denotes the identity on X, then $I_{V_1} \otimes \cdots \otimes I_{V_k} = I_{V_1 \otimes \cdots \otimes V_k}$*

Proof. Let $\phi : W_1 \times \cdots \times W_k \to W_1 \otimes \cdots \otimes W_k$ be the map from the definition of the tensor product. It is multilinear by Proposition 9.5***. Let $L : V_1 \times \cdots \times V_k \to W_1 \otimes \cdots \otimes W_k$ be defined by $L = \phi \circ (f_1, \cdots, f_k)$. We show that L is multilinear since each f_i is linear and ϕ is multilinear.

$$\begin{aligned}
L(v_1, \cdots, v_{i-1}, &av_i + bv_i', v_{i+1}, v_k) \\
&= f_1(v_1) \otimes \cdots \otimes f_{i-1}(v_{i-1}) \otimes f_i(av_i + bv_i') \otimes f_{i+1}(v_{i+1}) \otimes \cdots \otimes f(v_k) \\
&= f_1(v_1) \otimes \cdots \otimes f_{i-1}(v_{i-1}) \otimes (af_i(v_i) + bf_i(v_i')) \otimes f_{i+1}(v_{i+1}) \otimes \cdots \otimes f(v_k) \\
&= af_1(v_1) \otimes \cdots \otimes f_{i-1}(v_{i-1}) \otimes (f_i(v_i)) \otimes f_{i+1}(v_{i+1}) \otimes \cdots \otimes f(v_k) \\
&\qquad + bf_1(v_1) \otimes \cdots \otimes f_{i-1}(v_{i-1}) \otimes (f_i(v_i')) \otimes f_{i+1}(v_{i+1}) \otimes \cdots \otimes f(v_k) \\
&= aL(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k) + bL(v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_k)
\end{aligned}$$

There is an induced linear map $\tilde{L}$ by Theorem 9.7*** and this map is $f_1 \otimes \cdots \otimes f_k$.

The first property follows from the definition of the induced map. We check the second property,

$$
\begin{aligned}
I_{V_1} \otimes \cdots \otimes I_{V_k}(v_1 \otimes \cdots \otimes v_k) &= I_{V_1}(v_1) \otimes \cdots \otimes I_{V_k}(v_k) \\
&= v_1 \otimes \cdots \otimes v_k,
\end{aligned}
$$

which verifies the second property. $\square$

We have the following corollary.

Corollary 9.12*.** *Suppose $V_1, V_2, \cdots, V_k$ are vector spaces. Let $\{e^i_j \mid j = 1, \cdots, n_i\}$ be a basis for V_i and $\{(e^i_j)^* \mid j = 1, \cdots, n_i\}$ the dual basis for V_i^*. Using the isomorphism from Example 9.10***, $\mathbf{R} \otimes \cdots \otimes \mathbf{R} \cong \mathbf{R}$, we have that*

$$
(e^1_{j_1})^* \otimes \cdots \otimes (e^k_{j_k})^* = (e^1_{j_1} \otimes \cdots \otimes e^k_{j_k})^*,
$$

which gives an isomorphism $V^* \otimes \cdots \otimes V^* \cong (V \otimes \cdots \otimes V)^*$.

Proof. We check this formula on the basis given in Theorem 9.8***.

$$
(e^1_{j_1})^* \otimes \cdots \otimes (e^k_{j_k})^*(e^1_{i_1} \otimes \cdots \otimes e^k_{i_k}) = \begin{cases} 1 \text{ if } i_1 = j_1, \cdots, i_k = j_k \\ 0 \text{ if } i_r \neq j_r \text{for some } r \end{cases}
$$

This formula follows from the definition of the induced map in Theorem 9.11*** and the definition of the dual basis. It shows that $\{e^1_{j_1})^* \otimes \cdots \otimes (e^k_{j_k})^*\}$ is the basis dual to $\{e^1_{j_1} \otimes \cdots \otimes (e^k_{j_k}\}$. $\square$

Example 9.13*.** *$Hom(V, W) = V^* \otimes W$. If $\{e_i \mid i = 1, \cdots n\}$ is a basis for V, $\{e'_i \mid i = 1, \cdots m\}$ is a basis for W, and $f \in Hom(V, W)$, then $f = \sum_{i=1}^m \sum_{j=1}^n a_{ij} e^*_j \otimes e'_i$ where (a_{ij}) is the matrix representing f in the given bases.*

We check the formula:

$$
\begin{aligned}
(\sum_{i=1}^n \sum_{j=1}^n a_{ij} e^*_j \otimes e'_i)(e_r) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}(e^*_j(e_r)) \otimes e'_i \\
&= \sum_{i=1}^n a_{ir} 1 \otimes e'_i \\
&= \sum_{i=1}^n a_{ir} e'_i
\end{aligned}
$$

Which is the r-th column vector of the matrix. Also notice the last line used the idenification from Example 9.10***.

Example 9.14*.** *Bilinear Maps*

The usual dot product in $\mathbf{R}^n$ is a bilinear map. In fact, any bilinear map $<,>: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ induces a linear map

$$\tilde{<,>} : \mathbf{R}^n \otimes \mathbf{R}^n \to \mathbf{R}.$$

Now, $\tilde{<,>} \in \mathrm{Hom}(\mathbf{R}^n \otimes \mathbf{R}^n, \mathbf{R})$ and

$$\begin{aligned} \mathrm{Hom}(\mathbf{R}^n \otimes \mathbf{R}^n, \mathbf{R}) &= (\mathbf{R}^n \otimes \mathbf{R}^n)^* \text{ by definition of the dual} \\ &= (\mathbf{R}^n)^* \otimes (\mathbf{R}^n)^* \text{ using the isomorphism in 9.12***.} \end{aligned}$$

Using the standard basis $\{e_i \mid i = 1, \cdots, n\}$ for $\mathbf{R}^n$, we can write $\tilde{<,>} = \sum_{i=1}^n \sum_{j=1}^n g_{ij} e_i^* \otimes e_j^*$. Readers who are familar with representing a bilinear form as a matrix should note that (g_{ij}) is the matrix for $<,>$. In practice the tilde is not used and we only use it here for clarification.

Example 9.15*.** *The Cross Product in* $\mathbf{R}^3$

The vector cross product in $\mathbf{R}^3$ is a bilinear map and so induces a map

$$- \times - : \mathbf{R}^3 \otimes \mathbf{R}^3 \to \mathbf{R}^3.$$

If we use the identifications $\mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}^3, \mathbf{R}^3) = (\mathbf{R}^3)^* \otimes (\mathbf{R}^3)^* \otimes \mathbf{R}^3$, then the cross product is

$$e_2^* \otimes e_3^* \otimes e_1 - e_3^* \otimes e_2^* \otimes e_1 + e_3^* \otimes e_1^* \otimes e_2 - e_1^* \otimes e_3^* \otimes e_2 + e_1^* \otimes e_2^* \otimes e_3 - e_2^* \otimes e_1^* \otimes e_3.$$

Example 9.16*.** *The Stress Tensor*

Consider a solid object and put it in the usual coordinate system so that the origin 0 is at an interior point. One problem is to understand and describe the stresses at a point (the origin). Since the interior point, the origin, isn't moving, all the forces must be in balance. This fact is a consequece of Newton's laws of motion. However, we can ask about the stress (force per unit area) on a surface through the origin. There may be forces perpendicular to the surface, e.g., from compression and forces along the surface, e.g., shearing forces from twisting.

Given a vector $u \in \mathbf{R}^3$, let S be the square with area $|u|$ that is perpendicular to u. Let $F(u)$ be the force on the square S from the side that u points so that $F(u)$ is a vector in $\mathbf{R}^3$. Define a bilinear function τ by

$$\tau(v, w) = v \cdot DF(0)(w)$$

where $DF(0)$ is the best linear approximation to F at the origin, i.e., the derivative and the dot is the usual dot product. The number $\tau(e_i, e_j)$ is the $i-th$ component of the force per unit area acting on the surface perpendicular to e_j. The stress tensor τ is a map

$$\tau : \mathbf{R}^3 \otimes \mathbf{R}^3 \to \mathbf{R}.$$

In terms of the identification $\mathrm{Hom}(\mathbf{R}^3 \otimes \mathbf{R}^3, \mathbf{R}) = (\mathbf{R}^3 \otimes \mathbf{R}^3)^* = (\mathbf{R}^3)^* \otimes (\mathbf{R}^3)^*$ we have that $\tau = \sum_{i=1}^3 \sum_{j=1}^3 \tau_{ij} e_i^* \otimes e_j^*$.

Definition 9.17*.** *Suppose V is a vector space. Elements of the space $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$, the r-fold tensor of V and the s-fold tensor of V^* are tensors of type (r, s)***.*

Remark 9.18*.** *A tensor is an object that transforms ...*

Since tensors are elements of a vector space, they are vectors. However, different terminology is used emphasizing a conceptual difference. The primary object of study is usually associated to the vector space V and a tensor of type (r, s) is then an auxilary object. This perspective is apparent when one changes coordinates. If V is an n dimensional vector space then the space of tensor of type (r, s) is an $n(r + s)$-dimensional vector space. A change of coordinates in V is given by a map in $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbf{R})$ and a change in coordinates in the space of tensor of type (r, s) is given by a map in $\mathrm{GL}(V \otimes \cdots \otimes V^*) \cong \mathrm{GL}(n(r + s), \mathbf{R})$. However the only change in coordinates allowed in the tensor product are those that are induced from a change in coordinates V via Theorem 9.11***. If $f : V \to V$, then $f^* : V^* \to V^*$ and

$$f \otimes \cdots \otimes f \otimes f^* \otimes \cdots \otimes f^* : V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^* \to V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$$

are the induced change of coordinates.

We give the specific formula. Suppose $e_1, \cdots, e_n$ is a basis for V and the map f is represented by the matrix (a_{ij}). Then

$$f(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j e_i$$
$$f(\sum_{i=1}^{n} y^i e_i^*) = \sum_{i=1}^{n} \sum_{j=1} a_{ji} y^j e_i^*$$

If $\sum_{i_1,\cdots,i_r,j_1,\cdots,j_s=1}^{n} T_{i_1,\cdots,i_r}^{j_1,\cdots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{j_1}^* \otimes \cdots \otimes e_{j_s}^*$ is a tensor of type (r, s)*** then its image under the induced map is
(2***)

$$\sum_{\substack{i_1,\cdots,i_r,j_1,\cdots,j_s \\ p_1,\cdots,p_r,q_1,\cdots,q_s=1}}^{n} T_{p_1,\cdots,p_r}^{q_1,\cdots,q_s} a_{i_1p_1} \cdots a_{i_rp_r} a_{q_1j_1} \cdots a_{q_rj_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{j_1}^* \otimes \cdots \otimes e_{j_s}^*$$

Formula (2)*** is often expressed in physics by saying that under the linear change of coordinates f, $T_{p_1,\cdots,p_r}^{q_1,\cdots,q_s}$ transforms to

$$\sum_{p_1,\cdots,p_r,q_1,\cdots,q_s=1}^{n} T_{p_1,\cdots,p_r}^{q_1,\cdots,q_s} a_{i_1p_1} \cdots a_{i_rp_r} a_{q_1j_1} \cdots a_{q_rj_s}.$$

Furthermore, the summation sign is often supressed.

We now turn our attention to functions that are alternating and multilinear.

Definition 9.19*.** *Suppose V and W are vector spaces. A function from the k-fold cross product to W $g : V \times V \times \cdots \times V \to W$ is called alternating if it is multilinear and if for all $v_1, \cdots, v_k \in V$, $f(v_1, \cdots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$.*

There is a common equivalent definition which we give as a proposition.

Proposition 9.20*.** *Suppose V and W are vector spaces. A function $g : V \times V \times \cdots \times V \to W$ is alternating if and only if it is multilinear and it satisfies the following: for all $v_1, \cdots, v_k \in V$ and permutations $\sigma \in \Sigma_k$,*

$$f(v_1, \cdots, v_k) = (-1)^{\text{sign } \sigma} f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}). \tag{3***}$$

Proof. Suppose f is an alternating function. We first show that (3)*** holds for permutations. For any $v_1, \cdots, v_k \in V$ and $i < j$ we have that

$$\begin{aligned} f(v_1, \cdots, v_i + v_j, \cdots, v_j + v_i, \cdots, v_k) =& f(v_1, \cdots, v_i, \cdots, v_i, \cdots, v_k) \\ &+ f(v_1, \cdots, v_j, \cdots, v_j, \cdots, v_k) \\ &+ f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &+ f(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \end{aligned}$$

since f is multilinear. The first two terms on the right side are zero and the left side is zero, because the function is alternating. Therefore

$$f(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = (-1)^{\text{sign } \tau} f(v_{\tau(1)}, \cdots, v_{\tau(i)}, \cdots, v_{\tau(j)}, \cdots, v_{\tau(k)})$$

for τ the permutation (i, j). Now suppose that σ is an arbitrary permutation. Then σ can be written as a composition of transpositions

$$\sigma = \tau_m \cdots \tau_1$$

where we have written this product as a composition of functions in functional notation (not group multiplication). Thus we have that

$$\begin{aligned} f(v_1, \cdots, v_k) =& (-1)^{\text{sign } \tau_1} f(v_{\tau_1(1)}, \cdots, v_{\tau_1(k)}) \\ =& (-1)^{\text{sign } \tau_2} (-1)^{\text{sign } \tau_1} f(v_{\tau_2\tau_1(1)}, \cdots, v_{\tau_2\tau_1(k)}) \\ &\vdots \\ =& (-1)^{\text{sign } \tau_m} \cdots (-1)^{\text{sign } \tau_2} (-1)^{\text{sign } \tau_1} f(v_{\tau_m \cdots \tau_2\tau_1(1)}, \cdots, v_{\tau_m \cdots \tau_2\tau_1(k)}) \\ =& (-1)^{\text{sign } \sigma} f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \end{aligned}$$

Therefore, (3)*** is satisfied.

The converse is easy to show. Suppose that f is a multilinear function which satisfies (3)***. Suppose that $v_1, \cdots, v_k \in V$, and $v_i = v_j$ for some $i \neq j$. Let τ be the transposition $\tau = (i, j)$. Then $v_{\tau(i)} = v_{\tau(j)}$ so, $(v_1, \cdots, v_k) = (v_{\tau(1)}, \cdots, v_{\tau(k)})$ and

$$\begin{aligned} f(v_1, \cdots, v_k) =& -1 f(v_{\tau(1)}, \cdots, v_{\tau(k)}) \\ =& - f(v_1, \cdots, v_k). \end{aligned}$$

Therefore $f(v_1, \cdots, v_k) = 0$ and f is alternating. $\square$

Example 9.21*.** *The determinant function using either row or column vectors.*

Suppose V is an m-dimensional vector space and $v_1, \cdots, v_m$ are vectors in V. Let $\det(v_1, \cdots, v_m)$ be the determinant of the matrix whose $i-th$ column vector is v_i. Then

$$\det : V \times \cdots \times V \to \mathbf{R}$$

is an alternating map. The determinant is linear in each column vector and if two column vectors are the same then the determinant vanishes.

We can do the same with row vectors. Let $\det \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$ be the determinant of the matrix whose i-th row vector is v_i. Then

$$\det : V \times \cdots \times V \to \mathbf{R}$$

is an alternating map. The determinant is linear in each row vector and if two row vectors are the same then the determinant vanishes.

Example 9.22*.** *The cross product in* $\mathbf{R}^3$ *and* $\mathbf{R}^m$

The cross product in $\mathbf{R}^3$ is familiar as a map

$$- \times - : \mathbf{R}^3 \otimes \mathbf{R}^3 \to \mathbf{R}^3$$

from Calculus and Example 9.15***. It is multilinear and it is also alternating since $v \times v = 0$. Recall that the direction of $v \times w$ is perpendicular to the plane containing v and w, and the direction is determined by the right hand rule.

Also recall from Calculus that the cross product $v \times w$ can be computed as

$$\det \begin{pmatrix} e_1 & e_2 & e_3 \\ & v & \\ & w & \end{pmatrix}$$

which is formally interpreted as expansion in the first row and e_1, e_2, e_3 is the standard basis in $\mathbf{R}^3$ often denoted i, j, k in Calculus and physics books. The vectors v and w are row vectors. If u is another vector in $\mathbf{R}^3$, then $u \cdot (v \times w) = \det \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

In $\mathbf{R}^m$ we can define the cross product of $m-1$ vectors.

$$- \times \cdots \times - : \mathbf{R}^m \otimes \mathbf{R}^m \to \mathbf{R}^m.$$

The cross product $v_1 \times v_2 \times \cdots \times v_{m-2} \times v_{m-1}$ can be formally computed as

$$\det \begin{pmatrix} e_1 & \cdots & e_m \\ & v_1 & \\ & \vdots & \\ & v_{m-1} & \end{pmatrix}$$

by expansion along the first row. If u is another vector in $\mathbf{R}^m$, then

$$u \cdot (v_1 \times v_2 \times \cdots \times v_{m-2} \times v_{m-1}) = \det \begin{pmatrix} u \\ v_1 \\ \vdots \\ v_{m-1} \end{pmatrix}.$$

To check this formula, expand the right side along the first row. Also notice that if $u \in \operatorname{span}\{v_1, \cdots, v_{m-1}\}$, then the determinant vanishes. Hence $v_1 \times v_2 \times \cdots \times v_{m-2} \times v_{m-1}$ is perpendicular to the $m-1$ dimensional subspace spanned by $\{v_1, \cdots, v_{m-1}\}$.

The cross product is a multilinear and alternating function since the determinant is multilinear and alternating.

Example 9.23*.** *$v_m \cdot (v_1 \times \cdots \times v_{m-1})$ for $v_1, \cdots, v_m \in \mathbf{R}^m$*

This function on the m-fold cross product of $\mathbf{R}^m$ to $\mathbf{R}$ is multilinear and alternating since it was shown to be the determinant

$$\det \begin{pmatrix} v_m \\ v_1 \\ \vdots \\ v_{m-1} \end{pmatrix}$$

in the last example.

Example 9.24*.** *Signed Volume in $\mathbf{R}^3$*

Given three linearly independent vectors $v_1, v_2, v_3 \in \mathbf{R}^3$, we may form a parallelepiped with these three vectors as sides. The parallelepiped is $\{av_1 + bv_2 + cv_3 \mid a, b, c \in [0,1]\}$. We wish to compute the volume of the parallelepiped as a function of the three vectors and obtain a multilinear function. Note that volume is always a positive number and the value of a multilinear function must include negative values (since constants always pull out). We will try to find a notion of signed volume whose absolute value gives the actual volume. Denote the signed volume of the parallelepiped by $\mathrm{Vol}_3(v_1, v_2, v_3)$. The volume of a parallelepiped is the area of the base times the height, so

$$|\mathrm{Vol}_3(v_1, v_2, v_3)| = |\mathrm{Vol}_2(v_1, v_2)||(v_3 \cdot n)|$$

where $|\mathrm{Vol}_2(v_1, v_2)|$ is the area of the parallelogram determined by v_1, v_2, and $n = \frac{v_1 \times v_2}{|v_1 \times v_2|}$ is a unit vector perpendicular to the plane containing the parallelogram. The signed volume is

$$\mathrm{Vol}_3(v_1, v_2, v_3) = |\mathrm{Vol}_2(v_1, v_2)|(v_3 \cdot n).$$

Note that the function is clearly linear in the third variable since the dot product is linear in its first variable. Permuting the variables keeps the correct formula up to the sign (so you can move any variable to the third spot). To check the sign under a permutation, one must observe that the cross product dotted with the last vector is alternating by the previous example.

Example 9.25*.** *Signed Volume in* $\mathbf{R}^m$.

This example is essentially the same as Example 9.24***. Given m linearly independent vectors $v_1, \cdots, v_m \in \mathbf{R}^m$, we may form a parallelepiped with these m vectors as sides. The parallelepiped is $\{a_1 v_1 + \cdots + a_m v_m \mid a_i \in [0,1], i = 1, \cdots m\}$. Again we wish to compute the volume of the parallelepiped as a function of the m vectors. We show that there is a multilinear alternating function that gives a signed volume. Denote the signed volume of the parallelepiped by $\mathrm{Vol}_m(v_1, \cdots, v_m)$. The volume of a parallelepiped is the area of the base times the height, so

$$|\mathrm{Vol}_m(v_1, \cdots, v_m)| = |\mathrm{Vol}_{m-1}(v_1, \cdots, v_{m-1})||(v_m \cdot n)|$$

where $|\mathrm{Vol}_{m-1}(v_1, \cdots, v_{m-1})|$ is the $m-1$ dimensional volume of the parallelepiped determined by $v_1, \cdots, v_{m-1}$, and $n = \frac{v_1 \times \cdots \times v_{m-1}}{|v_1 \times \cdots \times v_{m-1}|}$ is a unit vector perpendicular to the plane containing the parallelepiped (Example 9.22***). The signed volume is

$$|\mathrm{Vol}_m(v_1, \cdots, v_m)| = |\mathrm{Vol}_{m-1}(v_1, \cdots, v_{m-1})|(v_m \cdot n).$$

The linearity check is the same as in the last example. The function is clearly linear in the last variable since the dot product is linear in its first variable. Permuting the variables keeps the correct formula up to the sign (so you can move any variable to the last spot). To check the sign under a permutation, one must observe that the cross product dotted with the last vector is alternating by Example 9.23***. Notice that if two of the entries in $\mathrm{Vol}_m(v_1, \cdots, v_m)$ are the same then the parallelepiped is in an $m-1$ dimensional subspace and so has zero volume.

Example 9.26*.** *Minors*

Suppose that W is an m-dimensional vector space, V is an n-dimensional vector space, $f : W \to V$ is linear, and $\det : V \times \cdots \times V \to \mathbf{R}$ is the determinant in terms of column vectors from the n fold product V. Then the map

$$F(w_1, \cdots, w_n) = \det(f(w_1), \cdots, f(w_n)),$$

i.e., $F = \det \circ (f \times \cdots \times f)$ from the n fold product of W to $\mathbf{R}$ is an alternating map. Suppose that W has $e_1, \cdots, e_m$ as an ordered basis and $m \geq n$. Fix numbers $i_1 < i_2 < \cdots < i_n$ in $\{1, \cdots, m\}$. Let $p(\sum_{i=1}^m a_i e_i) = \sum_{j=1}^n a_{i_j} e_{i_j}$, or $p = \sum_{j=1}^n e_{i_j}^*$. Let

$$M_{i_1, \cdots, i_n}(w_1, \cdots, w_n) = \det \circ (p \times \cdots \times p)(w_1, \cdots, w_n).$$

Now express the vector $w_i \in W$ in terms of the basis: $w_i = \sum_{j=1}^m w_{ji} e_j$. The function $M_{i_1, \cdots, i_n}$ is the minor of the matrix

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & & \vdots \\ w_{m1} & w_{m2} & \cdots & w_{mn} \end{pmatrix}$$

obtained by choosing the rows $i_1, \cdots, i_n$, $M_{i_1, \cdots, i_n}(w_1, \cdots, w_n) = \det \begin{pmatrix} w_{i_1 1} & \cdots & w_{i_1 n} \\ \vdots & & \vdots \\ w_{i_n 1} & \cdots & w_{i_n n} \end{pmatrix}$.

We now find the appropriate vector space in which to describe alternating multilinear maps.

Definition 9.27*.** *Suppose that V is a vector space and let $A \subset V \otimes \cdots \otimes V$ be the subspace spanned by*

$$\{v_1 \otimes \cdots \otimes v_k \mid v_1, \cdots, v_k \in V, v_i = v_j \text{ for some } i \neq j\}.$$

The vector space $V \otimes \cdots \otimes V/A$ along with the map $\psi : V \times \cdots \times V \to V \otimes \cdots \otimes V/A$ is called the exterior product. The vector space $V \otimes \cdots \otimes V/A$ is denote $V \wedge \cdots \wedge V$ and $\psi(v_1, \cdots, v_n)$ is denoted $v_1 \wedge \cdots \wedge v_n$.

Recall that there is a unique induced linear map $\tilde{\psi} : V \otimes \cdots \otimes V \to V \wedge \cdots \wedge V$ such that $\tilde{\psi} \circ \phi = \psi$ given by Proposition 9.6***.

Proposition 9.28*.** *The induce map $\psi : V \times \cdots \times V \to V \wedge \cdots \wedge V$ is a multilinear alternating map.*

Proof. The induced map is a composition of the multilinear map ϕ from Proposition 9.5*** and the linear quotient map $\tilde{\psi}$. Hence $\psi = \tilde{\psi} \circ \phi$ is multilinear. The map is alternating since $\phi(v_1, \cdots, v_n) \in A$ if $v_i = v_j$ for some $i \neq j$.

The main property of the exterior product is the universal mapping property for multilinear alternating maps. It is stated in the following theorem.

Proposition 9.29*.** *Suppose V is vector a space. The exterior product $\psi : V \times \cdots \times V \to V \wedge \cdots \wedge V$ satisfies the following property, the universal mapping property for multilinear alternating maps:*

If W is a vector space and $g : V \times \cdots \times V \to W$ is an alternating multilinear map, then there is a unique linear map $g' : V_1 \wedge \cdots \wedge V \to W$ such that $g' \circ \psi = g$.

Proof. There is a unique linear map $\tilde{g} : V \otimes \cdots \otimes V \to W$. The map g is alternating so $g(v_1, \cdots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$. From Proposition 9.6***, $g = \tilde{g} \circ \phi$ If $v_i = v_j$ for some $i \neq j$, then

$$0 = \tilde{g} \circ \phi(v_1, \cdots, v_k) = \tilde{g}(v_1 \otimes \cdots \otimes v_k).$$

Therefore $A \subset \operatorname{Ker} \tilde{g}$ and the unique map $\tilde{g}$ determines a unique map $g' : V \otimes \cdots \otimes V/A \to W$. □

The ability of the exterior product to convert alternating multilinear maps into linear maps is an immediate consequence of Proposition 9.29***.

Theorem 9.30*.** *Suppose V and W are vector spaces. Linear maps $\tilde{g} : V \wedge \cdots \wedge V \to W$ are in one to one correspondence with alternating multilinear maps $g : V \times \cdots \times V \to W$.*

Proof. Given a multilinear map g, Proposition 9.29*** produces the unique linear map g'. Given a linear map g' let $g = g' \circ \psi$. The map g' is a composition of a linear map and an alternating multilinear map, Proposition 9.28***. The composition of a linear map and an alternating multilinear map is an multilinear alternating map. The reader should check this fact. □

Theorem 9.31*.** *Suppose V are vector spaces and dim $V = n$. Let $\{e_i \mid i = 1, \cdots, n\}$ be a basis for V. Then the dimension of the k-fold wedge product $\dim V \wedge \cdots \wedge V = \binom{n}{k}$ and $\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} \mid 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}$ is a basis for the exterior product $V \wedge \cdots \wedge V$.*

Proof. We first show that $\dim V \wedge \cdots \wedge V \geq \binom{n}{k}$. Let W be the vector space with basis $\{E_{j_1,\cdots,j_k} \mid 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}$. Let $L : V \times \cdots \times V \to W$ be the map from the k-fold cross product of V to W defined by

$$L(v_1, \cdots, v_k) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} M_{j_1,\cdots,j_k}(v_1, \cdots, v_k) E_{j_1,\cdots,j_k}$$

where $M_{j_1,\cdots,j_k}$ is the multilinear alternating function from Example 9.26***. The function L is multilinear and alternating since each $M_{j_1,\cdots,j_k}$ is multilinear and alternating. The function L hits a whole basis to W since $L(e_{j_1}, \cdots, e_{j_k}) = E_{j_1,\cdots,j_k}$. There is an induced linear map $L' : V \wedge \cdots \wedge V \to W$ which is onto since it hits a basis to W. Therefore $\dim V \wedge \cdots \wedge V \geq \dim W = \binom{n}{k}$.

We next show that $\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} \mid 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}$ is a spanning set (the same size as the dimension of W). Hence it is a basis. To observe it is a spanning set first note that $\{e_{i_1} \otimes \cdots \otimes e_{i_k} \mid i_1, \cdots, i_k \in \{1, \cdots, n\}\}$ is a spanning set for the k-fold tensor product $V \otimes \cdots \otimes V$ by Theorem 9.8***. Therefore $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1, \cdots, i_k \in \{1, \cdots, n\}\}$ is a spanning set for the exterior product. Finally note that if σ is the permutation or $\{1, \cdots, k\}$ so that $i_{\sigma(1)} < \cdots < i_{\sigma(k)}$ then $e_{i_1} \wedge \cdots \wedge e_{i_k} = \pm e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(k)}} = \pm e_{j_1} \wedge \cdots \wedge e_{j_k}$ where the last wedge product is in the desired set. See Exercise 2*** □

The reader should note that the proof of Theorem 9.31*** actually shows that $v_1 \wedge \cdots \wedge v_k = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} M_{j_1,\cdots,j_k}(v_1, \cdots, v_k) e_{j_1} \wedge \cdots \wedge e_{j_k}$.

We now establish the existence of induced maps and two properties of induced maps.

Theorem 9.32*.** *Suppose V and W are vector spaces and $f : V \to W$ is a linear map. Then there is an induced linear map $\wedge^k f : V \wedge \cdots \wedge V \to W \wedge \cdots \wedge W$ defined by $\wedge^k f(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k)$. These induced maps satisfy the following two properties*

(1) *If Q is another vector space and $g : W \to Q$ is another linear map, then $\wedge^k g \circ \wedge^k f = \wedge^k(g \circ f)$*
(2) *If I_X denotes the identity on X, then $I_V \wedge \cdots \wedge I_V = I_{V \wedge \cdots \wedge V}$*

Proof. We show that $\wedge^k f$ exists and is defined by the formula in the statement of the theorem. Let $\psi : W \times \cdots \times W \to W \wedge \cdots \wedge W$ be the multilinear alternating map in from Definition 9.27*** an Lemma 9.28***. The function $\psi \circ (f \times \cdots \times f)$ is a multilinear alternating map from $V \times \cdots \times V$ to $W \wedge \cdots \wedge W$ since ψ is multilinear alternating and f is linear. Therefore there is a unique induced linear map by Proposition 9.29***. Call the map $\wedge^k f : V \wedge \cdots \wedge V \to W \wedge \cdots \wedge W$. Since

$$\psi \circ (f \times \cdots \times f)(v_1, \cdots, v_k) = \psi(f(v_1), \cdots, f(v_k)) = f(v_1) \wedge \cdots \wedge f(v_k),$$

we have that

$$\wedge^k f(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k).$$

The two properties follow from the formula for $\wedge^k f$. □

Theorem 9.*.** *Suppose V is an n dimensional vector space, $e_1, \cdots, e_n$ is a basis for V, and $f : V \to V$ is a linear map that has matrix $A = (a_{ij})$ in the basis $e_1, \cdots, e_n$. Let $c_1, \cdots, c_n$ be the column vectors of A so that $c_j = \sum_{i=1}^n a_{ij} e_i$. Then the induced linear map $\wedge^k f : V \wedge \cdots \wedge V \to V \wedge \cdots \wedge V$ on the k-fold wedge product is*

$$\wedge^k f(e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{1 \le i_1 < \cdots < i_k \le n} M_{j_1, \cdots, j_k}(c_{i_1}, \cdots, c_{i_k}) e_{i_1} \wedge \cdots \wedge e_{i_k}$$

*where $M_{j_1, \cdots, j_k}(c_{i_1}, \cdots, c_{i_k})$ is the minor obtained from A by choosing the rows numbered $j_1, \cdots, j_k$ and the columns numbered $i_1, \cdots, i_k$, see Example 9.26***. The elements $e_{j_1} \wedge \cdots \wedge e_{j_k}$ such that $1 \le j_1 < \cdots < j_k \le n$ form a basis for the wedge product by Theorem 9.31***.*

As a special case, if $k = n$, then $\wedge^n f(e_1 \wedge \cdots \wedge e_n) = det(f) e_1 \wedge \cdots \wedge e_n$.

Proof. The proof is a short computation.

$$\begin{aligned}
\wedge^k f(e_{j_1} \wedge \cdots \wedge e_{j_k}) &= \sum_{i_1=1}^n a_{i_1 j_1} e_{i_1} \wedge \cdots \wedge \sum_{i_k=1}^n a_{i_k j_k} e_{i_k} \\
&= \sum_{j_1, \cdots, j_k=1}^n a_{i_1 j_1} \cdots a_{i_k j_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \\
&= \sum_{1 \le j_1 < \cdots < j_k \le n} \sum_{\sigma \in \Sigma_k} (-1)^{\text{sign } \sigma} a_{i_{\sigma(1)} j_1} \cdots a_{i_{\sigma(k)} j_k} e_{i_1} \wedge \cdots \wedge e_{i_k}
\end{aligned}$$

The last line follows since the wedge is zero if the vectors are not distinct and by use of Exercise 9.2***. To complete the proof, we observe that

$$M_{j_1, \cdots, j_k}(c_{i_1}, \cdots, c_{i_k}) = \sum_{\sigma \in \Sigma_k} (-1)^{\text{sign } \sigma} a_{i_{\sigma(1)} j_1} \cdots a_{i_{\sigma(k)} j_k}.$$

□

According to Theorem 9.30*** the multilinear alternating functions from $V \times \cdots \times V$ to $\mathbf{R}$ can be interpreted as $(V \wedge \cdots \wedge V)^*$ (See Exercise 9.7***. We wish to construct an isomorphism between $\wedge^k(V^*)$ and $(\wedge^k V)^*$ analogous to the identification for tensor products. Unfortunately, there are two common choices in use. We discuss both. One is used when the author is primarily interested in using both tensor and exterior products. The other is used when the author is primarily interested in integration.

The notation used below is now given. We use $\wedge^k V$ and $\otimes^k V$ for the k fold wedge and tensor products. The symbols ϕ and ψ are used for the maps in Definitions 9.4*** and 9.27*** as well as the maps they induce. The subscript indicates to the reader the vector used to produce the tensor or exterior product.

Isomorphism I. If we are required to be consistent with the maps already defined, then there would be no choice. We already have the following maps:

$$\begin{tikzcd}
(\wedge^k V)^* \arrow[r, "(\psi_V)^*"] & (\otimes^k V)^* \arrow[d] \\
(\wedge^k V^*) & (\otimes^k V^*) \arrow[l, "\psi_{V^*}"]
\end{tikzcd}$$

The top map is the interesting map to observe. Take

$$\begin{aligned}(\psi_V)^*((e_{i_1}\wedge\cdots\wedge e_{i_k})^*)(e_{i_1}\otimes\cdots\otimes e_{i_k}) &= (e_{j_1}\wedge\cdots\wedge e_{j_k})^*(\psi_V(e_{i_1}\otimes\cdots\otimes e_{i_k}))\\ &= \begin{cases}\pm 1 \text{ if } i_1,\cdots,i_k \text{ is a permutation of } j_1,\cdots,j_k\\ 0 \text{ otherwise}\end{cases}\end{aligned}$$

since $e_{i_1}\wedge\cdots\wedge e_{i_k} = (-1)^{\mathrm{sign}\tau} e_{i_{\tau(1)}}\wedge\cdots\wedge e_{i_{\tau(k)}}$.

Hence, we have $(\phi_V)^*((e_{i_1}\wedge\cdots\wedge e_{i_k})^*) = (\sum_{\sigma\in\Sigma_k}(-1)^\sigma e_{i_{\sigma(1)}}\otimes\cdots\otimes e_{i_{\sigma(k)}})^*$, and the induced isomorphism $(\wedge^k V)^*\to(\wedge^k V^*)$ is

$$(e_{i_1}\wedge\cdots\wedge e_{i_k})^*\mapsto (k!)e^*_{i_1}\wedge\cdots\wedge e^*_{i_k}$$

Isomorphism II. The second map is convenient for area. The second map is

$$(e_{i_1}\wedge\cdots\wedge e_{i_k})^*\mapsto e^*_{i_1}\wedge\cdots\wedge e^*_{i_k}$$

Note that $(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y})^*(\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix})$ is the area of the parallelogram determined by $\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}c\\d\end{pmatrix}$. If we write $(\frac{\partial}{\partial x})^* = dx$ and $(\frac{\partial}{\partial y})^* = dy$, then under the first isomorphism $dx\wedge dy$ does not determine the area of the parallelogram but $\frac{1}{2!}$ times the area, however, the second pairing $dx\wedge dy$ does determine the area. Since the measuring of area is the central feature of integration, the second pairing lends itself to integration.

Exercise 9.1*.** *Write out the details to demonstrate the validity of Formula (2)****

Exercise 9.2*.** *Show that $A \subset V \otimes \cdots \otimes V$ defined in Definition 9.*** contains the elements*

$$v_1 \otimes \cdots \otimes v_k - (-1)^{\text{sign } \sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

for all permutations $\sigma \in \Sigma_k$. Show that $v_1 \wedge \cdots \wedge v_k = (-1)^{\text{sign } \sigma} v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}$.

Exercise 9.3*.** *Suppose that V_1, V_2, and V_3 are vector spaces. Show that there are well-defined isomorphisms*

$$f : V_1 \otimes V_2 \otimes V_3 \to (V_1 \otimes V_2) \otimes V_3$$

such that $f(u \otimes v \otimes w) = (u \otimes v) \otimes w$ and

$$g : V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$$

such that $g(u \otimes v \otimes w) = u \otimes (v \otimes w)$.

Exercise 9.4*.** *Suppose that V is a vector space. Show that there are well-defined isomorphisms*

$$f : V \wedge V \wedge V \to (V \wedge V) \wedge V$$

such that $f(u \wedge v \wedge w) = (u \wedge v) \wedge w$ and

$$g : V \wedge V \wedge V \to V \wedge (V \wedge V)$$

such that $g(u \wedge v \wedge w) = u \wedge (v \wedge w)$.

Exercise 9.5*.** *Suppose that $f : \mathbf{R}^3 \to \mathbf{R}^3$ is represented by the matrix $\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix}$ in the ordered basis e_1, e_2, e_3 and that $g : \mathbf{R}^3 \to \mathbf{R}^3$ is represented by the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}$ in the same ordered basis. Find the matrix for $f \otimes g$ in the ordered basis $e_1 \otimes e_1$, $e_2 \otimes e_1$, $e_3 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_2$, $e_3 \otimes e_2$, $e_1 \otimes e_3$, $e_2 \otimes e_3$, $e_3 \otimes e_3$.*

Exercise 9.6*.** *Suppose that $V_1, \cdots, V_k$ are vector spaces and let $Mult(V_1 \times \cdots \times V_k, \mathbf{R})$ denote the multilinear functions $f : V_1 \times \cdots \times V_k \to \mathbf{R}$. Show that $Mult(V_1 \times \cdots \times V_k, \mathbf{R})$ is a vector space under the usual addition an multiplication in $\mathbf{R}$. Show that there is a canonical isomorphism*

$$Mult(V_1 \times \cdots \times V_k, \mathbf{R}) \cong (V_1 \otimes \cdots \otimes V_k)^*.$$

Exercise 9.7*.**

a. Suppose that V is a vector space and let $Alt(V \times \cdots \times V, \mathbf{R})$ denote the k variable multilinear alternating functions $f : V \times \cdots \times V \to \mathbf{R}$. Show that $Alt(V \times \cdots \times V, \mathbf{R})$ is a vector space under the usual addition an multiplication in $\mathbf{R}$. Show that there is a canonical isomorphism

$$Alt(V \times \cdots \times V, \mathbf{R}) \cong (V \wedge \cdots \wedge V)^*.$$

b. If $\dim V = n$, then show that any multilinear alternating function in n-variables $f : V \times \cdots \times V \to \mathbf{R}$ is a multiple of det. *Show that*

$$Vol_n(v_1, \cdots, v_n) = \det(v_1, \cdots, v_n).$$

Exercise 9.8*.** *Suppose that $V_1, \cdots, V_k$ and Q are vector spaces and $\rho : V_1 \times \cdots \times V_k \to \mathbf{Q}$ is a multilinear function. Show that if $\rho : V_1 \times \cdots \times V_k \to \mathbf{Q}$ satisfies the universal property for multilinear functions (Proposition 9.6***), then there is a unique isomorphism $\iota : V_1 \otimes \cdots \otimes V_k \to Q$ such that $\iota \circ \rho = \phi$.*

Exercise 9.8*.** *Suppose that V and Q are vector spaces and $\rho : V \times \cdots \times V \to \mathbf{Q}$ is a multilinear function in k variables. Show that if $\rho : V \times \cdots \times V \to \mathbf{Q}$ satisfies the universal property for altermating multilinear functions (Proposition 9.29***), then there is a unique isomorphism $\iota : V \wedge \cdots \wedge V \to Q$ such that $\iota \circ \rho = \psi$.*