Complex Analysis

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Contents

List of Figures

1 Properties of Complex Numbers

Let us review the idea of complex numbers. Recall that the square root of -1 is represented with the letter i. So by definition

$$i = \sqrt{-1},$$

and therefore

$$i^2 = -1.$$

Sometimes in electrical engineering j is used in place of i, so as not to conflict with the traditional use of i for electrical current. Real multiples of i are called imaginary numbers. A complex number is a sum of a real number and an imaginary number. For example

$$2 + 5i,$$

is a complex number, where 2 is called the real part, and 5 the imaginary part of this complex number. Complex numbers $z = x + iy$ are represented in the two dimensional xy plane, which is called the complex plane, where the real number part is plotted horizontally in the x axis direction, and the imaginary number part plotted vertically in the y axis direction. So the point representing

$$2 + 5i,$$

is plotted two units to to the right of the vertical y axis, and 5 units above the horizontal x axis.

Complex numbers are added and multiplied using the usual laws of algebra, so for addition we have

$$(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i,$$

and for multiplication

$$(x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + y_1 x_2 i + y_1 y_2 i^2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The magnitude or absolute value of a complex number

$$z = x + iy,$$

is the distance to the origin, written as

$$|z| = \sqrt{x^2 + y^2}.$$

The conjugate of a complex number

$$z = x + iy,$$

is obtained by changing the sign of the complex part, and written with an over-line.

$$\overline{z} = x - yi.$$

We also write the conjugate of a complex expression with an over-line.
Exercise, Show that

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2},$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$$

and

$$z\bar{z} = |z|^2.$$

Thus to divide two complex numbers

$$w = \frac{z_1}{z_2},$$

multiply numerator and denominator by the conjugate of z_2, getting

$$w = \frac{z_1 \bar{z}_2}{|z_2|^2}.$$

Complex numbers also have a polar representation

$$z = |z|(\cos(\theta) + i\sin(\theta)) = |z|e^{i\theta}.$$

The number $|z|$ is called the modulus, and angle θ the argument. We have

$$z_1 z_2 = |z_1|e^{i\theta_1}|z_2|e^{i\theta_2} = |z_1||z_2|e^{i(\theta_1+\theta_2)}.$$

The product of two complex numbers has modulus (or magnitude) equal to the product of the moduli, and argument equal to the sum of the arguments. Now the argument function $arg(z)$ is any angle θ such that

$$|z|(\cos(\theta) + i\sin(\theta)) = |z|e^{i\theta} = z,$$

which can be any angle

$$\theta + n2\pi,$$

for any natural number n. So the function $arg(z)$ is multiple valued, a feature of complex analysis, which contradicts the usual definition of a function. This difficulty leads to the need to extend the complex plane to larger domains known as Riemann Surfaces. This historically lead to the advanced mathematical idea of a manifold, which now is a fundamental part of differential geometry and advanced physics, including string theory.

2 Roots of Complex Numbers

We shall see that every complex number a has n roots

$$a_1, a_2, ..., a_n,$$

so that

$$a_k^n = a.$$

If the magnitude of a is $|a|$ and the argument of a is $0 \le \theta < 2\pi$ then

$$a_k = |a|^{1/n} \exp(i(\theta + 2\pi k)/n),$$

is a distinct nth root of a for each $k = 0, 1, ..., n-1$. To show that these numbers are distinct suppose j and k are in the set $\{0, 1, ..., n-1\}$ with $j \neq k$, then the difference of the arguments of a_j and a_k is

$$(\theta + 2\pi j)/n - (\theta + 2\pi k)/n = 2\pi(j-k)/n,$$

and

$$0 < |(j-k)/n| < 1.$$

So the two numbers a_j and a_k have arguments, whose difference is not a multiple of 2π, and so must be distinct.

Equivalently, every polynomial of the form

$$p(z) = z^n - a = 0,$$

has n roots. If a_1 is a root of this polynomial, we can divide by $z - a_1$ and get a quotient polynomial $q(z)$ and a remainder r.

$$p(z) = q(z)(z - a_1) + r.$$

But r must be zero. Indeed $p(a_1) = 0$ because a_1 is a root, and from the division $p(a_1) = r$. Therefore $r = 0$, and we have

$$p(z) = q(z)(z - a_1).$$

Then continuing by dividing again by $z - a_k$, for $k = 2, 3, ..., n$, we arrive at the factorization

$$z^n - a = (z - a_1)(z - a_2)...(z - a_n).$$

So clearly there can be no more than n roots, because the degree of the polynomial is n.

3 Open and Closed Sets

An open disk of radius r and center z_c is the set

$$D_r(z_c) = \{z : |z - z_c| < r\},$$

which is the interior points of a circle of radius r.

An open set U is a set such that if $z \in U$, then there exists a number $r > 0$, and an open disk $D_r(z)$ so that $D_r(z) \subset U$.

A closed set is the compliment of an open set, an open set does not contain its boundary points, a closed set does.

4 The Heine-Borel Theorem

The Heine-Borel Theorem and the related Bolzano-Weierstrass Theorem are tied up with the topological concepts of, closed sets, bounded sets, compactness, and completeness. We supply here a little information about these things, which is not at all intended to be sufficient for anyone who has not had previous contact with these things.

There are various ways to define a compact set. (1) A set S is compact if for every family of open sets that cover S, there is a finite subset of the family that also covers S. Equivalently, (2) a space is compact if every sequence in the space has a convergent subsequence.

A space is complete if every Cauchy sequence in the space converges to a point in the space. A set is closed if every convergent sequence in the set converges to a point in the set.

The original Heine-Borel Theorem pertained to the real numbers and the complex numbers, and was stated for complex numbers in a form equivalent to the following:
Heine-Borel Theorem. Every bounded closed set of complex numbers is compact.

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Heine, E. "Die Elemente der Functionenlehre."
J. reine angew. Math. 74, 172-188, 1871.
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Completeness Theorem. The Complex Numbers form a Complete Space. These are tangential topics, which we don't want to divert attention to here. See books on topology, metric spaces, real and complex analysis, for the details.

From these ideas it follows that a sequence of nested closed sets

$$\{A_n : n = 1, 2, 3, ...\},$$

whose diameters converge to zero, have an intersection containing a single limit point z_0. This will be used in the proof of the Cauchy Integral Theorem given below.

Also relevant is the following theorem.

Bolzano-Weierstrass Theorem Every bounded infinite sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers has a cluster point, a point ζ such that every open neighborhood of ζ contains points of $\{z_n\}_{n=1}^{\infty}$.

5 A Complex Function of a Complex Variable

We write a complex function as

$$f(z) = w,$$

where z and w are complex numbers

$$z = x + iy$$

$$w = u + iv.$$

$$f(x + yi) = u(x + iy) + v(x + iy)i = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ can be considered real valued functions.

6 The Complex Derivative

The derivative is defined as

$$\frac{df}{dz} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Notice that for the derivative to exist the limits must be equal as z approaches z_0 from all directions.

So if the derivative exists, then holding $y = y_0$, we have

$$\frac{df}{dz} = \lim_{x \to x_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Similarly holding $x = x_0$, we have

$$\frac{df}{dz} = \lim_{y \to y_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + i \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{i(y - y_0)}$$

$$= \frac{1}{i} \left[\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right] = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Equating these two derivative representations we get the Cauchy-Riemann Equations, which characterize an analytic function.

7 The Cauchy-Riemann Equations

If the derivative of

$$f(x + yi) = u(x, y) + iv(x, y)$$

exists, by considering limits as $z = x + iy_0$ goes to z_0, and $z = x_0 + iy$ goes to z_0, as we did in the previous section, and by equating the two expressions for the derivative, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

8 Both the Real Part and the Imaginary Part of an Analytic Function, Are Solutions to Laplace's Equation in Two Dimensions.

Indeed, by differentiating the Cauchy-Riemann equations partially we find that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

9 Analytic Functions and Entire Functions

Functions that have a derivative everywhere in an open region are called analytic functions, (sometimes called regular functions or holomorphic functions). Then one can show that such functions have derivatives of all orders in that region, and thus they have convergent Taylor power series expansions at each point of that open region, with a radius of convergence equal to the distance to the closest singular point. Some functions are analytic everywhere in the complex plane, and so they have a power series representation with an infinite radius of convergence. An example is the complex exponential function defined by

$$exp(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

This function is called the exponential function because is satisfies the law of exponents

$$exp(z_1 + z_2) = e^{z_1+z_2} = e_1^z e_2^z.$$

This property can be proven by multiplying power series together to get a new power series.

Definition Entire Functions. An entire function is a function that is analytic in the whole complex plane.

The exponential function is an entire function.

10 The Complex Logarithm

The complex logarithm is defined as the inverse function of the exponential function, as it is in the real variable case. Consider the function

$$f(z) = \ln(|z|) + arg(z)i,$$

where $\ln(|z|)$ is the real natural logarithm of the magnitude of z. Let $\theta = arg(z)$ Then

$$e^{f(z)} = e^{\ln(|z|)+\theta i} = e^{\ln(|z|)}e^{\theta i} = |z|(cos(\theta) + \sin(\theta)i) = z.$$

Therefore by the definition of an inverse function, $f(z)$ is the inverse of e^z. So we define the complex logarithm as

$$\ln(z) = \ln(|z|) + arg(z)i.$$

11 Analytic Continuation

The logarithm function of the previous section is meant to be an analytic function. However it is not defined at zero, so is not an entire function. Also near the positive real axis if we take the standard value for the argument between 0 and 2π there is a jump in the argument as we cross from just below the positive real axis to the upper half plane, from a value near 2π to 0. So the function can not be analytic at a point of the real line. When such a thing happens we can extend the definition of an analytic function by using its power series representation to construct a new power series about a new point to get a new power series with a new radius of convergence to extend the function. In this case we generate a new sheet or copy of the complex plane lying above the old complex plane with a transition or cut at the positive real axis. If we continue this continuation everywhere we get an extended domain of definition of the logarithm consisting of a series of spiral sheets called a Riemann Surface for the logarithm. This discussion is rather vague, because in general the subject is rather complex, no pun intended.

12 Complex Integration

A complex integral is defined as a limit of sums similar to the definition of the real Riemann integral

$$\int_C f(z)dz = \lim_{n\to\infty, \Delta z_i \to 0} \sum_{i=1}^{n} f(\zeta_i)\Delta z_i,$$

where the Δz comes from a subdivision of points on the curve C, and where ζ_i is an arbitrary point between the pair of points z_k, z_{k+1}, where $\Delta z = z_{k+1} - z_k$. If the curve in the complex plane is a function of a real parameter t this is equivalent to the integral along the curve C, from starting point $C(a)$ to ending point $C(b)$,

$$\int_a^b f(z)\frac{dz}{dt}dt,$$

Example. We integrate the function $f(z) = 1/z$ around a circle with center at the origin, of radius $r = 1$. The curve C is described by $C(t) = z(t) = \cos(t) + i\sin(t) = \exp(it)$. We have

$$\int_C f(z)dz = \int_C \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{z}\frac{dz}{dt}dt.$$

We have

$$\frac{dC}{dt} = \frac{dz}{dt} = i\exp(it)$$

and

$$f(z) = \frac{1}{z} = \frac{1}{\exp(it)}.$$

So

$$\int_C \frac{1}{z}dz = \int_C \frac{i\exp(it)}{\exp(it)}dt = i\int_0^{2\pi} dt = 2\pi i.$$

For later reference this integral is $2\pi i$ times the residue, which is 1, at the first order pole located at $z = 0$.

13 A Simply Connected Region

A simply connected region is a region where any closed curve can be shrunk to a point without leaving the region. Thus a region consisting of a disk with a smaller interior disk subtracted from it, is not simply connected, because a circle around the smaller disk when shrunk to a point must penetrate the smaller disk.

14 Cauchy's Integral Theorem

Theorem. If a function $f(z)$ is analytic in a simply connected region Υ then for a closed curve C in Υ
$$\int_C f(z)dz = 0.$$

This implies that the integral between two points is independent of the path joining the points.

Proof.

This can be proved by first showing than this is true around a triangle. One can do this by subdividing the triangle into four triangles, and repeating this subdivision n times, showing that after the nth stage the original integral is less than 4^n times the integral of an nth stage triangle. The triangle subdivisions are nested, so that as n goes to infinity they will contain a common limit point z_0 by the Heini-Borel theorem. At this point z_0, because the function has a derivative there, given $\epsilon > 0$, there exist a δ so that if $|z - z_0| < \delta$, then
$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$
or
$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \epsilon|z - z_0|.$$
Let $\eta(z)$ be defined by
$$f(z) = f(z_0) - zf'(z_0) + z_0 f'(z_0) + \eta(z)(z - z_0).$$
We claim that for every z such that $|z - z_0| < \delta$ we have
$$|\eta(z)| < \epsilon.$$
For assume there is a z so that
$$|\eta(z)| \geq \epsilon.$$
Then
$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| = |\eta(z)||(z - z_0)| \geq \epsilon|z - z_0|.$$

If we integrate
$$f(z) = f(z_0) - zf'(z_0) + z_0 f'(z_0) + \eta(z)(z - z_0).$$

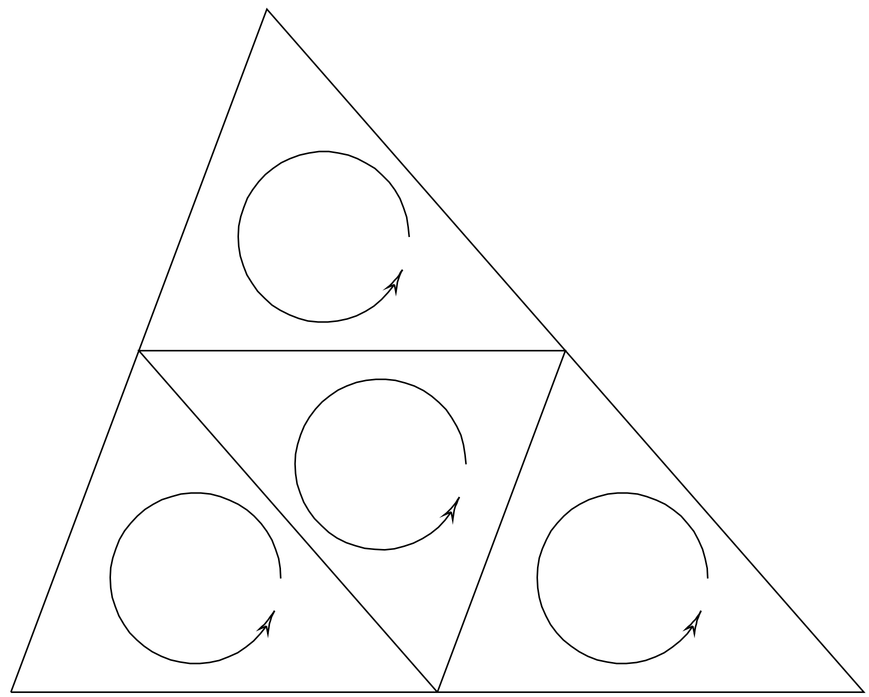

Figure 1: The Cauchy Integral Theorem can be established by proving it for a triangle, and then realizing that the interior of a general curve can be triangulated to any accuracy. Since the integral is zero in each triangle it is zero on the closed curve. So we may decompose a triangle into four similar subtriangles, and the integral around the triangle is equal to the sum of the integrals around the subtriangles because of cancellation on the internal edges. The absolute value of of the integral around the triangle is less than four times the absolute value of the largest absolute value of the four subtriangles. We choose this subtriangle and subdivide it. And then continue this process, getting an infinite set of nested triangles of smaller and smaller size.

we get

$$\int_{T_n} f(z)dz = 0 + 0 + 0 + \int_{T_n} \eta(z)(z - z_0)dz,$$

because the integrals of the first three terms are zero. Then considering bounds on the size of the perimeter of triangle T_n, and the relation to the original triangle perimeter, and on the $|z - z_0|$ the integral of the original triangle is less than

$$\frac{\epsilon}{2}s^2,$$

where s is the perimeter of the original triangle. It follows that the integral around the original triangle is zero.

For the general case, the interior of a curve bounding a region can be approximated by triangles. Thus the result follows.

See Konrad Knopp, **The Theory of Functions**, Volume I, p49, for details.

15 Cauchy's Integral Formula

A plane closed curve C is said to have the counterclockwise orientation when traversing the curve in the forward direction, the direction is opposite to the direction of the motion of a clock hand on the clock face. For a curve such as a circle or an ellipse, this is pretty obvious. However, one can stretch and distort a circle with twists and turns in such a way that this idea is not quite so obvious. However, in all cases such a curve will enclose an interior region, and the points of the interior near the curve will always lie to the left of the curve direction. So this is what is meant by a counterclockwise oriented closed curve. Similarly a clockwise oriented closed curve has near interior points to the right of the curve direction. **Theorem.** Let a closed curve C, be inside an open region where $f(z)$ is analytic, C has a clockwise orientation, and C encloses an interior point z_0, then

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}d\zeta = f(z_0).$$

That is, every value of an analytic function is determined by the values of the function on a bounding curve of a region containing the point.

Proof. We have

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}d\zeta = \frac{1}{2\pi i}\int_C \frac{f(\zeta_0)}{\zeta - z_0}d\zeta + \frac{1}{2\pi i}\int_C \frac{f(\zeta) - f(\zeta_0)}{\zeta - z_0}d\zeta.$$

By Cauchy's Integral Theorem the integrals on the right can be replaced by integrals on a circle C_r of radius r about center z_0,

$$\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta_0)}{\zeta - z_0}d\zeta$$

and

$$\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta) - f(\zeta_0)}{\zeta - z_0}d\zeta.$$

The first is equal to $f(z_0)$. For the second, because $f(z)$ is analytic at z_0 and thus continuous, for any ϵ there exists a circle of radius r where $|f(\zeta) - f(\zeta_0)| < \epsilon$, and so where

$$\left|\frac{f(\zeta) - f(\zeta_0)}{\zeta - z_0}\right| < \frac{\epsilon}{r}$$

Then

$$\left|\frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta) - f(\zeta_0)}{\zeta - z_0}d\zeta\right| < \frac{\epsilon 2\pi r}{2\pi r} = \epsilon,$$

where $2\pi r$ is the length of the circle C_r. Therefore this second integral is zero, and the result follows.

16 Cauchy's Integral Formulas for Derivatives

Theorem Let a closed curve C with counterclockwise orientation, be inside an open region where $f(z)$ is analytic, and let it contain the point z_0, then the nth derivative of f at z_0 is

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta.$$

Proof. Let us first prove this for the first derivative and then apply mathematical induction to prove it for all n.

The derivative at z is

$$\frac{df(z_0)}{dz} = \lim_{z' \to z_0}\frac{f(z') - f(z_0)}{z' - z_0}.$$

So if we can show that the following limit is zero,

$$\lim_{z'\to z_0}\left[\frac{f(z')-f(z_0)}{z'-z_0}-\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z_0)^2}d\zeta\right]=0,$$

then we will have proved the integral formula for the first derivative. So using the Cauchy Integral formula we have

$$f(z')-f(z_0)=\frac{1}{2\pi i}\left[\int_C\frac{f(\zeta)(z'-z_0)}{(\zeta-z')(\zeta-z_0)}d\zeta\right].$$

So

$$\frac{f(z')-f(z_0)}{(z'-z_0)}=\frac{1}{2\pi i}\left[\int_C\frac{f(\zeta)}{(\zeta-z')(\zeta-z_0)}d\zeta\right].$$

So

$$\begin{aligned}
&\left[\frac{f(z')-f(z_0)}{z'-z_0}-\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z_0)^2}d\zeta\right]\\
&=\left[\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z')(\zeta-z_0)}d\zeta-\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z_0)^2}d\zeta\right]\\
&=\left[\frac{1}{2\pi i}\int_C f(\zeta)\left(\frac{1}{(\zeta-z')(\zeta-z_0)}-\frac{1}{(\zeta-z_0)^2}\right)d\zeta\right]\\
&=\left[\frac{1}{2\pi i}\int_C f(\zeta)\frac{(\zeta-z_0)-(\zeta-z')}{(\zeta-z')(\zeta-z_0)^2}d\zeta\right]\\
&=\left[\frac{(z'-z_0)}{2\pi i}\int_C f(\zeta)\frac{1}{(\zeta-z')(\zeta-z_0)^2}d\zeta\right]
\end{aligned}$$

Let r be the minimum distance from z_0 to the curve C, then

$$|\zeta-z_0|\geq r,$$

so

$$\frac{1}{|\zeta-z_0|}\leq\frac{1}{r}.$$

Given $0<\epsilon<r/2$, there exists a disk D centered at z_0 of radius less than epsilon such that if $z'\in D$ then $|\zeta-z'|>r/2$, for otherwise there exists a $\zeta\in C$ such that

$$|z_0-\zeta|\leq|z_0-z'|+|z'-\zeta|<r/2+r/2<r$$

which contradicts that r is the minimum distance from z_0 to C. Hence if $z' \in D$, then

$$|\frac{1}{\zeta - z'}| < r/2.$$

Let M be an upper bound on $|f(\zeta)|$ on curve C and let λ be the length of curve C (We assume that C is a compact rectifiable curve). Then

$$\left|\frac{(z'-z_0)}{2\pi i}\int_C f(\zeta)\frac{1}{(\zeta - z')(\zeta - z_0)^2}d\zeta\right| < \frac{2\epsilon M\lambda}{r^3}.$$

Since ϵ can be made arbitrarily small, we conclude that

$$lim_{z'\to z_0}\left|\frac{f(z')-f(z_0)}{z'-z_0} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^2}d\zeta\right|$$

$$= lim_{z'\to z_0}\left|\frac{(z'-z_0)}{2\pi i}\int_C f(\zeta)\frac{1}{(\zeta - z')(\zeta - z_0)^2}d\zeta\right| = 0.$$

This completes the proof.

Using a very similar argument, by using mathematical induction, we can prove the formula for all n.

17 Taylor Series Expansion

The stategy in proving the following theorem is to start with Cauchy's integral theorem, then manipulate

$$\frac{1}{\zeta - z}$$

to get it in the form of the sum of a finite geometric series. Then write it as a sum and an error term. Multiply by

$$\frac{1}{2\pi i}f(\zeta)$$

and integrate. The sum becomes the nth partial sum of the Taylor Series, with the derivatives given by the Cauchy integral formulas. Using the fact that the function $f(\zeta)$ is bounded on the contour path and some other bounds, show that the error term goes to zero as n goes to ∞.

Taylor's Theorem Let $f(z)$ be an analytic function in an open circle C_0 with center z_0 and radius r_0, then at all points z in the circle the following series converges and

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

Proof. Let z be any point inside of C_0, and let $r = |z - z_0|$. Let C_1 be a circle of radius r_1 with center z_0, with

$$r < r_1 < r_0,$$

By Cauchy's integral theorem

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We have

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)}$$

$$= \frac{1}{\zeta - z_0} \left[\frac{1}{1 - (z - z_0)/(\zeta - z_0)} \right].$$

Let

$$a = \frac{z - z_0}{\zeta - z_0}.$$

then

$$s = 1 + a + a^2 + ... + a^{n-1}$$

$$sa = a + a^2 + ... + a^n$$

$$s(1 - a) = 1 - a^n$$

$$s = \frac{1}{1 - a} - \frac{a^n}{1 - a}$$

So

$$\frac{1}{1 - a} = s + \frac{a^n}{1 - a}$$

$$= \sum_{k=0}^{n-1} a^k + \frac{a^n}{1 - a}$$

So

$$\frac{1}{(\zeta - z)} = \frac{1}{(\zeta - z_0)} \frac{1}{1-a}$$

$$= \sum_{k=0}^{n-1} \frac{a^k}{(\zeta - z_0)} + \frac{1}{(\zeta - z_0)} \frac{a^n}{(1-a)}$$

$$= \sum_{k=0}^{n-1} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} + \frac{a^n}{(\zeta - z)}$$

$$= \sum_{k=0}^{n-1} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} + \frac{(z - z_0)^n}{(\zeta - z)(\zeta - z_0)^n}.$$

multiplying through by

$$\frac{f(\zeta)}{2\pi i},$$

and integrating with respect to ζ we get

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta$$

$$= \sum_{k=0}^{n-1} \left[\frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} dz \right] (z - z_0)^k + (z - z_0)^n \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} d\zeta$$

$$= \sum_{k=0}^{n-1} \frac{f^(k)(z_0)}{k!} (z - z_0)^k + (z - z_0)^n \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} d\zeta.$$

In the remainder term

$$R_n = (z - z_0)^n \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n} d\zeta,$$

$$\alpha = \frac{|z - z_0|}{|\zeta - z_0|} < 1,$$

so α^n goes to zero as n goes to ∞. $|f(\zeta)|$ is bounded by some constant $M > 0$ on the curve C_1, the length of the curve is C_1 is $2\pi r_1$, and if d is the minimum distance from z to C_1 then

$$\frac{1}{|\zeta - z|} < 1/d,$$

then the absolute value of R_n is less than some fixed number times α^n and so goes to zero as n goes to infinity. Therefore

$$f(z) = \sum_{k=0}^{\infty} \frac{f^(k)(z_0)}{k!}(z - z_0)^k,$$

for any z inside the circle C_0.
Corollary In the previous theorem, the circle can be expanded about z_0, until its boundary meets a singular point. The theorem holds in this expanded circle. If there is no singularity, then the function $f(z)$ is equal to the Taylor power series in the entire complex plane. So an entire function is defined everywhere by its Taylor power series.

18 The Elementary Functions

Define $\sin(z)$, $\cos(z)$, $\sinh(z)$, $\cosh(z)$ and $\exp(z)$ by the usual Taylor series:

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - ... = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}.$$

$$\cos(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - ... = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!}.$$

$$\sinh(z) = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + ...$$

$$\cosh(z) = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + ...$$

$$\exp(z) = 1 + \frac{1}{1!}z^1 + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + ...$$

Then

$$\begin{aligned}\exp(iz) &= 1 + i\frac{1}{1!}z - \frac{1}{2!}z^2 - i\frac{1}{3!}z^3 + ... \\ &= \cos(z) + i\sin(z).\end{aligned}$$

This is the famous **Euler's Formula**.

We also have

$$\sin(iz) = i\sinh(z)$$

$$\sinh(iz) = i\sin(z)$$
$$\cos(iz) = \cosh(z)$$
$$\cosh(iz) = \cos(z).$$

If $z = x + iy$ then

$$\sin(z) = \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$
$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

$$\cos(z) = (\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy)$$
$$= \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

19 Singular Points

Singular points of a function are points where a function is not analytic. The simple singular points x_0 are called poles and take the form

$$\frac{1}{(z-z_0)^k}, k = 1, 2, 3, ...$$

They are called poles because the magnitude of the function goes to infinity as $z \to z_0$, and a plot shows infinite peaks at z_0. An isolated singularity is a point z_0 where the function is analytic at any point in an open neighborhood of z_0 not including z_0 itself. An open set is a set not including its boundary.

20 The Laurent Expansion

Theorem. If f is analytic in an open region containing $r_2 < |z - z_0| < r_1$ then

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n$$
$$= \sum_{n=1}^{\infty} \frac{A_{-n}}{(z-z_0)^n} + A_0 + \sum_{n=1}^{\infty} A_n (z-z_0)^n,$$

where

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

and C is any counterclockwise curve in the annular region enclosing the small circle of radius r_2.

Proof. Consider a path consisting of a counterclockwise circle C_1 or radius r_1 and center z_0 together with a second counterclockwise circle C_2 of radius r_2 and center z_0 and a straight line joining the two circles. The complete path consists of the two circles and the line traced twice in opposite directions, see the figure. By Cauchy's Integral theorem, with z in the annulus between the two circles we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta - z)}d\zeta - \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta - z)}d\zeta.$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta - z)}d\zeta + \frac{1}{2\pi i}\int_{-C_2}\frac{f(\zeta)}{(\zeta - z)}d\zeta,$$

where by $-C_2$ we mean the curve C_2 having opposite direction.

Our strategy is to expand a term like

$$\frac{1}{\zeta - z}$$

by adding and subtracting z_0 as a geometric power series, and using uniform convergence, to integrate term by term, thereby obtaining the formula.

Hence

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)}$$

$$= \frac{1}{(\zeta - z_0)[1 - (z - z_0)/(\zeta - z_0)]}$$

$$= \frac{1}{(\zeta - z_0)}\sum_{k=0}^{\infty}\left[\frac{(z - z_0)}{(\zeta - z_0)}\right]^k.$$

$$= \sum_{k=0}^{\infty}\frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}.$$

This series converges for ζ on the curve C_1 because

$$\left|\frac{(z - z_0)^k}{(\zeta - z_0)^k}\right| < 1$$

there.

So by the Weierstrass M-test, the series

$$\sum_{k=0}^{\infty} f(\zeta)\frac{(z-z_0)^k}{(\zeta-z_0)^{k+1}},$$

converges uniformly, so can be integrated term by term. We get

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-z)}d\zeta$$

$$=\sum_{k=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta\right](z-z_0)^k$$

$$=\sum_{k=0}^{\infty} A_k(z-z_0)^k,$$

where

$$A_k=\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta, (k=0,1,2,3,...).$$

Similarly (fill in the details)

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-z)}d\zeta$$

$$=\sum_{k=1}^{\infty} A_{-k}\frac{1}{(z-z_0)^k},$$

where

$$A_{-k}=\frac{1}{2\pi i}\int_{C_2} f(\zeta)(\zeta-z_0)^{k-1}d\zeta, (k=1,2,3,...).$$

Because the integrands are analytic in the annular region, the integrals defining A_k and A_{-k} may be over any annular counterclockwise curve C.

21 Laurent Expansion and Singular Point Examples

(a)

$$\frac{e^{2z}}{(z-1)^3}$$

(b)

$$(z-3)\sin(1/(z+2))$$

(c)

$$\frac{z-\sin(z)}{z^3}$$

(d)

$$\frac{z}{(z+1)(z+2)}$$

(e)

$$\frac{1}{z^2(z-3)}$$

22 Residues

Let an analytic function be analytic in an open region except for a set of isolated singular points. The residue at one of these singular points z_k is the coefficient a_{-1} of the term

$$\frac{1}{(z-z_k)}$$

in the unique Laurent expansion of $f(z)$ about the point z_k.

23 The Residue Theorem

Theorem Let C be a closed counterclockwise contour in an open region where a function $f(z)$ is analytic, except at n isolated singular points inside of C, where the residues are $\alpha_1, \alpha_2, ..., \alpha_n$. Then

$$\int_C f(z)dz = 2\pi i(\alpha_1 + \alpha_2 + ... + \alpha_n).$$

Proof.

Consider the integration of the function $f(z) = 1/z$ around a circle C with center at the origin. The curve C is described by $z(t) = \cos(t) + i\sin(t) = \exp(it)$, with t in the closed interval $[0, 2\pi]$. We have

$$\int_C f(z)dz = \int_C \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{z}\frac{dz}{dt}dt.$$

On the contour C we have

$$\frac{dz}{dt} = i\exp(it)$$

and

$$f(z) = \frac{1}{z} = \frac{1}{\exp(it)}.$$

So

$$\int_C \frac{1}{z}dz = \int_C \frac{i\exp(it)}{\exp(it)}dt = i\int_0^{2\pi} dt = 2\pi i.$$

Now onsider the integration of the function $f(z) = 1/z^n$, where n is an integer $n > 1$, around the circle C with center at the origin. The curve C is described by $z(t) = \cos(t) + i\sin(t) = \exp(it)$. We have

$$\int_C f(z)dz = \int_C \frac{1}{z^n}dz = \int_0^{2\pi} \frac{1}{z^n}\frac{dz}{dt}dt.$$

We have

$$\frac{dz}{dt} = i\exp(it)$$

and

$$\frac{1}{z^n} = \frac{1}{(\exp(it))^n} = \frac{1}{(\exp(int))}.$$

So because $n-1$ is greater than 0, we have

$$\int_C \frac{1}{z^n}dz = \int_0^{2\pi} \frac{i\exp(it)}{\exp(int)}dt$$

$$= i\int_0^{2\pi} \frac{1}{\exp(i(n-1)t)}dt$$

$$= i\int_0^{2\pi} \frac{1}{\cos((n-1)t) + i\sin((n-1)t)}dt$$

$$= i\int_0^{2\pi} (\cos((n-1)t) - i\sin((n-1)t))dt = 0$$

We get zero for this integral because integrating both $\cos((n-1)t)$ and $\sin((n-1)t)$ from 0 to 2π by the nature of these functions the positive areas equal the negative areas.

This zero result is true for any contour surrounding a single pole because the function is analytic except at the pole. The result also holds in exactly the same way for an isolated singularity a not at the origin by considering a unit circle surrounding a, and the finction $1/(z-a)^n$. So considering the Laurent expansion of $f(z)$ about an isolated singular point a, the integral of a contour curve surrounding a single such point has value $2\pi i$ times the residue, which is the coefficient of the term $1/(z-a)$ in the Laurent expansion of $f(z)$. Further if there are multiple such isolated singular points, the value of the integral of $f(z)$ around a contour curve C containing these points, the value is $2\pi i$ times the sum of the residues at all of the singular points. This proves the theorem.

24 Calculating Residues

We can find the residues without computing the Laurent expansions. Indeed let $f(z)$ have a simple pole of order n at the isolated singular point z_0. Then

$$\phi(z) = (z - z_0)^n f(z),$$

is analytic in a neighborhood of z_0. Then the $n-1$ coefficient of the Taylor expansion of $\phi(z)$ gives the residue a_{-1} of $f(z)$,

$$a_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}\phi(z)}{dz^{n-1}}.$$

For example suppose a function $f(z)$ had a pole of order 3 at $z = 0$, so

$$f(z) = \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k$$

then

$$z^3 f(z) = a_{-3} + a_{-2}z + a_{-1}z^2 + \sum_{k=3}^{\infty} a_{k-3} z^k.$$

So differentiating two times and evaluating we get $2!a_{-1}$, so dividing by $2!$ we obtain the residue of $f(z)$ at $z = 0$.

25 Evaluating Contour Integrals and Real Integrals Using the Residue Theorem

Problem 1. Calculate the value of the integral

$$\int_C \frac{e^{-z}}{(z-1)^2} dz,$$

where C is the circle of radius 2, about the center $z = 1$.
Solution. There is an isolated singular point at $z = 1$. The Taylor series for e^{-z} about $z = 1$ is

$$e^{-z} = \sum_{k=0}^{\infty} e^{-1}(-1)^k \frac{(z-1)^k}{k!}.$$

Dividing by $(z-1)^2$, we have the unique Laurent expansion, about the singular point $z = 1$, of our function

$$f(z) = \frac{e^{-z}}{(z-1)^2}.$$

This is

$$f(z) = \frac{1}{(z-1)^2}\left[\sum_{k=0}^{\infty} e^{-1}(-1)^k \frac{(z-1)^k}{k!}\right]$$

$$= \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{(z-1)} + \left[\sum_{k=2}^{\infty} e^{-1}(-1)^k \frac{(z-1)^k}{k!}\right].$$

By definition, the residue at the singularity $z = 1$, is the coefficient of the term $1/(z-1)$ in the unique Laurent expansion, which is here $-e^{-1}$. The residue theorem says that the integral of the function is $2\pi i$ times the sum of the residues, at all of the isolated singularities enclosed by the curve C. In our case there is only one such isolated singularity. Thus our integral is

$$\int_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i(-e^{-1}) = -\frac{2\pi i}{e}.$$

For a little added amusement, let us use a computer program to try to approximate this result numerically.
The output of our numerical calculation:

```
pi= 3.141592653589793
e= 2.718281828459045
i =(0.000000000000000E+000,1.000000000000000)
z=exp(i * pi/3.)= (1.000000000000000,1.732050807568877)
z conjugate= (1.000000000000000,-1.732050807568877)
magnitude of z = 2.000000000000000
real part of z = 1.000000000000000
imaginary part of z = 1.732050807568877
```

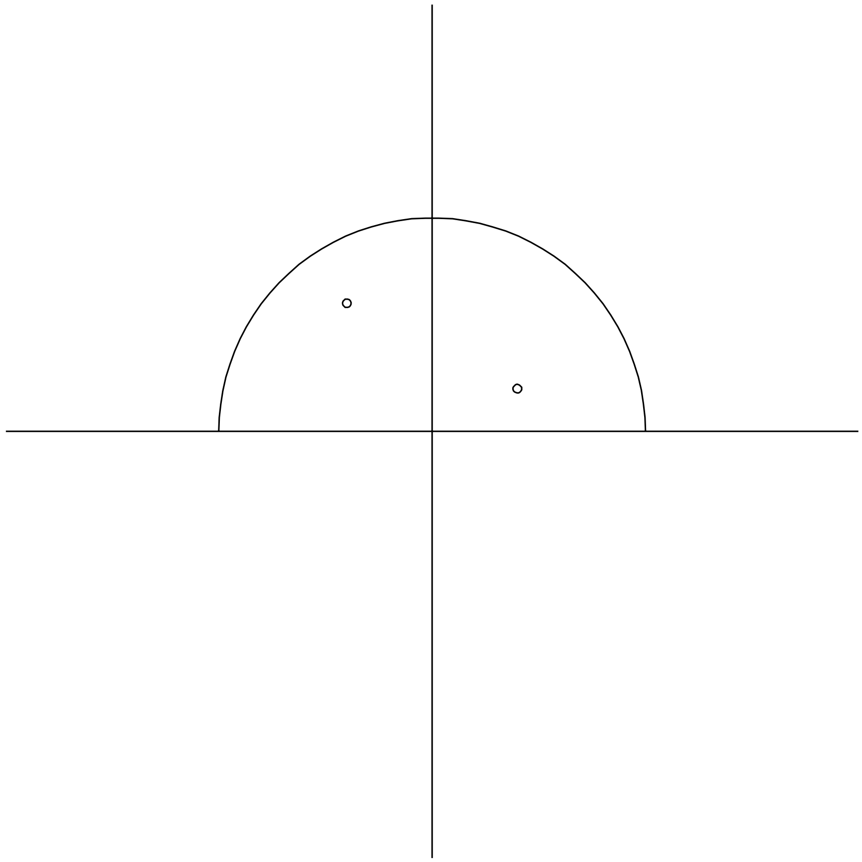

Figure 2: This figure shows a common case of evaluating a real integral from $-\infty$ to ∞, by integrating around a semicircle of radius R, and then letting R go to infinity. This contour integral from point $-R$ on the real axis to R on the real axis, and then around the semicircle of radius R is determined by the sum of the residues at the isolated singular points inside the contour, two shown here. Now if the integrand $f(z)$ is bounded by $1/R^n$, where $n \geq 2$, the integral around the semicircle will go to zero as R goes to infinity, so the real integral is equal to $2\pi i$ times the sum of the residues.

```
 steps= 10000
 center= (1.000000000000000,0.000000000000000E+000)
 actual integral= -2.311454699581844 i
 numerical calculation= (-8.068545779847255E-016,-2.311454775625656)
 = 0. -2.311455 i
```

So there is agreement to about seven decimal places.

Listing of the Fortran Program:

```
c contourint.ftn,
c Integration of f(z)= e^{-z}/(z-1)^2, about
c a circle, with center(1,0) and radius=2
      implicit real*8(a-h,o-z)
      complex*16 z,z1,z2,i,c,s,f
      external f
      one=1.
      c1=0.d0
      pi=4.*atan(one)
      write(*,*)' pi= ',pi
      e=exp(one)
      write(*,*)' e= ',e
      i=(0.d0,1.d0)
      i=cmplx(zero,one)
      write(*,*)' i = ',i

c     The next six lines are not part of the calculation. They are
c     just to illustrate some complex number functions in Fortran.
      z=2.*exp(i * pi/3.)
      write(*,*)' z=exp(i * pi/3.)= ',z
      write(*,*)' z conjugate=',conjg(z)
      write(*,*)' magnitude of z =',abs(z)
      write(*,*)' real part of z =',real(z)
      write(*,*)' imaginary part of z =',aimag(z)
c
      n=10000
      write(*,*)' steps= ',n
      c=1.d0
```

```
      write(*,*)' center= ',c
      r=2.
      angle=2.*pi
      do k=1,n
       t1=(k-1)*angle/n
       z1=c+r*exp(i*t1)
       t2=k*angle/n
       z2=c+r*exp(i*t2)
       z=(z1+z2)/2.
       s=s+f(z)*(z2-z1)
      enddo
      w=-2.*pi/e
      write(*,*)' actual integral= ',w,' i'
      write(*,*)' numerical calculation= ',s
      write(*,'(a,f5.0,f10.6,a)')'  =',real(s),aimag(s),' i'
      end
c
      complex*16 function f(z)
      implicit real*8(a-h,o-z)
      complex*16 z
      f=exp(-z)/(z-1.)**2
      return
      end
```

Problem 2. Evaluate

$$\int_0^\infty \frac{\cos(x)}{x^2+1}dx.$$

This equals

$$\frac{1}{2}\int_{-\infty}^\infty \frac{\cos(x)}{x^2+1}dx,$$

because $\cos(x)$ is an even function.

We have

$$e^{iz} = e^{i(x+yi)} = e^{ix-y} = e^{ix}e^{-y} = \frac{(cos(x) + i\sin(x))}{e^y},$$

So in the upper half plane, $y > 0$, so

$$|e^{iz}| = \frac{1}{e^y} \le 1.$$

The real part of e^{iz} on the x axis, where $y = 0$ is $\cos(x)$. So we replace our integrand by

$$\frac{e^{iz}}{z^2+1}.$$

So we shall integrate this on the contour consisting of the line on the x axis from $-R$ to R and then on the half circle in the upper half plane of radius R. Then we shall take the limit as R goes to infinity.

On the half circle the magnitude is less than

$$\frac{1}{e^y(R^2+1)} < \frac{1}{R^2}$$

and the length of the half circle is πR, so the integral on the half circle is less than

$$\pi/R$$

which goes to zero as R goes to infinity. Hence the real part of the contour integral as R goes to ∞ is

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos(x)}{x^2+1}dx.$$

Now

$$f(z) = \frac{e^{iz}}{z^2+1}$$

has isolated singularities at $z = i$ and $z = -i$, only $z = i$ inside our contour. So by the residue theorem our contour integral equals $2\pi i$ times the residue at that singularity. This is a first order pole, so the residue is

$$\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{e^{iz}}{z+i}$$

$$= \frac{e^{-1}}{2i}$$

So the value of the contour integral is

$$\frac{2\pi i}{2ei} = \frac{\pi}{e}$$

The value of our original integral is one half of this, so finally we find

$$\int_0^{\infty}\frac{\cos(x)}{x^2+1}dx = \frac{\pi}{2e}.$$

26 The Inversion of the Laplace Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)e^{-\phi t}e^{-i\omega t}dt \\ &= \hat{g}_\phi(\omega). \end{aligned}$$

Formally applying the Fourier inversion theorem, we have

$$f(t)e^{-\phi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}_\phi(\omega)e^{i\omega t}d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{i\omega t}d\omega.$$

Then

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{\phi t}e^{i\omega t}d\omega. \\ &= \frac{1}{2\pi i}\int_{C_\phi} F(s)e^{st}ds, \end{aligned}$$

where C_ϕ is the Bromwich contour defined by

$$\{\phi + i\omega : -\infty < \omega < \infty\}.$$

Note that i appears in the expression $2\pi i$ because

$$ds = id\omega.$$

In general we will find that if we define a closed curve consisting of a finite line of length $2R$ on the bromwich contour, and a semicircle of radius R to the left, then as R goes to infinity, the integral over the semicircle goes to zero, so that the total integral over the curve is equal to the integral on the Bromwich line, which is thus equal to $2\pi i$ times the residues of $F(s)e^{st}$ in the left halfspace bounded by the contour. Our inversion expression is therefore equal to the sum of the residues themselves. We get the single sided Laplace transform from the double when $f(t)$ is equal to zero for $t \leq 0$.

Example: Consider

$$F(s) = \frac{1}{s-1},$$

for $\Re(s) > 1$. The residue of $F(s)e^{st}$ is

$$\lim_{s \to 1}(s-1)F(s)e^{st} = e^t.$$

Therefore

$$f(t) = e^t.$$

Example: Consider

$$F(s) = \frac{1}{s^2+1} = \frac{1}{(s-i)(s+i)},$$

for $\Re(s) > 0$. The residues of $F(s)e^{st}$ are

$$\lim_{s \to i}(s-i)F(s)e^{st} = \frac{e^{it}}{2i},$$

and

$$\lim_{s \to -i}(s+i)F(s)e^{st} = \frac{e^{-it}}{-2i},$$

Therefore

$$f(t) = \frac{e^{it} - e^{-it}}{2i} = \sin(t).$$

27 Properties of the Elementary Functions

$$f(z) = itan(iz/2)$$

28 Conformal Mapping

A conformal mapping is a mapping that preserves angles between intersecting lines. Analytic functions serve as solutions to the Laplace partial differential equation in two dimensions in an analytic region. It is sometimes possible to map such a region into a new region with boundaries that match a required Laplace boundary value problem, thereby solving the problem.

29 Riemann Surfaces

See the Alfors books.

30 Partial Fractions

See Knopp, **Theory of Functions**.

31 The Special Functions

E. T. Whittaker and G. N. Watson, **A Course of Modern Analysis,** has a very long history, but is still a primary source on this subject.

32 Cauchy's Inequality

Given a power series representation for an analytic function $f(z)$, the nth coefficient is given by

$$a_n = \frac{f^{(n)}(z_o)}{n!} = \frac{1}{n!2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is a circle about Z_0 contained in the region of regularity of f. Let ρ be the radius of the circle, and let M be the maximum value of $|f(z)|$ on C. Then

$$|a_n| \leq \frac{M2\pi}{n!2\pi\rho^{n+1}} \leq \frac{M}{n!\rho^{n+1}}.$$

33 Liouville's Theorem. *A Bounded Entire Function is a Constant.*

Theorem. A bounded entire function is a constant.
Proof. Let the entire function have a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Suppose $f(z)$ is bounded by a number $M > 0$. Then using Cauchy's inequality

$$|a_k| \leq \frac{M}{k! \rho^{n+1}}.$$

But because f is an entire function, the radius ρ may be taken arbitrarily large, so the right side can be made arbitrarily small. Therefore a_k is zero for $k > 0$, so

$$f(z) = a_0,$$

a constant.

34 A Polynomial is Unbounded

Given a polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n,$$

we have

$$p(z) - (a_0 + a_1 z + a_2 z^2 + ... + a_{n-1} z^{n-1}) = a_n z^n.$$

Thus

$$|p(z)| + |a_0| + |a_1||z| + ... + |a_{n-1}||z^{n-1}| \geq |a_n||z^n|.$$

Let $r = |z|$, then

$$|p(z)| \geq |a_n| r^n - (|a_0| + |a_1| r^n + ... + |a_{n-1}| r^{n-1}).$$

So

$$|p(z)| \geq r^n (|a_n| - (\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r})).$$

Clearly

$$\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r}$$

goes to zero as r goes to infinity. So there is some $R > 1$ so that if $r > R$ then

$$|a_n| - (\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r}) > |a_n|/2.$$

So if $r > R$, then

$$|p(z)| \geq r^n(|a_n|/2.$$

Then given an arbitrarily large M, an $r > R$ can be chosen so that

$$r^n|a_n|/2 > M.$$

Hence given any $M > 0$, there exists a circle with center at the origin with radius r so that for all z outside of this circle.

$$p(z) > M.$$

Theorem. Given a polynomial $p(z)$ and a number $M > 0$ there exists a circle about the origin so that $\forall z$ outside of this circle.

$$|p(z)| > M.$$

35 A Proof of the Fundamental Theorem of Algebra

A bounded entire function is a constant. Given a non-constant polynomial $p(z)$. Suppose $p(z)$ does not have a root. Then

$$\frac{1}{p(z)}$$

is an entire function. But because $p(z)$ is a polynomial, $1/|p(z)|$ is say less than 1 for all points outside of some circle. That is it is bounded, and so a bounded entire function, and so a constant. This is a contradiction. Therefore $p(z)$ has a root.

36 The Winding Number

The winding number measures how many times a curve winds around a point. Reference Alfors. See geometry.tex, and subroutines and functions in my mathlib.ftn and mathlib.c

37 Appendix A: Complex Numbers in Electrical Engineering

37.1 Steady State Alternating Currents And The Concept of Impedance

Consider the RLC circuit with a voltage source. The equation for this circuit consisting of, a resistance R, an inductance L, and a capacitance C in series with an alternating current voltage source v, is

$$L\frac{di}{dt} + Ri + \frac{q}{C} = v,$$

where i is the current in the circuit. Differentiating this equation we get a second order differential equation

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv}{dt}.$$

Let

$$i = I_0 \exp j\omega t = I_0(\cos(\omega t) + \sin(\omega t)j),$$

and

$$v = V_0 \exp j\omega t = V_0(\cos(\omega t) + \sin(\omega t)j).$$

We let I_0 and V_0 be complex numbers to allow i and v to be out of phase. Then

$$[-\omega^2 L + Rj\omega + \frac{1}{C}]I_0 \exp j\omega t = V_0 j\omega \exp j\omega t.$$

Then

$$[-\omega^2 L + Rj\omega + \frac{1}{C}]I_0 = V_0 j\omega.$$

Dividing by $j\omega$

$$[-\frac{\omega^2 L}{j\omega} + R + \frac{1}{Cj\omega}]I_0 = V_0.$$

Then

$$[R + (\omega L - \frac{1}{\omega C})j]I_0 = V_0.$$

Then

$$I_0 = \frac{V_0}{Z},$$

where

$$Z = R + (\omega L - \frac{1}{\omega C})j = R + Xj$$

is the impedance. The imaginary part of the impedance X is called the reactance. The inductive reactance is

$$X_L = \omega L,$$

and the capacitive reactance is

$$X_C = -\frac{1}{\omega C}.$$

If

$$I_0 = |I_0| \exp(j\theta_I),$$

$$V_0 = |V_0| \exp(j\theta_V),$$

and

$$Z = |Z| \exp(j\theta_Z,)$$

then

$$i = |I_0| \exp(j(\omega t + \theta_I)),$$

and

$$v = |V_0| \exp(j(\omega t + \theta_V)).$$

Dropping the subscript, we can write complex numbers in boldface and so

$$\mathbf{I} = I \exp((\omega t + \theta_I)j),$$

where $\mathbf{I}$ is the complex current, and I is the magnitude of $\mathbf{I}$. And we can write similar expressions for $\mathbf{V}$ and $\mathbf{Z}$. The complex current $\mathbf{I}$ can be thought of as a vector rotating around at angular velocity ω, and the physical current i the projection of $\mathbf{I}$ to the real axis, that is

$$i = I \cos(\omega t + \theta_I).$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}}.$$

If we are only interested in phase differences between the various rotating complex vectors, and not the actual time dependence, we can omit the ωt in the expression for $\mathbf{I}$ and $\mathbf{V}$, and retain only the magnitudes and the phase angles. So because the vectors rotate at the same frequency, the phase and magnitude relation between them is the same for each time. That is we could let t be some constant say $t = 0$. Because as t varies the same magnitudes and phase relationships are maintained.

So for example consider the voltage across an inductor. We might specify the current to have magnitude 10 and phase angle say 0 degrees. We would specify this phaser in polar notation as a magnitude and an angle

$$\mathbf{I} = 10\angle 0.$$

Now suppose the inductive reactance is 5, so that the impedance is

$$5j,$$

which in terms of magnitude and angle is

$$\mathbf{Z} = 5\angle 90.$$

Then the voltage across the inductor is

$$\mathbf{V} = \mathbf{IZ} = (10\angle 0)(5\angle 90) = (10)(5)\angle(0+90) = 50\angle 90.$$

So the voltage leads the current by 90 degrees. This makes sense because when an alternating current passes through zero, the rate of change of current is a maximum and so the inductive voltage is a maximum. Similarly we can show that for a capacitor the voltage across the capacitor lags the current by 90 degrees.

If the peak value of the current i is I, then the average power dissipated in a resistor R is

$$P = \frac{1}{T}\int_0^T i^2 R dt = \frac{I^2 R}{T}\int_0^T \cos(\omega t)^2 dt$$
$$= \frac{I^2 R}{2} = I_{eff}^2 R,$$

where

$$I_{eff}^2 R = \frac{I^2 R}{2}.$$

So

$$I_{eff} = \frac{I}{\sqrt{2}}.$$

38 Appendix B: The Laplace Transform

38.1 The Laplace Transform

One of the primary uses of the Laplace Transform is in the solution of differential equations. So differential equations are mapped to algebraic equations, and often these algebraic equations are often easier to solve than the original equations.

The Laplace transform maps a function $f(t)$ of a real variable t to a function $Lf(s)$ of a complex variable s. The transform is given by

$$Lf(s) = \int_0^{\infty} \mathrm{f}(\, t\,)\, \mathrm{e}^{(-s\,t)}\, dt$$

Sometimes we write the transform of a function f by capitalizing. So we write

$$F(s) = Lf(s).$$

The Laplace transform of f in the symbolic computer algebra program Maple is specified as

$$\mathrm{laplace}(\, \mathrm{f}(\, t\,), t, s\,).$$

$f(t)$ is a function of a real variable, but s is a complex variable, so Lf is a complex valued function of a complex variable. Here are a few Laplace transforms.

$$\int_0^{\infty} \sin(\, t\,)\, \mathrm{e}^{(-s\,t)}\, dt = \frac{1}{s^2+1}$$

$$\int_0^{\infty} \cos(\, t\,)\, \mathrm{e}^{(-s\,t)}\, dt = \frac{s}{s^2+1}$$

$$\int_0^{\infty} t^a\, \mathrm{e}^{(-s\,t)}\, dt = \frac{\Gamma(\, a+1\,)}{s^{(\,a+1\,)}}$$

$\Gamma(x)$ is the Gamma function:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

$$\lim_{x \to 0} \Gamma(x) = \infty.$$

$$\Gamma(x) = \frac{1}{x} \Gamma(x+1).$$

If n is an integer then

$$\Gamma(n+1) = n!.$$

So if n is an integer,

$$\int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}}.$$

The Laplace transform of the derivative of a function f is obtained by integrating by parts. We find

$$Lf'(s) = \int_0^\infty \left(\frac{d}{dt}\,\mathrm{f}(t)\right) \mathrm{e}^{(-st)}\, dt = s\int_0^\infty \mathrm{f}(t)\, \mathrm{e}^{(-st)}\, dt - \mathrm{f}(0) = sLf - \mathrm{f}(0)$$

So the transform of a second derivative is

$$Lf'' = sLf' - f'(0) = s(sLf - f(0)) - f'(0) = s^2 Lf - sf(0) - f'(0)$$

and so on for higher derivatives.

If $f(t) = A$ is constant then

$$Lf(s) = \int_0^\infty Ae^{-st} dt = \left[-\frac{A}{s}e^{-st}\right]_0^\infty = \frac{A}{s}.$$

Suppose $f(t) = e^{-at}$ then

$$Lf(s) = \int_0^\infty e^{-at}e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = \frac{1}{s+a}.$$

Suppose

$$f(t) = \int_0^t g(x)dx.$$

Then $f'(t) = g(t)$, so integrating by parts we have

$$Lf(s) = \int_0^\infty f(t)e^{-st} dt$$

$$= \left[-f(t)\frac{e^{-st}}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty -e^{-st} f'(t)dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st} g(t)dt$$

$$= \frac{Lg(s)}{s}.$$

We have used

$$u = f(t)$$

$$dv = e^{-st}dt$$

and

$$udv = d(uv) - vdu.$$

Let us compute $L\sin(s)$. Integrating by parts we have

$$L\sin(s) = \int_0^\infty \sin(t)e^{-st}dt$$

$$= \left[-\frac{\sin(t)e^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty \cos(t)e^{-st}dt$$

$$= \frac{1}{s}\int_0^\infty \cos(t)e^{-st}dt$$

$$= \frac{1}{s}L\cos(s).$$

Similarly we compute $L\cos(s)$

$$L\cos(s) = \int_0^\infty \cos(t)e^{-st}dt$$

$$= \left[-\frac{\cos(t)e^{-st}}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty \sin(t)e^{-st}dt$$

$$= \frac{1 - L\sin(s)}{s}.$$

From above we have

$$L\sin(s) = \frac{1}{s}L\cos(s) = \frac{1}{s}\left[\frac{1 - L\sin(s)}{s}\right] = \frac{1 - L\sin(s)}{s^2}.$$

Solving for $L\sin(s)$, we find

$$L\sin(s) = \frac{1}{s^2+1},$$

and

$$L\cos(s) = sL\sin(s) = \frac{s}{s^2+1}.$$

Let $U(t)$ be the unit step function with step at $t = 0$. The unit step function at t_0 is

$$U_{t_0}(t) = U(t - t_0).$$

Proposition

$$L(U_{t_0}(t) f(t - t_0)) = e^{-st_0} L(f(t)).$$

Proof.

$$\begin{aligned} L(U(t-t_0)f(t-t_0)) &= \int_0^\infty e^{-st} U(t-t_0) f(t-t_0) dt \\ &= \int_{t_0}^\infty e^{-st} f(t-t_0) dt \\ &= \int_0^\infty e^{-s(t+t_0)} f(t) dt \\ &= e^{-st_0} L(f(t)). \end{aligned}$$

Example. Suppose the forcing function on the right side of the following equation is an impulse function at the point t_0. Then

$$x'' + k^2 x = \delta(t - t_0)$$

$$Lx(s)(s^2 + k^2) = e^{-t_0 s}$$

$$\begin{aligned} Lx(s) &= \frac{e^{-t_0 s}}{s^2 + k^2} = e^{-t_0 s} L(\sin(t)) \\ &= L(U(t - t_0)\sin(t - t_0)) \end{aligned}$$

So the solution to the differential equation is

$$x(t) = U_{t_0} \sin(t - t_0),$$

assuming the initial conditions are $x(0) = 0, x'(0) = 0$.

Example.

$$y'''(t) - y''(t) + y'(t) - y(t) = F(t), y(0) = y'(0) = y''(0) = 0.$$

Applying the Laplace transform, we have

$$L(y(t))(s^3 - s^2 + s - 1) = L(y(t))(s - 1)(s^2 + 1) = L(F(t)).$$

So

$$L(y(t)) = L(F(t)) \frac{1}{(s-1)(s^2+1)}.$$

Using partial fractions

$$2\frac{1}{(s-1)(s^2+1)} = \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

So

$$2L^{-1}\frac{1}{(s-1)(s^2+1)} = e^t - \cos(t) - \sin(t).$$

Let

$$g(t) = e^t - \cos(t) - \sin(t).$$

Then we have

$$2L(y(t)) = L(F(t))L(g(t)).$$

The Laplace transform of the convolution of two functions is the product of the transforms. Thus

$$2L(y(t)) = L(F * g(t)).$$

So

$$2y(t) = F * g(t) = \int_0^t F(t-\tau)g(\tau)d\tau = \int_0^t F(t-\tau)(e^\tau - \cos(\tau) - \sin(\tau)d\tau.$$

38.2 Bessel Functions

The Bessel function of the first kind of order ν is

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{\nu+2m}}{2^{\nu+2m} m! \Gamma(\nu+m+1)}.$$

This may also be written as

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-t^2/4)^k}{k!\Gamma(\nu+k+1)}.$$

38.3 Relation to the Fourier Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Some authors define it with a constant multiplier in front. The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

The single sided definition follows from this if $f(t)$ is zero for $t \leq 0$. Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)e^{-\phi t}e^{-i\omega t}dt \\ &= \hat{g}_\phi(\omega). \end{aligned}$$

For more on this see the section on the inversion of the transform.

38.4 Laplace Transform Table

`http://www.vibrationdata.com/math/Laplace_Transforms.pdf`

`or local file:`

`c:/je/pdf/Laplace_Transforms.pdf`

38.5 The Laplace Transform in Maple

See my documents **maple.tex** and **mapletwelve.tex**, titled **Quintessential Maple V** and **Quintessential Maple XII**. The computer Algebra program Maple, like many software programs changes a bit from time to time, so new documentation is required.

38.6 Solving a Differential Equation With The Laplace Transform Using Maple

This section has been made compatible with Maple 12. We read the following file into Maple:

```
% cat mlaplace
with(invtrans)
de:=diff(y(x),x,x)+2*diff(y(x),x)+y(x) = sin(2*x);
dsolve({de,y(0)=1,D(y)(0)=1},y(x));
laplace(de,x,s);
subs(laplace(y(x),x,s)=G,%);
solve(",G);
subs({D(y)(0)=1,y(0)=1},%);
invlaplace(%,s,x);
```

The above code was pasted into Maple 12. The laplace transform wouldnot work, until I blundered onto some information that the laplace transform and inverse laplace transform are in the inttrans package that must be loaded. Also the previous expression representation had to be changed to per cent sign from the double quote sign. Maple 12 gives equivalent though different forms for the results calculated by Maple 5, and which are listed here. The session is as follows:

```
> de:=diff(y(x),x,x)+2*diff(y(x),x)+y(x) = sin(2*x);
```

$$de := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(\,x\,)\right) + 2\,\left(\frac{\partial}{\partial x}\,\mathrm{y}(\,x\,)\right) + \mathrm{y}(\,x\,) = \sin(\,2\,x\,)$$

```
> dsolve({de,y(0)=1,D(y)(0)=1},y(x));
```

$$\mathrm{y}(\,x\,) = -\,\frac{4}{25}\cos(\,2\,x\,) - \frac{3}{25}\sin(\,2\,x\,) + \frac{29}{25}\,\mathrm{e}^{(\,-x\,)} + \frac{12}{5}\,\mathrm{e}^{(\,-x\,)}\,x$$

```
> laplace(de,x,s);
```

$$\begin{aligned}&(\,\mathrm{laplace}(\,\mathrm{y}(\,x\,), x, s\,)\,s - \mathrm{y}(\,0\,)\,)\,s - \mathrm{D}(\,y\,)(\,0\,) + 2\,\mathrm{laplace}(\,\mathrm{y}(\,x\,), x, s\,)\,s\\ &\quad - 2\,\mathrm{y}(\,0\,) + \mathrm{laplace}(\,\mathrm{y}(\,x\,), x, s\,) = 2\,\frac{1}{s^2+4}\end{aligned}$$

```
> subs(laplace(y(x),x,s)=G,%);
```

$$(\,G\,s-\mathrm{y}(\,0\,)\,)\,s-\mathrm{D}(\,y\,)(\,0\,)+2\,G\,s-2\,\mathrm{y}(\,0\,)+G=2\,\frac{1}{s^2+4}$$

```
> solve(%,G);
```

$$-\frac{-s\,\mathrm{y}(\,0\,)-\mathrm{D}(\,y\,)(\,0\,)-2\,\mathrm{y}(\,0\,)-2\,\dfrac{1}{s^2+4}}{s^2+2\,s+1}$$

```
> subs({D(y)(0)=1,y(0)=1},%);
```

$$-\frac{-s-3-2\,\dfrac{1}{s^2+4}}{s^2+2\,s+1}$$

```
> invlaplace(%,s,x);
```

$$-\frac{4}{25}\cos(\,2\,x\,)-\frac{3}{25}\sin(\,2\,x\,)+\frac{29}{25}\,\mathrm{e}^{(\,-x\,)}+\frac{12}{5}\,\mathrm{e}^{(\,-x\,)}\,x$$

The solution using dsolve, and the solution using the Laplace transform method are the same.

38.7 Solving Circuit Problems With the Laplace Transform

Resistor Capacitor Circuit Let a circuit consist of a constant voltage source V be in series with a Resistor R and a capacitor C. The voltage loop equation is

$$Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau+\frac{q_0}{C}=V,$$

where $i(t)$ is the current, and q_0 is the initial charge on the capacitor. We have

$$i(t)+\frac{1}{RC}\int_0^t i(\tau)d\tau+\frac{q_0}{RC}=\frac{V}{R}.$$

Taking the Laplace Transform

$$Li(s) + \frac{1}{RC}\frac{Li(s)}{s} + \frac{q_0}{RC}\frac{1}{s} = \frac{V}{R}\frac{1}{s}.$$

Then

$$Li(s)\left(1 + \frac{1}{RCs}\right) = \frac{CV - q_0}{RCs}$$

and so

$$Li(s) = \frac{CV - q_0}{RCs + 1} = \frac{CV/(RC) - q_0/(RC)}{s + 1/(RC)}$$

Then

$$Li(s) = (V/R - q_0/(RC))\frac{1}{s + 1/(RC)}.$$

So taking the inverse transform

$$i(t) = (V/R - q_0/(RC))e^{-t/(RC)}.$$

To find the charge we integrate

$$q(t) = (V/R - q_0/(RC))\int e^{-t/(RC)}$$

$$= (V/R - q_0/(RC))(-RC)e^{-t/(RC)} + K,$$

where K is a constant. So

$$q(t) = (q_0 - VC)e^{-t/(RC)} + K$$

At zero

$$q_0 = q(0) = (q_0 - VC) + K,$$

so $K = VC$. Finally

$$q(t) = q_0e^{-t/(RC)} + VC(1 - e^{-t/(RC)}).$$

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40 Index