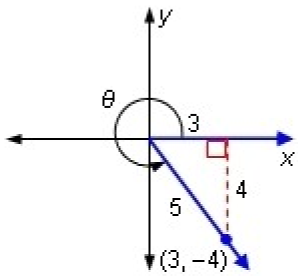
5-5 Multiple-Angle and Product-to-Sum Identities

Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval.

1. $\cos\theta = \frac{3}{5}$, (270°, 360°)

SOLUTION:

Since $\cos\theta = \frac{3}{5}$ on the interval (270°, 360°), one point on the terminal side of θ has x-coordinate 3 and a distance of 5 units from the origin as shown. The y-coordinate of this point is therefore $-\sqrt{5232}$ or −4.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $-\frac{4}{5}$ and $\tan\theta = \frac{y}{x}$ or $-\frac{4}{3}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2θ, cos 2θ, and tan 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

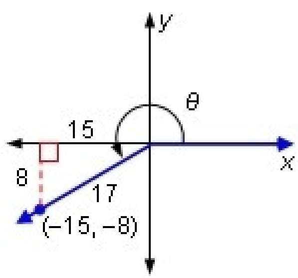
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{3}{5}\right)^2 - 1 \\ &= 2\left(\frac{9}{25}\right) - 1 \\ &= \frac{18}{25} - \frac{25}{25} \\ &= -\frac{7}{25}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}\\ &= \frac{-\frac{8}{3}}{1-\frac{16}{9}}\\ &= \frac{-\frac{8}{3}}{-\frac{7}{9}}\\ &= \frac{24}{7}\end{aligned}$$

2. $\tan\theta = \frac{8}{15}$, $(180°, 270°)$

SOLUTION:

Since $\tan\theta = \frac{8}{15}$ on the interval (180°, 270°), one point on the terminal side of θ has x-coordinate −15 and y-coordinate −8 as shown. The distance from the point to the origin is $\sqrt{82+152}$ or 17.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $-\frac{8}{17}$ and $\cos\theta = \frac{x}{r}$ or $-\frac{15}{17}$. Now use the double-angle identities for sine, cosine, and tangent to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{8}{17}\right)\left(-\frac{15}{17}\right)\\ &= \frac{240}{289}\end{aligned}$$

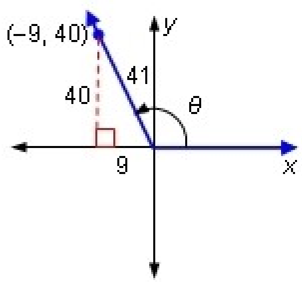
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{15}{17}\right)^2 - 1\\ &= 2\left(\frac{225}{289}\right) - 1\\ &= \frac{450}{289} - \frac{289}{289}\\ &= \frac{161}{289}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(\frac{8}{15}\right)}{1-\left(\frac{8}{15}\right)^2}\\ &= \frac{\frac{16}{15}}{1-\frac{64}{225}}\\ &= \frac{\frac{16}{15}}{\frac{161}{225}}\\ &= \frac{240}{161}\end{aligned}$$

3. $\cos\theta = -\frac{9}{41}$, (90°, 180°)

SOLUTION:

Since $\cos\theta = -\frac{9}{41}$ on the interval (90°, 180°), one point on the terminal side of θ has x-coordinate −9 and a distance of 41 units from the origin as shown. The y-coordinate of this point is therefore $\sqrt{41292}$ or 40.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $\frac{40}{41}$ and $\tan\theta = \frac{y}{x}$ or $-\frac{40}{9}$. Now use the double-angle identities for sine, cosine, and tangent to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{40}{41}\right)\left(-\frac{9}{41}\right)$$
$$= -\frac{720}{1681}$$

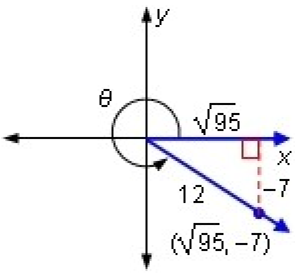
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$= 2\left(-\frac{9}{41}\right)^2 - 1$$
$$= 2\left(\frac{81}{1681}\right) - 1$$
$$= \frac{162}{1681} - \frac{1681}{1681}$$
$$= -\frac{1519}{1681}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \frac{2\left(-\frac{40}{9}\right)}{1-\left(-\frac{40}{9}\right)^2}$$
$$= \frac{-\frac{80}{9}}{1-\frac{1600}{81}}$$
$$= \frac{-\frac{80}{9}}{-\frac{1519}{81}}$$
$$= \frac{720}{1519}$$

4. $\sin\theta = -\frac{7}{12}$, $\left(\frac{3\pi}{2}, 2\pi\right)$

SOLUTION:

Since $\sin\theta = -\frac{7}{12}$ on the interval $\left(\frac{3\pi}{2}, 2\pi\right)$, one point on the terminal side of θ has *y*-coordinate −7 and a distance of 12 units from the origin as shown. The *x*-coordinate of this point is therefore $\sqrt{12^2 - 7^2}$ or $\sqrt{95}$.



Using this point, we find that $\cos\theta = \frac{x}{r}$ or $\frac{\sqrt{95}}{12}$ and $\tan\theta = \frac{y}{x}$ or $-\frac{7\sqrt{95}}{95}$. Now use the double-angle identities for sine and cosine to find sin 2θ and cos 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{7}{12}\right)\left(\frac{\sqrt{95}}{12}\right)\\ &= -\frac{7\sqrt{95}}{72}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{\sqrt{95}}{12}\right)^2 - 1\\ &= 2\left(\frac{95}{144}\right) - 1\\ &= \frac{95}{72} - \frac{72}{72}\\ &= \frac{23}{72}\end{aligned}$$

Use the definition of tangent to find tan 2θ.

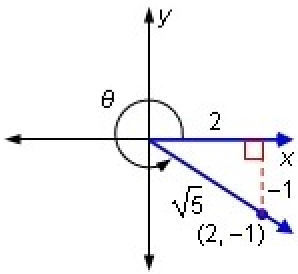
$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(-\frac{7\sqrt{95}}{95}\right)}{1-\left(-\frac{7\sqrt{95}}{95}\right)^2}\\ &= \frac{-\frac{14\sqrt{95}}{95}}{1-\left(\frac{49}{95}\right)}\\ &= \frac{-\frac{14\sqrt{95}}{95}}{\frac{46}{95}}\\ &= -\frac{7\sqrt{95}}{23}\end{aligned}$$

5-5 Multiple-Angle and Product-to-Sum Identities

5. $\tan\theta = -\frac{1}{2}$, $\left(\frac{3\pi}{2}, 2\pi\right)$

SOLUTION:

Since $\tan\theta = -\frac{1}{2}$ on the interval $\left(\frac{3\pi}{2}, 2\pi\right)$, one point on the terminal side of θ has *x*-coordinate 2 and *y*-coordinate −1 as shown. The distance from the point to the origin is $\sqrt{2^2+1^2}$ or $\sqrt{5}$.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $-\frac{\sqrt{5}}{5}$ and $\cos\theta = \frac{x}{r}$ or $\frac{2\sqrt{5}}{5}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2θ, cos 2θ, and tan 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{\sqrt{5}}{5}\right)\left(\frac{2\sqrt{5}}{5}\right)\\ &= -\frac{4}{5}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{2\sqrt{5}}{5}\right)^2 - 1\\ &= 2\left(\frac{20}{25}\right) - 1\\ &= \frac{40}{25} - \frac{25}{25}\\ &= \frac{3}{5}\end{aligned}$$

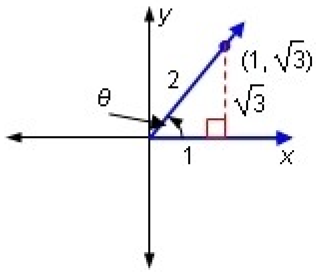
$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)^2}\\ &= \frac{-1}{1-\frac{1}{4}}\\ &= \frac{-1}{\frac{3}{4}}\\ &= -\frac{4}{3}\end{aligned}$$

5-5 Multiple-Angle and Product-to-Sum Identities

6. $\tan\theta = \sqrt{3}$, $\left(0, \frac{\pi}{2}\right)$

SOLUTION:

If $\tan\theta = \sqrt{3}$, then $\tan\theta = \frac{\sqrt{3}}{1}$. Since $\tan\theta = \frac{\sqrt{3}}{1}$ on the interval $\left(0, \frac{\pi}{2}\right)$, one point on the terminal side of θ has x-coordinate 1 and y-coordinate $\sqrt{3}$ as shown. The distance from the point to the origin is $\sqrt{\left(\sqrt{3}\right)^2 + 1^2}$ or 2.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{x}{r}$ or $\frac{1}{2}$. Now use the double-angle identities for sine, cosine, and tangent to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

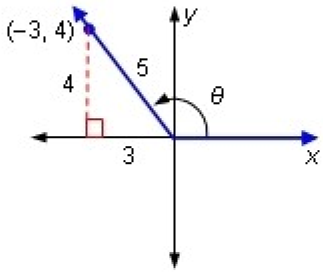
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{1}{2}\right)^2 - 1\\ &= 2\left(\frac{1}{4}\right) - 1\\ &= \frac{1}{2} - \frac{2}{2}\\ &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(\sqrt{3}\right)}{1-\left(\sqrt{3}\right)^2}\\ &= \frac{2\sqrt{3}}{-2}\\ &= -\sqrt{3}\end{aligned}$$

7. $\sin\theta = \frac{4}{5}$, $\left(\frac{\pi}{2}, \pi\right)$

SOLUTION:

Since $\sin\theta = \frac{4}{5}$ on the interval $\left(\frac{\pi}{2}, \pi\right)$, one point on the terminal side of θ has y-coordinate 4 and a distance of 5 units from the origin as shown. The x-coordinate of this point is therefore $-\sqrt{5^2 - 4^2}$ or −3.



Using this point, we find that $\cos\theta = \frac{x}{r}$ or $-\frac{3}{5}$ and $\tan\theta = \frac{y}{x}$ or $-\frac{4}{3}$. Now use the double-angle identities for sine and cosine to find sin 2θ and cos 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(-\frac{3}{5}\right)^2 - 1 \\ &= 2\left(\frac{9}{25}\right) - 1 \\ &= \frac{18}{25} - \frac{25}{25} \\ &= -\frac{7}{25}\end{aligned}$$

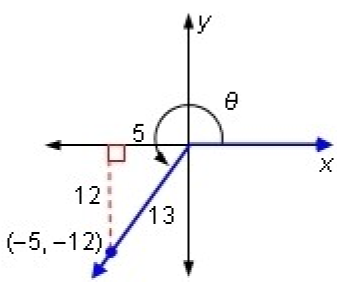
Use the definition of tangent to find tan 2θ.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} = \frac{-\frac{8}{3}}{1-\left(\frac{16}{9}\right)} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{24}{7}$$

8. $\cos\theta = -\frac{5}{13}, \left(\pi, \frac{3\pi}{2}\right)$

SOLUTION:

Since $\cos\theta = -\frac{5}{13}$ on the interval $\left(\pi, \frac{3\pi}{2}\right)$, one point on the terminal side of θ has *x*-coordinate −5 and a distance of 13 units from the origin as shown. The *y*-coordinate of this point is therefore $-\sqrt{13252}$ or −12.



Using this point, we find that $\sin\theta = \frac{y}{r}$ or $-\frac{12}{13}$ and $\tan\theta = \frac{y}{x}$ or $\frac{12}{5}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2θ, cos 2θ, and tan 2θ.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{12}{13}\right)\left(-\frac{5}{13}\right) = \frac{120}{169}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{5}{13}\right)^2 - 1\\ &= 2\left(\frac{25}{169}\right) - 1\\ &= \frac{50}{169} - \frac{169}{169}\\ &= -\frac{119}{169}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^2}\\ &= \frac{\frac{24}{5}}{1-\frac{144}{25}}\\ &= \frac{\frac{24}{5}}{-\frac{119}{25}}\\ &= -\frac{120}{119}\end{aligned}$$

Solve each equation on the interval [0, 2π).

9. sin 2 θ = cos θ

SOLUTION:

$$\begin{aligned}\sin 2\theta &= \cos\theta\\ 2\sin\theta\cos\theta &= \cos\theta\\ 2\sin\theta\cos\theta - \cos\theta &= 0\\ \cos\theta(2\sin\theta - 1) &= 0\end{aligned}$$

$$2\sin\theta - 1 = 0$$

$\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

On the interval [0, 2π), cos θ = 0 when $\theta = \frac{ð}{2}$ and $\theta = \frac{3ð}{2}$ and sin $\theta = \frac{1}{2}$ when $\theta = \frac{ð}{6}$ and $\theta = \frac{5ð}{6}$.

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10. $\cos 2\theta = \cos \theta$

SOLUTION:

$$\begin{aligned} \cos 2\theta &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \end{aligned}$$

$$2\cos\theta + 1 = 0 \qquad \text{or} \qquad \cos\theta - 1 = 0$$
$$\cos\theta = -\frac{1}{2} \qquad\qquad \cos\theta = 1$$

On the interval [0, 2π), $\cos\theta = -\frac{1}{2}$ when $\theta = \frac{2\text{ð}}{3}$ and $\theta = \frac{4\text{ð}}{3}$ and $\cos\theta = 1$ when $\theta = 0$.

11. $\cos 2\theta - \sin\theta = 0$

SOLUTION:

$$\begin{aligned} \cos 2\theta - \sin\theta &= 0 \\ 1 - 2\sin^2\theta - \sin\theta &= 0 \\ 2\sin^2\theta + \sin\theta - 1 &= 0 \\ (\sin\theta + 1)(2\sin\theta - 1) &= 0 \end{aligned}$$

$$\sin\theta + 1 = 0 \qquad \text{or} \qquad 2\sin\theta - 1 = 0$$
$$\sin\theta = -1 \qquad\qquad \sin\theta = \frac{1}{2}$$

On the interval [0, 2π), $\sin\theta = -1$ when $\theta = \frac{3\text{ð}}{2}$ and $\sin\theta = \frac{1}{2}$ when $\theta = \frac{\text{ð}}{6}$ and $\theta = \frac{5\text{ð}}{6}$.

12. $\tan 2\theta - \tan 2\theta \tan^2\theta = 2$

SOLUTION:

$$\begin{aligned} \tan 2\theta - \tan 2\theta \tan^2\theta &= 2 \\ \tan 2\theta(1 - \tan^2\theta) &= 2 \\ \frac{2\tan\theta}{1 - \tan^2\theta}(1 - \tan^2\theta) &= 2 \\ 2\tan\theta &= 2 \\ \tan\theta &= 1 \end{aligned}$$

On the interval [0, 2π), $\tan\theta = 1$ when $\theta = \frac{\text{ð}}{4}$ and $\theta = \frac{5\text{ð}}{4}$.

5-5 Multiple-Angle and Product-to-Sum Identities

13. $\sin 2\theta \csc \theta = 1$

SOLUTION:

$$\begin{aligned} \sin 2\theta \csc\theta &= 1 \\ 2\sin\theta\cos\theta\csc\theta &= 1 \\ 2\sin\theta\cos\theta \cdot \frac{1}{\sin\theta} &= 1 \\ 2\cos\theta &= 1 \\ \cos\theta &= \frac{1}{2} \end{aligned}$$

On the interval [0, 2π), $\cos \theta = \frac{1}{2}$ when $\theta = \frac{ð}{3}$ and $\theta = \frac{5ð}{3}$.

14. $\cos 2\theta + 4 \cos \theta = –3$

SOLUTION:

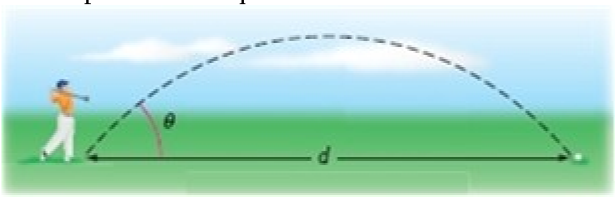
$$\begin{aligned} \cos 2\theta + 4\cos\theta &= -3 \\ 2\cos^2\theta - 1 + 4\cos\theta &= -3 \end{aligned}$$

c $2\cos^2\theta + 4\cos\theta + 2 = 0$

$$\begin{aligned} 2(\cos^2\theta + 2\cos\theta + 1) &= 0 \\ 2(\cos\theta + 1)^2 &= 0 \\ \cos\theta + 1 &= 0 \\ \cos\theta &= -1 \end{aligned}$$

On the interval [0, 2π), $\cos \theta = -1$ when $\theta = \pi$.

15. **GOLF** A golf ball is hit with an initial velocity of 88 feet per second. The distance the ball travels is found by $d = \frac{v02sin2\theta}{32}$, where v_0 is the initial velocity, θ is the angle that the path of the ball makes with the ground, and 32 is in feet per second squared.



a. If the ball travels 242 feet, what is θ to the nearest degree?
b. Use a double-angle identity to rewrite the equation for d.

SOLUTION:

a. Substitute $d = 242$ and $v_0 = 88$ into the equation.

$$
\begin{aligned}
d &= \frac{v_0^2 \sin 2\theta}{32} \\
242 &= \frac{(88)^2 \sin 2\theta}{32} \\
242 &= 242 \sin 2\theta \\
1 &= \sin 2\theta \\
90 &= 2\theta \\
45 &= \theta
\end{aligned}
$$

If the ball travels 242 feet, θ is 45°.

b.

$$
\begin{aligned}
d &= \frac{v_o^2 \sin 2\theta}{32} \\
&= \frac{v_o^2 (2 \sin\theta \cos\theta)}{32} \\
&= \frac{v_o^2 \sin\theta \cos\theta}{16}
\end{aligned}
$$

Rewrite each expression in terms with no power greater than 1.

16. $\cos^3 \theta$

SOLUTION:

$$
\begin{aligned}
\cos^3\theta &= \cos^2\theta \cos\theta \\
&= \left(\frac{1 + \cos 2\theta}{2}\right)\cos\theta \\
&= \frac{\cos\theta + \cos\theta\cos 2\theta}{2}
\end{aligned}
$$

17. $\tan^3 \theta$

SOLUTION:

$$\begin{aligned}\tan^3\theta &= \tan^2\theta\tan\theta\\ &= \left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)\tan\theta\\ &= \frac{\tan\theta-\tan\theta\cos 2\theta}{1+\cos 2\theta}\end{aligned}$$

18. $\sec^4 \theta$

SOLUTION:

$$\begin{aligned}\sec^4\theta &= (\sec^2\theta)^2\\ &= \left(\frac{1}{\cos^2\theta}\right)^2\\ &= \left(\frac{1}{\frac{1+\cos 2\theta}{2}}\right)^2\\ &= \left(\frac{2}{1+\cos 2\theta}\right)^2\\ &= \frac{4}{1+2\cos 2\theta+\cos^2 2\theta}\\ &= \frac{4}{1+2\cos 2\theta+\frac{1+\cos 2(2\theta)}{2}}\\ &= \frac{8}{2+4\cos 2\theta+1+\cos 4\theta}\\ &= \frac{8}{3+4\cos 2\theta+\cos 4\theta}\end{aligned}$$

19. $\cot^3 \theta$

SOLUTION:

$$\begin{aligned}\cot^3\theta &= \cot^2\theta\cot\theta\\ &= \left(\frac{1}{\tan^2\theta}\right)\cot\theta\\ &= \left(\frac{1}{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right)\cot\theta\\ &= \left(\frac{1+\cos 2\theta}{1-\cos 2\theta}\right)\cot\theta\\ &= \frac{\cot\theta+\cot\theta\cos 2\theta}{1-\cos 2\theta}\end{aligned}$$

20. $\cos^4 \theta - \sin^4 \theta$

SOLUTION:

$$\begin{aligned}\cos^4\theta-\sin^4\theta &= (\cos^2\theta)^2-(\sin^2\theta)^2\\ &= \left(\frac{1+\cos 2\theta}{2}\right)^2-\left(\frac{1-\cos 2\theta}{2}\right)^2\\ &= \frac{1+2\cos 2\theta+\cos^2 2\theta}{4}-\frac{1-2\cos 2\theta+\cos^2 2\theta}{4}\\ &= \frac{4\cos 2\theta}{4}\\ &= \cos 2\theta\end{aligned}$$

21. $\sin^2 \theta \cos^3 \theta$

SOLUTION:

$$\begin{aligned}\sin^2\theta\cos^3\theta &= \sin^2\theta\cos^2\theta\cos\theta\\ &= \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)\cos\theta\\ &= \frac{1-\cos^2 2\theta}{2}\cos\theta\\ &= \frac{1-\frac{1+\cos 2(2\theta)}{2}}{2}\cos\theta\\ &= \frac{2-1-\cos 4\theta}{4}\cos\theta\\ &= \frac{\cos\theta-\cos\theta\cos 4\theta}{4}\end{aligned}$$

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22. $\sin^2\theta - \cos^2\theta$

SOLUTION:

$$\begin{aligned}\sin^2\theta - \cos^2\theta &= \frac{1-\cos 2\theta}{2} - \frac{1+\cos 2\theta}{2}\\ &= \frac{-2\cos 2\theta}{2}\\ &= -\cos 2\theta\end{aligned}$$

23. $\frac{sin4\theta}{cos2\theta}$

SOLUTION:

$$\begin{aligned}\frac{\sin^4\theta}{\cos^2\theta} &= \frac{(\sin^2\theta)^2}{\cos^2\theta}\\ &= \frac{\left(\frac{1-\cos 2\theta}{2}\right)^2}{\frac{1+\cos 2\theta}{2}}\\ &= \frac{\frac{1-2\cos 2\theta+\cos^2 2\theta}{4}}{\frac{1+\cos 2\theta}{2}}\\ &= \frac{1-2\cos 2\theta+\cos^2 2\theta}{4}\cdot\frac{2}{1+\cos 2\theta}\\ &= \frac{1-2\cos 2\theta+\cos^2 2\theta}{2(1+\cos 2\theta)}\\ &= \frac{1-2\cos 2\theta+\frac{1+\cos 2(2\theta)}{2}}{2+2\cos 2\theta}\\ &= \frac{2-4\cos 2\theta+1+\cos 4\theta}{4+4\cos 2\theta}\\ &= \frac{3-4\cos 2\theta+\cos 4\theta}{4+4\cos 2\theta}\end{aligned}$$

5-5 Multiple-Angle and Product-to-Sum Identities

Solve each equation.

24. $1 - \sin^2\theta - \cos 2\theta = \frac{1}{2}$

SOLUTION:

$$\begin{aligned}1-\sin^2\theta-\cos 2\theta &= \frac{1}{2}\\ 1-\frac{1-\cos 2\theta}{2}-\cos 2\theta &= \frac{1}{2}\\ 2-(1-\cos 2\theta)-2\cos 2\theta &= 1\\ 1-\cos 2\theta &= 1\\ \cos 2\theta &= 0\\ 2\theta &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2}\\ \theta &= \frac{\pi}{4} \text{ or } \frac{3\pi}{4}\end{aligned}$$

The graph of $y = \cos 2\theta$ has a period of π, so the solutions are $\frac{\pi}{4} + n\pi$, $\frac{3\pi}{4} + n\pi$, where n is an integer.

25. $\cos^2\theta - \frac{3}{2}\cos 2\theta = 0$

SOLUTION:

$$\begin{aligned}\cos^2\theta-\frac{3}{2}\cos 2\theta &= 0\\ \frac{1+\cos 2\theta}{2}-\frac{3}{2}\cos 2\theta &= 0\\ 1+\cos 2\theta-3\cos 2\theta &= 0\\ 1-2\cos 2\theta &= 0\\ -2\cos 2\theta &= -1\\ \cos 2\theta &= \frac{1}{2}\\ 2\theta &= \frac{\pi}{3} \text{ or } \frac{5\pi}{3}\\ \theta &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6}\end{aligned}$$

The graph of $y = \cos 2\theta$ has a period of π, so the solutions are $\frac{\pi}{6} + n\pi$, $\frac{5\pi}{6} + n\pi$, where n is an integer.

26. $\sin^2\theta - 1 = \cos^2\theta$

SOLUTION:

$$\begin{aligned}
\sin^2\theta - 1 &= \cos^2\theta \\
\frac{1-\cos 2\theta}{2} - 1 &= \frac{1+\cos 2\theta}{2} \\
1 - \cos 2\theta - 2 &= 1 + \cos 2\theta \\
-2\cos 2\theta &= 2 \\
\cos 2\theta &= -1 \\
2\theta &= \pi \\
\theta &= \frac{\pi}{2}
\end{aligned}$$

The graph of $y = \cos 2\theta$ has a period of π, so the solutions are $\frac{\pi}{2} + n\pi$, where n is an integer.

27. $\cos^2\theta - \sin\theta = 1$

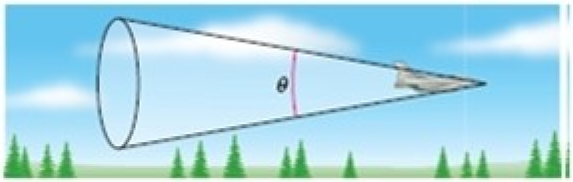
SOLUTION:

$$\begin{aligned}
\cos^2\theta - \sin\theta &= 1 \\
\frac{1+\cos 2\theta}{2} - \sin\theta &= 1 \\
1 + \cos 2\theta - 2\sin\theta &= 2 \\
1 + 1 - 2\sin^2\theta - 2\sin\theta &= 2 \\
-2\sin\theta(\sin\theta + 1) &= 0
\end{aligned}$$

$$\begin{array}{lcl}
-2\sin\theta = 0 & & \sin\theta + 1 = 0 \\
\sin\theta = 0 & \text{or} & \sin\theta = -1 \\
\theta = 0 \text{ or } \pi & & \theta = \frac{3\pi}{2}
\end{array}$$

The graph of $y = \sin\theta$ has a period of 2π. The solutions $0 + 2n\pi$ and $\pi + 2n\pi$ can be combined to $n\pi$. So, the solutions are $n\pi$, $\frac{3\pi}{2} + 2n\pi$, where n is an integer.

28. **MACH NUMBER** The angle θ at the vertex of the cone-shaped shock wave produced by a plane breaking the sound barrier is related to the mach number M describing the plane's speed by the half-angle equation $\sin \frac{\theta}{2} = \frac{1}{M}$.



a. Express the mach number of the plane in terms of cosine.

b. Use the expression found in part **a** to find the mach number of a plane if $\cos \theta = \frac{4}{5}$.

SOLUTION:

a.

$$\sin \frac{\theta}{2} = \frac{1}{M}$$
$$\pm\sqrt{\frac{1-\cos\theta}{2}} = \frac{1}{M}$$

Since the mach number describing a plane's speed cannot be negative, $\sqrt{\frac{1-\cos\theta}{2}} = \frac{1}{M}$.

b. Substitute $\cos \theta = \frac{4}{5}$ into the expression found in part **a**.

$$\sqrt{\frac{1-\cos\theta}{2}} = \frac{1}{M}$$
$$\sqrt{\frac{1-\frac{4}{5}}{2}} = \frac{1}{M}$$
$$\sqrt{\frac{\frac{1}{5}}{2}} = \frac{1}{M}$$
$$0.316 \approx \frac{1}{M}$$
$$3.2 \approx M$$

The mach number of the plane is about 3.2.

5-5 Multiple-Angle and Product-to-Sum Identities

Find the exact value of each expression.

29. sin 67.5°

SOLUTION:

Notice that 67.5° is half of 135°. Therefore, apply the half-angle identity for sine, noting that since 67.5° lies in Quadrant I, its sine is positive.

$$
\begin{aligned}
\sin 67.5^\circ &= \sin \frac{135^\circ}{2} \\
&= \sqrt{\frac{1-\cos 135^\circ}{2}} \\
&= \sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}} \\
&= \sqrt{\frac{2+\sqrt{2}}{4}} \\
&= \frac{\sqrt{2+\sqrt{2}}}{2}
\end{aligned}
$$

30. $\cos \frac{\eth}{12}$

SOLUTION:

Notice that $\frac{\eth}{12}$ is half of $\frac{\eth}{6}$. Therefore, apply the half-angle identity for cosine, noting that since $\frac{\eth}{12}$ lies in Quadrant I, its cosine is positive.

$$
\begin{aligned}
\cos \frac{\pi}{12} &= \cos \frac{\frac{\pi}{6}}{2} \\
&= \sqrt{\frac{1+\cos \frac{\pi}{6}}{2}} \\
&= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\
&= \sqrt{\frac{2+\sqrt{3}}{4}} \\
&= \frac{\sqrt{2+\sqrt{3}}}{2}
\end{aligned}
$$

31. tan 157.5°

SOLUTION:

Notice that 157.5° is half of 315°. Therefore, apply the half-angle identity for tangent, noting that since 157.5° lies in Quadrant III, its tangent is positive.

$$\begin{aligned}\tan 157.5^\circ &= \tan \frac{315^\circ}{2}\\ &= \frac{1-\cos 315^\circ}{\sin 315^\circ}\\ &= \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}\\ &= \frac{2-\sqrt{2}}{-\sqrt{2}}\\ &= -\frac{2-\sqrt{2}}{\sqrt{2}}\\ &= -\frac{2\sqrt{2}-2}{2}\\ &= \frac{2-2\sqrt{2}}{2}\\ &= 1-\sqrt{2}\end{aligned}$$

32. $\sin \frac{11\eth}{12}$

SOLUTION:

Notice that $\frac{11\eth}{12}$ is half of $\frac{11\eth}{6}$. Therefore, apply the half-angle identity for sine, noting that since $\frac{11\eth}{12}$ lies in Quadrant II, its sine is positive.

$$\begin{aligned}\sin \frac{11\pi}{12} &= \sin \frac{\frac{11\pi}{6}}{2}\\ &= \sqrt{\frac{1-\cos \frac{11\pi}{6}}{2}}\\ &= \sqrt{\frac{1-\left(\frac{\sqrt{3}}{2}\right)}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Solve each equation on the interval [0, 2π).

33. $\sin \frac{\theta}{2} + \cos \theta = 1$

SOLUTION:

$$\begin{aligned}
\sin\frac{\theta}{2}+\cos\theta &= 1\\
\sin\frac{\theta}{2} &= 1-\cos\theta\\
\pm\sqrt{\frac{1-\cos\theta}{2}} &= 1-\cos\theta\\
\left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right)^2 &= (1-\cos\theta)^2\\
\frac{1-\cos\theta}{2} &= 1-2\cos\theta+\cos^2\theta\\
1-\cos\theta &= 2-4\cos\theta+2\cos^2\theta\\
0 &= 2\cos^2\theta-3\cos\theta+1\\
0 &= (2\cos\theta-1)(\cos\theta-1)
\end{aligned}$$

$$2\cos\theta-1=0 \quad \text{or} \quad \cos\theta-1=0$$
$$2\cos\theta=1 \qquad \cos\theta=1$$
$$\cos\theta=\frac{1}{2}$$

On the interval [0, 2π), $\cos\theta=\frac{1}{2}$ when $\theta=\frac{\pi}{3}$ and $\theta=\frac{5\pi}{3}$ and $\cos\theta=1$ when $\theta=0$. Check each of these solutions in the original equation.

$$\begin{aligned}
\sin\frac{\theta}{2}+\cos\theta&=1\\
\sin\frac{0}{2}+\cos 0&=1\\
\sin 0+\cos 0&=1\\
0+1&=1
\end{aligned}$$

$$\begin{aligned}
\sin\frac{\theta}{2}+\cos\theta&=1\\
\sin\frac{\frac{\pi}{3}}{2}+\cos\frac{\pi}{3}&=1\\
\sin\frac{\pi}{6}+\cos\frac{\pi}{3}&=1\\
\frac{1}{2}+\frac{1}{2}&=1
\end{aligned}$$

$$\begin{aligned}
\sin\frac{\theta}{2}+\cos\theta&=1\\
\sin\frac{\frac{5\pi}{3}}{2}+\cos\frac{5\pi}{3}&=1\\
\sin\frac{5\pi}{6}+\cos\frac{5\pi}{3}&=1\\
\frac{1}{2}+\frac{1}{2}&=1
\end{aligned}$$

All solutions are valid. Therefore, the solutions to the equation on [0, 2π) are 0, $\frac{\pi}{3}$, and $\frac{5\pi}{3}$.

34. $\tan \frac{\theta}{2} = \sin \frac{\theta}{2}$

SOLUTION:

$$\begin{aligned}
\tan \frac{\theta}{2} &= \sin \frac{\theta}{2} \\
\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
\left(\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)^2 &= \left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right)^2 \\
\frac{1-\cos\theta}{1+\cos\theta} &= \frac{1-\cos\theta}{2} \\
(1+\cos\theta)(1-\cos\theta) &= 2(1-\cos\theta) \\
1+\cos\theta &= 2 \\
\cos\theta &= 1
\end{aligned}$$

On the interval [0, 2π), $\cos\theta = 1$ when $\theta = 0$. Check this solution in the original equation.

$$\begin{aligned}
\tan \frac{\theta}{2} &= \sin \frac{\theta}{2} \\
\tan \frac{0}{2} &= \sin \frac{0}{2} \\
\tan 0 &= \sin 0 \\
0 &= 0
\end{aligned}$$

The solution is valid. Therefore, the solution to the equation on [0, 2π) is 0.

35. $2 \sin \frac{\theta}{2} = \sin \theta$

SOLUTION:

$$
\begin{aligned}
2\sin\frac{\theta}{2} &= \sin\theta \\
\sin\frac{\theta}{2} &= \frac{1}{2}\sin\theta \\
\left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right) &= \frac{1}{2}\sin\theta \\
\left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right)^2 &= \left(\frac{1}{2}\sin\theta\right)^2 \\
\frac{1-\cos\theta}{2} &= \frac{1}{4}\sin^2\theta \\
1-\cos\theta &= \frac{1}{2}\sin^2\theta \\
1-\cos\theta &= \frac{1}{2}(1-\cos^2\theta) \\
2-2\cos\theta &= 1-\cos^2\theta \\
\cos^2\theta-2\cos\theta+1 &= 0 \\
(\cos\theta-1)^2 &= 0 \\
\cos\theta-1 &= 0 \\
\cos\theta &= 1
\end{aligned}
$$

On the interval [0, 2π), $\cos\theta = 1$ when $\theta = 0$. Check this solution in the original equation.

$$
\begin{aligned}
2\sin\frac{\theta}{2} &= \sin\theta \\
2\sin\frac{0}{2} &= \sin 0 \\
2\sin 0 &= \sin 0 \\
2\sin 0 &= \sin 0 \\
0 &= 0
\end{aligned}
$$

The solution is valid. Therefore, the solution to the equation on [0, 2π) is 0.

36. $1-\sin^2\frac{\theta}{2}-\cos\frac{\theta}{2}=\frac{3}{4}$

SOLUTION:

$$1-\sin^2\frac{\theta}{2}-\cos\frac{\theta}{2}=\frac{3}{4}$$
$$1-\left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right)^2\pm\sqrt{\frac{1+\cos\theta}{2}}=\frac{3}{4}$$
$$1-\frac{1-\cos\theta}{2}\pm\sqrt{\frac{1+\cos\theta}{2}}=\frac{3}{4}$$
$$1-\frac{3}{4}-\frac{1-\cos\theta}{2}=\mp\sqrt{\frac{1+\cos\theta}{2}}$$
$$\frac{1}{4}-\frac{1-\cos\theta}{2}=\mp\sqrt{\frac{1+\cos\theta}{2}}$$
$$\frac{1-2-2\cos\theta}{4}=\mp\sqrt{\frac{1+\cos\theta}{2}}$$
$$\frac{-2\cos\theta-1}{4}=\mp\sqrt{\frac{1+\cos\theta}{2}}$$
$$\left(\frac{-2\cos\theta-1}{4}\right)^2=\left(\mp\sqrt{\frac{1+\cos\theta}{2}}\right)^2$$
$$\frac{4\cos^2\theta+4\cos\theta+1}{16}=\frac{1+\cos\theta}{2}$$
$$4\cos^2\theta+4\cos\theta+1=8+8\cos\theta$$
$$4\cos^2\theta-4\cos\theta-7=0$$
$$(2\cos\theta+1)(2\cos\theta-7)=0$$
$$2\cos\theta+1=0 \quad\text{or}\quad 2\cos\theta-7=0$$
$$2\cos\theta=-1 \qquad 2\cos\theta=7$$
$$\cos\theta=-\frac{1}{2} \qquad \cos\theta=\frac{7}{2}$$

On the interval [0, 2π), $\cos\theta=-\frac{1}{2}$ when $\theta=\frac{2\pi}{3}$ and $\theta=\frac{4\pi}{3}$. The equation $\cos\theta=\frac{7}{2}$ has no solution since $\frac{7}{2}$ > 1. Check each of these solutions in the original equation.

$$
\begin{aligned}
1-\sin^2\frac{\theta}{2}-\cos\frac{\theta}{2}&=\frac{3}{4}\\
1-\sin^2\frac{\frac{2\pi}{3}}{2}-\cos\frac{\frac{2\pi}{3}}{2}&=\frac{3}{4}\\
1-\sin^2\frac{\pi}{3}-\cos\frac{\pi}{3}&=\frac{3}{4}\\
1-\left(\frac{\sqrt{3}}{2}\right)^2-\frac{1}{2}&=\frac{3}{4}\\
\frac{1}{2}-\frac{3}{4}&=\frac{3}{4}\\
-\frac{1}{4}&\neq\frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
1-\sin^2\frac{\theta}{2}-\cos\frac{\theta}{2}&=\frac{3}{4}\\
1-\sin^2\frac{\frac{4\pi}{3}}{2}-\cos\frac{\frac{4\pi}{3}}{2}&=\frac{3}{4}\\
1-\sin^2\frac{2\pi}{3}-\cos\frac{2\pi}{3}&=\frac{3}{4}\\
1-\left(\frac{\sqrt{3}}{2}\right)^2-\left(-\frac{1}{2}\right)&=\frac{3}{4}\\
1-\frac{3}{4}+\frac{1}{2}&=\frac{3}{4}\\
\frac{1}{4}+\frac{1}{2}&=\frac{3}{4}\\
\frac{3}{4}&=\frac{3}{4}
\end{aligned}
$$

The only valid solution is $\frac{4\pi}{3}$. Therefore, the solution to the equation on $[0, 2\pi)$ is $\frac{4\pi}{3}$.

Rewrite each product as a sum or difference.

37. $\cos 3\theta \cos \theta$

SOLUTION:

$$
\begin{aligned}
\cos 3\theta\cos\theta&=\frac{1}{2}[\cos(3\theta-\theta)+\cos(3\theta+\theta)]\\
&=\frac{1}{2}(\cos 2\theta+\cos 4\theta)\\
&=\frac{1}{2}\cos 2\theta+\frac{1}{2}\cos 4\theta
\end{aligned}
$$

38. $\cos 12x \sin 5x$

SOLUTION:

$$
\begin{aligned}
\cos 12x\sin 5x&=\frac{1}{2}[\sin(12x+5x)-\sin(12x-5x)]\\
&=\frac{1}{2}(\sin 17x-\sin 7x)\\
&=\frac{1}{2}\sin 17x-\frac{1}{2}\sin 7x
\end{aligned}
$$

39. $\sin 3x \cos 2x$

SOLUTION:

$$\begin{aligned}\sin 3x \cos 2x &= \frac{1}{2}[\sin(3x + 2x) + \sin(3x - 2x)]\\ &= \frac{1}{2}(\sin 5x + \sin x)\\ &= \frac{1}{2}\sin 5x + \frac{1}{2}\sin x\end{aligned}$$

40. $\sin 8\theta \sin \theta$

SOLUTION:

$$\begin{aligned}\sin 8\theta \sin \theta &= \frac{1}{2}[\cos(8\theta - \theta) - \cos(8\theta + \theta)]\\ &= \frac{1}{2}(\cos 7\theta - \cos 9\theta)\\ &= \frac{1}{2}\cos 7\theta - \frac{1}{2}\cos 9\theta\end{aligned}$$

Find the exact value of each expression.

41. $2 \sin 135° \sin 75°$

SOLUTION:

$$\begin{aligned}2\sin 135^\circ \sin 75^\circ &= 2 \cdot \frac{1}{2}[\cos(135^\circ - 75^\circ) - \cos(135^\circ + 75^\circ)]\\ &= \cos 60^\circ - \cos 210^\circ\\ &= \frac{1}{2} - \left(-\frac{\sqrt{3}}{2}\right)\\ &= \frac{1 + \sqrt{3}}{2}\end{aligned}$$

42. $\cos \frac{7ð}{12} + 3 \cos \frac{ð}{12}$

SOLUTION:

$$\begin{aligned}\cos\frac{7\pi}{12}+\cos\frac{\pi}{12} &= 2\cos\left(\frac{\frac{7\pi}{12}+\frac{\pi}{12}}{2}\right)\cos\left(\frac{\frac{7\pi}{12}-\frac{\pi}{12}}{2}\right)\\ &= 2\cos\frac{\pi}{3}\cos\frac{\pi}{4}\\ &= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

43. $\frac{2}{3}\sin 172.5^\circ \ \sin 127.5^\circ$

SOLUTION:

$$\begin{aligned}\frac{2}{3}\sin 172.5^\circ \sin 127.5^\circ &= \frac{2}{3}\cdot\frac{1}{2}[\cos(172.5^\circ-127.5^\circ)-\cos(172.5^\circ+127.5^\circ)]\\ &= \frac{1}{3}(\cos 45^\circ-\cos 300^\circ)\\ &= \frac{1}{3}\left(\frac{\sqrt{2}}{2}-\frac{1}{2}\right)\\ &= \frac{\sqrt{2}-1}{6}\end{aligned}$$

44. $\sin 142.5^\circ \cos 352.5^\circ$

SOLUTION:

$$\begin{aligned}\sin 142.5^\circ \cos 352.5^\circ &= \frac{1}{2}[\sin(142.5^\circ+352.5^\circ)+\sin(142.5^\circ-352.5^\circ)]\\ &= \frac{1}{2}\left[\sin 495^\circ+\sin\left(-210^\circ\right)\right]\\ &= \frac{1}{2}\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right)\\ &= \frac{1+\sqrt{2}}{4}\end{aligned}$$

5-5 Multiple-Angle and Product-to-Sum Identities

45. $\sin 75^\circ + \sin 195^\circ$

SOLUTION:

$$\begin{aligned}\sin 75^\circ + \sin 195^\circ &= 2\sin\left(\frac{75^\circ + 195^\circ}{2}\right)\cos\left(\frac{75^\circ - 195^\circ}{2}\right)\\ &= 2\sin 135^\circ \cos(-60^\circ)\\ &= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

46. $2\cos 105^\circ + 2\cos 195^\circ$

SOLUTION:

$$\begin{aligned}2\cos 105^\circ + 2\cos 195^\circ &= 2(\cos 105^\circ + \cos 195^\circ)\\ &= 2\left[2\cos\left(\frac{105^\circ + 195^\circ}{2}\right)\cos\left(\frac{105^\circ - 195^\circ}{2}\right)\right]\\ &= 4\cos 150^\circ \cos(-45^\circ)\\ &= 4\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \sqrt{6}\end{aligned}$$

47. $3\sin\frac{17ð}{12} - 3\sin\frac{ð}{12}$

SOLUTION:

$$\begin{aligned}3\sin\frac{17\pi}{12} - 3\sin\frac{\pi}{12} &= 3\left(\sin\frac{17\pi}{12} - \sin\frac{\pi}{12}\right)\\ &= 3\left[2\cos\left(\frac{\frac{17\pi}{12} + \frac{\pi}{12}}{2}\right)\sin\left(\frac{\frac{17\pi}{12} - \frac{\pi}{12}}{2}\right)\right]\\ &= 6\cos\frac{3\pi}{4}\sin\frac{2\pi}{3}\\ &= 6\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{3\sqrt{6}}{2}\end{aligned}$$

48. $\cos \frac{13\delta}{12} + \cos \frac{5\delta}{12}$

SOLUTION:

$$\begin{aligned}\cos\frac{13\pi}{12}+\cos\frac{5\pi}{12}&=2\cos\left(\frac{\frac{13\pi}{12}+\frac{5\pi}{12}}{2}\right)\cos\left(\frac{\frac{13\pi}{12}-\frac{5\pi}{12}}{2}\right)\\&=2\cos\frac{3\pi}{4}\cos\frac{\pi}{3}\\&=2\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\&=-\frac{\sqrt{2}}{2}\end{aligned}$$

Solve each equation.

49. $\cos\theta - \cos 3\theta = 0$

SOLUTION:

$$\begin{aligned}\cos\theta-\cos 3\theta&=0\\-2\sin\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{\theta-3\theta}{2}\right)&=0\\-2\sin 2\theta\sin\theta&=0\\\sin 2\theta\sin\theta&=0\\\sin 2\theta=0\quad&\text{or}\quad\sin\theta=0\\2\theta=\pi\quad&\quad\theta=0\ \text{ or }\ \theta=\pi\\\theta=\frac{\pi}{2}&\end{aligned}$$

The graph of $y = \sin 2\theta$ has a period of π. The solutions $0 + 2n\pi$ and $\pi + 2n\pi$ can be combined to $n\pi$. So, the solutions are $n\pi$, $\frac{\pi}{2} + 2n\pi$, where n is an integer.

50. $2 \cos 4\theta + 2 \cos 2\theta = 0$

SOLUTION:

$$\begin{aligned}
2\cos 4\theta + 2\cos 2\theta &= 0 \\
\cos 4\theta + \cos 2\theta &= 0 \\
2\cos\left(\frac{4\theta + 2\theta}{2}\right)\cos\left(\frac{4\theta - 2\theta}{2}\right) &= 0 \\
2\cos 3\theta \cos\theta &= 0 \\
\cos 3\theta \cos\theta &= 0 \\
\cos 3\theta &= 0 \qquad \text{or} \qquad \cos\theta = 0 \\
3\theta = \frac{\pi}{2} \quad \text{or} \quad 3\theta &= \frac{3\pi}{2} \qquad \theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{3\pi}{2} \\
\theta = \frac{\pi}{6} \qquad \theta &= \frac{\pi}{2}
\end{aligned}$$

The graph of $y = \cos 3\theta$ has a period of $\frac{2ð}{3}$. The solutions $\frac{\pi}{2} + 2n\pi$ and $\frac{3\pi}{2} + 2n\pi$ can be combined to $\frac{\pi}{2} + n\pi$. So, the solutions are $\frac{\pi}{6} + \frac{2\pi}{3}n$, $\frac{\pi}{2} + \frac{2\pi}{3}n$, and $\frac{\pi}{2} + n\pi$, where n is an integer.

51. $\sin 3\theta + \sin 5\theta = 0$

SOLUTION:

$$\begin{aligned}
\sin 3\theta + \sin 5\theta &= 0 \\
2\sin\left(\frac{3\theta + 5\theta}{2}\right)\cos\left(\frac{3\theta - 5\theta}{2}\right) &= 0 \\
2\sin 4\theta \cos(-\theta) &= 0 \\
2\sin 4\theta \cos\theta &= 0 \\
\sin 4\theta \cos\theta &= 0 \\
\sin 4\theta &= 0 \qquad \text{or} \qquad \cos\theta = 0 \\
4\theta = 0 \quad \text{or} \quad 4\theta &= \pi \qquad \theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{3\pi}{2} \\
\theta = 0 \qquad \theta &= \frac{\pi}{4}
\end{aligned}$$

The graph of $y = \sin 4\theta$ has a period of $\frac{ð}{2}$. The solutions $0 + \frac{\pi}{2}n$ and $\frac{\pi}{4} + \frac{\pi}{2}n$ can be combined to $\frac{\pi}{4} + \frac{\pi}{2}n$. The solutions $\frac{\pi}{2} + 2n\pi$ and $\frac{3\pi}{2} + 2n\pi$ can be combined to $\frac{\pi}{2} + n\pi$. So, the solutions are $\frac{\pi}{2}n$ and $\frac{\pi}{4} + \frac{\pi}{2}n$, where n is an integer.

52. $\sin 2\theta - \sin\theta = 0$

SOLUTION:

$$\sin 2\theta - \sin\theta = 0$$
$$2\cos\left(\frac{2\theta+\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right) = 0$$
$$2\cos\frac{3\theta}{2}\sin\theta = 0$$
$$\cos\frac{3\theta}{2}\sin\theta = 0$$
$$\cos\frac{3\theta}{2} = 0 \quad \text{or} \quad \sin\theta = 0$$
$$\frac{3\theta}{2} = \frac{\pi}{2} \text{ or } \frac{3\theta}{2} = \frac{3\pi}{2} \qquad \theta = 0 \text{ or } \theta = \pi$$
$$\theta = \frac{\pi}{3} \qquad \theta = \pi$$

The graph of $y = \cos\frac{3è}{2}$ has a period of $\frac{2\pi}{\frac{3}{2}}$ or $\frac{4\pi}{3}$. The solutions $0 + 2n\pi$ and $\pi + 2n\pi$ can be combined to $n\pi$. Then the solutions $\frac{\pi}{3} + \frac{4\pi}{3}n$, $\pi + \frac{4\pi}{3}n$, and $n\pi$ can be combined to $\frac{\pi}{3} + 2\pi n$, πn, $\frac{5ð}{3} + 2\pi n$, where n is an integer.

53. $3\cos 6\theta - 3\cos 4\theta = 0$

SOLUTION:

$$3\cos 6\theta - 3\cos 4\theta = 0$$
$$\cos 6\theta - \cos 4\theta = 0$$
$$-2\sin\left(\frac{6\theta+4\theta}{2}\right)\sin\left(\frac{6\theta-4\theta}{2}\right) = 0$$
$$2\sin 5\theta\sin\theta = 0$$
$$\sin 5\theta\sin\theta = 0$$
$$\sin 5\theta = 0 \quad \text{or} \quad \sin\theta = 0$$
$$5\theta = 0 \text{ or } 5\theta = \pi \qquad \theta = 0 \text{ or } \theta = \pi$$
$$\theta = 0 \qquad \theta = \frac{\pi}{5}$$

The graph of $y = \sin 5\theta$ has a period of $\frac{2\pi}{5}$ and the graph of $y = \sin 2\theta$ has a period of π. So, the solutions are $\frac{2\pi n}{5}$, $\frac{\pi}{5} + \frac{2\pi n}{5}$, and $n\pi$, where n is an integer.

54. $4 \sin \theta + 4 \sin 3\theta = 0$

SOLUTION:

$$\begin{aligned} 4\sin\theta + 4\sin 3\theta &= 0 \\ \sin\theta + \sin 3\theta &= 0 \\ 2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right) &= 0 \\ 2\sin 2\theta\cos(-\theta) &= 0 \\ 2\sin 2\theta\cos\theta &= 0 \\ \sin 2\theta\cos\theta &= 0 \end{aligned}$$

$$\sin 2\theta = 0 \quad \text{or} \quad \cos\theta = 0$$

$$2\theta = 0 \text{ or } 2\theta = \pi \qquad \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}$$

$$\theta = 0 \qquad \theta = \frac{\pi}{2}$$

The graph of $y = \sin 2\theta$ has a period of π. The solutions $\frac{\pi}{2} + n\pi$, $\frac{3\pi}{2} + n\pi$ and $\frac{\pi}{2} + 2n\pi$, can be combined to $\frac{\pi}{2} + n\pi$. So, the solutions are $n\pi$ and $\frac{\pi}{2} + n\pi$, where n is an integer.

Simplify each expression.

55. $\sqrt{\frac{1+cos6x}{2}}$

SOLUTION:

If $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$, then $\sqrt{\frac{1+\cos x}{2}} = \pm\cos\frac{x}{2}$. Use this form of the Cosine Half-Angle Identity to simplify the given expression.

$$\begin{aligned} \sqrt{\frac{1+\cos 6x}{2}} &= \pm\cos\frac{6x}{2} \\ &= \pm\cos 3x \end{aligned}$$

56. $\sqrt{\frac{1-cos16\grave{e}}{2}}$

SOLUTION:

If $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$, then $\sqrt{\frac{1-\cos\theta}{2}} = \pm\sin\frac{\theta}{2}$. Use this form of the Sine Half-Angle Identity to simplify the given expression.

$$\begin{aligned} \sqrt{\frac{1-\cos 16\theta}{2}} &= \pm\sin\frac{16\theta}{2} \\ &= \pm\sin 8\theta \end{aligned}$$

5-5 Multiple-Angle and Product-to-Sum Identities

Write each expression as a sum or difference.

57. cos (a + b) cos (a – b)

SOLUTION:

$$\begin{aligned}\cos(a+b)\cos(a-b) &= \frac{1}{2}\{\cos[(a+b)-(a-b)]+\cos[(a+b)+(a-b)]\}\\ &= \frac{1}{2}[\cos(a+b-a+b)+\cos(a+b+a-b)]\\ &= \frac{1}{2}(\cos 2b+\cos 2a)\end{aligned}$$

58. sin (θ − π) sin (θ + π)

SOLUTION:

$$\begin{aligned}\sin(\theta-\pi)\sin(\theta+\pi) &= \frac{1}{2}\{\cos[(\theta-\pi)-(\theta+\pi)]-\cos[(\theta-\pi)+(\theta+\pi)]\}\\ &= \frac{1}{2}[\cos(\theta-\pi-\theta-\pi)-\cos(\theta-\pi+\theta+\pi)]\\ &= \frac{1}{2}[\cos(-2\pi)-\cos 2\theta]\\ &= \frac{1}{2}[1-\cos 2\theta]\end{aligned}$$

59. sin (b + θ) cos (b + π)

SOLUTION:

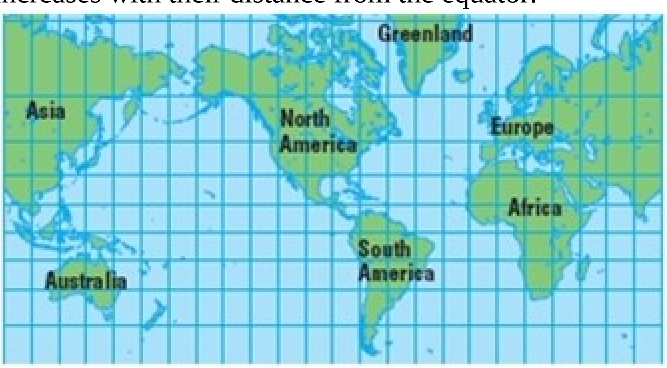
$$\begin{aligned}\sin(b+\theta)\cos(b+\pi) &= \frac{1}{2}\{\sin[(b+\theta)+(b+\pi)]+\sin[(b+\theta)-(b+\pi)]\}\\ &= \frac{1}{2}[\sin(b+\theta+b+\pi)+\sin(b+\theta-b-\pi)]\\ &= \frac{1}{2}[\sin(2b+\theta+\pi)+\sin(\theta-\pi)]\end{aligned}$$

60. cos ($a - b$) sin ($b - a$)

SOLUTION:

$$\begin{aligned}\cos(a-b)\sin(b-a) &= \frac{1}{2}\{\sin[(a-b)+(b-a)]-\sin[(a-b)-(b-a)]\}\\ &= \frac{1}{2}[\sin(a-b+b-a)-\sin(a-b-b+a)]\\ &= \frac{1}{2}[\sin 0-\sin(2a-2b)]\\ &= \frac{1}{2}[0-\sin(2a-2b)]\\ &= \frac{1}{2}[-\sin(2a-2b)]\\ &= -\frac{1}{2}\sin(2a-2b)\end{aligned}$$

61. **MAPS** A Mercator projection is a flat projection of the globe in which the distance between the lines of latitude increases with their distance from the equator.



The calculation of a point on a Mercator projection contains the expression $\tan\left(45 + \frac{\ell}{2}\right)$, where ℓ is the latitude of the point.

a. Write the expression in terms of sin ℓ and cos ℓ.

b. Find the value of this expression if $\ell = 60°$.

SOLUTION:

a.

$$\tan\left(45° + \frac{\ell}{2}\right) = \frac{\tan 45° + \tan\frac{\ell}{2}}{1 - \tan 45° \tan\frac{\ell}{2}}$$

$$= \frac{1 + \tan\frac{\ell}{2}}{1 - (1)\tan\frac{\ell}{2}}$$

$$= \frac{1 + \frac{\sin\ell}{1+\cos\ell}}{1 - \frac{\sin\ell}{1+\cos\ell}}$$

$$= \frac{\frac{1+\cos\ell}{1+\cos\ell} + \frac{\sin\ell}{1+\cos\ell}}{\frac{1+\cos\ell}{1+\cos\ell} - \frac{\sin\ell}{1+\cos\ell}}$$

$$= \frac{\frac{1+\cos\ell+\sin\ell}{1+\cos\ell}}{\frac{1+\cos\ell-\sin\ell}{1+\cos\ell}}$$

$$= \frac{1+\cos\ell+\sin\ell}{1+\cos\ell-\sin\ell}$$

b.

$$\frac{1+\cos\ell+\sin\ell}{1+\cos\ell-\sin\ell} = \frac{1+\cos 60°+\sin 60°}{1+\cos 60°-\sin 60°}$$

$$= \frac{1 + \frac{1}{2} + \frac{\sqrt{3}}{2}}{1 + \frac{1}{2} - \frac{\sqrt{3}}{2}}$$

$$= \frac{\frac{3+\sqrt{3}}{2}}{\frac{3-\sqrt{3}}{2}}$$

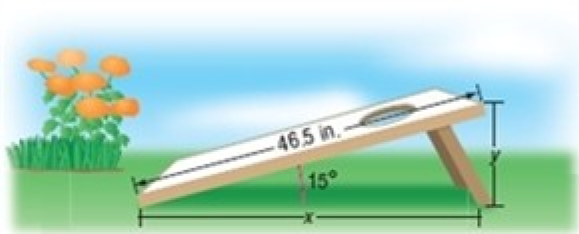
$$= \frac{3+\sqrt{3}}{3-\sqrt{3}} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}}$$

$$= \frac{9+6\sqrt{3}+3}{9-3}$$

$$= \frac{12+6\sqrt{3}}{6}$$

$$= 2+\sqrt{3}$$

62. **BEAN BAG TOSS** Ivan constructed a bean bag tossing game as shown in the figure below.



a. Exactly how far will the back edge of the board be from the ground?
b. Exactly how long is the entire setup?

SOLUTION:

a. Use the right triangle formed by the distance from the back edge to the ground y, the entire length of the setup x and the length of the board, 46.5 in. to find y.

$$\sin 15° = \frac{y}{46.5} \quad \text{Sine Ratio}$$

$$y = 46.5 \sin 15° \quad \text{Solve for } y.$$

$$y = 46.5 \sin\left(\frac{30°}{2}\right) \quad \text{Rewrite } 15° \text{ as } \frac{30°}{2}.$$

$$y = 46.5\left(\pm\sqrt{\frac{1-\cos 30°}{2}}\right) \quad \text{Sine Half-Angle Identity}$$

$$y = \pm 46.5\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \quad \cos 30° = \frac{\sqrt{3}}{2}$$

$$y = \pm 46.5\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} \quad \text{Simplify.}$$

$$y = \pm 46.5\sqrt{\frac{2-\sqrt{3}}{4}} \quad \text{Simplify.}$$

$$y = \pm 46.5\frac{\sqrt{2-\sqrt{3}}}{2} \quad \sqrt{4} = 2$$

$$y = \pm 23.25\sqrt{2-\sqrt{3}} \quad \text{Divide.}$$

Since length cannot be negative, the back edge is $23.25\sqrt{2-\sqrt{3}}$ in. from the ground.

b. Use the same right triangle to find x.

$\cos 15° = \dfrac{x}{46.5}$	Cosine Ratio
$x = 46.5 \sin 15°$	Solve for y.
$x = 46.5 \sin\left(\dfrac{30°}{2}\right)$	Rewrite 15° as $\dfrac{30°}{2}$.
$x = 46.5\left(\pm\sqrt{\dfrac{1+\cos 30°}{2}}\right)$	Sine Half-Angle Identity
$x = \pm 46.5\sqrt{\dfrac{1+\frac{\sqrt{3}}{2}}{2}}$	$\cos 30° = \dfrac{\sqrt{3}}{2}$
$x = \pm 46.5\sqrt{\dfrac{\frac{2+\sqrt{3}}{2}}{2}}$	Simplify.
$x = \pm 46.5\sqrt{\dfrac{2+\sqrt{3}}{4}}$	Simplify.
$x = \pm 46.5\dfrac{\sqrt{2+\sqrt{3}}}{2}$	$\sqrt{4} = 2$
$x = \pm 23.25\sqrt{2+\sqrt{3}}$	Divide.

Since length cannot be negative, the entire setup is $23.25\sqrt{2-\sqrt{3}}$ in. long.

PROOF Prove each identity.

63. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

SOLUTION:

$\cos 2\theta = \cos(\theta + \theta)$	$2\theta = \theta + \theta$
$= \cos\theta\cos\theta - \sin\theta\sin\theta$	Cosine Sum Identity
$= \cos^2\theta - \sin^2\theta$	Simplify.

64. $\cos 2\theta = 2\cos^2 \theta - 1$

SOLUTION:

$\cos 2\theta = \cos(\theta + \theta)$	$2\theta = \theta + \theta$
$= \cos\theta\cos\theta - \sin\theta\sin\theta$	Cosine Sum Identity
$= \cos^2\theta - \sin^2\theta$	Simplify.
$= \cos^2\theta - (1 - \cos^2\theta)$	Pythagoren Identity
$= 2\cos^2\theta - 1$	Combine like terms.

5-5 Multiple-Angle and Product-to-Sum Identities

65. $\tan 2\theta = \dfrac{2tanè}{1-tan2è}$

SOLUTION:

$$\begin{aligned}
\tan 2\theta &= \tan(\theta+\theta) && 2\theta = \theta+\theta \\
&= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} && \text{Tangent Sum Identity} \\
&= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Simplify.}
\end{aligned}$$

66. $\sin\dfrac{è}{2} = \pm\sqrt{\dfrac{1-cosè}{2}}$

SOLUTION:

Let $x = \dfrac{\theta}{2}$.

$$\begin{aligned}
\pm\sqrt{\frac{1-\cos\theta}{2}} &= \pm\sqrt{\frac{1-\cos\left(2\cdot\frac{\theta}{2}\right)}{2}} && \text{Rewrite } \theta \text{ as } 2\cdot\frac{\theta}{2}. \\
&= \pm\sqrt{\frac{1-\cos 2x}{2}} && \text{Substitute } x=\frac{\theta}{2}. \\
&= \pm\sqrt{\sin^2 x} && \text{Sine Power-Reducing Identity} \\
&= \sin x && \text{Simplify.} \\
&= \sin\frac{\theta}{2} && \text{Substitution.}
\end{aligned}$$

67. $\tan\dfrac{è}{2} = \pm\sqrt{\dfrac{1-cosè}{1+cosè}}$

SOLUTION:

Let $x = \dfrac{\theta}{2}$.

$$\begin{aligned}
\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} &= \pm\sqrt{\frac{1-\cos\left(2\cdot\frac{\theta}{2}\right)}{1+\cos\left(2\cdot\frac{\theta}{2}\right)}} && \text{Rewrite } \theta \text{ as } 2\cdot\frac{\theta}{2}. \\
&= \pm\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} && \text{Substitute } x=\frac{\theta}{2}. \\
&= \pm\sqrt{\tan^2 x} && \text{Tangent Power-Reducing Identity} \\
&= \tan x && \text{Simplify.} \\
&= \tan\frac{\theta}{2} && \text{Substitution.}
\end{aligned}$$

68. $\tan \frac{\grave{e}}{2} = \frac{sin\grave{e}}{1+\chi o\sigma\grave{e}}$

SOLUTION:

Let $x = \frac{\theta}{2}$.

$$
\begin{aligned}
\frac{\sin\theta}{1+\cos\theta} &= \frac{\sin\left(2\cdot\frac{\theta}{2}\right)}{1+\cos\left(2\cdot\frac{\theta}{2}\right)} && \text{Rewrite } \theta \text{ as } 2\cdot\frac{\theta}{2}.\\
&= \frac{\sin 2x}{1+\cos 2x} && \text{Substitute } x=\frac{\theta}{2}.\\
&= \frac{2\sin x\cos x}{1+\cos^2 x-\sin^2 x} && \text{Double-Angle Identities}\\
&= \frac{2\sin x\cos x}{1+\cos^2 x-(1-\cos^2 x)} && \text{Pythagorean Identity}\\
&= \frac{2\sin x\cos x}{2\cos^2 x} && \text{Simplify denominator.}\\
&= \frac{\sin x}{\cos x} && \text{Simplify fraction.}\\
&= \tan x && \text{Quotient Identity}\\
&= \tan\frac{\theta}{2} && \text{Substitution.}
\end{aligned}
$$

5-5 Multiple-Angle and Product-to-Sum Identities

Verify each identity by first using the power-reducing identities and then again by using the product-to-sum identities.

69. $2\cos^2 5\theta - 1 = \cos 10\theta$

SOLUTION:

i. $2\cos^2 5\theta - 1 = 2\left\{\frac{1}{2}[1 + \cos 2(5\theta)]\right\} - 1$ — Cosine Power-Reducing Identity

$= 2\left\{\frac{1}{2}[1 + \cos 10\theta]\right\} - 1$ — Simplify angle measure.

$= 1 + \cos 10\theta - 1$ — Multiply.

$= \cos 10\theta$ — Add.

ii. $2\cos^2 5\theta - 1 = 2\cos 5\theta \cos 5\theta - 1$ — Factor.

$= 2\left\{\frac{1}{2}[\cos(5\theta - 5\theta) + \cos(5\theta + 5\theta)]\right\} - 1$ — Product-to-Sum Identity

$= 2\left[\frac{1}{2}(\cos 0 + \cos 10\theta)\right] - 1$ — Simplify angle measures.

$= \cos 0 + \cos 10\theta - 1$ — Multiply.

$= 1 + \cos 10\theta - 1$ — $\cos 0 = 1$

$= \cos 10\theta$ — Simplify.

70. $\cos^2 2\theta - \sin^2 2\theta = \cos 4\theta$

SOLUTION:

i. $\cos^2 2\theta - \sin^2 2\theta$

$= \frac{1}{2}[1 + \cos 2(4\theta)] - \frac{1}{2}[1 - \cos 2(2\theta)]$	Cosine and Sine Power-Reducing Identities
$= \frac{1}{2}(1 + \cos 4\theta) - \frac{1}{2}(1 - \cos 4\theta)$	Simplify angle measures.
$= \frac{1}{2} + \frac{1}{2}\cos 4\theta - \frac{1}{2} + \frac{1}{2}\cos 4\theta$	Distributive Property
$= \cos 4\theta$	Combine like terms.

ii. $\cos^2 2\theta - \sin^2 2\theta$

$= \cos 2\theta \cos 2\theta - \sin 2\theta \sin 2\theta$	Factor.
$= \frac{1}{2}[\cos(2\theta - 2\theta) + \cos(2\theta + 2\theta)] - \frac{1}{2}[\cos(2\theta - 2\theta) - \cos(2\theta + 2\theta)]$	Product-to-Sum Identity
$= \frac{1}{2}(\cos 0 + \cos 4\theta) - \frac{1}{2}(\cos 0 - \cos 4\theta)$	Simplify angle measures.
$= \frac{1}{2}(1 + \cos 4\theta) - \frac{1}{2}(1 - \cos 4\theta)$	$\cos 0 = 1$
$= \frac{1}{2} + \frac{1}{2}\cos 4\theta - \frac{1}{2} - \frac{1}{2}\cos 4\theta$	Distributive Property
$= \cos 4\theta$	Combine like terms.

Rewrite each expression in terms of cosines of multiple angles with no power greater than 1.

71. $\sin^6 \theta$

SOLUTION:

$$\begin{aligned}
\sin^6\theta &= (\sin^2\theta)^3 \\
&= \left(\frac{1-\cos 2\theta}{2}\right)^3 \\
&= \frac{1}{2^3}(1-\cos 2\theta)^3 \\
&= \frac{1}{8}(1-\cos 2\theta)^2(1-\cos 2\theta) \\
&= \frac{1}{8}(1-2\cos 2\theta+\cos^2 2\theta)(1-\cos 2\theta) \\
&= \frac{1}{8}\left[1-2\cos 2\theta+\frac{1+\cos 2(2\theta)}{2}\right](1-\cos 2\theta) \\
&= \frac{1}{8}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)(1-\cos 2\theta) \\
&= \frac{1}{8}\left(\frac{2}{2}-\frac{4\cos 2\theta}{2}+\frac{1+\cos 4\theta}{2}\right)(1-\cos 2\theta) \\
&= \frac{1}{8}\left(\frac{2-4\cos 2\theta+1+\cos 4\theta}{2}\right)(1-\cos 2\theta) \\
&= \frac{1}{16}(3-4\cos 2\theta+\cos 4\theta)(1-\cos 2\theta) \\
&= \frac{1}{16}(3-4\cos 2\theta+\cos 4\theta-3\cos 2\theta+4\cos 2\theta\cos 2\theta-\cos 2\theta\cos 4\theta) \\
&= \frac{1}{16}(3-7\cos 2\theta+\cos 4\theta+4\cos^2 2\theta-\cos 2\theta\cos 4\theta) \\
&= \frac{1}{16}\left\{3-7\cos 2\theta+\cos 4\theta+4\left[\frac{1+\cos 2(2\theta)}{2}\right]-\cos 2\theta\cos 4\theta\right\} \\
&= \frac{1}{16}[3-7\cos 2\theta+\cos 4\theta+2(1+\cos 4\theta)-\cos 2\theta\cos 4\theta] \\
&= \frac{1}{16}[3-7\cos 2\theta+\cos 4\theta+2+2\cos 4\theta-\cos 2\theta\cos 4\theta] \\
&= \frac{1}{16}[5-7\cos 2\theta+3\cos 4\theta-\cos 2\theta\cos 4\theta]
\end{aligned}$$

72. $\sin^8 \theta$

SOLUTION:

$$
\begin{aligned}
\sin^8\theta &= \left[(\sin^2\theta)^2\right]^2 \\
&= \left[\left(\frac{1-\cos 2\theta}{2}\right)^2\right]^2 \\
&= \left[\frac{1}{4}(1-\cos 2\theta)^2\right]^2 \\
&= \frac{1}{16}\left[(1-\cos 2\theta)^2\right]^2 \\
&= \frac{1}{16}(1-2\cos 2\theta+\cos^2 2\theta)^2 \\
&= \frac{1}{16}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)^2 \\
&= \frac{1}{16}\left[\frac{1}{2}(2-4\cos 2\theta+1+\cos 4\theta)\right]^2 \\
&= \frac{1}{64}(3-4\cos 2\theta+\cos 4\theta)^2 \\
&= \frac{1}{64}(9-24\cos 2\theta+16\cos^2 2\theta+6\cos 4\theta-8\cos 2\theta\cos 4\theta+\cos^2 4\theta) \\
&= \frac{1}{64}\left(9-24\cos 2\theta+16\frac{1+\cos 4\theta}{2}+6\cos 4\theta-8\cos 2\theta\cos 4\theta+\frac{1+\cos 8\theta}{2}\right) \\
&= \frac{1}{64}\left(17-24\cos 2\theta+14\cos 4\theta-8\cos 2\theta\cos 4\theta+\frac{1+\cos 8\theta}{2}\right) \\
&= \frac{1}{128}(34-48\cos 2\theta+28\cos 4\theta-16\cos 2\theta\cos 4\theta+1+\cos 8\theta) \\
&= \frac{1}{128}(35-48\cos 2\theta+28\cos 4\theta+\cos 8\theta-16\cos 2\theta\cos 4\theta)
\end{aligned}
$$

5-5 Multiple-Angle and Product-to-Sum Identities

73. $\cos^7 \theta$

SOLUTION:

$$\cos^7\theta = (\cos^2\theta)^2\cos^2\theta\cos\theta$$
$$= \left(\frac{1+\cos 2\theta}{2}\right)^2\left(\frac{1+\cos 2\theta}{2}\right)\cos\theta$$
$$= \frac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta)\cdot\frac{1}{2}(1+\cos 2\theta)\cos\theta$$
$$= \frac{1}{8}(1+2\cos 2\theta+\cos^2 2\theta)(1+\cos 2\theta)\cos\theta$$
$$= \frac{1}{8}(1+2\cos 2\theta+\cos^2 2\theta)(\cos\theta+\cos\theta\cos 2\theta)$$
$$= \frac{1}{8}(\cos\theta+3\cos\theta\cos 2\theta+3\cos\theta\cos^2 2\theta+\cos\theta\cos 2\theta\cos^2 2\theta)$$
$$= \frac{1}{8}\left[\cos\theta+3\cos\theta\cos 2\theta+3\cos\theta\left(\frac{1+\cos 4\theta}{2}\right)+\cos\theta\cos 2\theta\left(\frac{1+\cos 4\theta}{2}\right)\right]$$
$$= \frac{1}{8}\left[\cos\theta+3\cos\theta\cos 2\theta+\frac{3}{2}\cos\theta(1+\cos 4\theta)+\frac{1}{2}\cos\theta\cos 2\theta(1+\cos 4\theta)\right]$$
$$= \frac{1}{16}[2\cos\theta+6\cos\theta\cos 2\theta+3\cos\theta(1+\cos 4\theta)+\cos\theta\cos 2\theta(1+\cos 4\theta)]$$
$$= \frac{1}{16}[2\cos\theta+6\cos\theta\cos 2\theta+3\cos\theta+3\cos\theta\cos 4\theta+\cos\theta\cos 2\theta+\cos\theta\cos 2\theta\cos 4\theta]$$
$$= \frac{1}{16}[5\cos\theta+7\cos\theta\cos 2\theta+3\cos\theta\cos 4\theta+\cos\theta\cos 2\theta\cos 4\theta]$$

74. $\sin^4 \theta \cos^4 \theta$

SOLUTION:

$$\begin{aligned}
\sin^4\theta\cos^4\theta &= (\sin^2\theta)^2(\cos^2\theta)^2 \\
&= \left(\frac{1-\cos 2\theta}{2}\right)^2\left(\frac{1+\cos 2\theta}{2}\right)^2 \\
&= \frac{1}{4}(1-\cos 2\theta)^2\frac{1}{4}(1+\cos 2\theta)^2 \\
&= \frac{1}{16}(1-\cos 2\theta)^2(1+\cos 2\theta)^2 \\
&= \frac{1}{16}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right) \\
&= \frac{1}{16}\cdot\frac{1}{2}(2-4\cos 2\theta+1+\cos 4\theta)\cdot\frac{1}{2}(2+4\cos 2\theta+1+\cos 4\theta) \\
&= \frac{1}{64}(3-4\cos 2\theta+\cos 4\theta)(3+4\cos 2\theta+\cos 4\theta) \\
&= \frac{1}{64}(9+6\cos 4\theta-16\cos^2 2\theta+\cos^2 4\theta) \\
&= \frac{1}{64}\left[9+6\cos 4\theta-16\left(\frac{1+\cos 4\theta}{2}\right)+\frac{1+\cos 8\theta}{2}\right] \\
&= \frac{1}{64}\left[9+6\cos 4\theta-8-8\cos 4\theta+\frac{1}{2}(1+\cos 8\theta)\right] \\
&= \frac{1}{64}\left[1+6\cos 4\theta-8\cos 4\theta+\frac{1}{2}(1+\cos 8\theta)\right] \\
&= \frac{1}{64}\cdot\frac{1}{2}[2+12\cos 4\theta-16\cos 4\theta+1+\cos 8\theta] \\
&= \frac{1}{128}[3-4\cos 4\theta+\cos 8\theta]
\end{aligned}$$

75. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate how graphs of functions can be used to find identities.

a. GRAPHICAL Use a graphing calculator to graph $f(x) = 4\left(\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}\right)$ on the interval $[-2\pi, 2\pi]$.

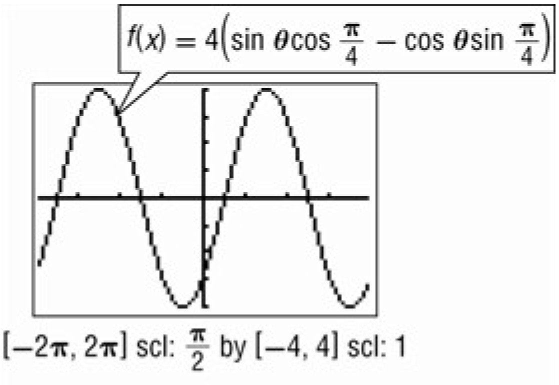
b. ANALYTICAL Write a sine function $h(x)$ that models the graph of $f(x)$. Then verify that $f(x) = h(x)$ algebraically.

c. GRAPHICAL Use a graphing calculator to graph $g(x) = \cos^2\left(\theta-\frac{\pi}{3}\right)-\sin^2\left(\theta-\frac{\pi}{3}\right)$ on the interval $[-2\pi, 2\pi]$.

d. ANALYTICAL Write a cosine function $k(x)$ that models the graph of $g(x)$. Then verify that $g(x) = k(x)$ algebraically.

SOLUTION:

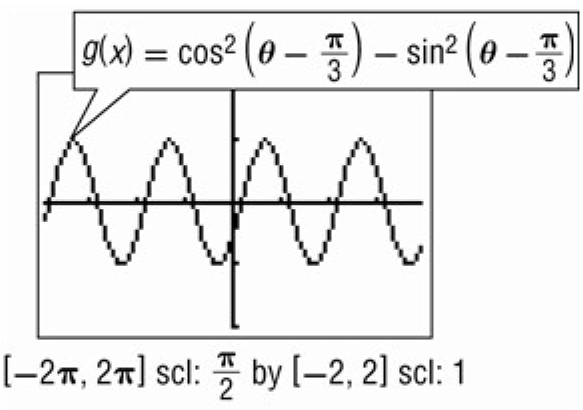
a.



b. Using the CALC maximum feature on the graphing calculator, you can determine that the function has a maximum of 4 at $x \approx 2.356197$ or $x = \frac{3\pi}{4}$. Since the maximum height of $y = \sin x$ is 1, a function $h(x) = a\sin(x + c)$ that models the graph of $f(x)$ has an amplitude of 4 times that of $f(x)$ or $b = 4$. Also, since the first maximum that occurs for $y = \sin x$, $x > 0$, is at $x = \frac{\pi}{2}$, the phase shift of the graph is about $\frac{\pi}{2} - \frac{3\pi}{4}$ or $-\frac{\pi}{4}$, so $c = -\frac{\pi}{4}$. Therefore, a function in terms of sine that models this graph is $h(x) = 4\sin\left(\theta - \frac{\pi}{4}\right)$.

$4\sin\left(\theta - \frac{\pi}{4}\right) = 4\left(\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right)$ Sine Difference Identity

c.



d. Using the CALC maximum feature on the graphing calculator, you can determine that the function has a maximum of 1 at $x \approx 1.0471978$ or $x = \frac{\pi}{3}$. Since the maximum height of $y = \cos x$ is 1, a function $k(x) = a\cos(bx + c)$ that models the graph of $f(x)$ also has an amplitude of 1, so $a = 1$. Since the graph completes 1 cycle on $[0, \pi]$ the frequency of the graph is $\frac{1}{\pi}$ which is twice the frequency of the cosine function, so $b = 2$. Since the first maximum for $y = \cos x$, $x > 0$, is at $x = \pi$, the phase shift of the graph is about $\pi - \frac{\pi}{3}$ or $\frac{2\pi}{3}$, so $c = \frac{2\pi}{3}$. Therefore, a function in terms of sine that models this graph is $k(x) = \cos\left(2\theta - \frac{2\pi}{3}\right)$.

$$\cos\left(2\theta - \frac{2\pi}{3}\right) = \cos\left[2\left(\theta - \frac{\pi}{3}\right)\right] \quad \text{Factor.}$$
$$= \cos^2\left(\theta - \frac{\pi}{3}\right) - \sin^2\left(\theta - \frac{\pi}{3}\right) \quad \text{Cosine Double-Angle Identity}$$

76. **CHALLENGE** Verify the following identity.
$\sin 2\theta \cos\theta - \cos 2\theta \sin\theta = \sin\theta$

SOLUTION:

$\sin 2\theta \cos\theta - \cos 2\theta \sin\theta$

$= 2\sin\theta\cos\theta\cos\theta - (1 - 2\sin^2\theta)\sin\theta$ Sine and Cosine Double-Angle Identities

$= 2\sin\theta\cos^2\theta - \sin\theta(1 - 2\sin^2\theta)$ Simplify.

$= 2\sin\theta - (1 - \sin^2\theta) - \sin\theta(1 - 2\sin^2\theta)$ Pythagorean Identity

$= 2\sin\theta - 2\sin^3\theta - \sin\theta + 2\sin^3\theta$ Distributive Property

$= \sin\theta$ Simplify.

REASONING Consider an angle in the unit circle. Determine what quadrant a double angle and half angle would lie in if the terminal side of the angle is in each quadrant.

77. I

SOLUTION:

If an angle θ lies in Quadrant I, then $0° < \theta < 90°$. If $0° < \theta < 45°$ then $2(0°) < 2\theta < 2(45°)$ or $0° < 2\theta < 90°$, which is Quadrant I. If $45° < \theta < 90°$ then $2(45°) < 2\theta < 2(90°)$ or $90° < 2\theta < 180°$, which is Quadrant II. If $\theta = 45°$, then $2\theta = 2(45°)$ or $90°$ and the angle is quadrantal, falling between Quadrants I and II. If $0° < \theta < 90°$, then $\frac{0°}{2} < \frac{\theta}{2} < \frac{90°}{2}$ or $0° < \theta < 45°$, which is still in Quadrant I.

78. II

SOLUTION:

If an angle θ lies in Quadrant II, then $90° < \theta < 180°$. If $90° < \theta < 135°$ then $2(90°) < 2\theta < 2(135°)$ or $180° < 2\theta < 270°$, which is Quadrant III. If $135° < \theta < 180°$ then $2(135°) < 2\theta < 2(180°)$ or $270° < 2\theta < 360°$, which is Quadrant IV. If $\theta = 135°$, then $2\theta = 2(135°)$ or $270°$ and the angle is quadrantal, falling between Quadrants III and IV. If $90° < \theta < 180°$, then $\frac{90°}{2} < \frac{\theta}{2} < \frac{180°}{2}$ or $45° < \theta < 90°$, which are in Quadrant I.

79. III

SOLUTION:

If an angle θ lies in Quadrant III, then $180° < \theta < 270°$. If $180° < \theta < 225°$ then $2(180°) < 2\theta < 2(225°)$ or $360° < 2\theta < 450°$. Using coterminal angles, this is equivalent to $0° < 2\theta < 90°$, which is Quadrant I. If $225° < \theta < 270°$ then $2(225°) < 2\theta < 2(270°)$ or $450° < 2\theta < 540°$. Using coterminal angles, this is equivalent to $90° < 2\theta < 180°$ which is Quadrant II. If $\theta = 225°$, then $2\theta = 2(225°)$ or $450°$ or $90°$ and the angle is quadrantal, falling between Quadrants I and II. If $180° < \theta < 270°$, then $\frac{180°}{2} < \frac{\theta}{2} < \frac{270°}{2}$ or $90° < \theta < 135°$, which are in Quadrant II.

5-5 Multiple-Angle and Product-to-Sum Identities

CHALLENGE Verify each identity.

80. $\sin 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$

SOLUTION:

$\sin 4\theta = \sin 2(2\theta)$	$4\theta = 2(2\theta)$
$= 2\sin 2\theta\cos 2\theta$	Sine Double-Angle Identity
$= 2(2\sin\theta\cos\theta)(1 - 2\sin^2\theta)$	Sine and Cosine Double-Angle Identities
$= 2(2\sin\theta\cos\theta - 4\sin^3\theta\cos\theta)$	Multiply.
$= 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$	Multiply.

81. $\cos 4\theta = 1 - 8\sin^2\theta\cos^2\theta$

SOLUTION:

$\cos 4\theta = \cos 2(2\theta)$	$4\theta = 2\theta$
$= 1 - 2\sin^2 2\theta$	Cosine Double-Angle Identity
$= 1 - 2(\sin 2\theta)(\sin 2\theta)$	$\sin^2 2\theta = (\sin 2\theta)(\sin 2\theta)$
$= 1 - 2(2\sin\theta\cos\theta)(2\sin\theta\cos\theta)$	Sine Double-Angle Identity
$= 1 - 8\sin^2\theta\cos^2\theta$	Multiply.

PROOF Prove each identity.

82. $\cos^2\theta = \dfrac{1 + cos2è}{2}$

SOLUTION:

$\dfrac{1+\cos 2\theta}{2} = \dfrac{1 + (2\cos^2\theta - 1)}{2}$	Double-Angle Identity for Cosine
$= \dfrac{2\cos^2\theta}{2}$	Simplify.
$= \cos^2\theta$	Simplify.

83. $\tan^2\theta = \dfrac{1 - cos2è}{1 + cos2è}$

SOLUTION:

$\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$	Quotient Identity
$= \dfrac{\frac{1-\cos 2\theta}{2}}{\frac{1+\cos 2\theta}{2}}$	Sine Power-Reducing Identity
$= \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$	Simplify.

84. $\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

SOLUTION:

$$\begin{aligned}\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] &= \frac{1}{2}(\cos\alpha\cos\beta+\sin\alpha\sin\beta+\cos\alpha\sin\beta-\sin\alpha\sin\beta)\\ &= \frac{1}{2}(2\cos\alpha\cos\beta)\\ &= \cos\alpha\cos\beta\end{aligned}$$

85. $\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

SOLUTION:

$$\begin{aligned}\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] &= \frac{1}{2}(\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\ &= \frac{1}{2}(2\sin\alpha\cos\beta)\\ &= \sin\alpha\cos\beta\end{aligned}$$

86. $\cos \alpha \sin \beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

SOLUTION:

$$\begin{aligned}\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] &= \frac{1}{2}[\sin\alpha\cos\beta+\cos\alpha\sin\beta-(\sin\alpha\cos\beta-\cos\alpha\sin\beta)]\\ &= \frac{1}{2}(\sin\alpha\cos\beta+\cos\alpha\sin\beta-\sin\alpha\cos\beta+\cos\alpha\sin\beta)\\ &= \frac{1}{2}(2\cos\alpha\sin\beta)\\ &= \cos\alpha\sin\beta\end{aligned}$$

87. $\cos \alpha + \cos \beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

SOLUTION:

Let $x = \left(\frac{\alpha+\beta}{2}\right)$ and let $y = \left(\frac{\alpha-\beta}{2}\right)$.

$$\begin{aligned}2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) &= 2\cos x\cos y\\ &= 2\left\{\frac{1}{2}[\cos(x+y)+\cos(x-y)]\right\}\\ &= \cos(x+y)+\cos(x-y)\\ &= \cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)\\ &= \cos\alpha+\cos\beta\end{aligned}$$

88. $\sin \alpha - \sin \beta = 2 \cos\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)$

SOLUTION:

Let $x = \left(\frac{\alpha+\beta}{2}\right)$ and let $y = \left(\frac{\alpha-\beta}{2}\right)$.

$$\begin{aligned}
2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) &= 2\cos x \sin y \\
&= 2\left\{\frac{1}{2}[\sin(x+y)-\sin(x-y)]\right\} \\
&= \sin(x+y)-\sin(x-y) \\
&= \sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)-\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right) \\
&= \sin\alpha-\sin\beta
\end{aligned}$$

89. $\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)$

SOLUTION:

Let $x = \left(\frac{\alpha+\beta}{2}\right)$ and let $y = \left(\frac{\alpha-\beta}{2}\right)$.

$$\begin{aligned}
-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) &= -2\sin x \sin y \\
&= -2\left\{\frac{1}{2}[\cos(x-y)-\cos(x+y)]\right\} \\
&= -\cos(x-y)+\cos(x+y) \\
&= -\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right) \\
&= -\cos\beta+\cos\alpha \\
&= \cos\alpha-\cos\beta
\end{aligned}$$

90. ***Writing in Math*** Describe the steps that you would use to find the exact value of cos 8θ if $\cos\theta = \frac{\sqrt{2}}{5}$.

SOLUTION:

Sample answer: Since $8\theta = 2(4\theta)$, use the identity $\cos 2\theta = 2\cos^2\theta - 1$ to find an expression for $\cos 8\theta$. Then, since $4\theta = 2(2\theta)$ use the same double-angle identity again to find an expression for $\cos 4\theta$. Finally, use the same double-angle identity to replace the remaining double-angle expression $\cos 2\theta$. The result will be an expression in terms of just $\cos\theta$. Substitute $\frac{\sqrt{2}}{5}$ for $\cos\theta$ in this expression and simplify.

5-5 Multiple-Angle and Product-to-Sum Identities

Find the exact value of each trigonometric expression.

91. $\cos \frac{\pi}{12}$

SOLUTION:

Write $\frac{\pi}{12}$ as the sum or difference of angle measures with cosines that you know.

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

92. $\cos \frac{19\pi}{12}$

SOLUTION:

Write $\frac{19\pi}{12}$ as the sum or difference of angle measures with cosines that you know.

$$\begin{aligned}\cos\frac{19\pi}{12} &= \cos\left(\frac{11\pi}{6}-\frac{\pi}{4}\right)\\ &= \cos\frac{11\pi}{6}\cos\frac{\pi}{4}-\sin\frac{11\pi}{6}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)-\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

93. $\sin \frac{5\pi}{6}$

SOLUTION:

Write $\frac{5\pi}{6}$ as the sum or difference of angle measures with sines that you know.

$$\begin{aligned}\sin\frac{5\pi}{6} &= \sin\left(\pi-\frac{\pi}{6}\right)\\ &= \sin\pi\cos\frac{\pi}{6}-\cos\pi\sin\frac{\pi}{6}\\ &= 0\left(\frac{\sqrt{3}}{2}\right)-(-1)\left(\frac{1}{2}\right)\\ &= 0+\frac{1}{2}\\ &= \frac{1}{2}\end{aligned}$$

94. $\sin \frac{13\pi}{12}$

SOLUTION:

Write $\frac{13\pi}{12}$ as the sum or difference of angle measures with sines that you know.

$$\begin{aligned}\sin\frac{13\pi}{12} &= \sin\left(\frac{4\pi}{3}-\frac{\pi}{4}\right)\\ &= \sin\frac{4\pi}{3}\cos\frac{\pi}{4}-\cos\frac{4\pi}{3}\sin\frac{\pi}{4}\\ &= -\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)-\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

95. $\cos\left(-\frac{7\pi}{6}\right)$

SOLUTION:

Write $-\frac{7\pi}{6}$ as the sum or difference of angle measures with cosines that you know.

$$\begin{aligned}\cos\left(-\frac{7\pi}{6}\right) &= \cos\left(\frac{\pi}{2}-\frac{5\pi}{3}\right)\\ &= \cos\frac{\pi}{2}\cos\frac{5\pi}{3}+\sin\frac{\pi}{2}\sin\frac{5\pi}{3}\\ &= 0\left(\frac{1}{2}\right)+1\left(-\frac{\sqrt{3}}{2}\right)\\ &= 0-\frac{\sqrt{3}}{2}\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

96. $\sin\left(-\frac{7\pi}{12}\right)$

SOLUTION:

Write $-\frac{7\pi}{12}$ as the sum or difference of angle measures with sines that you know.

$$\begin{aligned}\sin\left(-\frac{7\pi}{12}\right) &= \sin\left(\frac{5\pi}{4}-\frac{11\pi}{6}\right)\\ &= \sin\frac{5\pi}{4}\cos\frac{11\pi}{6}-\cos\frac{5\pi}{4}\sin\frac{11\pi}{6}\\ &= -\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)\\ &= -\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

97. **GARDENING** Eliza is waiting for the first day of spring in which there will be 14 hours of daylight to start a flower garden. The number of hours of daylight H in her town can be modeled by $H = 11.45 + 6.5 \sin(0.0168d - 1.333)$, where d is the day of the year, $d = 1$ represents January 1, $d = 2$ represents January 2, and so on. On what day will Eliza begin gardening?

SOLUTION:

Let $H = 14$, and solve for d.

$$\begin{aligned} H &= 11.45 + 6.5\sin(0.0168d - 1.333) \\ 14 &= 11.45 + 6.5\sin(0.0168d - 1.333) \\ 2.55 &= 6.5\sin(0.0168d - 1.333) \\ \frac{2.55}{6.5} &= \sin(0.0168d - 1.333) \\ \sin^{-1}\frac{2.55}{6.5} &= 0.0168d - 1.333 \\ 0.4031 &\approx 0.0168d - 1.333 \\ 1.736 &\approx 0.0168d \\ 103.3 &\approx d \end{aligned}$$

Therefore, Eliza will begin gardening on the 104th day of the year. Because there are 31 days in January, 28 days in February (on a non-leap year), and 31 days in March, the 104th day corresponds to April 14.

Find the exact value of each expression. If undefined, write *undefined*.

98. $\csc\left(-\frac{\pi}{3}\right)$

SOLUTION:

$$\begin{aligned} \csc\left(-\frac{\pi}{3}\right) &= \csc\left(-\frac{\pi}{3} + 2\pi\right) && \text{Rewrite } -\frac{\pi}{3} \text{ as the sum of a number and an integer multiple of } 2\pi. \\ &= \csc\frac{5\pi}{3} && -\frac{\pi}{3} \text{ and } \frac{5\pi}{3} \text{ map to the same point } (x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \text{ on the unit circle.} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \text{ or } -\frac{2\sqrt{3}}{3} && \csc t = \frac{1}{y} \text{ and } y = -\frac{\sqrt{3}}{2} \text{ when } t = \frac{5\pi}{3}. \end{aligned}$$

99. tan 210°

SOLUTION:

210° corresponds to the point $(x, y) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

$\tan t = \frac{y}{x}$	Definition of $\tan t$
$\tan 210° = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$	$x = -\frac{\sqrt{3}}{2}$ and $y = -\frac{1}{2}$, when $t = 210°$.
$= \frac{\sqrt{3}}{3}$	Simplify.

100. $\sin \frac{19\pi}{4}$

SOLUTION:

$\sin \frac{19\pi}{4} = \sin\left(\frac{19\pi}{4} + 2(-2)\pi\right)$	Rewrite $\frac{19\pi}{4}$ as the sum of a number and an integer multiple of 2π.
$= \sin \frac{3\pi}{4}$	$\frac{19\pi}{4}$ and $\frac{3\pi}{4}$ map to the same point $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle.
$= \frac{\sqrt{2}}{2}$	$\sin t = y$ and $y = \frac{\sqrt{2}}{2}$ when $t = \frac{3\pi}{4}$.

101. cos (–3780°)

SOLUTION:

$\cos(-3780°) = \cos(-3780° + 360(11)°)$	Rewrite $-3780°$ as the sum of a number and an integer multiple of
$= \cos 180°$	$-3780°$ and $180°$ map to the same point $(x, y) = (-1, 0)$ on the unit c
$= -1$	$\cos t = x$ and $x = -1$ when $t = 180°$.

5-5 Multiple-Angle and Product-to-Sum Identities

Graph and analyze each function. Describe the domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.

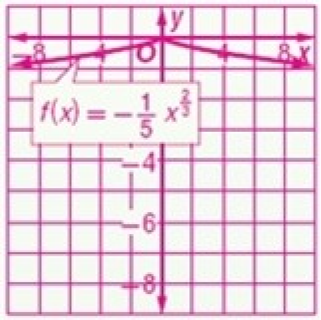
102. $f(x) = -\frac{1}{5}x^{\frac{2}{3}}$

SOLUTION:

Evaluate the function for several x-values in its domain.

x	−6	−4	−2	0	2	4	6
***f*(x)**	–0.66	–0.50	–0.32	0	–0.32	–0.50	–0.66

Use these points to construct a graph.



D = (−∞, ∞), R = (−∞, 0]; intercept: (0, 0); $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$; continuous for all real numbers; increasing on (−∞, 0), decreasing on (0, ∞)

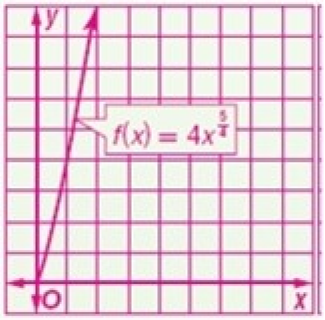
103. $f(x) = 4x^{\frac{5}{4}}$

SOLUTION:

Evaluate the function for several x-values in its domain.

x	0	1	2	3	4	5	6
***f*(x)**	0	4	9.51	15.8	22.6	29.9	37.6

Use these points to construct a graph.



Since the denominator of the power is even, the domain must be restricted to nonnegative values.

D = [0, ∞), R = [0, ∞); intercept: (0, 0); $\lim_{x \to \infty} f(x) = \infty$; continuous on [0, ∞); increasing on (0, ∞)

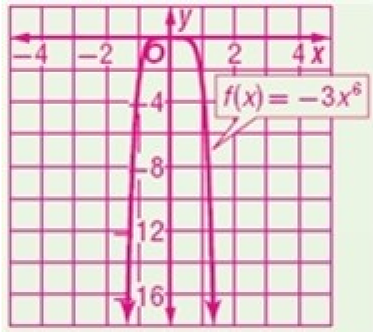
104. $f(x) = -3x^6$

SOLUTION:

Evaluate the function for several x-values in its domain.

x	–1.5	–1	–0.5	0	0.5	1	1.5
$f(x)$	–34.2	–3	–0.05	0	–0.05	–3	–34.2

Use these points to construct a graph.



D = $(-\infty, \infty)$, R = $(-\infty, 0]$; intercept: (0, 0); $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = -\infty$; continuous for all real numbers; increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$

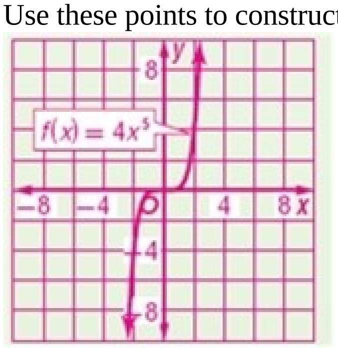
105. $f(x) = 4x^5$

SOLUTION:

Evaluate the function for several x-values in its domain.

x	–3	–2	–1	0	1	2	3
$f(x)$	–972	–128	–4	0	4	128	972

Use these points to construct a graph.



D = $(-\infty, \infty)$, R = $(-\infty, \infty)$; intercept: (0, 0); $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$; continuous for all real numbers; increasing on $(-\infty, \infty)$

5-5 Multiple-Angle and Product-to-Sum Identities

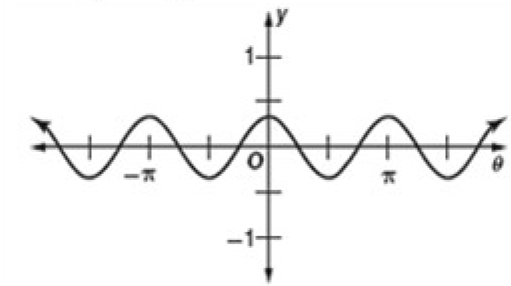
106. **REVIEW** Identify the equation for the graph.

A $y = 3\cos 2\theta$

B $y = \frac{1}{3}\cos 2\theta$

C $y = 3\cos \frac{1}{2}\theta$

D $y = \frac{1}{3}\cos \frac{1}{2}\theta$



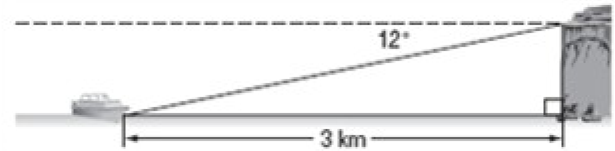
SOLUTION:

Half of the distance between the maximum and minimum values of the function appears to be about $\frac{1}{3}$ of a unit, so $a = \frac{1}{3}$. The graph appears to complete one cycle on the interval [0, π], so the period is π. Use the period to find *b*.

$$\begin{aligned} \text{period} &= \frac{2\pi}{|b|} \\ \pi &= \frac{2\pi}{|b|} \\ |b|\pi &= 2\pi \\ |b| &= \frac{2\pi}{\pi} \text{ or } 2 \end{aligned}$$

So, one sinusoidal equation that corresponds to the graph is $y = \frac{1}{3}\cos 2\theta$. Therefore, the correct answer is B.

107. **REVIEW** From a lookout point on a cliff above a lake, the angle of depression to a boat on the water is 12°. The boat is 3 kilometers from the shore just below the cliff. What is the height of the cliff from the surface of the water to the lookout point?



F $\dfrac{3}{sin 12°}$

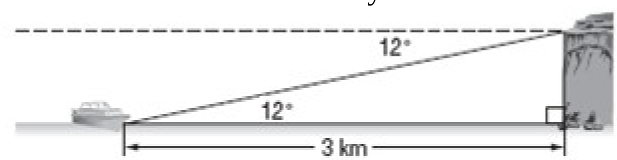
G $\dfrac{3}{tan 12°}$

H $\dfrac{3}{cos 12°}$

J 3 tan 12°

SOLUTION:

The angle of elevation from the boat to the top of the cliff is congruent to the angle of depression from the top of the cliff to the boat because they are alternate interior angles of parallel lines.

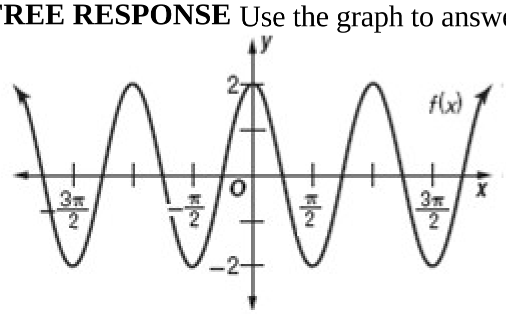


Because the side adjacent to the 12º angle of elevation is given, the tangent function can be used to find the height of the cliff.

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan 12° = \frac{x}{3}$$
$$3 \tan 12° = x$$

Therefore, the correct answer is J.

108. **FREE RESPONSE** Use the graph to answer each of the following.



a. Write a function of the form $f(x) = a \cos(bx + c) + d$ that corresponds to the graph.

b. Rewrite $f(x)$ as a sine function.

c. Rewrite $f(x)$ as a cosine function of a single angle.

d. Find all solutions of $f(x) = 0$.

e. How do the solutions that you found in part **d** relate to the graph of $f(x)$?

SOLUTION:

a. Sample answer: Half of the distance between the maximum and minimum values of the function appears to be 2 units, so a = 2. The graph appears to complete one cycle on the interval [0, π], so the period is π. Use the period to find b.

$$\text{period} = \frac{2\pi}{|b|}$$
$$\pi = \frac{2\pi}{|b|}$$
$$|b|\pi = 2\pi$$
$$|b| = 2$$

So, $f(x) = 2\cos 2x$.

b. Sample answer: Because sine and cosine are confunctions, $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$. So, substituting $2x$ for θ, $f(x) = 2\cos 2x$ can be rewritten in terms of sine as $f(x) = 2 \cdot \sin\left(\frac{\pi}{2} - 2x\right)$ or $f(x) = 2\sin\left(\frac{\pi}{2} - 2x\right)$.

c. Sample answer: Use the double-angle identity for cosine to write $2\cos 2x$ in terms of a single angle.

$$f(x) = 2\cos 2x$$
$$= 2(2\cos^2 x - 1)$$
$$= 4\cos^2 x - 2$$

d. Sample answer: Let $f(x) = 0$ and solve for x.

$$4\cos^2 x - 2 = 0$$
$$4\cos^2 x = 2$$
$$\cos^2 x = \frac{1}{2}$$
$$\cos x = \frac{\sqrt{2}}{2}$$
$$x = \cos^{-1}\frac{\sqrt{2}}{2}$$
$$x = \pm\frac{\pi}{4}$$

The period of $f(x) = 2\cos 2x$ is π, so the solutions are $x = \frac{\pi}{4} + n\pi$, where n is an integer.

e. Sample answer: The solutions or zeros of a function are the x-intercepts of the graph of that function. So, the solutions of $f(x) = 4\cos^2 x - 2$ are the x-intercepts of the graph of $f(x)$.