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A NOTE ON A PAPER OF ERCAN AND ÖNAL

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Abstract. In this short note we formulate and prove a Banach space version of the Banach-Stone theorem, obtained recently by Ercan and Onal ([4]) for the case of lattice-valued continuous functions.

1. Introduction

Let X, Y be compact Hausdorff spaces and E a Banach lattice and F be an abstract M-space with unit. Let $\pi : C(X, E) \to C(Y, F)$ be a Riesz isomorphism such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$ for each $f \in C(X, E)$. Ercan and Önal have proved in [4] that E is Riesz isomorphic to F and X is homeomorphic to Y. In this paper we show that similar conclusion can be drawn when E is a complex Banach space and π is a surjective isometry. It is well-known that an additional condition is necessary on π to ensure this conclusion. See Corollary 7.3 of [1] and the remark following it. Also see [2] for topological conditions which make the component spaces homeomorphic.

We also consider this problem for certain M-spaces without unit, i.e, $C_0(X)$ for a locally compact, non-compact set X.

For a Banach space E, let E_1 denote the closed unit ball and $\partial_e E_1$ denote the set of extreme points.

We recall that $e \in E_1$ is a strong extreme point if $e_k \in E$, $\|e \pm e_k\| \to 1 \implies e_k \to 0$. It is easy to see that $1 \in C(X)$ is a strong extreme point.

2. Main Result

We recall that from [7] (Chapter 1) that by the Kakutani' representation theorem

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F is isometric to $C(N)$ for a compact Hausdorff space N. In what follows we use the well-known identification of $C(Y, C(N))$ with $C(Y \times N)$. A Banach space E is said to be an L^1-predual if E^* is isometric to $L^1(\mu)$ for a positive measure μ. See [7] Chapter 7 for properties of these spaces. For a compact Hausdorff space K, $C(K)$ is an L^1-predual. See [8] and [9] for a classification of these spaces among function spaces.

Theorem 1. *Let E be a Banach space and let $\pi : C(X, E) \to C(Y, C(N))$ be a surjective isometry such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$ for each $f \in C(X, E)$. Then there exists a compact Hausdorff space M such that E is isometric to $C(M)$ and X, M are homeomorphic to Y, N respectively.*

Proof. Since E is isometric to the range of a projection of norm one in $C(X, E)$ it follows from the results in Chapter 6 in [7] that E is an L^1-predual space. To show that E is isometric to a $C(M)$, in view of the results from [8], (see [9] for the complex versions) we need to show that $\partial_e E_1 \neq \emptyset$ and $\partial_e E_1^*$ is a weak*-closed set.

Since $1 \in C(Y, C(N))$ is a strong extreme point, we have that $C(X, E)_1$ has a strong extreme point, say f. It follows from [3] that $f(x) \in \partial_e E_1$ for all $x \in X$. Since π is a surjective isometry, we have that $\partial_e C(X, E)_1^*$ is a weak*-closed set. Now let $\{e_\alpha^*\}_{\alpha \in \Delta} \subset \partial_e E_1^*$ be a net such that $e_\alpha^* \to e^*$, in the weak*-topology. For $x \in X$, as $\{\delta(x) \otimes e_\alpha^*\} \subset \partial_e C(X, E)_1^*$ and $\delta(x) \otimes e_\alpha^* \to \delta(x) \otimes e^*$ in the weak*-topology, we get that $\delta(x) \otimes e^* \in \partial_e C(X, E)_1^*$. Therefore $e^* \in \partial_e E_1^*$.

Hence E is isometric to $C(M)$ for a compact Hausdorff space M. After identifying $C(X, E)$ with $C(X \times M)$, it follows from the classical Banach-Stone theorem that there is a homeomorphism $\sigma : Y \times N \to X \times M$ such that $\pi(f) = \pi(1) f \circ \sigma$ for all $f \in C(X \times M)$.

Since $|\pi(1)| \equiv 1$, it is easy to see that the hypothesis implies that for each $y \in Y$ there exists a unique $x \in X$ such that, $\sigma(\{y\} \times N) = \{x\} \times M$. It therefore follows from Lemma 5 of [4] that X, M are homeomorphic to Y, N respectively. ■

We next state a more general version for injective tensor product spaces. This can be proved using arguments similar to the ones given during the proof of the above Theorem. For the extremal arguments, instead of the results from [3], one uses the results from [5]. The arguments will be symmetric for the component spaces. In what follows, we use the well-known fact that the injective product space, $E \otimes_\epsilon F$ can be identified as a subspace of the space of compact operators, $\mathcal{K}(E^*, F)$.

Theorem 2. *Let $\pi : E \otimes_\epsilon F \to C(Y, C(N))$ be a surjective isometry such that for $T \in E \otimes_\epsilon F$, $0 \notin \pi(T)(Y) \iff 0 \notin T^*(\partial_e F_1^*)$. Then there exist compact*

Hausdorff spaces X, M such that E, F are isometric to $C(X), C(M)$ respectively and X, M are homeomorphic to Y, N respectively.

For general Banach spaces E, F, G, for which the injective tensor product spaces $E \otimes_\epsilon F$ is isometric to $E \otimes_\epsilon G$, the cancelation theorems ask for conditions under which F is isometric to G. It is also reasonable to investigate what geometric properties of G are passed on to F. In view of the classification scheme in [8] and [9], one can fix E as one of the spaces in the classification scheme and let G be some other type of L^1-predual and ask for conditions on the isometry between $E \otimes_\epsilon F$ and $E \otimes_\epsilon G$ to ensure that F is the same type of L^1-predual as G. In what follows we do this for $C_0(X)$ type spaces.

In the above results we have used crucially the presence of extreme point in the unit ball of one of the tensor product spaces. We do not know how to prove a similar result in the case of $C_0(X)$ spaces where X is locally compact and not compact.

Question 3. Let $\Phi : C_0(X, E) \to C_0(Y, C_0(N))$ be an isometry. Is E isometric to $C_0(M)$ for some locally compact space M? Under what additional condition on Φ do we get that X, M are homeomorphic to Y, N respectively?

Since $C_0(Y, C_0(N))$ can be identified with $C_0(Y \times N)$, we again have that E is an L^1-predual space and that $\partial_e E_1^* \cup \{0\}$ is weak*-closed.

It follows from the classification results of [9] that, now in order to show that E is a $C_0(M)$ space, one needs to show that for any maximal face F of the unit sphere $S(E^*)$, the convex hull $CO(F \cup \{0\})$ is a weak*-closed set. We have only a partial result in this direction.

Proposition 4. *Let X be a dispersed locally compact space. Let $\Phi : C_0(X, E) \to C_0(Y, C_0(N))$ be a surjective isometry. Then E is isometric to $C_0(M)$ for a locally compact space M.*

Proof. We again use the identification of $C_0(Y, C_0(N))$ with $C_0(Y \times N)$. Since X is dispersed, it has a dense set of isolated points. Let x_0 be an isolated point. It easy to see that $f \to \chi_{x_0} f$ is a projection in $C(X, E)$ such that $C(X, E)$ is isometric to $E \oplus_\infty E'$ (ℓ^∞-direct sum) for some Banach space E'. Now since $C(X, E)$ is a $C_0(.)$-space, it follows from Example 1.1.4(a) of [6] which describes ℓ^∞-summands in $C_0(.)$-spaces, that E is isometric to a $C_0(S)$ for a locally compact space S. ∎

Analogous to the above results for M-spaces, one can also formulate the corresponding L-space questions. It is possible that the L and M space duality theorem of Kakutani and the fact that an L spaces is the unique predual of its dual, is an approach to handle these problems.

Question 5. *Let* $(\Omega, \mathcal{A})$ *be a measure space. Let* λ, ν *be positive measures with* ν*,* σ*-finite. Suppose* $\Psi : L^1(\lambda, E) \to L^1(\lambda, L^1(\nu))$ *is an isometry. Under what assumptions on* Ψ *do we get that* E *is isometric to* $L^1(\nu)$*?*

Here again one has that $L^1(\lambda, L^1(\nu))$ is isometric to $L^1(\lambda \times \nu)$. Therefore $L^1(\lambda, E)$ is an L-space. Since E is isometric to the range of a projection of norm one in $L^1(\lambda, E)$, it follows from [7] that E is an abstract L space.

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