Robust Constrained Model Predictive Control using Linear Matrix Inequalities

Mayuresh V. Kothare Venkataramanan Balakrishnan* Manfred Morari†

Chemical Engineering, 210-41
California Institute of Technology
Pasadena, CA 91125

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Subtitle: We present a new technique for the synthesis of a robust model predictive control (MPC) law, using linear matrix inequalities (LMIs). The technique allows incorporation of a large class of plant uncertainty descriptions and is shown to be robustly stabilizing.

Key Words: Model predictive control, linear matrix inequalities, convex optimization, multivariable control systems, state-feedback, on-line operation, robust control, robust stability, state-feedback, time-varying systems.

Abstract

The primary disadvantage of current design techniques for model predictive control (MPC) is their inability to deal *explicitly* with plant model uncertainty. In this paper, we present a new approach for robust MPC synthesis which allows explicit incorporation of the description of plant uncertainty in the problem formulation. The uncertainty is expressed both in the time and frequency domains. The goal is to design, at each time step, a state-feedback control law which minimizes a "worst-case" infinite horizon objective function, subject to constraints on the control input and plant output. Using standard techniques, the problem of minimizing an upper bound on the "worst-case" objective function, subject to input and output constraints, is reduced to a convex optimization involving linear matrix inequalities (LMIs). It is shown that the feasible receding horizon state-feedback control design robustly stabilizes the set of uncertain plants. Several extensions, such as application to systems with time-delays, problems involving constant set-point tracking, trajectory tracking and disturbance rejection, which follow naturally from our formulation, are discussed. The controller design is illustrated with two examples.

*School of Electrical Engineering, Purdue University, West Lafayette, IN 47907-1285. This work was initiated when this author was affiliated with Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125.

†To whom all correspondence should be addressed: Institut für Automatik, the Swiss Federal Institute of Technology (ETH), Physikstrasse 3, ETH-Zentrum, 8092 Zürich, Switzerland; phone +411 632 7626, fax +411 632 1211, e-mail morari@aut.ee.ethz.ch

1 Introduction

Model Predictive Control (MPC), also known as Moving Horizon Control (MHC) or Receding Horizon Control (RHC), is a popular technique for the control of slow dynamical systems, such as those encountered in chemical process control in the petrochemical, pulp and paper industries, and in gas pipeline control. At every time instant, MPC requires the on-line solution of an optimization problem to compute optimal control inputs over a fixed number of future time instants, known as the "time horizon". Although more than one control move is generally calculated, only the first one is implemented. At the next sampling time, the optimization problem is reformulated and solved with new measurements obtained from the system. The on-line optimization can be typically reduced to either a linear program or a quadratic program.

Using MPC, it is possible to handle inequality constraints on the manipulated and controlled variables in a systematic manner during the design and implementation of the controller. Moreover, several process models as well as many performance criteria of significance to the process industries can be handled using MPC. A fairly complete discussion of several design techniques based on MPC and their relative merits and demerits can be found in the review article by Garcia et al. (1989) [16].

Perhaps the principal shortcoming of existing MPC-based control techniques is their inability to *explicitly* incorporate plant model uncertainty. Thus, nearly all known formulations of MPC minimize, on-line, a *nominal* objective function, using a single linear time-invariant (LTI) model to predict the future plant behavior. Feedback, in the form of plant measurement at the next sampling time, is expected to account for plant model uncertainty. Needless to say, such control systems which provide "optimal" performance for a particular model may perform very poorly when implemented on a physical system which is not exactly described by the model (for example, see [34]). Similarly, the extensive amount of literature on stability analysis of MPC algorithms [26, 21, 27, 33, 32, 31, 10, 9, 13] is by and large restricted to the nominal case, with no plant-model mismatch; the issue of the behavior of MPC algorithms in the face of uncertainty, i.e., "robustness", has been addressed to a much lesser extent. Broadly, the existing literature on robustness in MPC can be summarized as follows:

- *Analysis of robustness properties of MPC.* Garcia and Morari [13, 14, 15] have analyzed the robustness of unconstrained MPC in the framework of internal model control (IMC) and have developed tuning guidelines for the IMC filter to guarantee robust stability. Zafiriou (1990) [31] and Zafiriou and Marchal (1991) [32] have used the contraction properties of MPC to develop necessary/sufficient conditions for robust stability of MPC with input and output constraints. Given upper and lower bounds on the impulse response coefficients of a single-input-single-output (SISO) plant with Finite Impulse Responses (FIR), Genceli and Nikolaou (1993) [17] have presented robustness analysis of constrained ℓ_1-norm MPC algorithms. Polak and Yang (1993) [24, 25] have analyzed robust stability of their MHC algorithm for continuous-time linear systems with variable sampling times by using a contraction constraint on the state.

- *MPC with explicit uncertainty description.* The basic philosophy of MPC-based design algorithms which explicitly account for plant uncertainty [8, 2, 34] is the following:

  Modify the on-line constrained minimization problem to a min-max problem (minimizing the worst-case value of the objective function, where the worst-case is taken over the set of uncertain plants).

Based on this concept, Campo and Morari (1987) [8], Allwright and Papavasiliou (1992) [2] and Zheng and Morari (1993) [34] have presented robust MPC schemes for SISO FIR plants, given uncertainty bounds on the impulse response coefficients. For certain choices of the objective function, the on-line problem is shown to be reducible to a linear program.

One of the problems with this linear programming approach is that to simplify the on-line computational complexity, one must choose simplistic, albeit unrealistic model uncertainty descriptions, for e.g., fewer FIR coefficients. Secondly, this approach cannot be extended to unstable systems.

From the preceding review, we see that there has been progress in the *analysis* of robustness properties of MPC. But *robust synthesis*, i.e., the explicit incorporation of realistic plant uncertainty description in the problem formulation, has been addressed only in a restrictive framework for FIR models. There is a need for computationally inexpensive techniques for robust MPC synthesis which are suitable for on-line implementation and which allow incorporation of a broad class of model uncertainty descriptions.

In this paper, we present one such MPC-based technique for the control of plants with uncertainties. This technique is motivated by recent developments in the theory and application (to control) of optimization involving Linear Matrix Inequalities (LMIs) [6]. There are two reasons why LMI optimization is relevant to MPC. First, LMI-based optimization problems can be solved in polynomial-time, often in times comparable to that required for the evaluation of an analytical solution for a similar problem. Thus, LMI optimization can be implemented on-line. Secondly, it is possible to recast much of existing robust control theory in the framework of LMIs. The implication is that we can devise an MPC scheme where at each time instant, an LMI optimization problem (as opposed to conventional linear or quadratic programs) is solved, which incorporates input and output constraints and a description of the plant uncertainty and guarantees certain robustness properties.

The paper is organized as follows: In §2, we discuss background material such as models of systems with uncertainties, LMIs and MPC. In §3, we formulate the robust unconstrained MPC problem with state-feedback as an LMI problem. We then extend the formulation to incorporate input and output constraints, and show that the feasible receding horizon control law which we obtain is robustly stabilizing. In §4, we extend our formulation to systems with time-delays and to problems involving trajectory tracking, constant set-point tracking and disturbance rejection. In §5, we present two examples to illustrate the design procedure. Finally, in §6, we present concluding remarks.

2 Background

2.1 Models for uncertain systems

We present two paradigms for robust control which arise from two different modeling and identification procedures. The first is a "multi-model" paradigm, and the second is the more popular "linear system with a feedback uncertainty" robust control model. Underlying both these paradigms is a linear time-varying (LTV) system

$$
\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)u(k), \\
y(k) &= Cx(k), \\
[A(k) \quad B(k)] &\in \Omega,
\end{aligned} \tag{1}
$$

where $u(k) \in \mathcal{R}^{n_u}$ is the control input, $x(k) \in \mathcal{R}^{n_x}$ is the state of the plant and $y(k) \in \mathcal{R}^{n_y}$ is the plant output, and Ω is some prespecified set.

Polytopic or multi-model paradigm

For polytopic systems, the set Ω is the polytope

$$
\Omega = \mathrm{Co}\{[A_1 \ B_1], [A_2 \ B_2], \ldots, [A_L \ B_L]\}, \tag{2}
$$

where Co refers to the convex hull. In other words, if $[A \ B] \in \Omega$, then for some nonnegative $\lambda_1, \lambda_2, \ldots, \lambda_L$ summing to one, we have

$$
[A \ B] = \sum_{i=1}^{L} \lambda_i [A_i \ B_i].
$$

$L = 1$ corresponds to the nominal LTI system.

Polytopic system models can be developed as follows. Suppose that for the (possibly nonlinear) system under consideration, we have input/output data sets at different operating points, or at different times. From each data set, we develop a number of linear models (for simplicity, we assume that the various linear models involve the same state vector). Then, it is reasonable to assume that any analysis and design methods for the polytopic system (1), (2) with vertices given by the linear models will apply to the real system.

Alternatively, suppose the Jacobian $\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial u} \end{bmatrix}$ of a nonlinear discrete time-varying system $x(k+1) = f(x(k), u(k), k)$ is known to lie in the polytope Ω. Then it can be shown that every trajectory (x, u) of the original nonlinear system is also a trajectory of (1) for some LTV system in Ω [20]. Thus, the original nonlinear system can be approximated (possibly conservatively) by a polytopic uncertain LTV system. Similarly, it can be shown that bounds on impulse response coefficients of SISO FIR plants can be translated to a polytopic uncertainty description on the state-space matrices. Thus, this polytopic uncertainty description is suitable for several problems of engineering significance.

Structured feedback uncertainty

A second, more common paradigm for robust control consists of an LTI system with uncertainties or perturbations appearing in the feedback loop (see Figure 1-B):

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + B_p p(k), \\
y(k) &= Cx(k), \\
q(k) &= C_q x(k) + D_{qu} u(k), \\
p(k) &= (\Delta q)(k).
\end{aligned} \tag{3}
$$

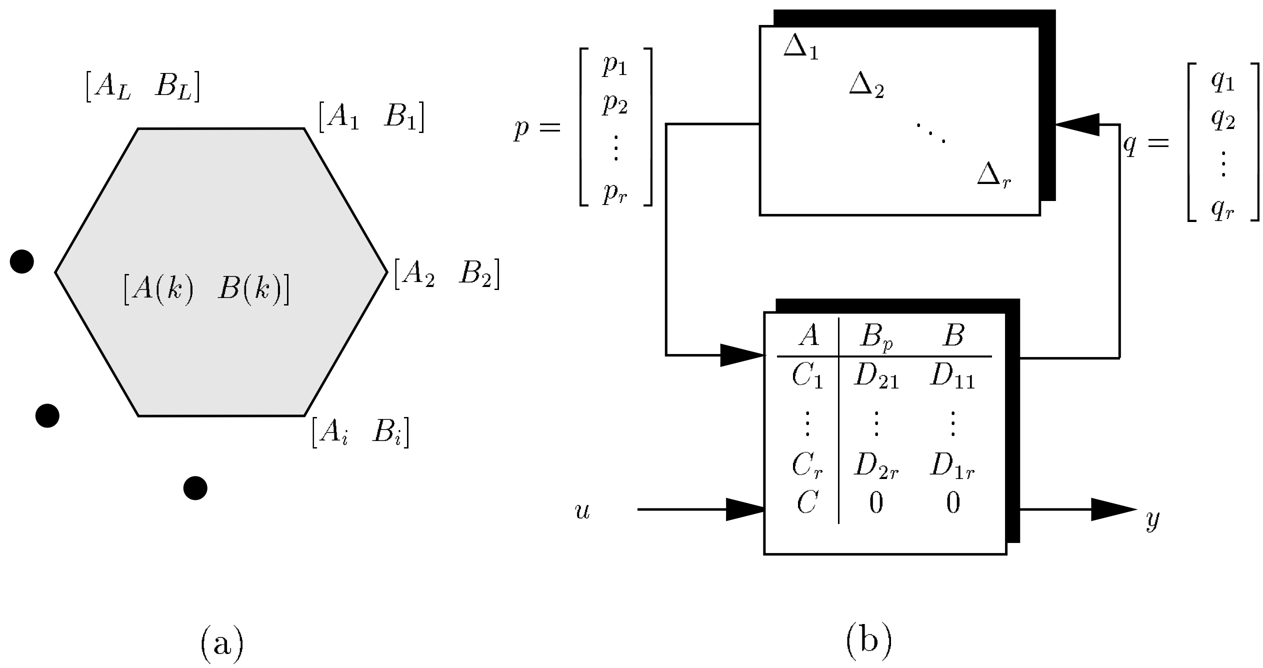

Figure 1: (a) Graphical representation of polytopic uncertainty; (b) Structured uncertainty.

The operator Δ is block diagonal:

$$\Delta = \begin{bmatrix} \Delta_1 & & & \\ & \Delta_2 & & \\ & & \ddots & \\ & & & \Delta_r \end{bmatrix} \tag{4}$$

with $\Delta_i : \mathcal{R}^{n_i} \longrightarrow \mathcal{R}^{n_i}$. Δ can represent either a memoryless time-varying matrix with $\|\Delta_i(k)\|_2 \equiv \bar{\sigma}(\Delta_i(k)) \leq 1, \quad i = 1, 2, \ldots, r, \quad k \geq 0$; or a convolution operator (for e.g., a stable LTI dynamical system) with the operator norm induced by the truncated ℓ_2-norm less than 1, i.e.,

$$\sum_{j=0}^{k} p_i(j)^T p_i(j) \quad \leq \quad \sum_{j=0}^{k} q_i(j)^T q_i(j), \;\; i = 1, \ldots, r, \quad \forall k \geq 0. \tag{5}$$

Each Δ_i is assumed to be either a *repeated scalar* block or a *full* block [23] and models a number of factors, such as nonlinearities, dynamics or parameters, that are unknown, unmodeled or neglected. A number of control systems with uncertainties can be recast in this framework [23]. For ease of reference, we will refer to such systems as systems with structured uncertainty. Note that in this case, the uncertainty set Ω is defined by (3) and (4).

When Δ_i is a stable LTI dynamical system, the quadratic sum constraint (5) is equivalent to the following frequency domain specification on the z-transform $\hat{\Delta}_i(z)$

$$\|\hat{\Delta}_i\|_{\mathcal{H}_\infty} \equiv \sup_{\theta \in [0, 2\pi)} \bar{\sigma}(\hat{\Delta}_i(e^{j\theta})) \leq 1.$$

Thus, the structured uncertainty description is allowed to contain both LTI and LTV blocks, with frequency domain and time-domain constraints respectively. We will, however, only consider the LTV case since the results we obtain are identical for the general mixed uncertainty case, with one exception, as pointed out in §3.2.2. The details can be found in [6, Sec. 8.2] and will be omitted here for brevity. For the LTV case, it is easy to show through routine algebraic manipulations that system (3) corresponds to system (1) with

$$\Omega = \left\{ \left[\begin{array}{cc} A + B_p \Delta C_q & B + B_p \Delta D_{qu} \end{array} \right] : \Delta \text{ satisfies (4) with } \bar{\sigma}(\Delta_i) \leq 1 \right\}. \tag{6}$$

$\Delta \equiv 0,\ p(k) \equiv 0,\ k \geq 0$, corresponds to the nominal LTI system.

The issue of whether to model a system as a polytopic system or a system with structured uncertainty depends on a number of factors, such as the underlying physical model of the system, available model identification and validation techniques etc. For example, nonlinear systems can be modeled either as polytopic systems or as systems with structured perturbations. We will not concern ourselves with such issues here; instead we will assume that one of the two models discussed thus far is available.

2.2 Model Predictive Control

Model Predictive Control is an open-loop control design procedure where at each sampling time k, plant measurements are obtained and a model of the process is used to predict future outputs of the system. Using these predictions, m control moves $u(k+i|k), i = 0, 1, \ldots, m-1$ are computed by minimizing a *nominal* cost $J_p(k)$ over a prediction horizon p as follows:

$$\min_{u(k+i|k), i=0,1,\ldots,m-1} J_p(k), \tag{7}$$

subject to constraints on the control input $u(k+i|k), i = 0, 1, \ldots, m-1$ and possibly also on the state $x(k+i|k), i = 0, 1, \ldots, p$ and the output $y(k+i|k), i = 1, 2, \ldots, p$. Here

$x(k+i\|k)$, $y(k+i\|k)$	:	state and output respectively, at time $k+i$, predicted based on the measurements at time k; $x(k\|k)$ and $y(k\|k)$ refer respectively to the state and output measured at time k.
$u(k+i\|k)$	:	control move at time $k+i$, computed by the optimization problem (7) at time k; $u(k\|k)$ is the control move to be implemented at time k.
p	:	*output* or *prediction* horizon
m	:	*input* or *control* horizon.

It is assumed that there is no control action after time $k+m-1$, i.e., $u(k+i|k) = 0,\ i \geq m$. In the receding horizon framework, only the first computed control move $u(k|k)$ is implemented. At the next sampling time, the optimization (7) is resolved with new measurements from the plant. Thus, both the control horizon m and the prediction horizon p *move* or *recede* ahead by one step as time moves ahead by one step. This is the reason why MPC is also sometimes referred to as Receding Horizon Control (RHC) or Moving Horizon Control (MHC). The purpose of taking new measurements at each time step is to compensate for unmeasured disturbances and model inaccuracy both of which cause the system output to be different from the one predicted by the model. We assume that exact measurement of the state of the system is available at each sampling time k, i.e.,

$$x(k|k) = x(k). \tag{8}$$

Several choices of the objective function $J_p(k)$ in the optimization (7) have been reported [17, 21, 16, 32] and have been compared in [7]. In this paper, we consider the following quadratic objective:

$$J_p(k) = \sum_{i=0}^{p} \left(x(k+i|k)^T Q_1 x(k+i|k) + u(k+i|k)^T R u(k+i|k) \right),$$

where $Q_1 > 0$ and $R > 0$ are symmetric weighting matrices. In particular, we will consider the case $p = \infty$ which is referred to as the infinite horizon MPC (IH-MPC). Finite horizon control laws have been known to have poor nominal stability properties [4, 26]. Nominal stability of finite horizon MPC requires imposition of a terminal state constraint ($x(k+i|k) = 0, i = m$) and/or use of the contraction mapping principle [31, 32] to tune Q_1, R, m and p for stability. But the terminal state constraint is somewhat artificial since only the first control move is implemented. Thus, in the closed loop, the states actually approach zero only asymptotically. Also, the computation of the contraction condition [31, 32] at all possible combinations of active constraints at the optimum of the on-line optimization can be extremely time consuming, and as such, this issue remains unaddressed. On the other hand, infinite horizon control laws have been shown to guarantee nominal stability [26, 21]. We therefore believe that rather than using the above methods to "tune" the parameters for stability, it is preferable to adopt the infinite horizon approach to guarantee at least nominal stability.

In this paper, we consider Euclidean norm bounds and component-wise peak bounds on the input $u(k+i|k)$, given respectively as

$$\|u(k+i|k)\|_2 \le u_{\max}, \quad k,\ i \ge 0, \tag{9}$$

and

$$|u_j(k+i|k)| \le u_{j,\max}, \quad k,\ i \ge 0, \quad j = 1, 2, \ldots, n_u. \tag{10}$$

Similarly, for the output, we consider the Euclidean norm constraint and component-wise peak bounds on $y(k+i|k)$, given respectively as

$$\|y(k+i|k)\|_2 \le y_{\max}, \quad k \ge 0,\ i \ge 1, \tag{11}$$

and

$$|y_j(k+i|k)| \le y_{j,\max}, \quad k \ge 0,\ i \ge 1,\ j = 1, 2, \ldots, n_y. \tag{12}$$

Note that the output constraints have been imposed strictly over the future horizon (i.e., $i \ge 1$) and not at the current time (i.e, $i = 0$). This is because the current output cannot be influenced by the current or future control action and hence imposing any constraints on y at the current time is meaningless. Note also that (11) and (12) specify "worst-case" output constraints. In other words, (11) and (12) must be satisfied for any time-varying plant in Ω used as a model for predicting the output.

Remark 1 Constraints on the input are typically *hard* constraints, since they represent limitations on process equipment (such as valve saturation) and as such cannot be relaxed or *softened.* Constraints on the output, on the other hand, are often performance goals; it is usually only required to make $y_{\max}$ and $y_{i,\max}$ as small as possible, subject to the input constraints.

2.3 Linear Matrix Inequalities

We give a brief introduction to Linear Matrix Inequalities and some optimization problems based on LMIs. For more details, we refer the reader to the book [6].

A linear matrix inequality or LMI is a matrix inequality of the form

$$F(x) = F_0 + \sum_{i=1}^{l} x_i F_i > 0,$$

where x_1, x_2, $\ldots$, x_l are the variables, $F_i = F_i^T \in \mathcal{R}^{n\times n}$ are given, and $F(x) > 0$ means that $F(x)$ is positive definite. Multiple LMIs $F_1(x) > 0, \ldots, F_n(x) > 0$ can be expressed as the single LMI

$$\mathrm{diag}(F_1(x), \ldots, F_n(x)) > 0.$$

Therefore we will make no distinction between a set of LMIs and a single LMI, i.e., "the LMI $F_1(x) > 0, \ldots, F_n(x) > 0$" will mean "the LMI $\mathrm{diag}(F_1(x), \ldots, F_n(x)) > 0$".

Convex quadratic inequalities are converted to LMI form using Schur complements: Let $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on x. Then the LMI

$$\begin{array}{c} \begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \\ \text{is equivalent to the matrix inequality} \\ R(x) > 0,\ Q(x) - S(x)R(x)^{-1}S(x)^T > 0 \\ \text{or equivalently,}\quad Q(x) > 0,\ R(x) - S(x)^T Q(x)^{-1} S(x) > 0. \end{array} \tag{13}$$

We often encounter problems in which the variables are matrices, for example, the constraint $P > 0$, where the entries of P are the optimization variables. In such cases we will not write out the LMI explicitly in the form $F(x) > 0$, but instead make clear which matrices are the variables.

The LMI-based problem of central importance to this paper is that of minimizing a linear objective subject to LMI constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) > 0. \end{array} \tag{14}$$

Here, F is a symmetric matrix that depends affinely on the optimization variable x, and c is a real vector of appropriate size. This is a convex nonsmooth optimization problem. For more on this and other LMI-based optimization problems, we refer the reader to [6]. The observation about LMI-based optimization that is most relevant to us is that

LMI problems are tractable.

LMI problems can be solved in polynomial time, which means that they have low computational complexity; from a practical standpoint, there are effective and powerful algorithms for the solution of these problems, that is, algorithms that rapidly compute the global optimum, with non-heuristic stopping criteria. Thus, on exit, the algorithms can prove that the global optimum has been obtained to within some prespecified accuracy [22, 5, 28, 1]. Numerical experience shows that these algorithms solve LMI problems with extreme efficiency.

The most important implication from the foregoing discussion is that LMI-based optimization is well-suited for on-line implementation which is essential for MPC.

3 Model Predictive Control using Linear Matrix Inequalities

In this section, we discuss the problem formulation for robust MPC. In particular, we modify the minimization of the *nominal* objective function, discussed in §2.2, to a minimization of the *worst-case* objective function. Following the motivation in §2.2, we consider the IH-MPC problem. We begin with the robust IH-MPC problem without input and output constraints and reduce it to a linear objective minimization problem. We then incorporate input and output constraints. Finally, we show that the feasible receding horizon state-feedback control law robustly stabilizes the set of uncertain plants Ω.

3.1 Robust Unconstrained IH-MPC

The system is described by (1) with the associated uncertainty set Ω (either (2) or (6)). Analogous to the familiar approach from linear robust control, we replace the minimization, at each sampling time k, of the nominal performance objective (given in (7)), by the minimization of a *robust* performance objective as follows:

$$\begin{aligned} &\min_{u(k+i|k),i=0,1,\ldots,m} \quad \max_{[A(k+i)\ B(k+i)]\in\Omega,\ i\geq 0} J_\infty(k), \\ &\text{where} \quad J_\infty(k) = \sum_{i=0}^{\infty} \left(x(k+i|k)^T Q_1 x(k+i|k) + u(k+i|k)^T R u(k+i|k) \right). \end{aligned} \tag{15}$$

This is a "min-max" problem. The maximization is over the set Ω and corresponds to choosing that time-varying plant $[A(k+i)\ B(k+i)] \in \Omega,\ i \geq 0$ which, if used as a "model" for predictions, would lead to the largest or "worst-case" value of $J_\infty(k)$ among all plants in Ω. This worst-case value is minimized over present and future control moves $u(k+i|k), i = 0, 1, \ldots, m$. This min-max problem, though convex for finite m, is not computationally tractable, and as such has not been addressed in the MPC literature. We address problem (15) by first deriving an upper bound on the robust performance objective. We then minimize this upper bound with a constant state feedback control law $u(k+i|k) = Fx(k+i|k),\ i \geq 0$.

Derivation of the upper bound

Consider a quadratic function $V(x) = x^T P x,\ P > 0$ of the state $x(k|k) = x(k)$ (see (8)) of the system (1) with $V(0) = 0$. At sampling time k, suppose V satisfies the following inequality for all $x(k+i|k),\ u(k+i|k),\ i \geq 0$ satisfying (1), and for any $[A(k+i)\ B(k+i)] \in \Omega,\ i \geq 0$:

$$\begin{aligned} &V(x(k+i+1|k)) - V(x(k+i|k)) \\ &\qquad \leq - \left(x(k+i|k)^T Q_1 x(k+i|k) + u(k+i|k)^T R u(k+i|k) \right). \end{aligned} \tag{16}$$

For the robust performance objective function to be finite, we must have $x(\infty|k) = 0$ and hence, $V(x(\infty|k)) = 0$. Summing (16) from $i = 0$ to $i = \infty$, we get

$$-V(x(k|k)) \leq -J_\infty(k).$$

Thus,

$$\max_{[A(k+i)\ B(k+i)]\in\Omega,\ i\geq 0} J_\infty(k) \leq V(x(k|k)). \tag{17}$$

This gives an upper bound on the robust performance objective. Thus, the goal of our robust MPC algorithm has been redefined to synthesize, at each time step k, a constant state-feedback control law $u(k+i|k) = Fx(k+i|k)$ to minimize this upper bound $V(x(k|k))$. As is standard in MPC, only the first computed input $u(k|k) = Fx(k|k)$ is implemented. At the next sampling time, the state $x(k+1)$ is measured and the optimization is repeated to recompute F. The following theorem gives us conditions for the existence of the appropriate $P > 0$ satisfying (16) and the corresponding state feedback matrix F.

Theorem 1 *Let $x(k) = x(k|k)$ be the state of the uncertain system (1) measured at sampling time k. Assume that there are no constraints on the control input and plant output.*

(A) *Suppose the uncertainty set Ω is defined by a polytope as in (2). Then, the state feedback matrix F in the control law $u(k+i|k) = Fx(k+i|k), i \geq 0$ which minimizes the upper bound $V(x(k|k))$ on the robust performance objective function at sampling time k is given by*

$$F = YQ^{-1}, \tag{18}$$

where $Q > 0$ and Y are obtained from the solution (if it exists) to the following linear objective minimization problem (this problem is of the same form as problem (14)):

$$\min_{\gamma,\, Q,\, Y} \quad \gamma \tag{19}$$

subject to

$$\begin{bmatrix} 1 & x(k|k)^T \\ x(k|k) & Q \end{bmatrix} \geq 0 \tag{20}$$

and

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_1^{\frac{1}{2}} Q & 0 & \gamma I & 0 \\ R^{\frac{1}{2}} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \quad j = 1, 2, \ldots, L. \tag{21}$$

(B) *Suppose the uncertainty set Ω is defined by a structured norm-bounded perturbation Δ as in (6). In this case, F is given by*

$$F = YQ^{-1}, \tag{22}$$

where $Q > 0$ and Y are obtained from the solution (if it exists) to the following linear objective minimization problem with variables γ, Q, Y and Λ:

$$\min_{\gamma,\, Q,\, Y,\, \Lambda} \quad \gamma \tag{23}$$

subject to

$$\begin{bmatrix} 1 & x(k|k)^T \\ x(k|k) & Q \end{bmatrix} \geq 0, \tag{24}$$

and

$$\begin{bmatrix} Q & Y^T R^{\frac{1}{2}} & Q Q_1^{\frac{1}{2}} & Q C_q^T + Y^T D_{qu}^T & Q A^T + Y^T B^T \\ R^{\frac{1}{2}} Y & \gamma I & 0 & 0 & 0 \\ Q_1^{\frac{1}{2}} Q & 0 & \gamma I & 0 & 0 \\ C_q Q + D_{qu} Y & 0 & 0 & \Lambda & 0 \\ AQ + BY & 0 & 0 & 0 & Q - B_p \Lambda B_p^T \end{bmatrix} \geq 0 \tag{25}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_r I_{n_r} \end{bmatrix} > 0. \tag{26}$$

Proof. See Appendix A. □

Remark 2 Part (A) of Theorem 1 can be derived from the results in [3] for quadratic stabilization of uncertain polytopic continuous-time systems and their extension to the discrete-time case [18]. Part (B) can be derived using the same basic techniques in conjunction with the $\mathcal{S}$-procedure (see [30] and the references therein).

Remark 3 Strictly speaking, the variables in the above optimization should be denoted by Q_k, F_k, Y_k etc. to emphasize that they are computed at time k. For notational convenience, we omit the subscript here and in the next section. We will, however, briefly utilize this notation in the robust stability proof (Theorem 3). Closed loop stability of the receding horizon state-feedback control law given in Theorem 1 will be established in §3.2.

Remark 4 For the nominal case, ($L = 1$ or $\Delta(k) \equiv 0, p(k) \equiv 0,\ k \geq 0$), it can be shown that we recover the standard discrete-time Linear Quadratic Regulator (LQR) solution (see Kwakernaak and Sivan (1972) [19] for the standard LQR solution).

Remark 5 The previous remark establishes that for the nominal case, the feedback matrix F computed from Theorem 1 is constant, independent of the state of the system. However, in the presence of uncertainty, even without constraints on the control input or plant output, F can show strong dependence on the state of the system. In such cases, using a receding horizon approach and recomputing F at each sampling time shows significant improvement in performance as opposed to using a static state feedback control law. This, we believe, is one of the key ideas in this paper and is illustrated with the following simple example. Consider the polytopic system (1), Ω being defined by (2) with

$$A_1 = \begin{bmatrix} 0.9347 & 0.5194 \\ 0.3835 & 0.8310 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.0591 & 0.2641 \\ 1.7971 & 0.8717 \end{bmatrix}, \quad B = \begin{bmatrix} -1.4462 \\ -0.7012 \end{bmatrix}.$$

Figure 2 (a) shows the initial state response of a time-varying system in the set Ω, using the receding horizon control law of Theorem 1 ($Q_1 = 1, R = 1$). Also included is the static state-feedback control law from Theorem 1, where the feedback matrix F is not recomputed at each time k. The response with the receding horizon controller is about five times faster. Figure 2 (b) shows the norm of F as a function of time for the two schemes and thus explains the significantly better performance of the receding horizon scheme.

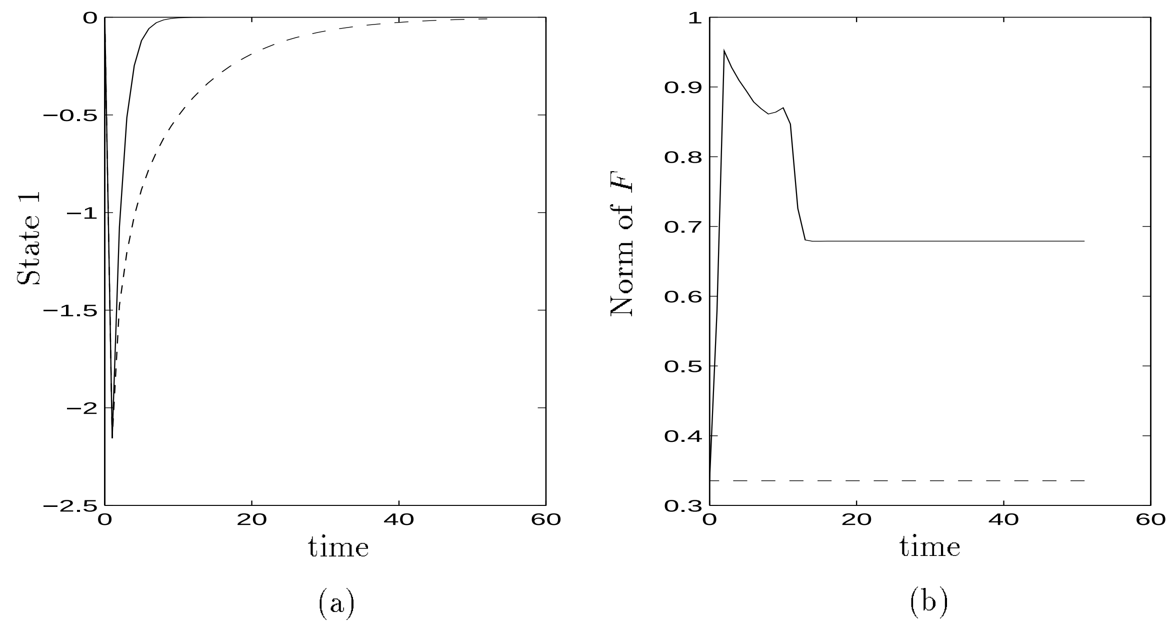

Figure 2: (a) Unconstrained closed-loop responses and (b) norm of the feedback matrix F; solid: using receding horizon state-feedback; dash: using robust static state-feedback

Remark 6 Traditionally, feedback in the form of plant measurement at each sampling time k is interpreted as accounting for model uncertainty and unmeasured disturbances (see §2.2). In our robust MPC setting, this feedback can now be reinterpreted as potentially reducing the conservatism in our worst-case MPC synthesis by recomputing F using new plant measurements.

Remark 7 The speed of the closed-loop response can be influenced by specifying a minimum decay rate on the state x ($\|x(k)\| \leq c\rho^k \|x(0)\|$, $0 < \rho < 1$) as follows:

$$x(k+i+1|k)^T P x(k+i+1|k) \quad \leq \quad \rho^2 x(k+i|k)^T P x(k+i|k), \quad i \geq 0 \tag{27}$$

for any $[A(k+i)\ B(k+i)] \in \Omega,\ i \geq 0$. This implies that

$$\|x(k+i+1|k)\| \leq \left[\frac{\bar{\sigma}(P)}{\underline{\sigma}(P)}\right]^{\frac{1}{2}} \rho \|x(k+i|k)\|, \quad i \geq 0$$

Following the steps in the proof of Theorem 1, it can be shown that requirement (27) reduces to the following LMIs for the two uncertainty descriptions:

Polytopic uncertainty

$$\begin{bmatrix} \rho^2 Q & (A_i Q + B_i Y)^T \\ A_i Q + B_i Y & Q \end{bmatrix} \quad \geq \quad 0, \quad i = 1, \ldots, L \tag{28}$$

Structured uncertainty

$$\begin{bmatrix} \rho^2 Q & (C_q Q + D_{qu} Y)^T & (AQ + BY)^T \\ C_q Q + D_{qu} Y & \Lambda & 0 \\ AQ + BY & 0 & Q - B_p \Lambda B_p^T \end{bmatrix} \geq 0 \tag{29}$$

where $\Lambda > 0$ is of the form (26).

Thus, an additional tuning parameter $\rho \in (0, 1)$ is introduced in the MPC algorithm to influence the speed of the closed-loop response. Note that with $\rho = 1$, the above two LMIs are trivially satisfied if (21) and (25) are satisfied.

3.2 Robust Constrained IH-MPC

In the previous section, we formulated the robust MPC problem without input and output constraints, and derived an upper bound on the robust performance objective. In this section, we show how input and output constraints can be incorporated as LMI constraints in the robust MPC problem. As a first step, we need to establish the following lemma which will also be required to prove robust stability.

Lemma 1 (Invariant ellipsoid) *Consider the system (1) with the associated uncertainty set Ω.*

(A) *Let Ω be a polytope described by (2). At sampling time k, suppose there exist $Q > 0$, γ and $Y = FQ$ such that (21) holds. Also suppose that $u(k+i|k) = Fx(k+i|k)$, $i \geq 0$.*

Then if

$$x(k|k)^T Q^{-1} x(k|k) \leq 1 \ (\textit{or equivalently, } x(k|k)^T P x(k|k) \leq \gamma \textit{ with } P = \gamma Q^{-1}),$$

then

$$\max_{[A(k+j)\ \ B(k+j)] \in \Omega,\ j \geq 0} x(k+i|k)^T Q^{-1} x(k+i|k) < 1, \quad i \geq 1, \tag{30}$$

or equivalently,

$$\max_{[A(k+j)\ \ B(k+j)] \in \Omega,\ j \geq 0} x(k+i|k)^T P x(k+i|k) < \gamma, \ i \geq 1. \tag{31}$$

Thus, $\mathcal{E} = \left\{ z | z^T Q^{-1} z \leq 1 \right\} = \left\{ z | z^T P z \leq \gamma \right\}$ is an invariant ellipsoid for the predicted states of the uncertain system.

(B) *Let Ω be described by (6) in terms of a structured Δ block as in (4). At sampling time k, suppose there exist $Q > 0$, γ, $Y = FQ$ and $\Lambda > 0$ such that (25) and (26) hold. If $u(k+i|k) = Fx(k+i|k)$, $i \geq 0$, then the result in* **(A)** *holds as well for this case.*

Remark 8 The maximization in (30) and (31) is over the set Ω of time-varying models that can be used for prediction of the future states of the system. This maximization leads to the "worst-case" value of $x(k+i|k)^T Q^{-1} x(k+i|k)$ (equivalently, $x(k+i|k)^T P x(k+i|k)$) at every instant of time $k+i$, $i \geq 1$.

Proof. See Appendix B. □

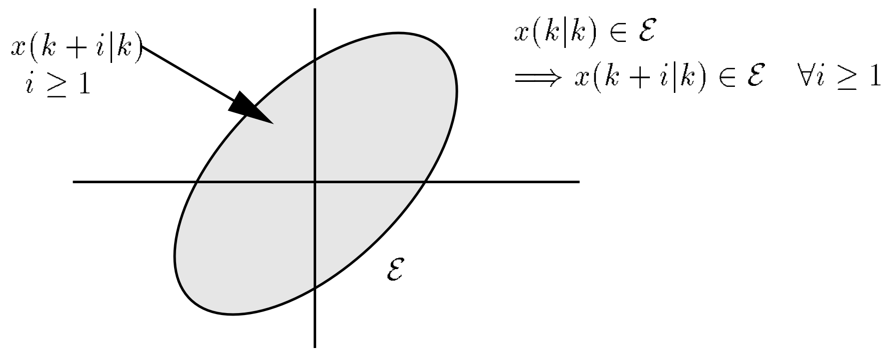

Figure 3: Graphical representation of the state-invariant ellipsoid $\mathcal{E}$ in 2−dimensions

3.2.1 Input Constraints

Physical limitations inherent in process equipment invariably impose *hard* constraints on the manipulated variable $u(k)$. In this section, we show how limits on the control signal can be incorporated into our robust MPC algorithm as *sufficient* LMI constraints. The basic idea of the discussion that follows can be found in Boyd et al. (1994) [6] in the context of continuous time systems. We present it here to clarify its application in our (discrete-time) robust MPC setting and also for completeness of exposition. We will assume for the rest of this section that the postulates of Lemma 1 are satisfied so that $\mathcal{E}$ is an invariant ellipsoid for the predicted states of the uncertain system.

At sampling time k, consider the Euclidean norm constraint (9):

$$\|u(k+i|k)\|_2 \leq u_{\max}, \quad i \geq 0.$$

The constraint is imposed on the present and the entire horizon of future manipulated variables, although only the first control move $u(k|k) = u(k)$ is implemented. Following [6], we have

$$\begin{aligned} \max_{i\geq 0} \|u(k+i|k)\|_2^2 &= \max_{i\geq 0} \|YQ^{-1}x(k+i|k)\|_2^2 \\ &\leq \max_{z\in\mathcal{E}} \|YQ^{-1}z\|_2^2 \\ &= \lambda_{\max}(Q^{-\frac{1}{2}}Y^TYQ^{-\frac{1}{2}}). \end{aligned}$$

Using (13), we see that $\|u(k+i|k)\|_2^2 \leq u_{\max}^2$, $i \geq 0$ if

$$\begin{bmatrix} u_{\max}^2 I & Y \\ Y^T & Q \end{bmatrix} \geq 0. \tag{32}$$

This is an LMI in Y and Q. Similarly, let us consider peak bounds on each component of $u(k+i|k)$ at sampling time k (10):

$$|u_j(k+i|k)| \leq u_{j,\max}, \quad i \geq 0, \; j = 1, 2, \ldots, n_u.$$

Now,

$$\begin{aligned}
\max_{i\geq 0} |u_j(k+i|k)|^2 &= \max_{i\geq 0} |\left(YQ^{-1}x(k+i|k)\right)_j|^2 \\
&\leq \max_{z\in\mathcal{E}} |\left(YQ^{-1}z\right)_j|^2 \\
&\leq \|\left(YQ^{-\frac{1}{2}}\right)_j\|_2^2 \quad \text{(using the Cauchy Schwarz inequality)} \\
&= \left(YQ^{-1}Y^T\right)_{jj}.
\end{aligned}$$

Thus, the existence of a symmetric matrix X such that

$$\begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \geq 0, \quad \text{with } X_{jj} \leq u_{j,\max}^2, \quad j = 1, 2, \ldots, n_u, \tag{33}$$

guarantees that $|u_j(k+i|k)| \leq u_{j,\max}, \quad i \geq 0, \quad j = 1, 2, \ldots, n_u$. These are LMIs in X, Y and Q. Note that (33) is a slight generalization of the result derived in [6].

Remark 9 Inequalities (32) and (33) represent sufficient LMI constraints which guarantee that the specified constraints on the manipulated variables are satisfied. In practice, these constraints have been found to be not too conservative, at least in the nominal case.

3.2.2 Output Constraints

Performance specifications impose constraints on the process output $y(k)$. As in §3.2.1, we derive sufficient LMI constraints for both the uncertainty descriptions (see (2) and (3),(4)) which guarantee that the output constraints are satisfied.

At sampling time k, consider the Euclidean norm constraint ((11)):

$$\max_{[A(k+j)\ B(k+j)]\in\Omega,\, j\geq 0} \|y(k+i|k)\|_2 \leq y_{\max}, \quad i \geq 1.$$

As discussed in §2.2, this is a worst-case constraint over the set Ω and is imposed strictly over the future prediction horizon ($i \geq 1$).

Polytopic uncertainty

In this case, Ω is given by (2). As shown in Appendix C, if

$$\begin{bmatrix} Q & (A_jQ + B_jY)^T C^T \\ C(A_jQ + B_jY) & y_{\max}^2 I \end{bmatrix} \geq 0, \quad j = 1, 2, \ldots, L, \tag{34}$$

then

$$\max_{[A(k+j)\ B(k+j)]\in\Omega,\, j\geq 0} \|y(k+i|k)\|_2 \leq y_{\max}, \quad i \geq 1.$$

Condition (34) represents a set of LMIs in Y and $Q > 0$.

Structured uncertainty

In this case, Ω is described by (3), (4) in terms of a structured Δ block. As shown in Appendix C, if

$$\begin{bmatrix} y_{\max}^2 Q & (C_q Q + D_{qu} Y)^T & (AQ + BY)^T C^T \\ C_q Q + D_{qu} Y & T^{-1} & 0 \\ C(AQ + BY) & 0 & I - C B_p T^{-1} B_p^T C^T \end{bmatrix} \geq 0 \tag{35}$$

with

$$T = \begin{bmatrix} t_1 I_{n_1} & & & \\ & t_2 I_{n_2} & & \\ & & \ddots & \\ & & & t_r I_{n_r} \end{bmatrix} > 0,$$

then

$$\max_{[A(k+j)\ B(k+j)] \in \Omega,\, j \geq 0} \|y(k+i|k)\|_2 \ \leq\ y_{\max}, \quad i \geq 1.$$

Condition (35) is an LMI in $Y, Q > 0$ and $T^{-1} > 0$.

In a similar manner, component-wise peak bounds on the output (see (12)) can be translated to sufficient LMI constraints. The development is identical to the preceding development for the Euclidean norm constraint if we replace C by C_l and T by T_l, $l = 1, 2, \ldots, n_y$ in (34), (35), where

$$y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_{n_y}(k) \end{bmatrix} = Cx(k) = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{n_y} \end{bmatrix} x(k).$$

T_l is in general different for each $l = 1, 2, \ldots, n_y$.

Note that for the case with mixed Δ blocks, we can satisfy the output constraint over the current and future horizon $\max_{i \geq 0} \|y(k+i|k)\|_2 \leq y_{\max}$ and not over the (strict) future horizon ($i \geq 1$) as in (11). The corresponding LMI is derived as follows:

$$\begin{aligned} \max_{i \geq 0} \|Cx(k+i|k)\|_2^2 &\leq \max_{z \in \mathcal{E}} \|Cz\|_2^2 \\ &= \lambda_{\max}(Q^{\frac{1}{2}} C^T C Q^{\frac{1}{2}}) \end{aligned}$$

Thus, $CQC^T \leq y_{\max}^2 I \Longrightarrow \|y(k+i|k)\|_2 \leq y_{\max},\ i \geq 0$. For component-wise peak bounds on the output, we replace C by C_l, $l = 1, \ldots, n_y$.

3.2.3 Robust Stability

We are now ready to state the main theorem for robust MPC synthesis with input and output constraints and establish robust stability of the closed-loop.

Theorem 2 *Let $x(k) = x(k|k)$ be the state of the uncertain system (1) measured at sampling time k.*

(A) *Suppose the uncertainty set Ω is defined by a polytope as in (2). Then, the state feedback matrix F in the control law $u(k+i|k) = Fx(k+i|k), i \geq 0$, which minimizes the upper bound*

$V(x(k|k))$ on the robust performance objective function at sampling time k and satisfies a set of specified input and output constraints is given by

$$F = YQ^{-1},$$

where, $Q > 0$ and Y are obtained from the solution (if it exists) to the following linear objective minimization problem:

$$\min \{\gamma \mid \gamma, Q, Y \text{ and variables in the LMIs for input and output constraints}\},$$

subject to (20), (21), either (32) or (33), depending on the input constraint to be imposed, and (34) with either C and T, or C_l and T_l, $l = 1, 2, \ldots, n_y$, depending on the output constraint to be imposed.

(B) *Suppose the uncertainty set Ω is defined by (6) in terms a structured perturbation Δ as in (4). In this case, F is given by*

$$F = YQ^{-1},$$

where, $Q > 0$ and Y are obtained from the solution (if it exists) to the following linear objective minimization problem:

$$\min \{\gamma \mid \gamma, Q, Y, \Lambda \text{ and variables in the LMIs for input and output constraints}\}$$

subject to (24), (25), (26), either (32) or (33) depending on the input constraint to be imposed, and (35) with either C and T, or C_l and T_l, $l = 1, 2, \ldots, n_y$, depending on the output constraint to be imposed.

Proof. From Lemma 1, we know that (21) and (24,25) imply respectively for the polytopic and structured uncertainties, that $\mathcal{E}$ is an invariant ellipsoid for the predicted states of the uncertain system (1). Hence, the arguments in §3.2.1 and §3.2.2 used to translate the input and output constraints to sufficient LMI constraints hold true. The rest of the proof is similar to the proof of Theorem 1. □

In order to prove robust stability of the closed loop, we need to establish the following lemma.

Lemma 2 (Feasibility) *Any feasible solution of the optimization in Theorem 2 at time k is also feasible for all times $t > k$. Thus, if the optimization problem in Theorem 2 is feasible at time k, then it is feasible for all times $t > k$.*

Proof. Let us assume that the optimization problem in Theorem 2 is feasible at sampling time k. The only LMI in the problem which depends explicitly on the measured state $x(k|k) = x(k)$ of the system is the following

$$\begin{bmatrix} 1 & x(k|k)^T \\ x(k|k) & Q \end{bmatrix} \geq 0.$$

Thus, to prove the lemma, we need only prove that this LMI is feasible for all future measured states $x(k+i|k+i) = x(k+i), i \geq 1$.

Now, feasibility of the problem at time k implies satisfaction of (21) and (24,25), which, using Lemma 1, in turn imply respectively for the two uncertainty descriptions that (30) is satisfied.

Thus, for any $[A(k+i) \ \ B(k+i)] \in \Omega, i \geq 0$ (where Ω is the corresponding uncertainty set), we must have

$$x(k+i|k)^T Q^{-1} x(k+i|k) < 1, \quad i \geq 1.$$

Since the state measured at $k+1$, that is, $x(k+1|k+1) = x(k+1)$, equals $(A(k)+B(k)F)\,x(k|k)$ for some $[A(k) \ \ B(k)] \in \Omega$, it must also satisfy this inequality, i.e.,

$$x(k+1|k+1)^T Q^{-1} x(k+1|k+1) < 1,$$

or

$$\begin{bmatrix} 1 & x(k+1|k+1)^T \\ x(k+1|k+1) & Q \end{bmatrix} > 0 \text{ (using 13).}$$

Thus, the feasible solution of the optimization problem at time k is also feasible at time $k+1$. Hence, the optimization is feasible at time $k+1$. This argument can be continued for time $k+2, k+3, \ldots$ to complete the proof. □

Theorem 3 (Robust stability) *The feasible receding horizon state feedback control law obtained from Theorem 2 robustly asymptotically stabilizes the closed loop system.*

Proof. In what follows, we will refer to the uncertainty set as Ω since the proof is identical for the two uncertainty descriptions.

To prove asymptotic stability, we will establish that $V(x(k|k)) = x(k|k)^T P_k x(k|k)$, where $P_k > 0$ is obtained from the *optimal* solution at time k, is a strictly decreasing Lyapunov function for the closed-loop.

First, let us assume that the optimization in Theorem 2 is feasible at time $k=0$. Lemma 2 then ensures feasibility of the problem at all times $k > 0$. The optimization being convex, therefore, has a unique minimum and a corresponding optimal solution (γ, Q, Y) at each time $k \geq 0$.

Next, we note from Lemma 2 that γ, $Q > 0$, Y (or equivalently, γ, $F = YQ^{-1}$, $P = \gamma Q^{-1} > 0$) obtained from the optimal solution at time k are feasible (of course, not necessarily optimal) at time $k+1$. Denoting the values of P obtained from the optimal solutions at time k and $k+1$ respectively by P_k and P_{k+1} (see Remark 3), we must have

$$x(k+1|k+1)^T P_{k+1} x(k+1|k+1) \leq x(k+1|k+1)^T P_k x(k+1|k+1). \tag{36}$$

This is because P_{k+1} is optimal whereas P_k is only feasible at time $k+1$.

And lastly, we know from Lemma 1 that if $u(k+i|k) = F_k x(k+i|k)$, $i \geq 0$ (F_k is obtained from the optimal solution at time k), then for any $[A(k) \ \ B(k)] \in \Omega$, we must have

$$x(k+1|k)^T P_k x(k+1|k) < x(k|k)^T P_k x(k|k), \quad (x(k|k) \neq 0) \tag{37}$$

(see (49) with $i = 0$).

Since the measured state $x(k+1|k+1) = x(k+1)$ equals $(A(k)+B(k)F_k)\,x(k|k)$ for some $[A(k) \ \ B(k)] \in \Omega$, it must also satisfy inequality (37). Combining this with inequality (36) we conclude that

$$x(k+1|k+1)^T P_{k+1} x(k+1|k+1) < x(k|k)^T P_k x(k|k), \quad (x(k|k) \neq 0).$$

Thus, $x(k|k)^T P_k x(k|k)$ is a strictly decreasing Lyapunov function for the closed-loop, which is bounded below by a positive definite function of $x(k|k)$ (see (17)). We therefore conclude that $x(k) \to 0$ as $k \to \infty$. □

Remark 10 The proof of Theorem 1 (the unconstrained case) is identical to the preceding proof if we recognize that Theorem 1 is only a special case of Theorem 2 without the LMIs corresponding to input and output constraints.

4 Extensions

The presentation up to this point was restricted to the infinite horizon regulator with a zero target. In this section, we extend the preceding development to several standard problems encountered in practice.

4.1 Reference Trajectory Tracking

In optimal tracking problems, the system output is required to track a reference trajectory $y_r(k) = C_r x_r(k)$ where the reference states x_r are computed from the following equation

$$x_r(k+1) = A_r x_r(k), \quad x_r(0) = x_{r0}.$$

The choice of $J_\infty(k)$ for the robust trajectory tracking objective in the optimization (15) is the following

$$\begin{aligned} J_\infty(k) = \sum_{i=0}^{\infty} \Big(& (Cx(k+i|k) - C_r x_r(k+i))^T Q_1 (Cx(k+i|k) - C_r x_r(k+i)) \\ & + u(k+i|k)^T R u(k+i|k) \Big), \quad Q_1 > 0,\ R > 0. \end{aligned}$$

As discussed in [19], the plant dynamics can be augmented by the reference trajectory dynamics to reduce the robust reference trajectory tracking problem (with input and output constraints) to the standard form as in §3. Due to space limitations, we will omit these details.

4.2 Constant Set-point Tracking

For uncertain linear *time-invariant* systems, the desired equilibrium state may be a constant point $x_s,\ u_s$ (called the *set-point*) in state-space, different from the origin. Consider (1) which we will now assume to represent an uncertain LTI system, i.e., $[A \quad B] \in \Omega$ are *constant* unknown matrices. Suppose that the system output y is required to track the target vector y_t by moving the system to the set-point x_s, u_s where

$$x_s = Ax_s + Bu_s, \quad y_t = Cx_s.$$

We assume that $x_s,\ u_s,\ y_t$ are feasible, i.e., they satisfy the imposed constraints. The choice of $J_\infty(k)$ for the robust set-point tracking objective in the optimization (15) is the following:

$$\begin{aligned} J_\infty(k) = \sum_{i=0}^{\infty} \Big(& (Cx(k+i|k) - Cx_s)^T Q_1 (Cx(k+i|k) - Cx_s) \\ & + (u(k+i|k) - u_s)^T R (u(k+i|k) - u_s) \Big), \quad Q_1 > 0,\ R > 0. \end{aligned} \tag{38}$$

As discussed in [19], we can define a shifted state $\tilde{x}(k) = x(k) - x_s$, a shifted input $\tilde{u}(k) = u(k) - u_s$ and a shifted output $\tilde{y}(k) = y(k) - y_t$ to reduce the problem to the standard form as in §3.

Component-wise peak bounds on the control signal u can be translated to constraints on $\tilde{u}$ as follows:

$$|u_j| \le |u_{j,\max}| \quad \Longleftrightarrow \quad |\tilde{u}_j + u_{s,j}| \le u_{j,\max} \quad \Longleftrightarrow \quad -u_{j,\max} - u_{s,j} \le \tilde{u}_j \le u_{j,\max} - u_{s,j}$$

Constraints on the transient deviation of $y(k)$ from the steady state value y_t, i.e., $\tilde{y}(k)$ can be incorporated in a similar manner.

4.3 Disturbance Rejection

In all practical applications, some disturbance invariably enters the system and hence it is meaningful to study its effect on the closed-loop response. Let an unknown disturbance $e(k)$, having the property $\lim_{k\to\infty} e(k) = 0$ enter the system (1) as follows

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + e(k) \\ y(k) &= Cx(k) \\ [A(k) \;\; B(k)] &\in \Omega. \end{aligned} \tag{39}$$

A simple example of such a disturbance is any energy bounded signal ($\sum_{i=0}^{\infty} e(i)^T e(i) < \infty$). Assuming that the state of the system $x(k)$ is measurable, we would like to solve the optimization problem (15). We will assume that the predicted states of the system satisfy the following equation

$$\begin{aligned} x(k+i+1|k) &= A(k+i)x(k+i|k) + B(k+i)u(k+i|k) \\ [A(k+i) \;\; B(k+i)] &\in \Omega. \end{aligned} \tag{40}$$

As in §3, we can derive an upper bound on the robust performance objective (15). The problem of minimizing this upper bound with a state-feedback control law $u(k+i|k) = Fx(k+i|k)$, $i > 0$, at the same time satisfying constraints on the control input and plant output, can then be reduced to a linear objective minimization as in Theorem 2. The following theorem establishes stability of the closed-loop for the system (39) with this receding horizon control law, in the presence of the disturbance $e(k)$.

Theorem 4 *Let $x(k) = x(k|k)$ be the state of the system (39) measured at sampling time k and let the predicted states of the system satisfy (40). Then, assuming feasibility at each sampling time $k \ge 0$, the receding horizon state feedback control law obtained from Theorem 2 robustly asymptotically stabilizes the system (39) in the presence of any asymptotically vanishing disturbance $e(k)$.*

Proof. It is easy to show that for sufficiently large time $k > 0$, $V(x(k|k)) = x(k|k)^T P x(k|k)$, where $P > 0$ is obtained from the optimal solution at time k, is a strictly decreasing Lyapunov function for the closed-loop. Due to lack of space, we will skip these details. □

4.4 Systems with delays

Consider the following uncertain discrete-time linear time-varying system with delay elements, described by the following equations:

$$\begin{aligned} x(k+1) &= A_0(k)x(k) + \sum_{i=1}^{m} A_i(k)x(k-\tau_i) + B(k)u(k-\tau), \\ y(k) &= Cx(k) \\ \text{with} \quad & [A_0(k) \;\; A_1(k)\ldots A_m(k) \;\; B(k)] \in \Omega. \end{aligned} \tag{41}$$

We will assume, without loss of generality, that the delays in the system satisfy $0 < \tau < \tau_1 < \ldots < \tau_m$. At sampling time $k \geq \tau$, we would like to design a state-feedback control law $u(k+i-\tau|k) = Fx(k+i-\tau|k), i \geq 0$, to minimize the following modified infinite horizon robust performance objective

$$\max_{[A(k+i) \;\; B(k+i)]\in\Omega, \; i\geq 0} J_\infty(k), \tag{42}$$

where

$$J_\infty(k) \;=\; \sum_{i=0}^{\infty} \left(x(k+i|k)^T Q_1 x(k+i|k) + u(k+i-\tau|k)^T R u(k+i-\tau|k) \right),$$

subject to input and output constraints. Defining an augmented state

$$w(k) = \left[x(k)^T \; x(k-1)^T \ldots x(k-\tau)^T \ldots x(k-\tau_1)^T \ldots x(k-\tau_m)^T \right]^T$$

which is assumed to be measurable at each time $k \geq \tau$, we can derive an upper bound on the robust performance objective (42) as in §3. The problem of minimizing this upper bound with the state-feedback control law $u(k+i-\tau|k) = Fx(k+i-\tau|k), k \geq \tau, i \geq 0$, subject to constraints on the control input and plant output, can then be reduced to a linear objective minimization as in Theorem 2. These details can be worked out in a straightforward manner and will be omitted here. Note, however, that the appropriate choice of the function $V(w(k))$, satisfying an inequality of the form (16) is the following

$$\begin{aligned} V(w(k)) \;&=\; x(k)^T P_0 x(k) + \sum_{i=1}^{\tau} x(k-i)^T P_\tau x(k-i) + \sum_{i=\tau+1}^{\tau_1} x(k-i)^T P_{\tau_1} x(k-i) \\ & \quad + \ldots + \sum_{i=\tau_{m-1}+1}^{\tau_m} x(k)^T P_{\tau_m} x(k) \;\;=\;\; w(k)^T P w(k) \end{aligned}$$

where P is appropriately defined in terms of $P_0, P_\tau, P_{\tau_1} \ldots, P_{\tau_m}$. The motivation for this modified choice of V comes from [11] where such a V is defined for continuous time systems with delays, and is referred to as a Modified Lyapunov-Krasovskii (MLK) functional.

5 Numerical Examples

In this section, we present two examples which illustrate the implementation of the proposed robust MPC algorithm. The examples also serve to highlight some of the theoretical results in the paper. For both these examples, the software LMI Control Toolbox [12] in the MATLAB environment was used to compute the solution of the linear objective minimization problem. No attempt was made to optimize the computation time. Also, it should be noted that the times required for computation of the closed-loop responses, as indicated at the end of each example, only reflect the state-of-the-art of LMI solvers. While these solvers are significantly faster than classical convex optimization algorithms, research in LMI optimization is still very active and substantial speed-ups can be expected in the future.

5.1 Example 1

The first example is a classical angular positioning system adapted from [19]. The system (see Figure 4) consists of a rotating antenna at the origin of the plane, driven by an electric motor. The control problem is to use the input voltage to the motor (u volts) to rotate the antenna so that it always points in the direction of a moving object in the plane. We assume that the angular positions of the antenna and the moving object (θ and θ_r radians respectively) and the angular velocity of the antenna ($\dot{\theta}$ rad/sec) are measurable. The motion of the antenna can be described by the following

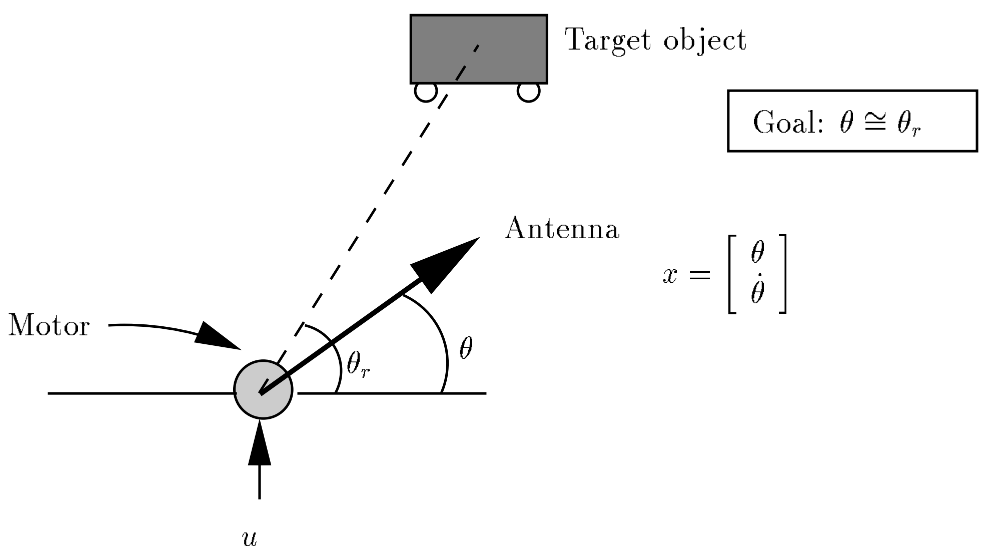

Figure 4: Angular positioning system

discrete-time equations obtained from their continuous-time counterparts by discretization, using a sampling time of 0.1 sec and Euler's first-order approximation for the derivative

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} \theta(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1-0.1\alpha(k) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.1\kappa \end{bmatrix} u(k) \\
&\triangleq A(k)x(k) + Bu(k) \\
y(k) &= [1 \;\; 0]\, x(k) \triangleq Cx(k)
\end{aligned}
$$

$\kappa = 0.787$ rad/(volts sec^2), 0.1 sec$^{-1} \le \alpha(k) \le 10$ sec^{-1}.
The parameter $\alpha(k)$ is proportional to the coefficient of viscous friction in the rotating parts of the

antenna and is assumed to be arbitrarily time-varying in the indicated range of variation. Since $0.1 \leq \alpha(k) \leq 10$, we conclude that $A(k) \in \Omega = \mathrm{Co}\{A_1,\ A_2\}$, where

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix}, \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix}.$$

Thus, the uncertainty set Ω is a polytope, as in (2). Alternatively, if we define

$$\delta(k) = \frac{\alpha(k) - 5.05}{4.95},\ A = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.495 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix},\ C_q = [0\ \ 4.95],\ D_{qu} = 0$$

then $\delta(k)$ is time-varying and norm-bounded with $|\delta(k)| \leq 1,\ k \geq 0$. The uncertainty can then be described as in (3) with

$$\Omega = \{[A + B_p \delta C_q] : |\delta| \leq 1\}.$$

Given an initially disturbed state $x(k)$, the robust IH-MPC optimization to be solved at each time k is the following

$$\min_{u(k+i|k)=Fx(k+i|k),\ i\geq 0}\ \max_{A(k+i)\in\Omega,\ i\geq 0} \left(J_\infty(k) = \sum_{i=0}^{\infty} \left(y(k+i|k)^2 + Ru(k+i|k)^2 \right) \right), \quad R = 0.00002$$

subject to $|u(k+i|k)| \leq 2$ volts, $\quad i \geq 0$.

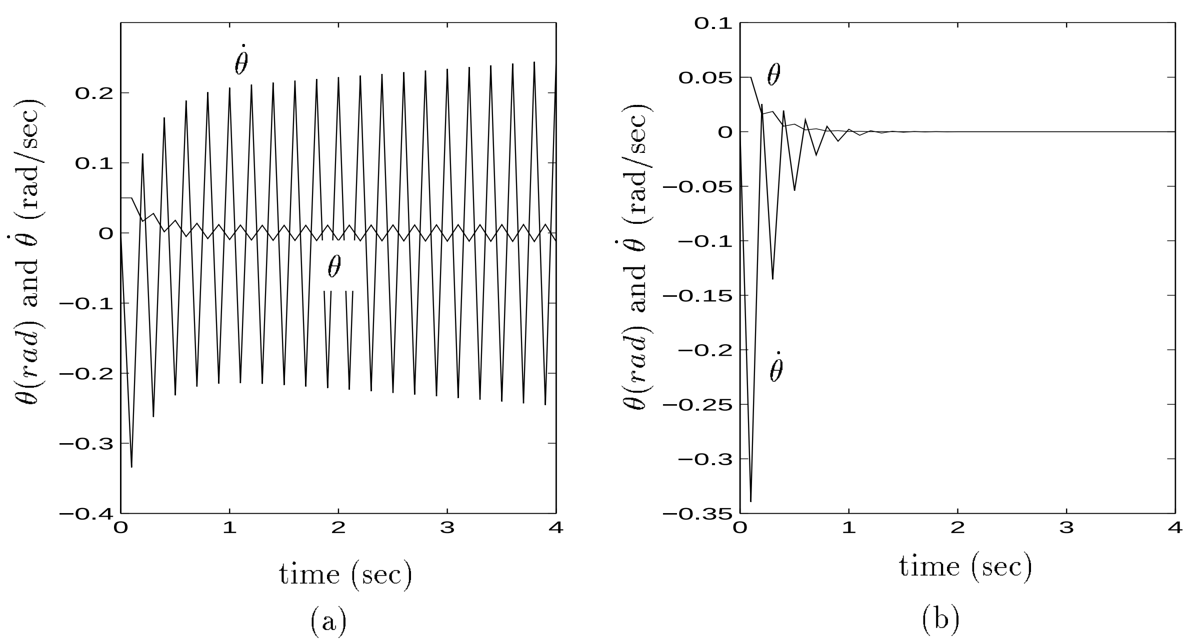

Figure 5: Unconstrained closed-loop responses for nominal plant ($\alpha(k) \equiv 9\ \mathrm{sec}^{-1}$); (a) using nominal MPC with $\alpha(k) \equiv 1\ \mathrm{sec}^{-1}$; (b) using robust LMI-based MPC.

No existing MPC synthesis technique can address this robust synthesis problem. If the problem is formulated without *explicitly* taking into account plant uncertainty, the output response could be unstable. Figure 5(a) shows the closed-loop response of the system corresponding to $\alpha(k) \equiv 9 \text{ sec}^{-1}$, given an initial state of $x(0) = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}$. The control law is generated by minimizing a *nominal* unconstrained infinite horizon objective function using a *nominal* model corresponding to $\alpha(k) \equiv \alpha_{\text{nom}} \equiv 1 \text{ sec}^{-1}$. The response is unstable. Note that the optimization is feasible at each time $k \geq 0$ and hence the controller cannot diagnose the unstable response via infeasibility, even though the horizon is infinite (see [26]). This is not surprising and shows that the prevalent notion that "feedback in the form of plant measurements at each time step k is expected to compensate for unmeasured disturbances and model uncertainty" is only an ad-hoc fix in MPC for model uncertainty without any guarantees of robust stability. Figure 5(b) shows the response using the control law derived from Theorem 1. Notice that the response is stable and the performance is very good. Figure 6(a) shows the closed-loop response of the system when $\alpha(k)$ is randomly time-

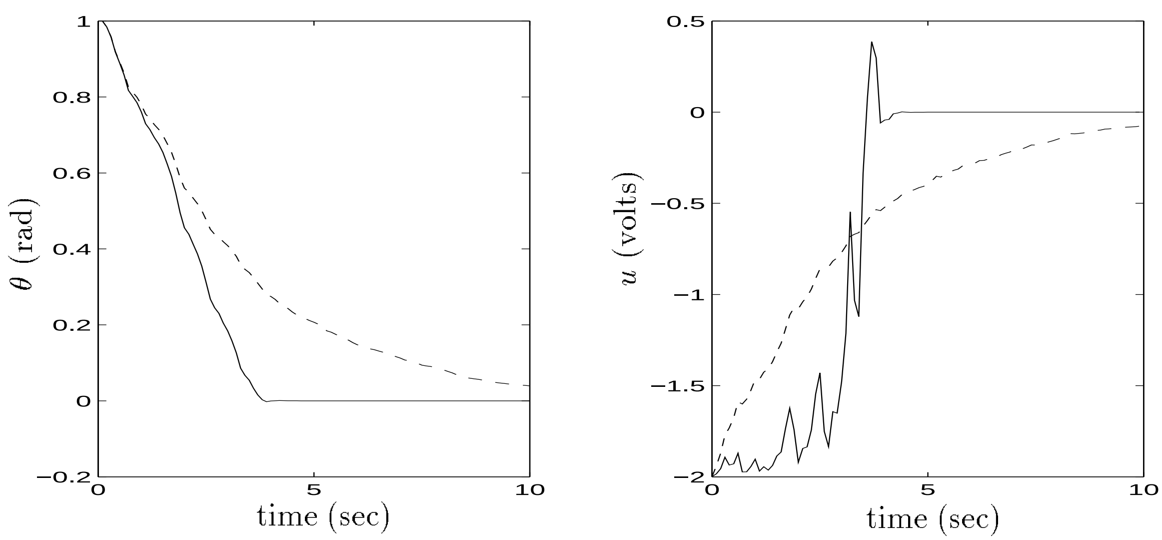

Figure 6: Closed-loop responses for the time-varying system with input constraint; solid: using robust receding horizon state-feedback; dash: using robust static state-feedback

varying between 0.1 and 10 sec^{-1}. The corresponding control signal is given in Figure 6(b). A control constraint of $|u(k)| \leq 2$ volts is imposed. The control law is synthesized according to Theorem 2. We see that the control signal stays close to the constraint boundary up to time $k \approx 3$ sec, thus shedding light on Remark 9. Also included in Figure 6 are the response and control signal using a *static* state-feedback control law, where the feedback matrix F computed from Theorem 2 at time $k = 0$ is kept constant for all times $k > 0$, i.e., it is not recomputed at each time k. The response is about four times slower than the response with the receding horizon state-feedback control law. This sluggishness can be understood if we consider Figure 7 which shows the norm of F as a function of time for the receding horizon controller and for the static state-feedback controller. To meet the constraint $|u(k)| = |Fx(k)| \leq 2$ volts for small k, F must be "small" since $x(k)$ is large for small k. But as $x(k)$ approaches 0, F can be made larger while still meeting the input constraint. This "optimal" use of the control constraint is possible only if F is recomputed at each time k, as in the receding horizon controller. The static state-feedback controller does not recompute F at each time $k \geq 0$ and hence shows a sluggish (though stable) response.

For the computations, the solution at time k was used as an initial guess for solving the optimization at time $k+1$. The total times required to compute the closed-loop responses in Figure 5 (b) (40 samples) and Figure 6 (100 samples) were about 27 and 77 seconds respectively (or equivalently, 0.68 and 0.77 seconds per sample), on a SUN SPARCstation 20, using MATLAB code. The actual CPU times were about 18 and 52 seconds (i.e., 0.45 and 0.52 seconds per sample) respectively. In both cases, nearly 95% of the time was required to solve the LMI optimization at each sampling time.

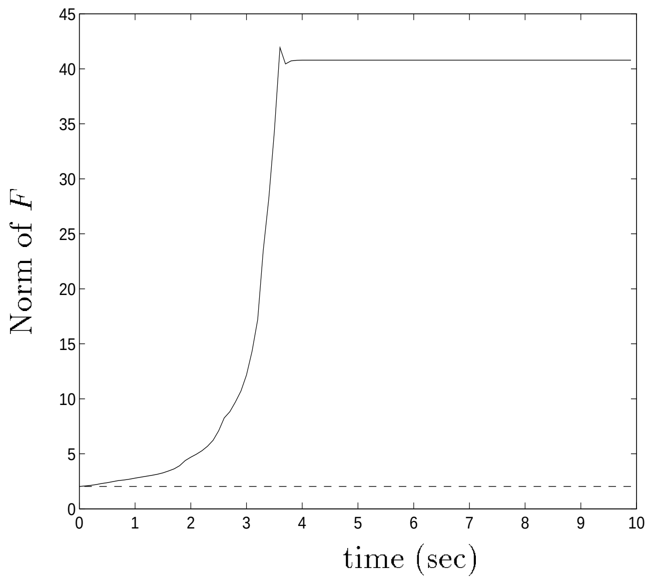

Figure 7: Norm of the feedback matrix F as a function of time; solid: using robust receding horizon state-feedback; dash: using robust static state-feedback

5.2 Example 2

The second example is adapted from Problem 4 of the benchmark problems described in [29]. The system consists of a two-mass-spring system shown in Figure 8. Using Euler's first-order approximation for the derivative and a sampling time of 0.1 sec, the following discrete-time state-

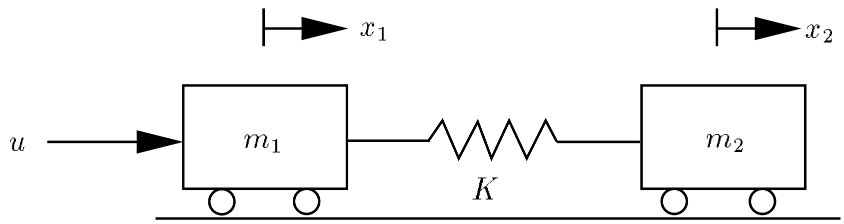

Figure 8: Coupled spring-mass system

space equations are obtained by discretizing the continuous-time equations of the system (see [29])

$$\begin{aligned}
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -\frac{0.1K}{m_1} & \frac{0.1K}{m_1} & 1 & 0 \\ \frac{0.1K}{m_2} & -\frac{0.1K}{m_2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{0.1}{m_1} \\ 0 \end{bmatrix} u(k) \\
y(k) &= x_2(k).
\end{aligned}$$

Here, x_1 and x_2 are the positions of body 1 and 2, and x_3 and x_4 are their velocities respectively. m_1 and m_2 are the masses of the two bodies and K is the spring constant. For the nominal system, $m_1 = m_2 = K = 1$ with appropriate units. The control force u acts on m_1. The performance specifications are defined in Problem 4 of [29] as follows:
Design a feedback/feedforward controller for a unit-step output command tracking problem for the output y with the following properties:

1. A control input constraint of $|u| \leq 1$ must be satisfied.

2. Settling time and overshoot are to be minimized.

3. Performance and stability robustness with respect to m_1, m_2, K are to be maximized.

We will assume for this problem that exact measurement of the state of the system, that is, $[x_1\ x_2\ x_3\ x_4]^T$ is available. We will also assume that the masses m_1 and m_2 are constant equal to 1, and that K is an uncertain constant in the range $K_{\min} \leq K \leq K_{\max}$. The uncertainty in K is modeled as in (3) by defining

$$\begin{aligned}
&\delta = \frac{K - K_{\text{nom}}}{K_{\text{dev}}},\ A = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1K_{\text{nom}} & 0.1K_{\text{nom}} & 1 & 0 \\ 0.1K_{\text{nom}} & -0.1K_{\text{nom}} & 0 & 1 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ 0.1 \end{bmatrix}, \\
&C_q = [K_{\text{dev}}\ \ -K_{\text{dev}}\ 0\ 0],\ D_{qu} = 0
\end{aligned}$$

where $K_{\text{nom}} = \frac{K_{\max} + K_{\min}}{2}$, $K_{\text{dev}} = \frac{K_{\max} - K_{\min}}{2}$.

For unit-step output tracking of y, we must have at steady state $x_{1s} = x_{2s} = 1$, $x_{3s} = x_{4s} = 0$, $u_s = 0$. As in §4.2, we can shift the origin to the steady state. The problem we would like to solve at each sampling time k is the following:

$$\min_{u(k+i|k)=Fx(k+i|k),\ i\geq 0} \ \max_{A(k+i)\in\Omega,\ i\geq 0} \ J_\infty(k)$$

subject to $|u(k+i|k)| \leq 1, \quad i \geq 0$. Here, $J_\infty(k)$ is given by (38) with $Q_1 = I,\ R = 1$. Figure 9

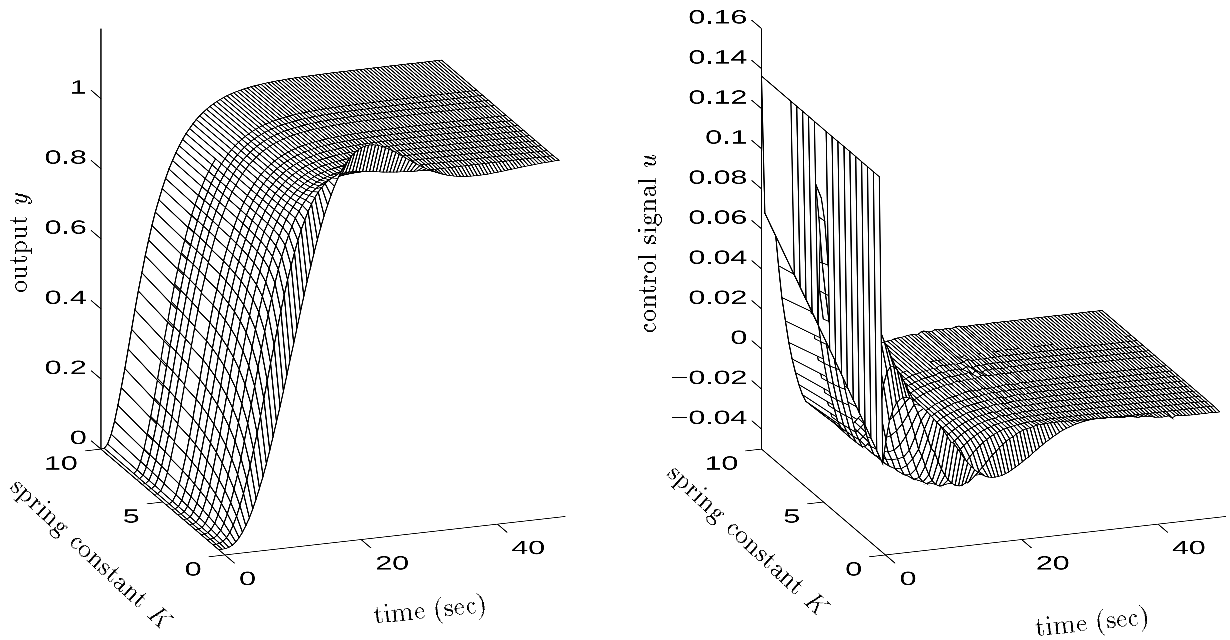

Figure 9: Position of body 2 and the control signal as functions of time for varying values of the spring constant.

shows the output and control signal as functions of time, as the spring constant K (assumed to be constant but unknown) is varied between $K_{\min} = 0.5$ and $K_{\max} = 10$. The control law is synthesized using Theorem 2. An input constraint of $|u| \leq 1$ is imposed. The output tracks the set-point to within 10% in about 25 sec for all values of K. Also, the worst-case overshoot (corresponding to $K = K_{\min} = 0.5$) is about 0.2. It was found that asymptotic tracking is achievable in a range as large as $0.01 \leq K \leq 100$. The response in that case was, as expected, much more sluggish than that in Figure 9.

The total time required to compute the closed-loop response in Figure 9 (500 samples) for each fixed value of the spring constant K was about 438 seconds (about 0.87 seconds per sample) on a SUN SPARCstation 20, using MATLAB code. The CPU time was about 330 seconds (about 0.66 seconds per sample). Of these times, nearly 94% was required to solve the LMI optimization at each sampling time.

6 Conclusions

Model Predictive Control (MPC) has gained wide acceptance as a control technique in the process industries. From a theoretical standpoint, the stability properties of nominal MPC have been studied in great detail in the past 7-8 years. Similarly, the analysis of robustness properties of MPC has also received significant attention in the MPC literature. However, robust synthesis for MPC has been addressed only in a restrictive sense for uncertain FIR models. In this article, we have described a new theory for robust MPC synthesis for two classes of very general and commonly encountered uncertainty descriptions. The development is based on the assumption of full state-feedback. The on-line optimization involves solution of an LMI-based linear objective minimization. The resulting time-varying state-feedback control law minimizes, at each time-step, an upper bound on the robust performance objective, subject to input and output constraints. Several extensions such as constant set-point tracking, reference trajectory tracking, disturbance rejection and application to delay systems complete the theoretical development. Two examples serve to illustrate application of the control technique.

Acknowledgments

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A Appendix A: Proof of Theorem 1

Minimization of $V(x(k|k)) = x(k|k)^T P x(k|k),\ P > 0$ is equivalent to

$$\begin{array}{rcl} \min\limits_{\gamma,P} & & \gamma \\ \text{subject to} \quad x(k|k)^T P x(k|k) & \leq & \gamma. \end{array}$$

Defining $Q = \gamma P^{-1} > 0$ and using (13), this is equivalent to

$$\begin{array}{rcl} \min\limits_{\gamma,Q} & & \gamma \\ \text{subject to} \quad \begin{bmatrix} 1 & x(k|k)^T \\ x(k|k) & Q \end{bmatrix} & \geq & 0, \end{array}$$

which establishes (19), (20), (23) and (24). It remains to prove (18), (21), (22), (25) and (26). We will prove these by considering **(A)** and **(B)** separately.

(A) The quadratic function V is required to satisfy (16). Substituting $u(k+i|k) = Fx(k+i|k),\ i \geq 0$ and the state space (1), inequality (16) becomes:

$$\begin{aligned} x(k+i|k)^T \Big(&(A(k+i) + B(k+i)F)^T P (A(k+i) + B(k+i)F) - P \\ &+ F^T R F + Q_1 \Big) x(k+i|k) \ \leq \ 0. \end{aligned}$$

This is satisfied for all $i \geq 0$ if

$$(A(k+i)+B(k+i)F)^T P(A(k+i)+B(k+i)F) - P + F^T RF + Q_1 \leq 0. \tag{43}$$

Substituting $P = \gamma Q^{-1}, Q > 0$, $Y = FQ$, pre- and post-multiplying by Q (which leaves the inequality unaffected), and using (13), we see that this is equivalent to

$$\begin{bmatrix} Q & QA(k+i)^T + Y^T B(k+i)^T & QQ_1^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ A(k+i)Q + B(k+i)Y & Q & 0 & 0 \\ Q_1^{\frac{1}{2}} Q & 0 & \gamma I & 0 \\ R^{\frac{1}{2}} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0. \tag{44}$$

Inequality (44) is affine in $[A(k+i) \;\; B(k+i)]$. Hence, it is satisfied for all

$$[A(k+i) \;\; B(k+i)] \in \Omega = \mathrm{Co}\{[A_1 \; B_1], [A_2 \; B_2], \ldots, [A_L \; B_L]\}$$

if and only if there exist $Q > 0, Y = FQ$ and γ such that

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_1^{\frac{1}{2}} Q & 0 & \gamma I & 0 \\ R^{\frac{1}{2}} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \quad j = 1, 2, \ldots, L.$$

The feedback matrix is then given by $F = YQ^{-1}$. This establishes (18) and (21).

(B) Let Ω be described by (3) in terms of a structured uncertainty block Δ as in (4). As in **(A)**, we substitute $u(k+i|k) = Fx(k+i|k)$, $i \geq 0$ and the state space equations (3) in (16) to get

$$\begin{bmatrix} x(k+i|k) \\ p(k+i|k) \end{bmatrix}^T \begin{bmatrix} (A+BF)^T P(A+BF) - P & (A+BF)^T PB_p \\ +F^T RF + Q_1 & \\ B_p^T P(A+BF) & B_p^T PB_p \end{bmatrix} \begin{bmatrix} x(k+i|k) \\ p(k+i|k) \end{bmatrix} \leq 0, \tag{45}$$

with

$$p_j(k+i|k)^T p_j(k+i|k) \leq x(k+i|k)^T (C_{q,j} + D_{qu,j}F)^T (C_{q,j} + D_{qu,j}F) x(k+i|k), \; j = 1, 2, \ldots, r. \tag{46}$$

It is easy to see that (45) and (46) are satisfied if $\exists \; \lambda_1', \lambda_2', \ldots, \lambda_r' > 0$ such that

$$\begin{bmatrix} (A+BF)^T P(A+BF) - P + F^T RF & (A+BF)^T PB_p \\ +Q_1 + (C_q + D_{qu}F)^T \Lambda' (C_q + D_{qu}F) & \\ B_p^T P(A+BF) & B_p^T PB_p - \Lambda' \end{bmatrix} \leq 0, \tag{47}$$

$$\text{where} \quad \Lambda' = \begin{bmatrix} \lambda_1' I_{n_1} & & & \\ & \lambda_2' I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_r' I_{n_r} \end{bmatrix} > 0. \tag{48}$$

Substituting $P = \gamma Q^{-1}$ with $Q > 0$, using (13) and after some straightforward manipulations, we see that this is equivalent to the existence of $Q > 0,\ Y = FQ,\ \Lambda' > 0$ such that

$$\begin{bmatrix} Q & Y^T R^{\frac{1}{2}} & Q Q_1^{\frac{1}{2}} & Q C_q^T + Y^T D_{qu}^T & Q A^T + Y^T B^T \\ R^{\frac{1}{2}} Y & \gamma I & 0 & 0 & 0 \\ Q_1^{\frac{1}{2}} Q & 0 & \gamma I & 0 & 0 \\ C_q Q + D_{qu} Y & 0 & 0 & \gamma \Lambda'^{-1} & 0 \\ AQ + BY & 0 & 0 & 0 & Q - B_p \gamma \Lambda'^{-1} B_p^T \end{bmatrix} \geq 0.$$

Defining $\Lambda = \gamma \Lambda'^{-1} > 0$ and $\lambda_i = \gamma \lambda_i'^{-1} > 0,\ i = 1, 2, \ldots, r$ then gives (22), (25) and (26) and the proof is complete. □

B Appendix B: Proof of Lemma 1

(A) From the proof of Theorem 1, Part **(A)**, we know that

$$(21) \Longleftrightarrow (44) \Longleftrightarrow (43) \Longrightarrow (16).$$

Thus,

$$\begin{aligned} & x(k+i+1|k)^T P x(k+i+1|k) - x(k+i|k)^T P x(k+i|k) \\ \leq \;& -x(k+i|k)^T Q_1 x(k+i|k) - u(k+i|k)^T R u(k+i|k) \\ < \;& 0 \quad \text{since } Q_1 > 0. \end{aligned}$$

Therefore,

$$x(k+i+1|k)^T P x(k+i+1|k) < x(k+i|k)^T P x(k+i|k), \quad i \geq 0, \quad (x(k+i|k) \neq 0). \tag{49}$$

Thus, if $x(k|k)^T P x(k|k) \leq \gamma$, then $x(k+1|k)^T P x(k+1|k) < \gamma$. This argument can be continued for $x(k+2|k), x(k+3|k), \ldots$ and this completes the proof. □

(B) From the proof of Theorem 1, Part **(B)**, we know that:

$$(25), (26) \Longleftrightarrow (47),\ (48) \Longrightarrow (45), (46) \Longleftrightarrow (16).$$

Arguments identical to case **(A)** then establish the result. □

C Appendix C: Output constraints as LMIs

As in §3.2.1, we will assume that the postulates of Lemma 1 are satisfied so that $\mathcal{E}$ is an invariant ellipsoid for the predicted states of the uncertain system (1).

Polytopic uncertainty

For any plant $[A(k+j) \;\; B(k+j)] \in \Omega,\ j \geq 0$, we have

$$\begin{aligned}
\max_{i\geq 1} \|y(k+i|k)\|_2 &= \max_{i\geq 0} \|C\,(A(k+i)+B(k+i)F)\,x(k+i|k)\|_2 \\
&\leq \max_{z\in\mathcal{E}} \|C\,(A(k+i)+B(k+i)F)\,z\|_2,\ i\geq 0 \\
&= \bar{\sigma}\left[C\,(A(k+i)+B(k+i)F)\,Q^{\frac{1}{2}}\right],\ i\geq 0.
\end{aligned}$$

Thus, $\|y(k+i|k)\|_2 \leq y_{\max},\ i\geq 1$ for any $[A(k+j) \;\; B(k+j)] \in \Omega,\ j\geq 0$ if

$$\bar{\sigma}\left[C\,(A(k+i)+B(k+i)F)\,Q^{\frac{1}{2}}\right] \leq y_{\max}, \quad i\geq 0,$$

$$\text{or}\ \ Q^{\frac{1}{2}}(A(k+i)+B(k+i)F)^T C^T C(A(k+i)+B(k+i)F)Q^{\frac{1}{2}} \leq y_{\max}^2 I, \quad i\geq 0,$$

which, in turn, is equivalent to

$$\left[\begin{array}{cc} Q & (A(k+i)Q+B(k+i)Y)^T C^T \\ C(A(k+i)Q+B(k+i)Y) & y_{\max}^2 I \end{array}\right] \geq 0,\ i\geq 0$$

(multiplying on the left and right by $Q^{\frac{1}{2}}$ and using (13)).

Since the last inequality is affine in $[A(k+i) \; B(k+i)]$, it is satisfied for all

$$[A(k+i) \; B(k+i)] \in \Omega = \mathrm{Co}\{[A_1 \; B_1], [A_2 \; B_2], \ldots, [A_L \; B_L]\}$$

if and only if

$$\left[\begin{array}{cc} Q & (A_jQ+B_jY)^T C^T \\ C(A_jQ+B_jY) & y_{\max}^2 I \end{array}\right] \geq 0, \quad j=1,2,\ldots,L.$$

This establishes (34).

Structured uncertainty

For any admissible $\Delta(k+i),\ i\geq 0$, we have

$$\begin{aligned}
\max_{i\geq 1} \|y(k+i|k)\|_2 &= \max_{i\geq 0} \|C(A+BF)x(k+i|k)+CB_p p(k+i|k)\|_2 \\
&\leq \max_{z\in\mathcal{E}} \|C(A+BF)z+CB_p p(k+i|k)\|_2,\ i\geq 0 \\
&= \max_{z^Tz\leq 1} \|C(A+BF)Q^{\frac{1}{2}}z+CB_p p(k+i|k)\|_2,\ i\geq 0.
\end{aligned}$$

We want $\|C(A+BF)Q^{\frac{1}{2}}z+CB_p p(k+i|k)\|_2 \leq y_{\max},\ i\geq 0$ for all $p(k+i|k), z$ satisfying

$$p_j(k+i|k)^T p_j(k+i|k) \leq z^T Q^{\frac{1}{2}}(C_{q,j}+D_{qu,j}F)^T(C_{q,j}+D_{qu,j}F)Q^{\frac{1}{2}}z, \quad j=1,2,\ldots,r$$

and $z^T z \leq 1$. This is satisfied if $\exists\ t_1, t_2, \ldots, t_r, t_{r+1} > 0$ such that for all $z, p(k+i|k)$

$$\begin{bmatrix} z \\ p(k+i|k) \end{bmatrix}^T \begin{bmatrix} \begin{array}{c} Q^{\frac{1}{2}}(A+BF)^T C^T C(A+BF)Q^{\frac{1}{2}} \\ +Q^{\frac{1}{2}}(C_q + D_{qu}F)^T T(C_q + D_{qu}F)Q^{\frac{1}{2}} \\ -t_{r+1}I \end{array} & Q^{\frac{1}{2}}(A+BF)^T C^T C B_p \\ B_p^T C^T C(A+BF)Q^{\frac{1}{2}} & B_p^T C^T C B_p - T \end{bmatrix} \begin{bmatrix} z \\ p(k+i|k) \end{bmatrix}$$

$$\leq y_{\max}^2 - t_{r+1},\ i \geq 0,$$

where

$$T = \begin{bmatrix} t_1 I_{n_1} & & & \\ & t_2 I_{n_2} & & \\ & & \ddots & \\ & & & t_r I_{n_r} \end{bmatrix} > 0.$$

Without loss of generality, we can choose $t_{r+1} = y_{\max}^2$. Then, the last inequality is satisfied for all $z, p(k+i|k)$ if

$$\begin{bmatrix} \begin{array}{c} Q^{\frac{1}{2}}(A+BF)^T C^T C(A+BF)Q^{\frac{1}{2}} \\ +Q^{\frac{1}{2}}(C_q + D_{qu}F)^T T(C_q + D_{qu}F)Q^{\frac{1}{2}} - y_{\max}^2 I \end{array} & Q^{\frac{1}{2}}(A+BF)^T C^T C B_p \\ B_p^T C^T C(A+BF)Q^{\frac{1}{2}} & B_p^T C^T C B_p - T \end{bmatrix} \leq 0,$$

$$\text{or equivalently,} \begin{bmatrix} y_{\max}^2 Q & (C_q Q + D_{qu} Y)^T & (AQ+BY)^T C^T \\ C_q Q + D_{qu} Y & T^{-1} & 0 \\ C(AQ+BY) & 0 & I - C B_p T^{-1} B_p^T C^T \end{bmatrix} \geq 0$$

(using (13) and after some simplification).

This establishes (35).

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