

The law of the wake in the turbulent boundary layer

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SUMMARY

After an extensive survey of mean-velocity profile measurements in various two-dimensional incompressible turbulent boundary-layer flows, it is proposed to represent the profile by a linear combination of two universal functions. One is the well-known law of the wall. The other, called the law of the wake, is characterized by the profile at a point of separation or reattachment. These functions are considered to be established empirically, by a study of the mean-velocity profile, without reference to any hypothetical mechanism of turbulence. Using the resulting complete analytic representation for the mean-velocity field, the shearing-stress field for several flows is computed from the boundary-layer equations and compared with experimental data.

The development of a turbulent boundary layer is ultimately interpreted in terms of an equivalent wake profile, which supposedly represents the large-eddy structure and is a consequence of the constraint provided by inertia. This equivalent wake profile is modified by the presence of a wall, at which a further constraint is provided by viscosity. The wall constraint, although it penetrates the entire boundary layer, is manifested chiefly in the sublayer flow and in the logarithmic profile near the wall.

Finally, it is suggested that yawed or three-dimensional flows may be usefully represented by the same two universal functions, considered as vector rather than scalar quantities. If the wall component is defined to be in the direction of the surface shearing stress, then the wake component, at least in the few cases studied, is found to be very nearly parallel to the gradient of the pressure.

I. COMPENDIUM

A. The law of the wall

Consider a turbulent shear flow which is steady and two-dimensional on the average. Let $u(x, y)$ and $v(x, y)$ be the mean velocities in the direction of increasing rectangular coordinates x and y respectively. Suppose that the flow exerts a shearing stress $\tau_w(x)$ on a smooth impermeable wall at rest at $y = 0$. For a fluid of constant density, define a friction velocity $u_\tau(x)$ by

$$\rho u_\tau^2 = \tau_w. \quad (1)$$

Experience with turbulent shear flow has shown that, under these conditions the mean-velocity profile in a considerable region near the surface is described by a relationship called the 'law of the wall':

$$\frac{u}{u_\tau} = f\left(\frac{yu_\tau}{\nu}\right). \quad (2)$$

The earliest formulation of the law of the wall was based on observations of pipe flow. The current view, however, is one first suggested by Ludwig & Tillmann (1949); the relationship (2) is taken for practical purposes as a unique and universal similarity law for every turbulent flow past a smooth surface.

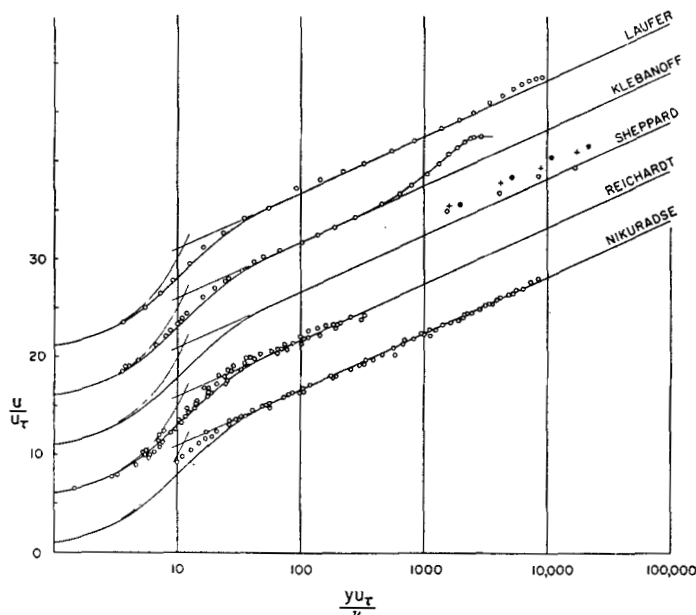


Figure 1. The law of the wall.

For the special case of steady two-dimensional mean flow of an incompressible fluid, the form of the universal law is well established. In particular, for values of yu_τ/ν greater than about 50, equation (2) takes the form

$$\frac{u}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{yu_\tau}{\nu}\right) + c, \quad (3)$$

in which κ and c are constants to be determined experimentally. On the other hand, the predominance of laminar shear very near the wall requires u/u_τ to approach yu_τ/ν as y approaches zero. To illustrate the presence of linear and logarithmic regions in the mean-velocity profile, experimental data from several sources are collected in figure 1. A brief discussion of these data will be found in Part II of the present paper.

B. The defect law and the equilibrium boundary layer

The description just given of the mean velocity profile in a turbulent shear flow may be summarized in the formula

$$\frac{u}{u_\tau} = f\left(\frac{yu_\tau}{\nu}\right) + h(x, y), \quad (4)$$

where the function h is arbitrary except that it is negligibly small in some finite region near the wall—say for y/δ less than about 0.1, where δ is the thickness of the shear flow.

For certain special cases frequently encountered (e.g. uniform pipe and channel flow and the boundary layer on a flat plate in a uniform stream), equation (4) is found experimentally to have the special form

$$\frac{u}{u_\tau} = f\left(\frac{yu_\tau}{\nu}\right) + g\left(\Pi, \frac{y}{\delta}\right), \quad (5)$$

where Π is a parameter which is independent of x and y . Profile similarity in terms of the argument y/δ is usually expressed by a relationship known as the velocity-defect law, or more properly the momentum-defect law. Outside the sublayer, it is an immediate consequence of the logarithmic variation of f in equation (5) that

$$\frac{u_1 - u}{u_\tau} = F\left(\Pi, \frac{y}{\delta}\right), \quad (6)$$

with $u = u_1$ at $y = \delta$.

Quite recently these contributions to the experimental definition of turbulent shear flows have been brilliantly extended by F. Clauser (1954), who generalized the idea of a defect law by showing experimentally the existence of boundary-layer flows for which the similarity laws (5) and (6) remain valid although the pressure gradient is positive.

Finally, the existence of a logarithmic region in the mean-velocity profile has been shown by Millikan (1938) to follow directly, without reference to any hypothetical mechanism of turbulent mixing, from the assumption of the simultaneous validity of the law of the wall in the form (2) and the defect law in the form (6).

C. The law of the wake

The relationships so far cited, especially the law of the wall, form the hard core of empirical knowledge of flow in turbulent boundary layers. Once they are accepted, it is natural to look for further inspiration in the inverse of Millikan's problem. That is, given a universal law of the wall which is logarithmic outside the sublayer, what other assumptions about the motion are necessary in order to establish a defect law? The most obvious answer is that the mean-velocity profile must have the form of equation (5). The purpose of the present paper, however, is to suggest a less superficial answer, one which has several immediately useful consequences and which at the same time appears to approach somewhat closer to the central problem of turbulence itself.

The essential element in this work is a study of the function $h(x, y)$ in the general mean-velocity formula (4). An extensive survey of experimental data at large Reynolds numbers, reported in Parts II and III, leads to the crucial conclusion that this function can be reduced directly to a second universal similarity law. Specifically, equation (4) may be written in the form

$$\frac{u}{u_\tau} = f\left(\frac{yu_\tau}{\nu}\right) + \frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right), \quad (7)$$

where Π is a profile parameter, as in equation (5), but the function $w(y/\delta)$ is now supposedly common to all two-dimensional turbulent boundary-layer flows.

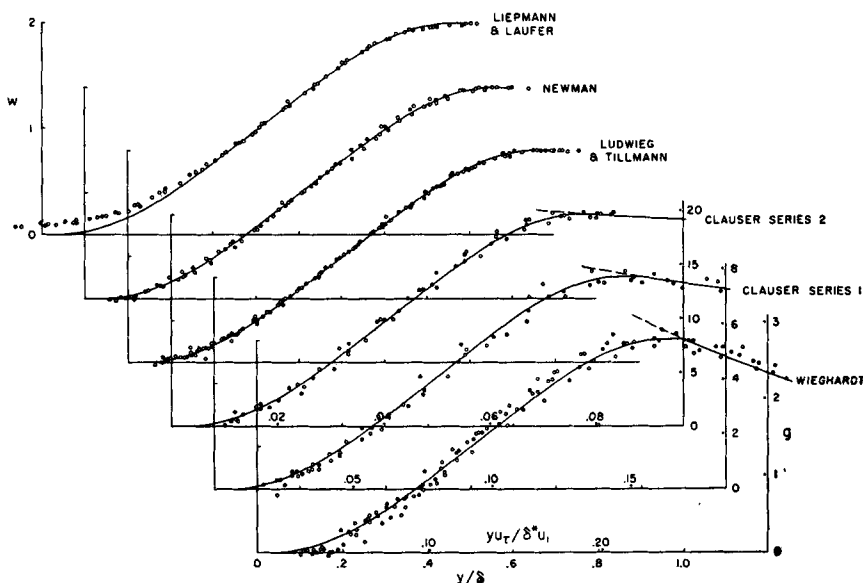


Figure 2. The law of the wake.

The introduction of a second universal function in the mean-velocity profile will be referred to as the wake hypothesis, and the function $w(y/\delta)$ in equation (7) will be referred to as the 'law of the wake'. Some experimental data which support the wake hypothesis are collected in figure 2. The measurements in question are included in the survey already mentioned, and the development leading to equation (7) and to figure 2 will be found in Part III.

In preparing figure 2 the wake function $w(y/\delta)$ in (7) has been subjected to the normalizing conditions $w(0) = 0$, $w(1) = 2$ and $\int_0^2 (y/\delta) dw = 1$. The parameter Π is then found to be related to the local friction coefficient $C_f = 2u_\tau^2/u_1^2$ by

$$\frac{u_1}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{\delta u_\tau}{\nu}\right) + c + \frac{2\Pi}{\kappa}, \quad (8)$$

to the displacement thickness δ^* by

$$\kappa \frac{\delta^* u_1}{\delta u_\tau} = 1 + \Pi, \quad (9)$$

and to the momentum thickness θ by

$$\frac{\kappa^2}{2} \left(\frac{\delta^* - \theta}{\delta} \right) \frac{u_1^2}{u_\tau^2} = 1 + \alpha \Pi + \beta \Pi^2, \quad (10)$$

where α and β are constants of order unity.

D. The equations of mean motion

To the extent that the similarity laws of the preceding sections are empirical, and not based on clear physical principles, these laws cannot be extended with confidence to conditions outside the range of observation. At the same time, these similarity laws go well beyond the usual limits of dimensional analysis.

In the first place, the thickness δ and the parameter Π are uniquely defined by equations (8) and (9), as is easily confirmed by eliminating either of these parameters between the equations in question. In the second place, the mean-velocity field, including the flow in the sublayer, is determined whenever the kinematic viscosity ν and three of the four parameters u_1 , u_τ , δ and Π are known. But a complete analytic description of the mean-velocity field implies a complete knowledge (at least within the boundary-layer approximation) of the streamline pattern, of the shearing-stress field, and of the local rate of transfer of energy from the mean flow to the turbulent secondary flow. Some typical calculations based on the formula (7) are presented in Part IV.

E. Physical interpretation

It is instructive to examine the nature of the wake component in equation (7) by letting u_τ approach zero (i.e. $\tau_w \rightarrow 0$) after expressing Π in terms of δ , δ^* , u_1 and u_τ with the aid of equation (9). The result is

$$\frac{u}{u_1} = \frac{1}{2} w \left(\frac{y}{\delta} \right),$$

which does not involve either κ or ν . It follows, since $w(y/\delta)$ is by hypothesis a universal function, that the flow at a point of separation or reattachment is locally a pure wake flow.

This observation suggests a simple physical interpretation for the law of the wall and the law of the wake, as illustrated in figure 3. The figure shows the profile $u(y)$ for various values of x in a hypothetical boundary layer, which is proceeding from separation to separation through a region of attached flow. The dashed lines in the figure denote the wake-like structure represented in equation (7) by the function $w(y/\delta)$. The associated velocity defect $u_1 - u$ is given by $\Pi u_\tau (2 - w)/\kappa$, and the intercept at $y = 0$ of the equivalent wake profile therefore differs from the velocity in the external stream by an amount $2\Pi u_\tau/\kappa$.

According to this interpretation, the law of the wake is to be viewed as a manifestation of a large-scale mixing process similar to flow in a wake, in that it is constrained primarily by inertia rather than by viscosity. When the flow is bounded by a wall, however, it is ultimately necessary to satisfy the boundary conditions of vanishing velocity and Newtonian friction at the surface. These conditions impose a further viscous constraint on the flow whose effect is to modify the mean-velocity distribution as shown by the solid lines in figure 3. Near the wall, where the wake mean velocity is nearly constant, the constraint provided by viscosity can be observed in the sublayer flow and in the similarity relationship known as the law of the wall.

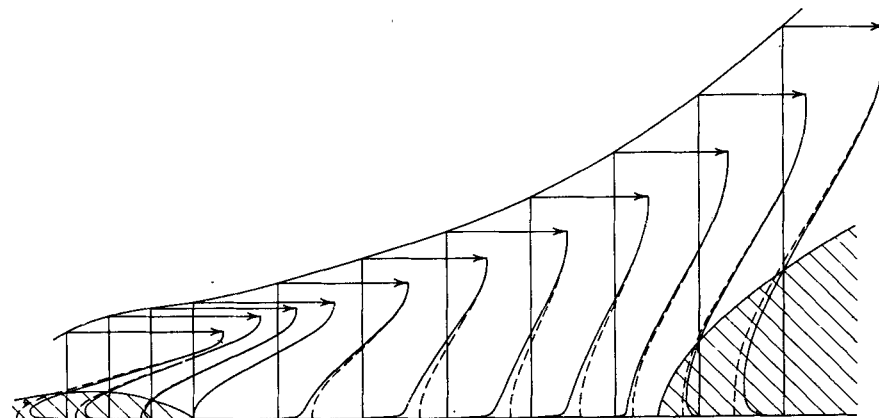


Figure 3. Mean-velocity field in a hypothetical boundary layer, including the equivalent wake.

II. THE LAW OF THE WALL

A. Historical development

1. *Evolution before 1949.* The early development of the law of the wall, in the hands of Prandtl, von Kármán and others, included a simple dimensional argument which has not lost its usefulness. Suppose that the mean-velocity profile in a turbulent shear flow is found to be adequately represented by a relationship $\phi(u, y, \delta, \tau_w, \mu, \rho) = 0$, in an obvious notation, and that this relationship is found in some region to be independent of the characteristic length δ . It follows from the principles of dimensional analysis, without any explicit assumptions about the nature of the turbulence, that in this region equation (2) must be valid.

Before the development of the mixing analogy, the function in equation (2) was sometimes taken as a power law, for lack of a better representation. The sublayer, that is, the region where viscous stress is predominant, was treated separately by means of the plausible assumption of a linear velocity profile very near the wall. In this approximation

$$\partial u / \partial y = u / y = \tau / \mu = \tau_w / \mu = u_\tau^2 / \nu,$$

and therefore $u / u_\tau = y u_\tau / \nu$.

By 1930 the present general formulation of the law of the wall had been achieved, in the sense that no distinction was made in equation (2) between the sublayer and the fully turbulent flow. Nikuradse (1930), at the suggestion of Prandtl, expressed the law of the wall in the present form during an analysis of some measurements in pipe flow. Equation (2) also appears in articles by Tollmien and by Schiller in Volume IV of *Handbuch der Experimentalphysik*.

At about the same time, the mixing analogy of Prandtl (1926) and the similarity hypothesis of von Kármán (1932) had provided an equation $\partial u(x, y)/\partial y = u_\tau(x)/\kappa y$ for the mean velocity in the fully turbulent region, with the integral $u/u_\tau = (1/\kappa) \ln [y/y_0(x)] + \text{constant}$. It is clear, however, that the choice of ν/u_τ for the unspecified characteristic length $y_0(x)$ is not properly a part of the mixing analogy, but rather a part of the dimensional argument already mentioned.

2. *Recent developments.* Until quite recently, the three most important elements in the development of the law of the wall have been, first, the dimensional argument leading to equation (2); second, the stipulation that the function f is linear at the wall; and third, the recognition that, for whatever reason, the function f is very nearly logarithmic in a certain region outside the sublayer.

A fourth important element was added when Ludwig & Tillmann (1949) found experimentally, using an ingenious heat-transfer technique for the indirect measurement of surface shearing stress, that for flow in a turbulent boundary layer the function f in equation (2) is apparently independent of pressure gradient. This result lies at the heart of the present study, virtually every part of which depends directly or indirectly on the hypothesis that the law of the wall is a unique and universal relationship for flow past a smooth surface.

As several writers in the field have pointed out, equation (2) is an implicit equation for u_τ (hence for τ_w) when ρ , μ , and $u(y)$ are given. The law of the wall thus provides a means for accurate determination of the elusive wall shearing stress, once the function f in (2) has been established by a survey of experimental data on flows for which τ_w is accurately known.

3. *Experimental data.* Nikuradse's classical pipe measurements (1930) confirmed the prediction of a logarithmic region in the mean-velocity profile. The lowest curve in figure 1 shows the data obtained, including a small correction for wall interference, in sixteen surveys at various Reynolds numbers of the region between the pipe wall and a value of y/δ of about 0.15, where δ is the pipe radius. Nikuradse did not obtain data on the sublayer, but inferred the validity of the general law of the wall from the argument already given in favour of the limiting form $u/u_\tau = yu_\tau/\nu$ at $y = 0$. This omission was partly repaired by Reichardt's measurements (1940) of mean velocity in the sublayer of a channel flow, and more completely repaired by the work of Laufer (1953) and Klebanoff (1954) on pipe flow and boundary-layer flow respectively.

The latter measurements, shown in figure 1, include data on the sublayer obtained in each instance with a hot-wire anemometer. For Laufer's experiments on pipe flow, the wall shearing stress has been derived from the observed axial pressure gradient. For Klebanoff's experiments on boundary layers, the wall shearing stress has been derived from the rate of momentum loss observed in other experiments with the same model (Klebanoff & Diehl 1951; see also Coles 1954). For Reichardt's experiments on channel flow, the wall shearing stress was not measured independently, as Reichardt's object was to interpolate experimentally in the region of the mean-velocity profile not studied by Nikuradse.

Finally, some measurements by Sheppard (1947) in a natural wind near the ground offer a striking example of a turbulent shear flow for which it would seem that no Reynolds number can be explicitly defined. However, difficulty in visualizing either an origin of coordinates or a second boundary does not prevent the presentation of these data in terms of the law of the wall. Sheppard observed the surface shearing stress directly, using the floating-element technique, together with the mean velocity at several points up to a height of two metres. The result of the measurements is given in figure 1. The agreement with wind-tunnel data is surprisingly good when it is considered that Sheppard's measurements were made over a concrete surface, and that the vertical temperature gradient may have differed significantly from the adiabatic lapse rate associated with neutral stability.

4. *Numerical evaluation.* Two empirical constants, κ and c , appear in equation (3). Throughout the present study, the numerical values given to these constants are

$$\begin{aligned}\kappa &= 0.40, \\ c &= 5.1.\end{aligned}$$

A great variety of other values, especially for κ , can be found in the experimental literature. However, in practically all cases where equation (3) is explicitly taken as a definition, κ is found to lie between 0.39 and 0.41. Values outside this range are usually the result of operations or assumptions which change the definition of κ and c . In any event, it is clear in figure 1 that the data outside the sublayer are well represented by equation (3) when κ and c are given the values already mentioned.

Within the sublayer, on the other hand, large fluctuations in velocity and cramped quarters for experimentation usually combine to make measurements of mean velocity somewhat uncertain. The available data in figure 1, therefore, should not be said to establish conclusively the uniqueness of the law of the wall in the sublayer, although these data have been used elsewhere (Coles 1955) in forming a tentative estimate of the function $f(yu_*'/\nu)$ and related functions.

B. Test of the wall law

1. *Momentum balance.* Except for a few applications of the floating-element technique in flows at constant pressure, values of surface stress in

turbulent boundary layers have usually been obtained indirectly from the observed pressure gradient and rate of momentum loss, using the momentum-integral equation of von Kármán (1921). For two-dimensional incompressible steady mean flow in a boundary layer, this equation is

$$\frac{\tau_w}{\rho} = \frac{d}{dx} u_1^2 \theta + u_1 \delta^* \frac{du_1}{dx}, \quad (11)$$

where $\tau_w(x)$ is the wall shearing stress, $u_1(x)$ is the velocity of the free stream outside the boundary layer, and $\delta^*(x)$ and $\theta(x)$ are the displacement and momentum thicknesses, defined respectively by

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{u_1}\right) dy, \quad (12)$$

and

$$\theta = \int_0^\delta \frac{u}{u_1} \left(1 - \frac{u}{u_1}\right) dy. \quad (13)$$

Experience in applying equation (11) to flows with positive pressure gradient has usually been that the values obtained for τ_w appear to increase rapidly when separation is imminent, contrary to the behaviour which is intuitively expected. In fact, the experiments of Ludwig & Tillmann (1949) were originally designed not to study the law of the wall, but rather to investigate the validity of the momentum-integral equation (11) by providing an independent estimate for τ_w .

It is therefore instructive to compare, for various experimental studies of flow in turbulent boundary layers, the values of wall shearing stress obtained from equation (11) with those inferred from the law of the wall. In order to avoid differentiation of measured quantities, equation (11) may be integrated with respect to x between two stations x_0 and x . Replacing τ_w by ρu^2 , the result of the integration can be expressed by the two equations

$$\Phi(x) = \frac{u_1^2 \theta}{u_0^2 \theta_0} - 1 + \frac{1}{2} \int_{x_0}^x \frac{\delta^*}{\theta_0} d\left(\frac{u_1^2}{u_0^2}\right), \quad (14)$$

and

$$\Phi(x) = \int_{x_0}^x \frac{u_\tau^2}{u_0^2} d\left(\frac{x}{\theta_0}\right), \quad (15)$$

where $u_0 = u_1(x_0)$ and $\theta_0 = \theta(x_0)$.

The dimensionless function $\Phi(x)$ may be evaluated experimentally from equation (14) if δ^* , θ , and u_1 are known as functions of x . Quite independently, the same function $\Phi(x)$ may be evaluated from equation (15) if the friction velocity u_τ is obtained by fitting the velocity profile to the hypothetical law of the wall. Agreement between the slopes of the two functions $\Phi(x)$ shows that the flow in question is not inconsistent with the logarithmic law of the wall displayed in figure 1.

2. *Presentation of data.* The discussion will be limited to data obtained at reasonably large Reynolds numbers in experiments at reasonably large

scale.* The experimental point of departure is flow at constant pressure, for which some mean-velocity measurements of Wieghardt (1943) for a free-stream velocity of 33 m/sec are shown in figure 4. Here and in the fifteen following figures, the mean-velocity profiles shown are typical of the measurements, although some of the profile data have occasionally been omitted for reasons of economy in the graphical presentation.

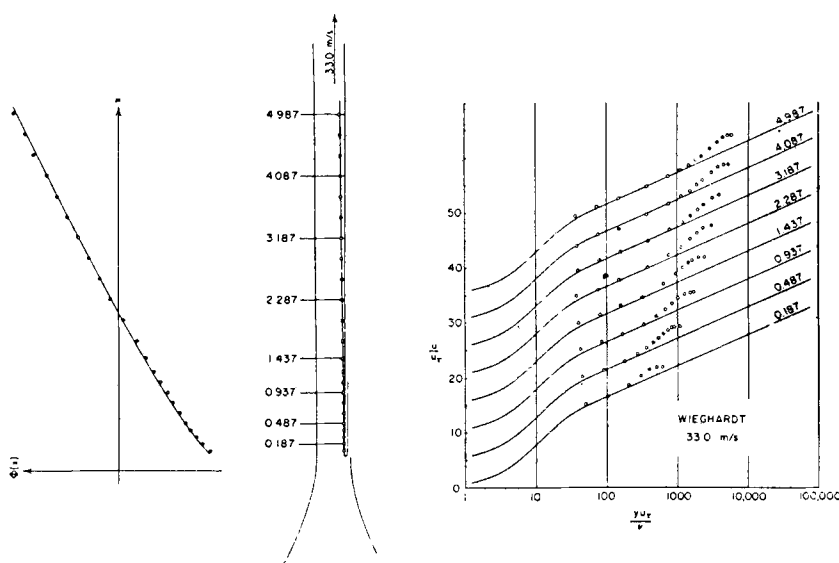


Figure 4. Data of Wieghardt (1943) for constant stream velocity; stations in metres.

All of the profiles, whether shown in the figures or not, have first been individually fitted to the logarithmic region of the law of the wall, i.e. to the formula $u/u_\tau = 5.75 \log_{10}(yu_\tau/\nu) + 5.10$, and the resulting values of u_1/u_τ have then been smoothed where necessary. In the fitting operation it has generally been assumed that the measurements for values of yu_τ/ν less than about 200 may be unreliable as a result of large fluctuations in velocity, wall interference, poor probe sensitivity at small mean velocities, probe position error, or uncertainty in the static pressure. For consistency, therefore, the contribution of the sublayer and of the logarithmic region to δ^* and θ have been computed for the function f of figure 1, i.e. for the solid lines in figures 4 to 19, rather than for the actual measurements.

* This survey of experimental data includes certain material which has not previously been reported in detail in the literature. For their courtesy in providing this material I am indebted to F. Clauser of the Johns Hopkins University, Baltimore; Smith J. DeFrance of the NACA Ames Aeronautical Laboratory, Moffett Field; W. Tillmann of the Max-Planck-Institut für Strömungsforschung, Göttingen; K. Wieghardt of the Institut für Schiffbau, Hamburg; and A. Kuethe of the University of Michigan, Ann Arbor. I am also indebted to F. Goddard of the Jet Propulsion Laboratory, Pasadena, for making available the services of the JPL computing section.

Each of figures 4 to 19 includes a sketch showing both the geometry of the experiment and the physical extent $\delta(x)$ of the shear flow. Finally, each of the figures includes a comparison of the two functions $\Phi(x)$ of equations (14) and (15), represented by open points and by a solid line respectively.

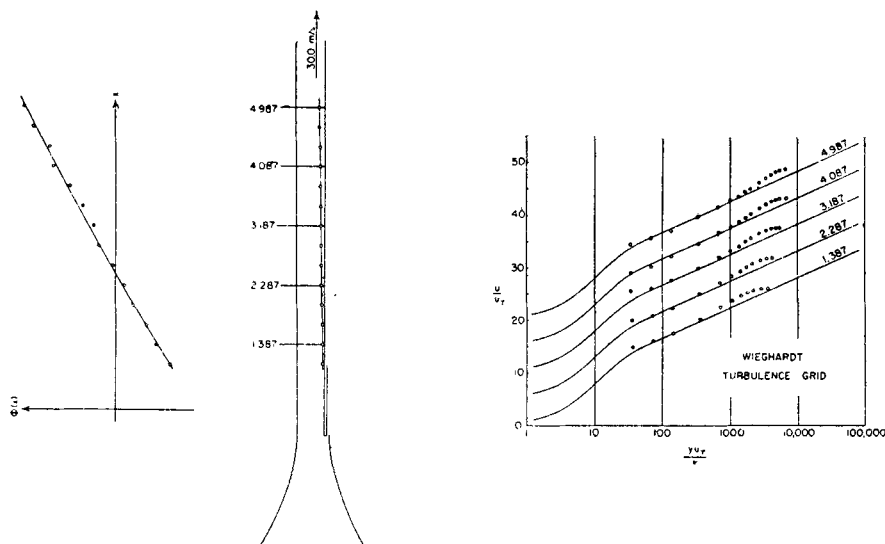


Figure 5. Data of Wieghardt (1944) with turbulence grid; stations in metres.

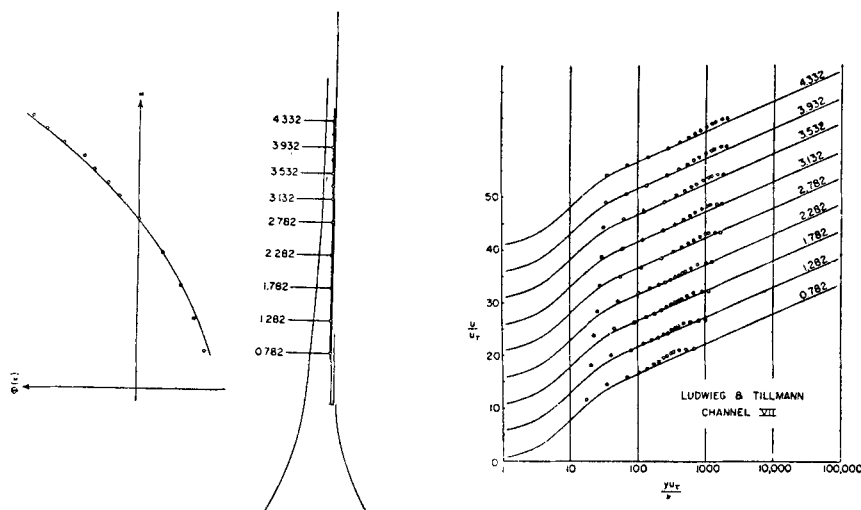


Figure 6. Data of Ludwig and Tillmann (1949) for falling pressure; stations in metres.

3. *Unseparated flows.* Wieghardt's measurements (1943) of boundary-layer growth for a uniform external stream have already been presented in figure 4. Wieghardt (1944) also investigated the effect of increasing the

tunnel turbulence level by means of a coarse screen, obtaining the mean-velocity profiles shown in figure 5. Using the same channel and instrumentation, Ludwig & Tillmann (1949) observed a turbulent boundary-layer flow in a region of a negative pressure gradient, with the result shown in figure 6. These particular profile measurements, and those of figures 13 and 14 below, are the ones originally cited by Ludwig & Tillmann in their cogent paper on the law of the wall.

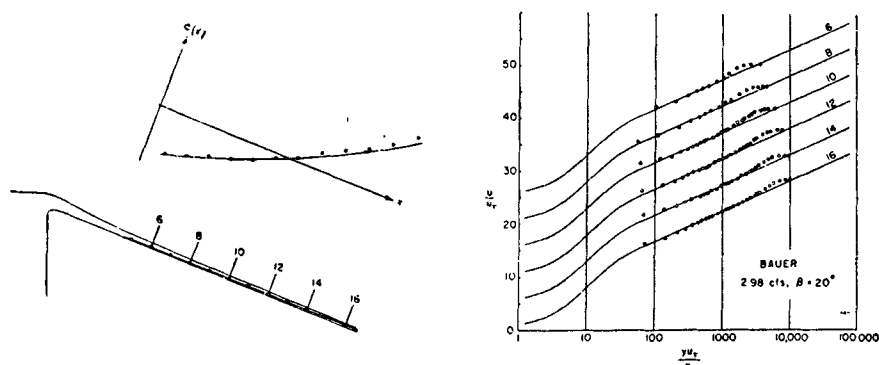


Figure 7. Data of Bauer (1951) for 20° slope; stations in feet.

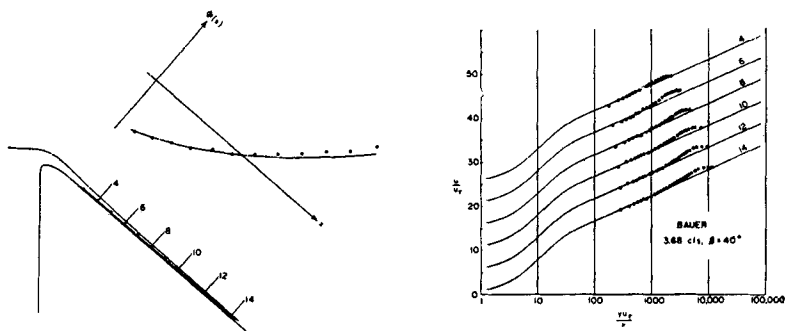


Figure 8. Data of Bauer (1951) for 40° slope; stations in feet.

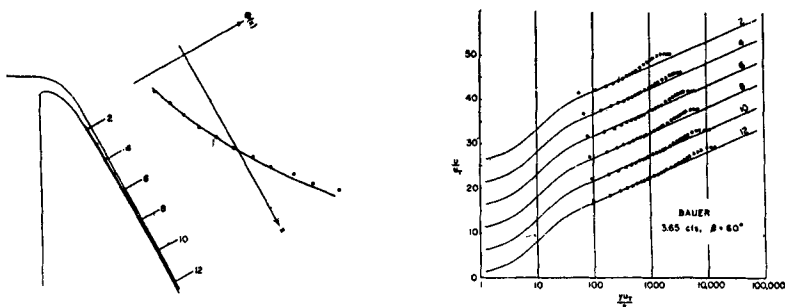


Figure 9. Data of Bauer (1951) for 60° slope; stations in feet.

Some measurements of flow near the lower boundary of a stream accelerating under the force of gravity at essentially constant pressure have been reported by Bauer (1951). The fluid is water traversing the face of a model spillway; data for flow over a smooth plane surface at angles of 20° , 40° , and 60° to the horizontal are reproduced in figures 7, 8, and 9. In each case the body force per unit volume $\rho g \sin \beta$ is independent of x and y and so plays the same role as a constant pressure force per unit volume dp/dx .

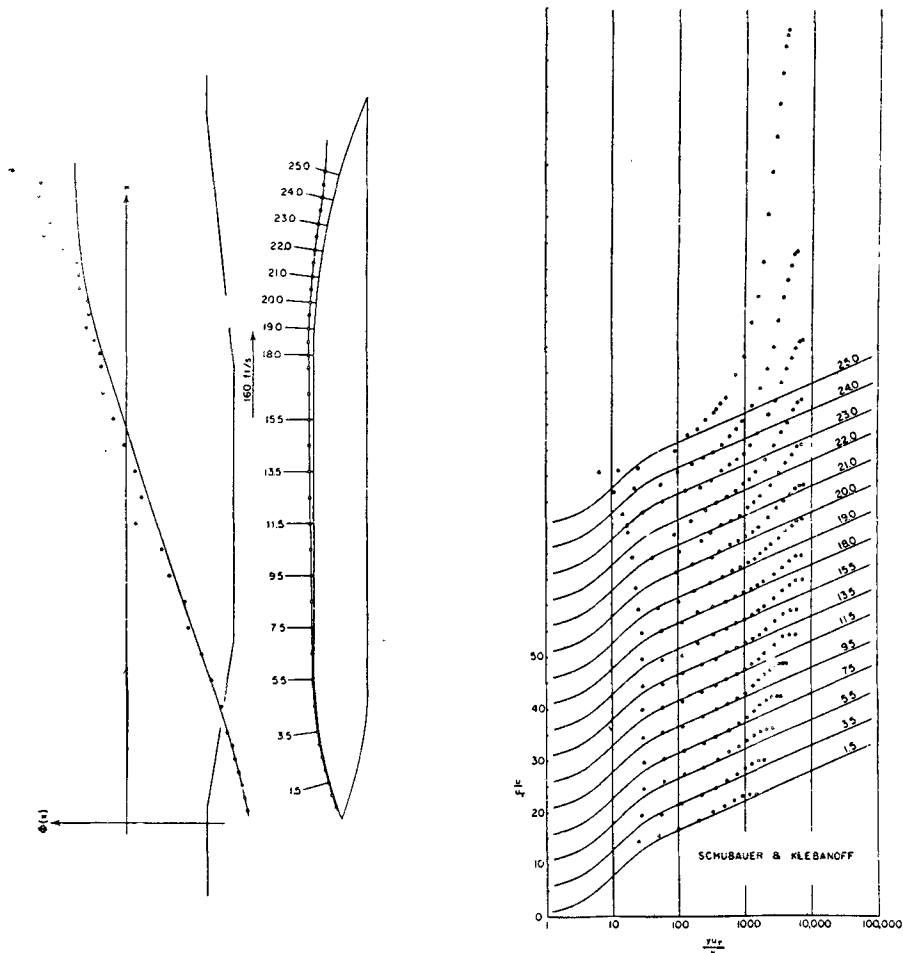


Figure 10. Data of Schubauer and Klebanoff (1950); stations in feet.

4. *Flows approaching separation.* Extensive measurements in a flow approaching separation have been made by Schubauer & Klebanoff (1950) on a large airfoil at the National Bureau of Standards. Typical mean-velocity profiles are shown in figure 10. Upstream of the 7 ft. station, the surface shearing stress and free-stream dynamic pressure both increase

in the direction of flow. Between $x = 7$ ft. and $x = 17.5$ ft. the stress at the wall and the pressure are nearly constant. The region of rising pressure begins at about $x = 18$ ft., and separation occurs at about $x = 25.7$ ft.

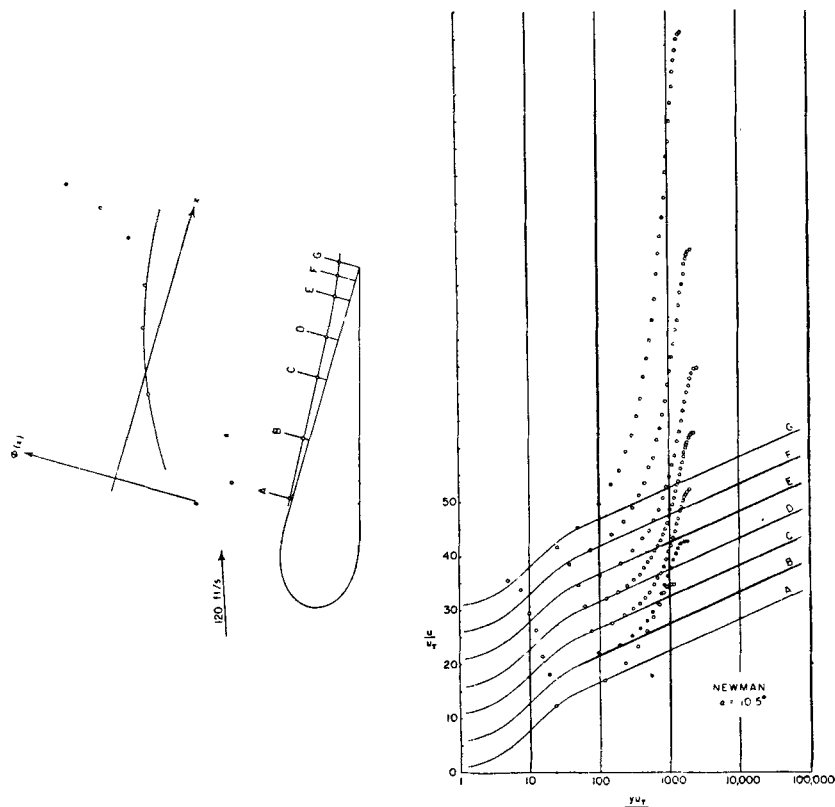


Figure 11. Data of Newman (1951); airfoil chord 50 inches.

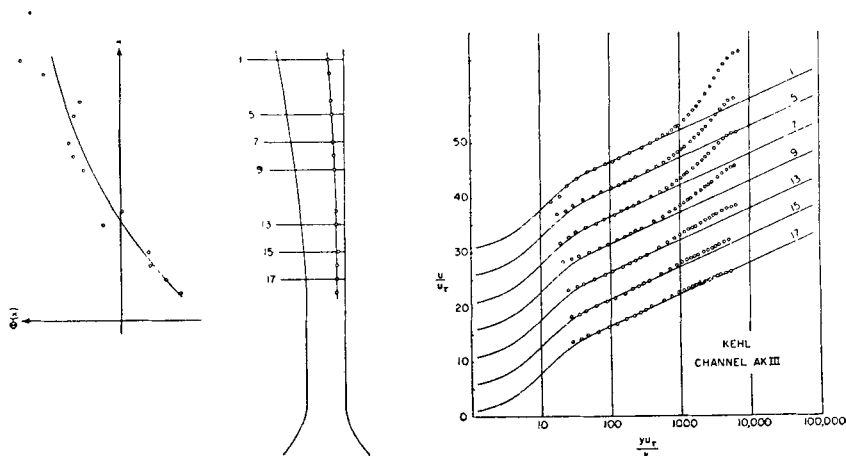


Figure 12. Data of Kehl (1943); stations in decimetres.

A similar study of a boundary layer near separation is available in some work of Newman (1951) for which the mean-velocity data are presented in figure 11. Because Newman measured the static-pressure variation within the boundary layer, and also made a plausible correction to his observed mean-velocity profiles for instrumental errors caused by turbulence, these data are among the most accurate and detailed which are available. Newman,

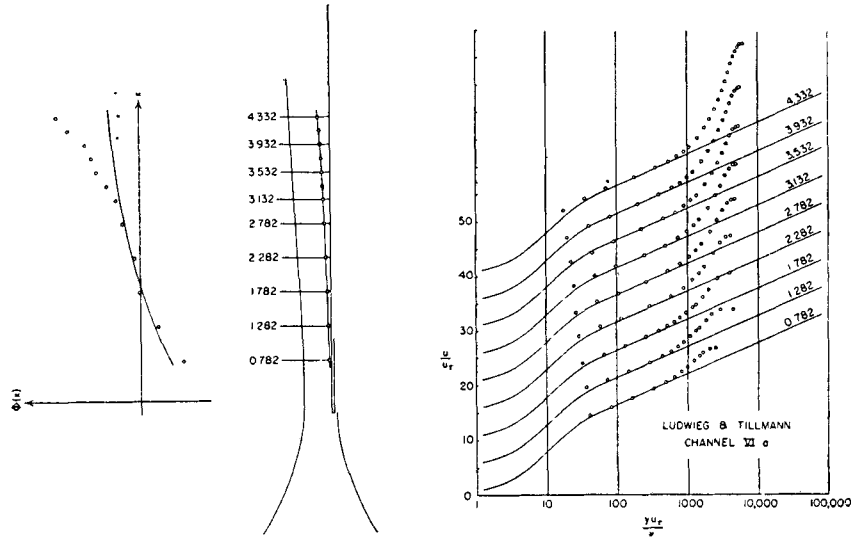


Figure 13. Data of Ludwig and Tillmann (1949) for moderately rising pressure; stations in metres.

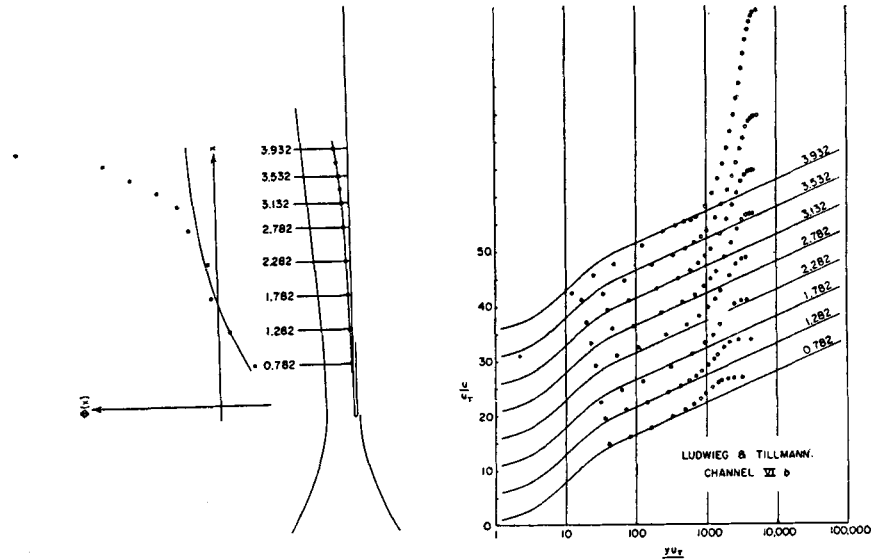


Figure 14. Data of Ludwig and Tillmann (1949) for strongly rising pressure; stations in metres.

like Schubauer & Klebanoff, also measured turbulence intensities and turbulent shearing stress within the shear flow.

Boundary-layer flow in diffusers of constant width and rectangular section has been studied by Kehl (1943), by Ludwig & Tillmann (1949), and by Clauser (1954), with the results shown in figures 12 to 16. In none of these five experiments did separation actually occur, and in the last two it was deliberately prevented.

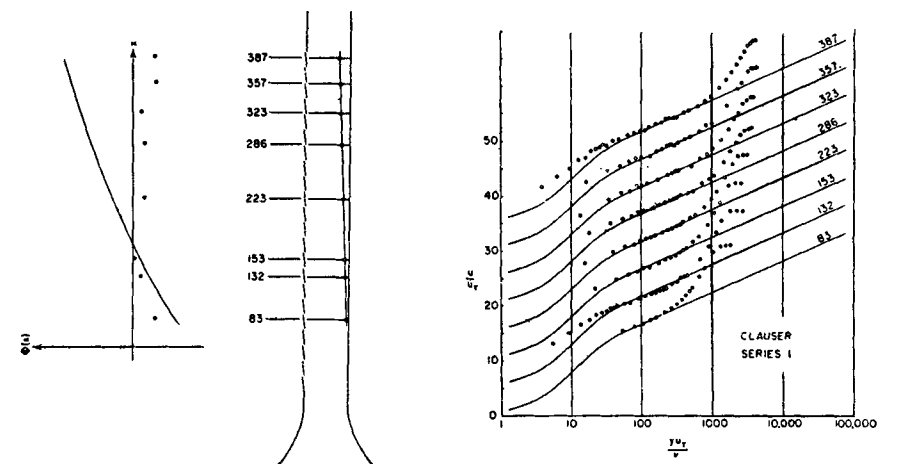


Figure 15. Data of Clauser (1954) for moderately rising pressure; stations in inches.

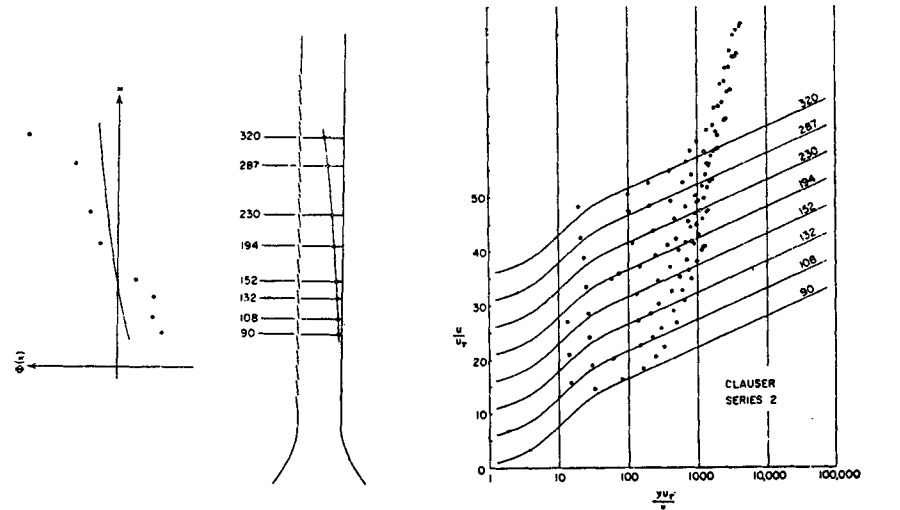


Figure 16. Data of Clauser (1954) for strongly rising pressure; stations in inches.

5. *Flows following reattachment.* Turbulent boundary-layer flow following reattachment downstream of a separation bubble has been investigated by McCullough & Gault (1949). Data for the upper surface of an NACA 64A006 airfoil section at an angle of attack of 5° are reproduced in figure 17.

Reattachment of separated flow downstream of a tripping device or spoiler, in flow with nominally constant pressure, is illustrated in figures 18 and 19 with some measurements of Klebanoff & Diehl (1951) and of Tillmann (1945). In the experiment of Klebanoff & Diehl transition occurred at the spoiler, which was a 1/4-in. diameter rod at the 4 ft. station of the plate.

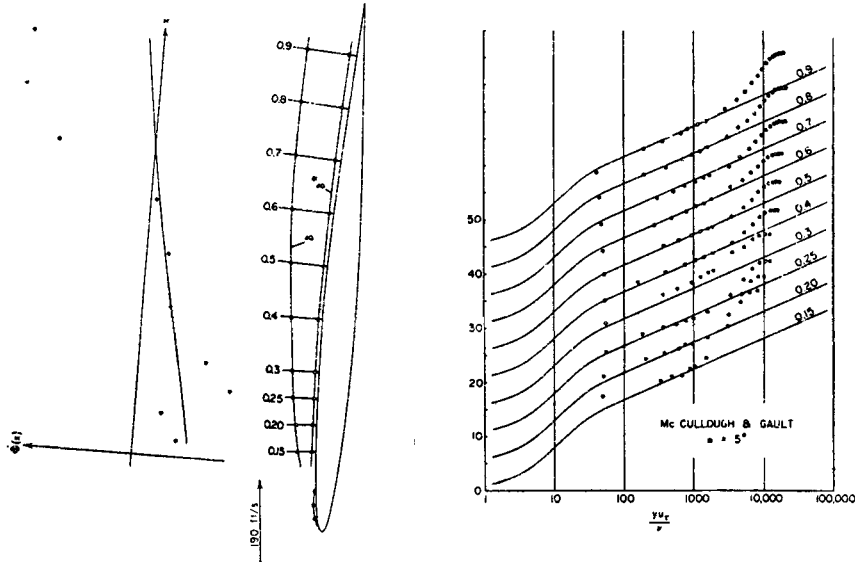


Figure 17. Data of McCullough and Gault (1949) for 5° angle of attack; airfoil chord 5 feet.

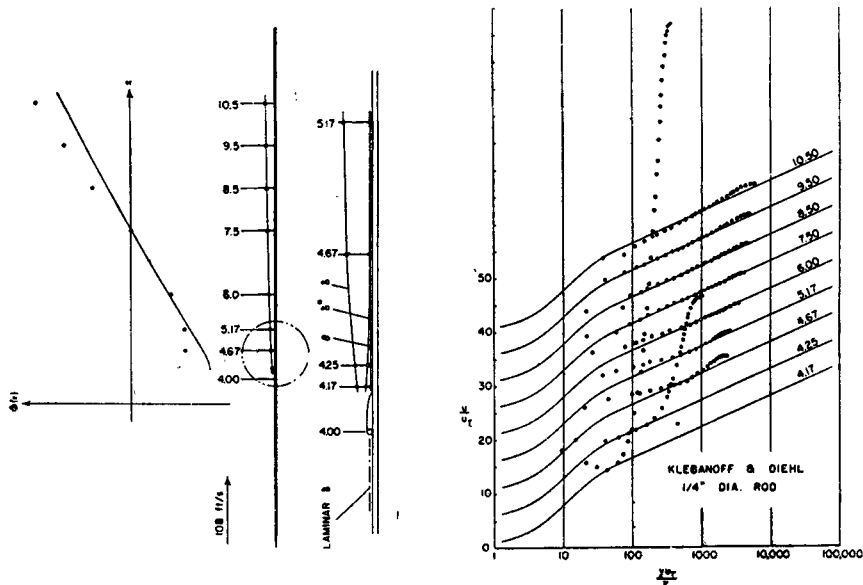


Figure 18. Data of Klebanoff and Diehl (1951) for 1/4-inch rod; stations in feet.

In the experiment of Tillmann the boundary layer was turbulent well upstream of the spoiler, which was a rectangular ledge 1.2 cm square at the 2.02 m station. In both cases it may be noted that the flow far downstream has apparently not recovered from the effects of the enforced separation, as the mean-velocity profiles do not resemble the profiles for flow at constant pressure shown in figure 4.

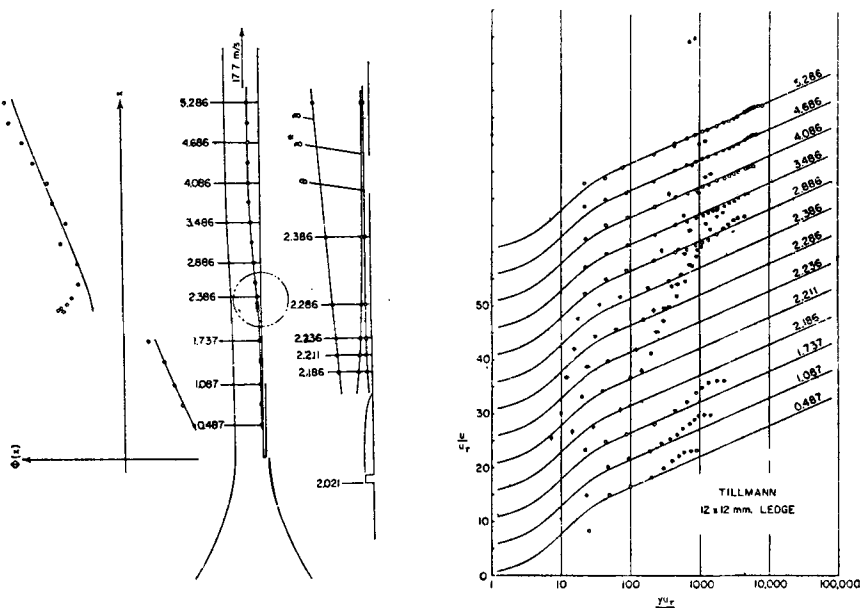


Figure 19. Data of Tillmann (1945) for 12 mm ledge; stations in metres.

6. *Evaluation.* The data cited in the previous section obviously do not provide a test of the hypothetical law of the wall alone, but provide a joint test of the law of the wall together with the turbulent boundary-layer approximation and the assumption of two-dimensional mean flow.

These data confirm that the momentum-integral equation (11) cannot be relied upon to give accurate values of surface shearing stress in the neighbourhood of separation, for the reason that the left side of (11) is then equal to a small difference between two large quantities on the right. Consequently, large errors in τ_w may be encountered, either as a result of inconspicuous departures from two-dimensional mean flow or as a result of the omission from the boundary-layer approximation of certain terms involving the Reynolds normal stresses and the pressure variation normal to the wall.

Both Ludwig & Tillmann (1949), using a surface heat-transfer technique, and Schubauer & Klebanoff (1950), using extrapolated values of the measured

turbulent stress, have found experimentally that the surface shearing stress decreases monotonically toward zero on approaching a point of separation. Consequently, whether or not the hypothesis of a universal similarity law is correct for the flows in question, it is certain from the evidence, for example, of figures 10, 13 and 14 that the momentum-integral equation in the form (11) is seriously in error.

Besides the work of Ludwig & Tillmann, perhaps the most convincing evidence for a universal law of the wall is simply that a distinct logarithmic region occurs in each of several hundred mean-velocity profiles examined here, with very few exceptions; a definite estimate for the wall shearing stress is readily obtained, and this estimate is entirely plausible.

At the same time, it is known (Coles 1955) that a necessary and sufficient condition for a universal law of the wall, given the boundary conditions of vanishing velocity and Newtonian friction at the surface, is that the ratio u/u_* is constant on streamlines of the mean flow. The edge of the sublayer, as usually defined, is therefore a mean streamline. This result must surely be considered in any search for a fundamental order and unity in the description of turbulent shear flows.

It should also be noted that the concept of a universal similarity law has recently been reinforced by work of Preston (1954), using an experimental technique which depends on the general validity of equation (2) in essentially the same way that the use of a Stanton tube depends on the existence of a linear profile very near the wall.

In view of these remarks, the hypothesis of a universal law of the wall will be accepted for the purposes of the present paper. In fact, it will eventually be suggested that the similarity laws of this and the next section may be concepts sufficiently powerful to allow a quantitative treatment not only of flows approaching or recovering from separation, but also of yawed flows and flows which are actually separated from the adjacent wall.

III. THE LAW OF THE WAKE

A. The defect law

1. *Historical development.* A special form of the defect law (6) was proposed by Darcy nearly a hundred years ago, and again by Stanton in 1911, to describe the mean-velocity profile in turbulent pipe flow. The defect law in the general form (6) was formulated independently by von Kármán (1932), who derived an approximate friction law involving two empirical constants for the turbulent boundary layer with constant pressure. According to experimental evidence from many sources, the defect function $F(\Pi, y/\delta)$ in a given flow is insensitive to roughness at the wall, provided that the origin for the normal coordinate y is properly chosen. On the other hand, it appears from a comparison of figures 4 and 5 that there is a small dependence of the defect law on the turbulence level in the external stream.

One well known and useful property of the defect law is that it avoids the awkward problem of defining the thickness δ for a boundary layer.

For, if the displacement thickness δ^* is computed from equation (12) for the profile (6), neglecting the departure of the flow in the sublayer from the logarithmic law,* there is obtained

$$\frac{\delta^* u_1}{\delta u_\tau} = \int_0^1 F\left(\Pi, \frac{y}{\delta}\right) d\left(\frac{y}{\delta}\right) = C(\Pi),$$

so that δ is proportional to $\delta^* u_1 / u_\tau$. This implicit notation for the parameter Π will eventually be replaced by an explicit formula.

2. *The equilibrium boundary layer.* The concept of a defect law has recently been generalized by F. Clauser (1954), who constructed experimentally two boundary-layer flows with positive pressure gradient, such that equations (5) and (6) remained valid. The flows in question have already been described in figures 15 and 16. Clauser used the term 'equilibrium flow' to denote a flow with a defect law, that is, in the present notation, a flow for which the parameter Π is constant.

Examination of figures 7, 8 and 9 suggests that the three spillway flows studied by Bauer may be equilibrium flows in the sense of Clauser's definition. So, at least approximately, is the flow with falling pressure studied by Ludwig & Tillmann and reported in figure 6.

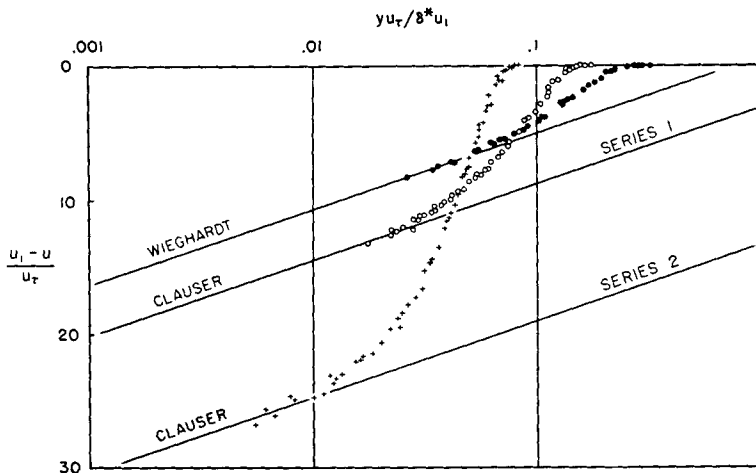


Figure 20. The defect law.

A comparison of the velocity-defect function F for three equilibrium flows can be found in figure 18 of Clauser's paper, and is repeated here in figure 20. This figure, unfortunately, sheds little light on the way in which the argument y/δ and the parameter Π are involved in the function $F(\Pi, y/\delta)$ of equation (6), and therefore does not immediately suggest any useful generalization of the defect law to non-equilibrium flows.

* This remark implies that, for example, $\int f dz$ is written $fz - \int z df$, and df is replaced by $dz/\kappa z$ as if the function f were logarithmic everywhere.

3. *The logarithmic region.* At this point a further important consequence of the defect law in equilibrium flow should be mentioned. In the first instance, the mixing analogy of Prandtl or von Kármán implies a logarithmic variation of the function $f(yu_\tau/\nu)$ in flows for which τ does not depend on y . However, if the law of the wall is universally valid, then the velocity distribution $u(y)$ can be expressed independently of the shear distribution $\tau(y)$; thus the argument based on the mixing analogy in favour of a logarithmic mean-velocity distribution is seriously weakened.

It is therefore instructive to consider another argument, first proposed by Millikan (1938) and based on the wall and defect laws, which also leads to a logarithmic function f in equation (2). From the law of the wall, $u/u_\tau = f(yu_\tau/\nu)$, it follows that

$$\frac{y}{u_\tau} \frac{\partial u}{\partial y} = \frac{yu_\tau}{\nu} f' \left(\frac{yu_\tau}{\nu} \right).$$

From the defect law, $(u_1 - u)/u_\tau = F(\Pi, y/\delta)$, the corresponding expression is

$$\frac{y}{u_\tau} \frac{\partial u}{\partial y} = - \frac{y}{\delta} F' \left(\frac{y}{\delta} \right),$$

where the constant parameter Π has been suppressed for an equilibrium flow. Now suppose that there is a finite region in which the wall and defect laws are simultaneously valid. In this region, the last two equations require that

$$\frac{yu_\tau}{\nu} f' \left(\frac{yu_\tau}{\nu} \right) = - \frac{y}{\delta} F' \left(\frac{y}{\delta} \right) = \frac{y}{u_\tau} \frac{\partial u}{\partial y} = \frac{1}{\kappa(x, y)},$$

say. Obviously, $\kappa(x, y)$ is fixed when either of the two variables yu_τ/ν or y/δ is specified. But these variables are formally independent of each other, since their ratio $\delta u_\tau/\nu$ may be chosen arbitrarily. It follows that κ must be a constant. Furthermore, on integrating the expression for $\partial u/\partial y$ over the region in question, it is found that

$$f \left(\frac{yu_\tau}{\nu} \right) = \frac{1}{\kappa} \ln \frac{yu_\tau}{\nu} + \text{const.},$$

in agreement with equation (3).

B. The wake hypothesis

1. *The wake function.* The defect law (6) has at least a limited physical interpretation, in that the loss of momentum is expressed independently of the viscosity. This property, being consistent with the idea of a turbulent rather than a viscous transport process, can reasonably be assumed, as in Millikan's argument, to apply everywhere outside the sublayer. Furthermore, the observed sensitivity of the momentum defect to external turbulence level and the observed insensitivity to wall roughness are not surprising.

On the other hand, the various mean-velocity profiles so far studied are quite systematic in coordinates $(u/u_\tau, yu_\tau/\nu)$ which involve the viscosity

of the fluid. In fact, once the universal law of the wall is accepted, it is difficult to escape the conviction that an arbitrarily chosen profile is completely determined when the free-stream point ($u_1/u_\tau, \delta u_\tau/\nu$) is specified. To illustrate this remark, figure 21 shows several mean-velocity profiles selected from various boundary-layer flows described earlier.* These profiles have essentially the same defect law (that is to say, the same value of the parameter Π) in spite of wide variations in environment.

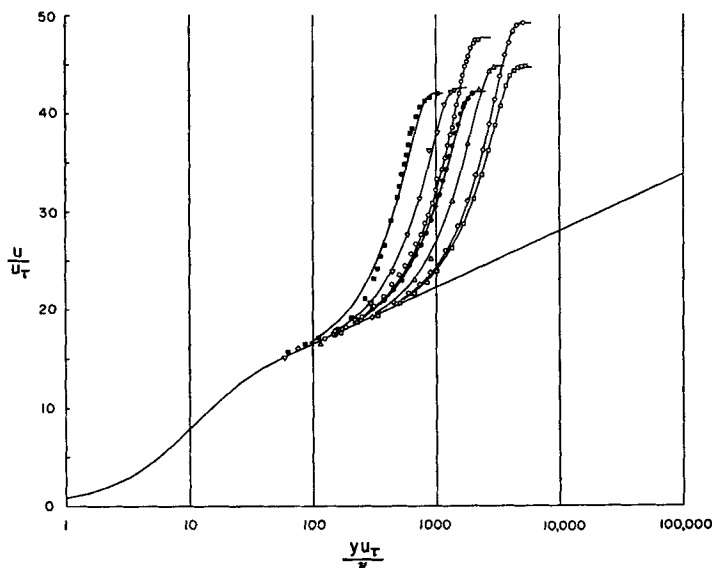


Figure 21. Two-parameter similarity in the mean-velocity profile.

Taking these properties together, it is something of an anticlimax to discover that the puzzle of the defect law is apparently a simple one. The key lies not in a study of the defect function F of equation (6), but in a study of the original function $g(\Pi, y/\delta)$ of equation (5), which gives the departure of the mean-velocity profile from the logarithmic law of the wall. For the three equilibrium flows of figure 20, this departure is shown in figure 2, using a linear scale for yu_τ/δ^*u_1 . There is a striking resemblance between the three curves, including a common anti-symmetry about a midpoint. But this resemblance is obviously not confined to equilibrium flows if, as has just been suggested in figure 21, an arbitrary profile in non-equilibrium flow coincides in the coordinate system ($u/u_\tau, yu_0/\nu$) with a profile from some member of the one-parameter family of equilibrium flows.

* Reading from left to right, the profiles are taken from figure 18 (Klebanoff & Diehl 1951, station 4.25), figure 16 (Clauser 1954, Series 2, station 108), figure 11 (Newman 1951, stations D and C), figure 16 (Clauser 1954, Series 2, station 230), and figure 14 (Ludwig & Tillmann 1949, Channel VI b, stations 3.73 and 3.53).

In view of these remarks and especially in view of figure 2, the mean-velocity profile may tentatively be written in the form already presented as equation (7),

$$\frac{u}{u_\tau} = f\left(\frac{yu_\tau}{\nu}\right) + \frac{\Pi(x)}{\kappa} w\left(\frac{y}{\delta}\right),$$

where the function $w(y/\delta)$, for reasons which will become apparent, will be called the law of the wake. If Π does not depend on x , then both $g(\Pi, y/\delta)$ in equation (5) and $\Pi w(y/\delta)$ in equation (7) are functions of y/δ only. This is the property assigned to equilibrium flows by Clauser. The present formulation of the mean-velocity profile is, however, more general than equation (5), in the sense that the law of the wake, although it occurs as a restricted form of (5) which is itself a special form of equation (4), is here assumed to apply for non-equilibrium flows.

2. *Normalizing conditions.* In order to test the hypothesis of a universal wake function in equation (7), it is necessary first to define the thickness δ and to specify some normalizing factor for w . To this end, the displacement thickness δ^* may be computed from the definition, equation (12), for the particular profile given by (7). Neglecting the departure of the flow in the sublayer from the logarithmic wall law, we obtain

$$\delta^* = \delta \frac{u_\tau}{u_1} \left(\frac{1}{\kappa} + \frac{\Pi}{\kappa} \int_0^{w_1} \frac{y}{\delta} dw \right),$$

where w_1 is tentatively defined as the maximum value of w . It is therefore convenient to take as a first normalizing condition

$$\int_0^{w_1} \frac{y}{\delta} dw = 1. \quad (16)$$

A second normalizing condition is suggested by the nearly anti-symmetric form of the curves in figure 2. The maximum value of w will occur very nearly at $y/\delta = 1$, provided that

$$w_1 = w(1) = 2. \quad (17)$$

Now $w(y/\delta)$ is by hypothesis a universal function, so that the boundary-layer thickness δ is uniquely defined in terms of δ^* by the integral condition (16) and the maximum condition (17). That is, the two relationships (8) and (9), together with the identity

$$\left(\frac{\delta^*}{\delta}\right)\left(\frac{u_1}{u_\tau}\right) = \left(\frac{\delta^* u_1}{\nu}\right) / \left(\frac{\delta u_\tau}{\nu}\right),$$

are sufficient to determine all five of the dimensionless parameters u_1/u_τ , δ^*/δ , $\delta u_\tau/\nu$, $\delta^* u_1/\nu$, and Π (or $\delta^* u_1/\delta u_\tau$) when any two are given. For example, if the two known quantities are u_1/u_τ and $\delta^* u_1/\nu$, as is supposedly the case for the data cited in Part II of this study, then equations (8) and (9) lead to a simple transcendental equation for Π ;

$$2\Pi - \ln(1 + \Pi) = \kappa \frac{u_1}{u_\tau} - \ln \frac{\delta^* u_1}{\nu} - \kappa c - \ln \kappa. \quad (18)$$

3. *Profile at separation.* Equation (7) as originally written is not in a convenient form for use near a point of separation or reattachment. However, on multiplication by u_τ/u_1 and the elimination of Π by means of (9), equation (7) leads to

$$\frac{u}{u_1} = \frac{u_\tau}{u_1} f\left(\frac{\gamma u_\tau}{\nu}\right) + \left(\frac{\delta^*}{\delta} - \frac{1}{\kappa} \frac{u_\tau}{u_1}\right) w\left(\frac{y}{\delta}\right).$$

If u_τ is put equal to zero in this expression, the result is evidently

$$\frac{u}{u_1} = \frac{\delta^*}{\delta} w\left(\frac{y}{\delta}\right) = \frac{1}{2} w\left(\frac{y}{\delta}\right).$$

Moreover, certain numerical values can be assigned in advance to the ratios δ^*/δ , θ/δ and δ^*/θ at separation or reattachment. For example, the definition of δ adopted here takes account of the nearly anti-symmetric variation of w by requiring that $\delta^*/\delta = \frac{1}{2}$ at separation. A corollary, anticipating numerical values of the next section, is $\theta/\delta = 0.12$ (approximately). On the other hand, the prediction $\delta^*/\theta = 4.2$ (approximately) at separation or reattachment is a result which is relatively free of preconceptions about the form of the wake function w . It should also be noted that this prediction is based on examination of the mean-velocity profile in flows which need not be close to separation.

Table 1. The wake function $w(\zeta)$ and related functions.

ζ	$w(\zeta)$	$\int_0^w \zeta dw$	$\omega_1(\Pi, \zeta)$	$\omega_2(\Pi, \zeta)$
0	0	0	$1+0.000\Pi$	$1+0.000\Pi+0.000\Pi^2$
0.05	0.004	0	$1+0.002\Pi$	
0.10	0.029	0.002	$1+0.022\Pi$	$1+0.027\Pi+0.000\Pi^2$
0.15	0.084	0.009	$1+0.062\Pi$	
0.20	0.168	0.024	$1+0.119\Pi$	$1+0.154\Pi+0.008\Pi^2$
0.25	0.272	0.047	$1+0.190\Pi$	
0.30	0.396	0.082	$1+0.272\Pi$	$1+0.360\Pi+0.044\Pi^2$
0.35	0.535	0.127	$1+0.363\Pi$	
0.40	0.685	0.183	$1+0.458\Pi$	$1+0.614\Pi+0.127\Pi^2$
0.45	0.838	0.248	$1+0.552\Pi$	
0.50	0.994	0.322	$1+0.645\Pi$	$1+0.880\Pi+0.257\Pi^2$
0.55	1.152	0.405	$1+0.737\Pi$	
0.60	1.307	0.495	$1+0.824\Pi$	$1+1.143\Pi+0.426\Pi^2$
0.65	1.458	0.589	$1+0.906\Pi$	
0.70	1.600	0.685	$1+0.978\Pi$	$1+1.380\Pi+0.610\Pi^2$
0.75	1.729	0.778	$1+1.037\Pi$	
0.80	1.840	0.863	$1+1.079\Pi$	$1+1.561\Pi+0.765\Pi^2$
0.85	1.926	0.935	$1+1.100\Pi$	
0.90	1.980	0.981	$1+1.090\Pi$	$1+1.640\Pi+0.823\Pi^2$
0.95	1.999	0.999	$1+1.051\Pi$	
1.00	2.000	1.000	$1+1.000\Pi$	$1+1.600\Pi+0.761\Pi^2$

4. *Test of the hypothesis.* For obvious reasons, the existence and form of the hypothetical wake function $w(y/\delta)$ are most readily investigated in flows with a large wake component. Figure 2 shows the function in question,

subject to the normalizing conditions (16) and (17), for several mean-velocity profiles taken from the present survey. The difference in apparent scatter between the non-equilibrium and equilibrium data is the result of applying the normalizing conditions to individual profiles in the first case, but to the average of several profiles in the second. For unseparated flows at least, the wake hypothesis appears to be a useful concept, and a tentative determination of the wake function $w(y/\delta)$ is therefore tabulated in table 1 and plotted in figures 2 and 21.

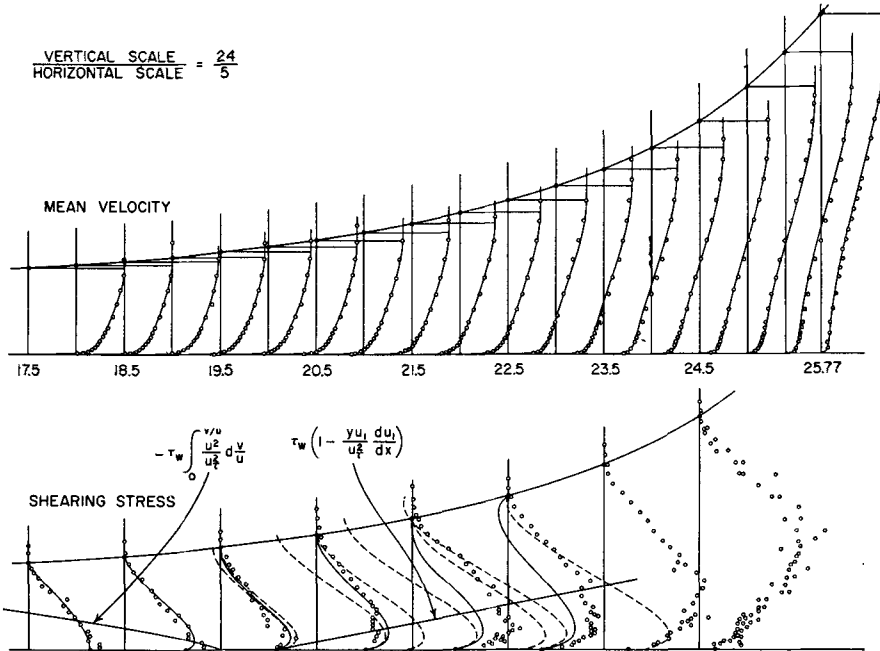


Figure 22. The separating flow of Schubauer and Klebanoff (1950) (measured values of shearing stress have been reduced by 31 percent).

The two-component profile, equation (7), can be made to represent quite well the data of Schubauer & Klebanoff, as shown in figure 22; of Newman, as shown in figure 23; of Kehl; of Clauser; of Ludwig & Tillmann; of McCullough & Gault at 5° angle of attack, as shown in figure 24; and of Tillmann for reattaching flow, as shown in figure 25. The solid lines in these figures are computed from equation (7), using values for u_1 , u_r , δ , and Π which vary smoothly with x . Occasionally it has been found desirable to make slight revisions in the original values for the profile parameters obtained from consideration of the law of the wall alone.

A few exceptions to the law of the wake can be found among the numerous profiles presented here. For example, it is impossible to find satisfactory values for the parameters u_1 , u_r , δ , and Π in the general formula (7) such that the profile at the left in figure 21 can be represented within the apparent

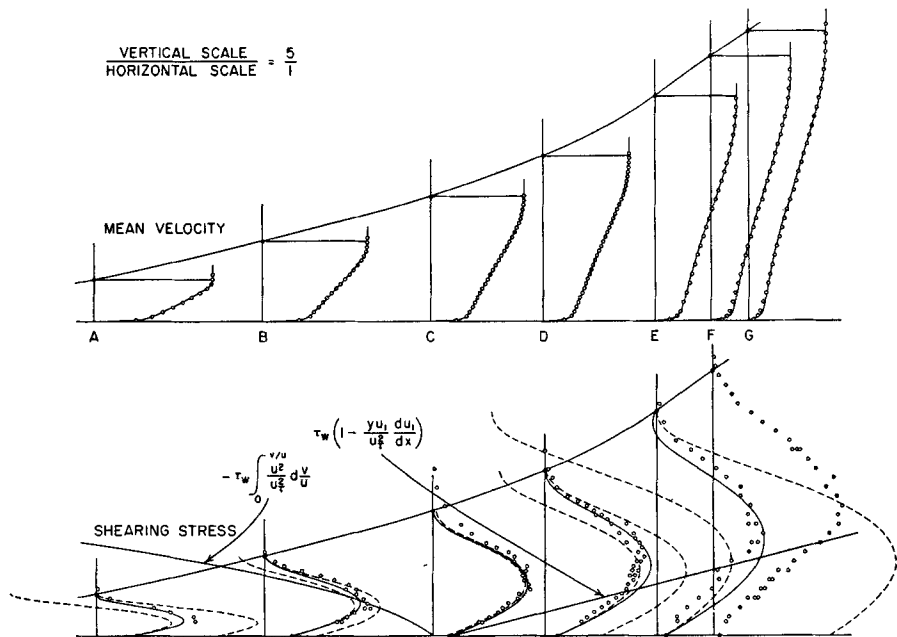


Figure 23. The separating flow of Newman (1951).

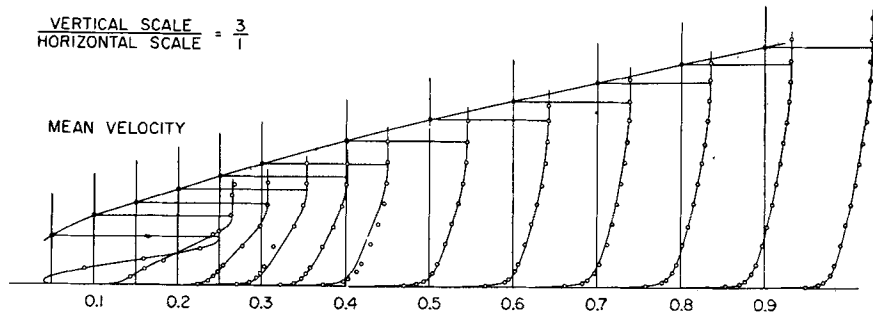


Figure 24. The reattaching flow of McCullough and Gault (1949).

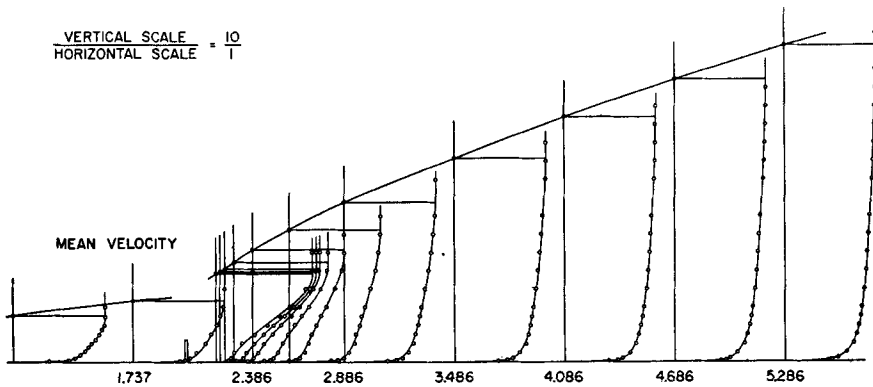


Figure 25. The reattaching flow of Tillmann (1945).

experimental accuracy. This profile, however, was obtained about two boundary-layer thicknesses downstream of reattachment, in a region where the real accuracy of the profile measurements is unknown.

Finally, it may be remarked that Wieghardt's experiments in figure 5 indicate that there is a definite change in the shape as well as the amplitude of the wake component in flow at constant pressure when the free-stream turbulence level is increased. The term 'universal function' applied to the law of the wake therefore implies that the external turbulence level is low, much as the same term applied to the law of the wall implies negligible surface roughness.

IV. DISCUSSION

A. The equations of mean motion

1. *The turbulent shearing stress.* In the past an important aim of phenomenological theories of turbulent shear flow has usually been to develop *a priori* some relationship connecting shearing stress and mean velocity, by analogy with the relationship $\tau = \mu \partial u / \partial y$ for laminar flow. It is now generally accepted that such a relationship probably does not exist in any practical sense. However, the same purpose is ultimately served by the equations of mean motion, which provide a valid relationship between shearing stress and mean velocity. In this respect the present analytic representation (7) of the mean-velocity profile may be exploited immediately.

The continuity equation may first be satisfied by introducing a stream function $\psi(x, y)$ such that $u = \partial \psi / \partial y$ and $v = -\partial \psi / \partial x$; ψ is then constant on streamlines of the mean flow. But, if the departure of the profile in the sublayer from the logarithmic law of the wall is neglected, equation (7) requires that

$$\frac{\psi}{v} = \frac{1}{v} \int_0^y u \, dy = \frac{yu_\tau}{v} \left(\frac{u}{u_\tau} - \frac{1}{\kappa} \right) - \frac{\Pi}{\kappa} \frac{\delta u_\tau}{v} \int_0^w \frac{y}{\delta} \, dw. \quad (19)$$

A formula for the normal component of velocity may be obtained from $v = -\partial \psi / \partial x$, using equation (19) for $\psi(x, y)$, or directly from the continuity equation $\partial u / \partial x + \partial v / \partial y = 0$, using equation (7) for $u(x, y)$. In either case it is found that

$$\frac{v}{u} = -\frac{y}{u_\tau} \frac{du_\tau}{dx} - \frac{yu_\tau}{\kappa} \frac{w}{u} \frac{d\Pi}{dx} + \frac{1}{\kappa u} \frac{d(\Pi \delta u_\tau)}{dx} \int_0^w \frac{y}{\delta} \, dw. \quad (20)$$

This expression is an exact consequence of equation (7), in the sublayer as elsewhere. Its application in the calculation of the shearing-stress profile is most easily shown by putting $-\partial v / \partial y$ for $\partial u / \partial x$ in the boundary-layer momentum equation

$$\tau = \tau_w - y \rho u_1 \frac{du_1}{dx} + \rho \int_0^y \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) dy,$$

thereby obtaining

$$\frac{\tau}{\tau_w} = 1 - \frac{yu_1}{u_\tau^2} \frac{du_1}{dx} - \int_0^{v/u} \frac{u^2}{u_\tau^2} d\left(\frac{v}{u}\right). \quad (21)$$

2. *Flow at constant pressure.* For flow at constant pressure, i.e. for Π and u_1 constant, the expression (21) will be evaluated explicitly for the profile given by equation (7). Making the usual approximation in the sublayer, the result can be written in the form

$$\kappa^2 \left(1 - \frac{\tau}{\tau_w}\right) = -\frac{y}{u_\tau} \frac{du_\tau}{dx} \left[\left(\kappa \frac{u}{u_\tau}\right)^2 - 2 \left(\kappa \frac{u}{u_\tau}\right) \omega_1 + \left(\kappa \frac{u}{u_\tau}\right) \left(\kappa \frac{u_1}{u_\tau}\right) (\omega_1 - 1) - 2 \left(\kappa \frac{u_1}{u_\tau}\right) (\omega_2 - \omega_1) + 2\omega_2 \right], \quad (22)$$

where ω_1 and ω_2 are auxiliary functions which depend on an argument $\zeta = y/\delta$ and a parameter Π ; that is,

$$\omega_1(\Pi, \zeta) = 1 + \frac{\Pi}{\zeta} \int_0^{w(\zeta)} \zeta' dw',$$

$$\omega_2(\Pi, \zeta) = \omega_1 + \frac{\Pi}{\zeta} \int_0^{w(\zeta)} \zeta' \ln \frac{\zeta}{\zeta'} dw' + \frac{\Pi^2}{\zeta} \int_0^{w(\zeta)} \zeta' (w - w') dw',$$

where $w(\zeta)$ is the wake function. The quantities $\omega_1(\Pi, \zeta)$ and $\omega_2(\Pi, \zeta)$ are tabulated together with w in table 1.

Finally, the derivative du_τ/dx in equation (22) may be disposed of by putting $y = \delta$ to obtain*

$$\kappa^2 = -\frac{\delta}{u_\tau} \frac{du_\tau}{dx} \left[\left(\kappa \frac{u_1}{u_\tau}\right)^2 \Omega_1 - 2 \left(\kappa \frac{u_1}{u_\tau}\right) \Omega_2 + 2\Omega_2 \right]$$

with

$$\Omega_1(\Pi) = \omega_1(\Pi, 1) = 1 + \Pi$$

and

$$\Omega_2(\Pi) = \omega_2(\Pi, 1) = 1 + 1.600\Pi + 0.761\Pi^2.$$

Klebanoff (1954) has recently measured the turbulent shearing-stress distribution in a boundary layer with constant pressure such that

$$\delta^* u_1/\nu = 9,700,$$

$$\delta^* = 0.400 \text{ in.}$$

The experimental mean-velocity profile for this flow has already been shown in figure 1. However, in order to make the present calculation independent of Klebanoff's measurements, it is convenient to estimate the parameter Π from the data of Wieghardt in figure 4. Using either equation (7) and the observed maximum excursion from the logarithmic law, or equation (18) and the experimental values for u_1/u_τ and $\delta^* u_1/\nu$, it is found that Π is very nearly 0.55 for flow at constant pressure. Then, from equations (9) and (8), with $\delta^* u_1/\nu = 9,700$ and $\Pi = 0.55$,

$$\delta u_\tau/\nu = 2,500, \quad u_1/u_\tau = 27.4, \quad \delta^*/\delta = 0.142.$$

Furthermore, with $\delta^* = 0.400 \text{ in.}$, $\delta = 2.83 \text{ in.}$

The profile u/u_τ computed as a function of $y u_\tau/\nu = (\delta u_\tau/\nu)(y/\delta)$ from equation (7) is plotted in figure 1. The excellent agreement with Klebanoff's

* This expression, like equation (37) of a previous paper (Coles 1954), is a special form of the momentum-integral equation (11), and may be used to compute a length Reynolds number. The present definition of the thickness δ , however, requires the numerical constants $\phi(1) = 7.90$, $C_1 = 4.05$, and $C_2 = 29.0$ of the cited paper to be replaced (anticipating the value $\Pi = 0.55$) by $c + 2\Pi/\kappa = 7.85$, $\Omega_1/\kappa = 3.88$, and $2\Omega_2/\kappa^2 = 26.4$.

measurements is to be expected in view of the efforts made at the National Bureau of Standards to insure that the flow in question would be typical of the fully-developed boundary layer.

The profile $\tau/q = (\tau/\tau_w)(2u_\tau^2/u_1^2) = -2\overline{u'v'}/u^2$ computed from equation (22) is plotted against y/δ in figure 26, together with measured data using the present estimate for δ of 2.83 in. Finally, to establish the extent of the region of intermittent turbulence in terms of the coordinate y/δ occurring in the law of the wake, figure 26 also shows Klebanoff's measurements of the intermittency factor γ , defined as the fraction of the time that the flow is turbulent. The mean position of the turbulent boundary, i.e. the point $\gamma = \frac{1}{2}$, is found to be at $y/\delta = 0.825$, with a standard deviation of 0.148 about the mean.

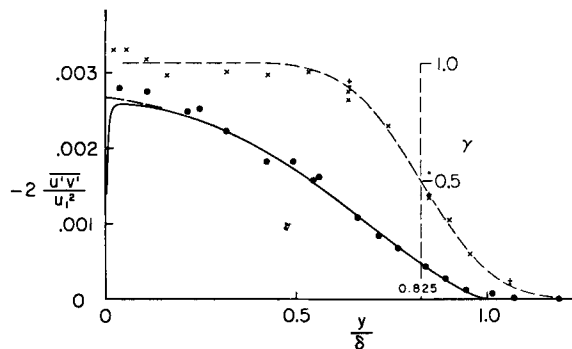


Figure 26. Shearing-stress profile in the flow of Klebanoff (1954).

3. *Flow approaching separation.* Schubauer & Klebanoff (1950) and Newman (1951) have recently carried out hot-wire measurements of turbulent shearing-stress in flows approaching separation. The representation of the mean-velocity field by the general profile equation (7) has already been illustrated in figures 22 and 23 for the flows in question. The corresponding shearing stress profiles, computed with the aid of equations (20) and (21), are shown in the same figures together with the profiles determined experimentally. As noted in the figure, the values of $\overline{u'v'}$ reported by Schubauer & Klebanoff have been reduced by 31% in view of the excessively large values obtained for τ_w when $\tau(y)$ is extrapolated to $y = 0$. It is likely that the hot-wire data are in fact too high, most probably as a result of over-compensation.

Unfortunately, these calculations of shearing stress can be attempted only for regions in which the mean-velocity field is in reasonable agreement with the momentum-integral condition expressed by equation (11). Elsewhere in the flow it is found to be possible to satisfy only one of the two boundary conditions $\tau = 0$ at $y = \delta$ or $\tau = \rho u_\tau^2$ at $y = 0$, u_τ being derived from a fit to the law of the wall. The alternative calculations are shown in figures 22 and 23 by the dashed lines. Because equation (21) is precisely

equivalent to equation (11), the values obtained for τ_w on putting $\tau = 0$ at $y = \delta$ are obviously the same as the values implied by the slope of the function $\Phi(x)$ which is defined by equation (14) and is denoted by the open points in figures 10 and 11 for the data of Schubauer & Klebanoff and of Newman respectively. The discrepancies encountered in τ_w appear to be too large to be caused entirely by the neglect in equation (21) of the Reynolds normal stresses and the pressure variation $\partial p/\partial y$, although this question is still open.

Another explanation for the failure of the two-dimensional momentum-integral equation (11) has been suggested on experimental grounds by several writers. The displacement of fluid in a boundary layer is assumed to be described by the streamline slope v/u , obtained by integrating the continuity equation $\partial u/\partial x + \partial v/\partial y = 0$. This displacement will certainly be affected by lateral convergence or divergence of the general flow, although the presence of three-dimensional effects would not necessarily be revealed by a study of the mean-velocity profile $u(y)$ measured at various lateral stations.

If τ_w is known but the values taken for v/u are even slightly in error, the right-hand side of equation (21) will not vanish at the outer edge of the boundary layer. Conversely, large errors in τ_w may be encountered on putting $\tau = 0$ at $y = \delta$. At one station in each of figures 22 and 23, the quantities

$$1 - \frac{yu_1}{u_\tau^2} \frac{du_1}{dx} \quad \text{and} \quad - \int_0^{v/u} \frac{u^2}{u_\tau^2} d\left(\frac{v}{u}\right)$$

are plotted separately in order to illustrate the extreme sensitivity of their sum τ/τ_w to small variations in v/u or in du_1/dx . Finally, as a tentative correction for three-dimensional flow the streamline slopes v/u have been multiplied by suitable constant factors close to unity in order to satisfy both boundary conditions on τ . The corrected shearing stress profiles are shown by the solid lines in figures 22 and 23.

B. Physical interpretation

1. *Two-dimensional flow.* In this paper the term 'wake function' has consistently been used to denote the function $w(y/\delta)$ in equation (7). The reason for this choice of terminology can be found in some measurements by Liepmann & Laufer (1947) of a plane half-wake or half-jet; that is, in the wedge-shaped region of turbulent mixing between a uniform flow and a fluid at rest.

The dimensionless mean-velocity profiles for the fully developed half-wake were originally reported by Liepmann & Laufer to satisfy a similarity law corresponding to a linear growth of the shear flow. These same profiles, after a further translation and change of scale for the coordinates,* are compared in figure 2 with the wake function w for flow

* The present variable $\zeta = y/\delta$ is related to the original variable $\sigma y/x$ in figure 13 of Liepmann & Laufer's paper by the expression $\zeta = 0.505 + 0.331\sigma y/x$. When $\zeta = 0$, $\sigma y/x = -1.52$; and when $\zeta = 1$, $\sigma y/x = 1.50$. Then $\sigma\delta/x = 3.02$, and it follows that, if $\sigma = 12$, $\delta/x = 0.252$ rads or 14.4° .

in a boundary layer. The residual velocity near $y/\delta = 0$ in the figure is the normal component of velocity associated with fluid entrainment, and agrees in magnitude with the value $v/u_1 = 0.03$ implied at $y/\delta = 0$ by the equations of mean motion for the half-wake.

Although the motion represented by the upper curve in figure 2 is bounded on the low-velocity side by fluid at rest, rather than by a solid wall, there can be little doubt that the similarity between the various experimental curves in the figure is more than accidental, and that substantially the same physical phenomenon is involved. The interpretation of the wake component in a boundary layer as a large-scale mixing process has already been mentioned in Part I in connection with figure 3.

It goes without saying that if the streamwise mean-velocity distribution in a turbulent boundary layer can in fact be expressed as a linear combination of wall and wake components, as in equation (7), then so can the normal mean-velocity component v , the mean stream function ψ , and the displacement thickness δ^* , all of which are obtained from $u(x, y)$ by linear operations. This is not so, however, for the flow inclination v/u , the momentum thickness θ , or the shearing stress τ ; and it is most emphatically not so for the turbulent fluctuations, except in so far as the wake and wall components might be expected to contribute more strongly to the small and large wave number regions of the spectrum respectively. On the other hand, a foundation has been laid in the present paper for the comparison, at corresponding points in various free and bound shear layers, of measurements of intermittency factor as well as of spectra and intensity of various fluctuating quantities.

Finally, in the event that the parameter Π is constant in a turbulent boundary layer, a certain balance is implied between the constraints imposed by inertia and by viscosity, and between the large-scale and small-scale mixing processes. This balance is precisely measured in equation (9) by the magnitude of the parameter Π or alternatively of the combination $\delta^*u_1/\delta u_\tau$. From the point of view adopted in these paragraphs, therefore, Clauser's choice of the term 'equilibrium flow' to describe the situation when Π is constant may well be regarded as inspired.

It should be noted that Lees & Crocco (1952) have recently attempted a qualitative analysis of turbulent shear flows in which they visualize a continuous spectrum of mixing processes having both wake-like and boundary-layer-like properties. The concept of two flow components, one depending on friction and the other on the cumulative effect of pressure gradient, has also been independently advanced by Ross & Robertson (1951) and by Rotta (1950), both of whom used a term linear in y to represent what is called here the wake function $w(y/\delta)$. These authors did not give any interpretation for either of the mean-velocity functions, and were therefore able to recommend their formulation of the problem only as a useful engineering approach, although Rotta obtained relationships which anticipate the present work. As a matter of historical interest, it should also be noted that a much earlier attempt by Millikan (1938) to

examine the function denoted by $h(x, y)$ in equation (4) was unsuccessful because the experimental data studied were not sufficiently precise. These efforts have certainly contributed, if not to the specific concept called the law of the wake, at least to the atmosphere in which this concept was evolved.

2. *Yawed flow.* The present interpretation of the wake and wall components in a turbulent boundary layer, as manifestations of the constraints provided by inertia and viscosity respectively, is not necessarily restricted to two-dimensional flows, provided the two flow components are viewed as vector rather than scalar functions of position. Specifically, suppose that the general profile equation (7) is rewritten in the form

$$\mathbf{q} = \mathbf{q}_f + \mathbf{q}_w,$$

where

$$\mathbf{q}_f = \mathbf{q}_\tau f\left(\frac{y q_\tau}{\nu}\right),$$

and

$$\mathbf{q}_w = \frac{\Pi \mathbf{q}_\tau}{\kappa} w\left(\frac{y}{\delta}\right).$$

In these expressions, $f(y q_\tau / \nu)$ and $w(y / \delta)$ are to be identified with the scalar functions previously described for two-dimensional flow, and q_τ is defined as the magnitude of a friction velocity vector \mathbf{q}_τ taken parallel to the surface shearing stress $\boldsymbol{\tau}_w$, i.e.

$$\boldsymbol{\tau}_w = \rho q_\tau \mathbf{q}_\tau.$$

Furthermore, both \mathbf{q}_τ and Π are assumed to depend on two space coordinates, say x and z . Then, if \mathbf{q}_f and \mathbf{q}_w are not parallel vectors, the parameter $\Pi(x, z)$ should presumably be interpreted as a linear operator having the properties of a square matrix, i.e. as a tensor. It also follows that the generalized vector friction law is

$$\mathbf{q}_1 = \mathbf{q}_\tau f\left(\frac{\delta q_\tau}{\nu}\right) + \frac{2\Pi \mathbf{q}_\tau}{\kappa}.$$

This notation is highly tentative, and may have to be revised after more data on yawed flows become available. In particular, the present definition for q_τ is not perfectly consistent with the streamline hypothesis (Coles 1955) if $\boldsymbol{\tau}_w$ is an arbitrary continuous function of two coordinates on the surface. However, the notation does allow the interpretation already proposed, that the flow near the surface where the wake component is small should have the same direction and sense as the surface shearing stress. Thus the concept of a constraint provided by friction is a vector concept.

Given a mean-velocity profile in yawed flow, the vector nature of the wall and wake components can be tested in five steps. First, the direction of the mean flow near the surface, which is also by assumption the direction of the shearing-stress vector $\boldsymbol{\tau}_w$ and of the wall component \mathbf{q}_f , is noted. Second, the component of mean velocity in this direction is plotted in coordinates $(\mathbf{q} \cdot \mathbf{q}_\tau / q_\tau^2, y q_\tau / \nu)$ appropriate to the law of the wall; a fit to the function f then yields a value for q_τ . Third, the thickness δ is estimated

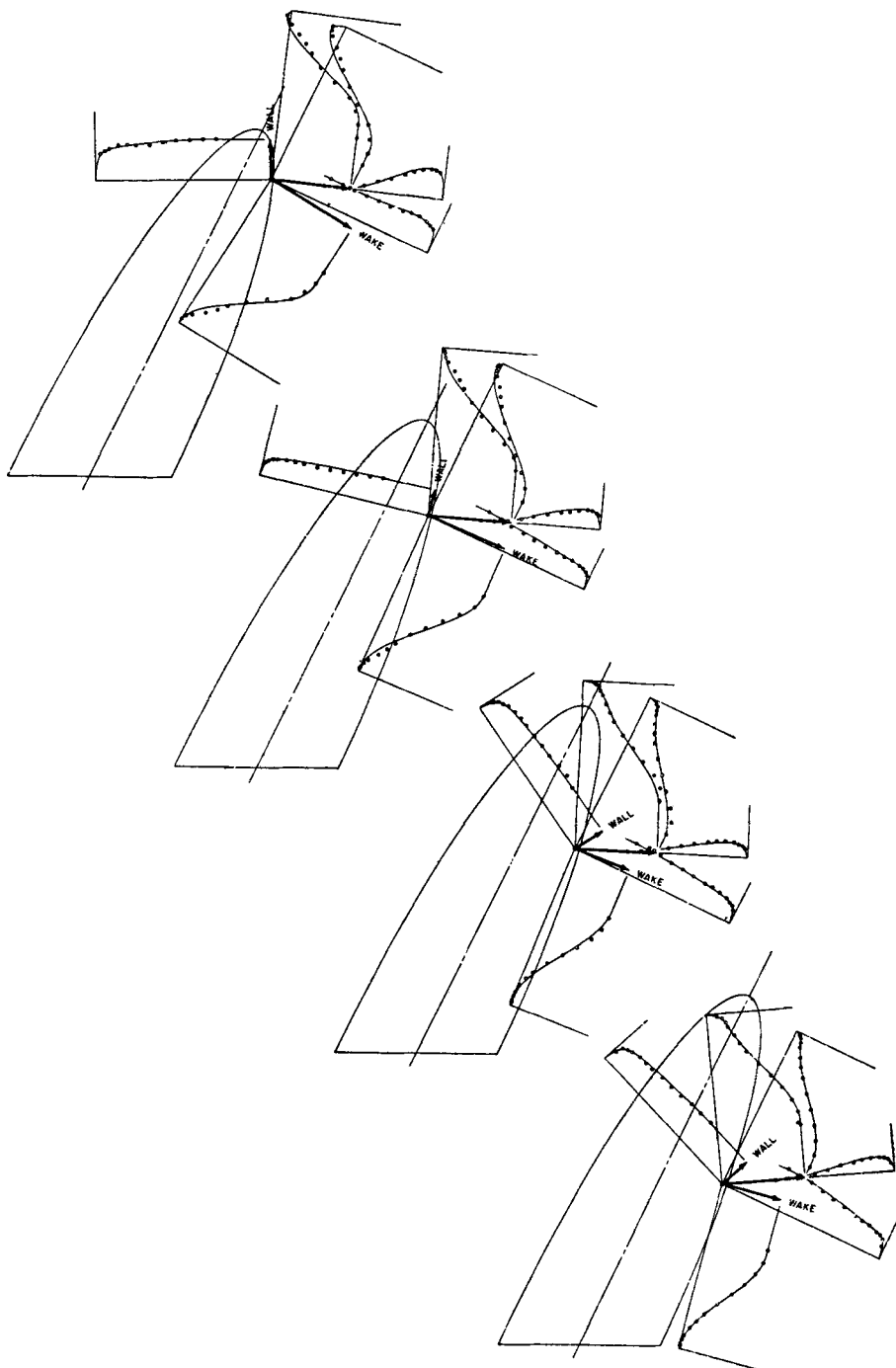


Figure 27. Vector resolution of the yawed flow of Kuethe, McKee, & Curry (1949).

from this same plot, for example as twice the value of y for which the profile reaches half of its maximum excursion from the law of the wall. Fourth, the 'wall' vector $\mathbf{q}_f = \mathbf{q}_r f(\delta q_r/\nu)$ is computed and subtracted from the free-stream vector \mathbf{q}_i to obtain the 'wake' vector $\mathbf{q}_w = 2\Pi\mathbf{q}_r/\kappa$. Finally, the profile is resolved in oblique coordinates determined by the direction of the wall and wake vectors.

An examination of the experimental data of Gruschwitz (1935) in a curved channel suggests that serious errors were introduced in the mean-velocity measurements near the surface by the use of a periscope probe. In particular, the profiles in the straight portion of the channel differ from the consensus of data obtained by other investigators under similar conditions. Fortunately, the hot-wire measurements of Kuethe, McKee & Curry (1949) on a swept airfoil, although carried out at relatively small Reynolds number, involve large angles of yaw within the boundary layer and therefore provide a useful test for the concept of vector similarity. In treating these data, incidentally, it has been assumed that the indication in some of the profiles of a sudden change in flow direction within the sublayer is fictitious.

The airfoil of Kuethe, McKee & Curry was of elliptical planform, of 18 in. chord and 96.5 in. span, with the major axis swept back at an angle of 25° . Four profiles obtained near the trailing edge of the airfoil at an angle of attack of 14° are plotted in figure 27 in terms of spanwise and chordwise components of mean velocity; in terms of streamwise and crossflow components; and finally in terms of wall and wake components.

The data in figure 27 can be fairly well represented by the characteristic wall and wake functions defined previously for unyawed flows, and it is difficult to say whether or not there is any systematic discrepancy. A more stimulating result, which may be coincidental or may illustrate an important intrinsic property of strongly wake-like yawed flows, is that the direction of the wake component is found in each case to be nearly the same as the direction of the gradient of the pressure field over the airfoil surface. That is, the final resolution is for practical purposes along the directions defined by the two vectors $\boldsymbol{\tau}_w$ and $-\text{grad } p$, so that $(\text{grad } p) \times (\Pi\boldsymbol{\tau}_w)$ vanishes everywhere. The inference that the constraint provided by inertia is also a vector constraint is potentially useful in investigating the nature of the tensor parameter Π when more suitable data become available from experiments carried out at larger Reynolds numbers in flow on a larger scale.

REFERENCES

- BAUER, W. 1951 The development of the turbulent boundary layer on steep slopes, Thesis: State Univ. of Iowa; abridged in *Proc. Amer. Soc. Civ. Engrs.*, *Separate* no. 281, 1953.
- CLAUSER, F. 1954 Turbulent boundary layers in adverse pressure gradients, *J. Aero. Sci.* **21**, 91–108.
- COLES, D. 1954 The problem of the turbulent boundary layer, *Z. angew. Math. Phys.* **5**, 181–203.

- COLES, D. 1955 The law of the wall in turbulent shear flow, *50 Jahre Grenzschichtforschung* (Ed. H. Görtler & W. Tollmien), pp. 153–163. Braunschweig: F. Vieweg und Sohn.
- GRUSCHWITZ, E. 1935 Turbulente Reibungsschichten mit Sekundärströmung, *Ing.-Arch.* **6**, 355–365.
- KÁRMÁN, T. v. 1921 Über laminare und turbulente Reibung, *Z. angew. Math. Mech.* **1**, 233–252; translated as On laminar and turbulent friction, *Nat. Adv. Comm. Aero., Wash., Tech. Mem.* no. 1092, 1946.
- KÁRMÁN, T. v. 1932 Theorie des Reibungswiderstandes, *Hydromechanische Probleme des Schiffsantriebs, Hamburg*.
- KEHL, A. 1943 Untersuchungen über konvergente und divergente turbulente Reibungsschichten, *Ing.-Arch.* **13**, 293–329; translated as Investigations on convergent and divergent turbulent boundary layers, *British Ministry of Supply R.T.P.* no. 2035, 1946.
- KLEBANOFF, P. & DIEHL, Z. 1951 Some features of artificially thickened fully developed turbulent boundary layers with zero pressure gradient, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 2475.
- KLEBANOFF, P. 1954 Characteristics of turbulence in a boundary layer with zero pressure gradient, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 3178.
- KUETHE, A., MCKEE, P. & CURRY, W. 1949 Measurements in the boundary layer of a yawed wing, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 1946.
- LAUFER, J. 1953 The structure of turbulence in fully developed pipe flow, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 2954.
- LEES, L. & CROCCO, L. 1952 A mixing theory for the interaction between dissipative flows and nearly isentropic streams, *J. Aero. Sci.* **19**, 649–676.
- LIEPMANN, H. & LAUFER, J. 1947 Investigations of free turbulent mixing, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 1257.
- LUDWIG, H. & TILLMANN, W. 1949 Untersuchungen über die Wandschubspannung in turbulenten Reibungsschichten, *Ing.-Arch.* **17**, 288–299; translated as Investigations of the wall shearing stress in turbulent boundary layers, *Nat. Adv. Comm. Aero., Wash., Tech. Mem.* no. 1285, 1950.
- MCCULLOUGH, G. & GAULT, D. 1949 Boundary-layer and stalling characteristics of the N.A.C.A. 64A006 airfoil section, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 1923.
- MILLIKAN, C. 1938 A critical discussion of turbulent flows in channels and circular tubes, *Proc. 5th Int. Cong. Appl. Mech., Cambridge*, 386–392.
- NEWMAN, B. 1951 Some contributions to the study of the turbulent boundary layer near separation, *Aust. Dept. Supply, Rep.* no. ACA-53.
- NIKURADSE, J. 1930 Widerstandsgesetz und Geschwindigkeitsverteilung von turbulenten Wasserströmung in glatten und rauhen Röhren, *Proc. 3rd Int. Cong. Appl. Mech., Stockholm*, 239–248.
- PRANDTL, L. 1926 Über die ausgebildete Turbulenz, *Proc. 2nd Int. Cong. Appl. Mech., Zürich*, 62–74; translated as Turbulent flow, *Nat. Adv. Comm. Aero., Wash., Tech. Mem.* no. 435, 1927.
- PRESTON, J. 1954 The determination of turbulent skin friction by means of pitot tubes, *J. R. Aero. Soc.* **58**, 109–121.
- REICHARDT, H. 1940 Die Wärmeübertragung in turbulenten Reibungsschichten, *Z. angew. Math. Mech.* **20**, 297–328; translated as Heat transfer through turbulent friction layers, *Nat. Adv. Comm. Aero., Wash., Tech. Mem.* no. 1047, 1943.

- ROSS, D. & ROBERTSON, J. 1951 A superposition analysis of the turbulent boundary layer in an adverse pressure gradient, *J. Appl. Mech.* **18**, 95–100.
- ROTTA, J. 1950 Über die Theorie der turbulenten Grenzschichten, *Mitt. Max-Planck-Inst., Göttingen*, No. 1; *translated as* On the theory of the turbulent boundary layer, *Nat. Adv. Comm. Aero., Wash., Tech. Mem.* no. 1344, 1953.
- SCHUBAUER, G. & KLEBANOFF, P. 1950 Investigation of separation of the turbulent boundary layer, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 2133.
- SHEPPARD, P. 1947 The aerodynamic drag of the earth's surface and the value of von Kármán's constant in the lower atmosphere, *Proc. Roy. Soc. A*, **188**, 208–222.
- TILLMANN, W. 1945 Untersuchungen über Besonderheiten bei turbulenten Reibungsschichten an Platten, *Z.W.B., K.W.I., Göttingen, U. & M.* 6627; *translated as* Investigations of some particularities of turbulent boundary layers on plates *British R. & T. MAP-VG-34*, 1946.
- WIEGHARDT, K. 1943 Über die Wandschubspannung in turbulenten Reibungsschichten bei veränderlichem Aussendruck, *Z.W.B., K.W.I., Göttingen, U. & M.* 6603. *See also* Wieghardt, K. & Tillmann, W., Zur turbulenten Reibungsschicht bei Druckanstieg, *Z.W.B., K.W.I., Göttingen, U. & M.* 6617, 1944; *translated as* On the turbulent friction layer for rising pressure, *Nat. Adv. Comm. Aero., Wash., Tech. Mem.* no. 1314, 1951.
- WIEGHARDT, K. 1944 Zum Reibungswiderstand rauher Platten, *Z.W.B., K.W.I., Göttingen, U. & M.* 6612.