CK-12 Trigonometry

Second Edition

CK-12 Trigonometry - Second Edition, Solution Key

CK-12 Foundation

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CHAPTER 1 Right Triangles and an Introduction to Trigonometry, Solution Key

Chapter Outline

1.1 The Pythagorean Theorem

1. $6^2+9^2 ? 13^2 \rightarrow 36+81 ? 169 \rightarrow 117 < 169$ The triangle is obtuse.
2. $9^2+10^2 ? 11^2 \rightarrow 81+100 ? 121 \rightarrow 181 > 121$ The triangle is acute.
3. $16^2+30^2 ? 34^2 \rightarrow 256+900 ? 1156 \rightarrow 1156 = 1156$ This is a right triangle.
4. $20^2+23^2 ? 40^2 \rightarrow 400+529 ? 1600 \rightarrow 929 < 1600$ The triangle is obtuse.
5. These lengths cannot make up the sides of a triangle. $11+16 < 29$
6. $\left(2\sqrt{6}\right)^2+\left(6\sqrt{3}\right)^2 ? \left(2\sqrt{33}\right)^2 \rightarrow (4\cdot 6)+(36\cdot 3) ? (4\cdot 33) \rightarrow 24+108 ? 132 \rightarrow 132 = 132$ This is a right triangle.
7.

$$\begin{aligned} 7^2+x^2 &= 18^2 \\ 49+x^2 &= 324 \\ x^2 &= 275 \\ x &= \sqrt{275} = 5\sqrt{11} \end{aligned}$$

8.

$$\begin{aligned} 5^2+\left(5\sqrt{3}\right)^2 &= x^2 \\ 25+(25\cdot 3) &= x^2 \\ 25+75 &= x^2 \\ 100 &= x^2 \\ 10 &= x \end{aligned}$$

9. Both legs are 11.

$$\begin{aligned} 11^2+11^2 &= x^2 \\ 121+121 &= x^2 \\ 242 &= x^2 \\ \sqrt{242} &= x \\ 11\sqrt{2} &= x \end{aligned}$$

10. Plug $n^2-m^2, 2nm, n^2+m^2$ into the Pythagorean Theorem.

$$\begin{aligned} (n^2-m^2)^2+(2nm)^2 &= (n^2+m^2)^2 \\ n^4-2n^2m^2+m^4+4n^2m^2 &= n^4+2n^2m^2+m^4 \\ -2n^2m^2+4n^2m^2 &= 2n^2m^2 \\ 4n^2m^2 &= 4n^2m^2 \end{aligned}$$

11. (a) (5, -6) and (18, 3)

$$\begin{aligned} d &= \sqrt{(5-18)^2+(-6-3)^2} \\ &= \sqrt{(-13)^2+(-9)^2} \\ &= \sqrt{169+81} \\ &= \sqrt{250} \\ &= 5\sqrt{10} \end{aligned}$$

(b) $\left(\sqrt{3},\, -\sqrt{2}\right)$ and $\left(-2\sqrt{3},\, 5\sqrt{2}\right)$

$$\begin{aligned} d &= \sqrt{\left(\sqrt{3}-\left(-2\sqrt{3}\right)\right)^2+\left(-\sqrt{2}-5\sqrt{2}\right)^2} \\ &= \sqrt{\left(3\sqrt{3}\right)^2+\left(-6\sqrt{2}\right)^2} \\ &= \sqrt{(9\cdot 3)+(36\cdot 2)} \\ &= \sqrt{27+72} \\ &= \sqrt{99} = 3\sqrt{11} \end{aligned}$$

1.2 Special Right Triangles

1. Each leg is $\frac{16}{\sqrt{2}} = \frac{16}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$.
2. Short leg is $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2}$ and hypotenuse is $2\sqrt{2}$.
3. Short leg is $\frac{12}{\sqrt{3}} = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$ and hypotenuse is $8\sqrt{3}$.
4. The hypotenuse is $4\sqrt{10} \cdot \sqrt{2} = 4\sqrt{20} = 8\sqrt{5}$.
5. Each leg is $\frac{5\sqrt{2}}{\sqrt{2}} = 5$.
6. The short leg is $\frac{15}{2}$ and the long leg is $\frac{15\sqrt{3}}{2}$.
7. If the diagonal of a square is 6 ft, then each side of the square is $\frac{6}{\sqrt{2}}$ or $3\sqrt{2} \approx 4.24\ ft$.
8. These are not dimensions for a special right triangle, so to find the diagonal (both are the same length) do the Pythagorean Theorem:

$$\begin{aligned} 10^2 + 20^2 &= d^2 \\ 100 + 400 &= d^2 \\ \sqrt{500} &= d \\ 10\sqrt{5} &= d \end{aligned}$$

   So, if each diagonal is $10\sqrt{5}$, two diagonals would be $20\sqrt{5} \approx 45\ ft$. Pablo needs 45 ft of lights for his yard.
9. $2 : 2 : 2\sqrt{3}$ does not fit into either ratio, so it is not a special right triangle. To see if it is a right triangle, plug these values into the Pythagorean Theorem:

$$\begin{aligned} 2^2 + 2^2 &= \left(2\sqrt{3}\right)^2 \\ 4 + 4 &= 12 \\ 8 &< 12 \end{aligned}$$

   this is not a right triangle, it is an obtuse triangle.
10. $\sqrt{5} : \sqrt{15} : 2\sqrt{5}$ is a $30-60-90$ triangle. The long leg is $\sqrt{5} \cdot \sqrt{3} = \sqrt{15}$ and the hypotenuse is $2\sqrt{5}$.

1.3 Basic Trigonometric Functions

1. Answers:
   - $\sin A = \frac{9}{15} = \frac{3}{5}$
   - $\cos A = \frac{12}{15} = \frac{4}{5}$
   - $\tan A = \frac{9}{12} = \frac{3}{4}$
   - $\csc A = \frac{15}{9} = \frac{5}{3}$
   - $\sec A = \frac{15}{12} = \frac{5}{4}$
   - $\cot A = \frac{12}{9} = \frac{4}{3}$
2. The hypotenuse is 17 $\left(\sqrt{15^2+8^2} = \sqrt{225+64} = \sqrt{289} = 17\right)$.
   - $\sin T = \frac{15}{17}$
   - $\cos T = \frac{8}{17}$
   - $\tan T = \frac{15}{8}$
   - $\csc T = \frac{17}{15}$
   - $\sec T = \frac{17}{8}$
   - $\cot T = \frac{8}{15}$
3. Answers:
   a. The hypotenuse is 13 $\left(\sqrt{5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13\right)$.
   b. $\sin X = \frac{12}{13}, \cos X = \frac{5}{13}, \tan X = \frac{12}{5}, \csc X = \frac{13}{12}, \sec X = \frac{13}{5}, \cot X = \frac{5}{12}$
   c. $\sin Z = \frac{5}{13}, \cos Z = \frac{12}{13}, \tan Z = \frac{5}{12}, \csc Z = \frac{13}{5}, \sec Z = \frac{13}{12}, \cot Z = \frac{12}{5}$
4. From #3, we can conclude that:
   - $\sin X = \cos Z$
   - $\cos X = \sin Z$
   - $\tan X = \cot Z$
   - $\cot X = \tan Z$
   - $\csc X = \sec Z$
   - $\sec X = \csc Z$
   - Yes, this can be generalized for all complements.
5. The hypotenuse is $2\sqrt{2}$. Each angle is 45°, so the sine, cosine, and tangent are the same for both angles.
   - $\sin 45^\circ = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
   - $\cos 45^\circ = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
   - $\tan 45^\circ = \frac{2}{2} = 1$
6. If the legs are length x, then the hypotenuse is $x\sqrt{2}$. For 45°, the sine, cosine, and tangent are:
   - $\sin 45^\circ = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
   - $\cos 45^\circ = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
   - $\tan 45^\circ = \frac{x}{x} = 1$

   This tells us that regardless of the length of the sides of an isosceles right triangle, the sine, cosine and tangent of 45° are always the same.

7. If the hypotenuse is 10, then the short leg is 5 and the long leg is $5\sqrt{3}$. Recall, that 30° is opposite the short side, or 5, and 60° is opposite the long side, or $5\sqrt{3}$.
   - $\sin 30^\circ = \frac{5}{10} = \frac{1}{2}$
   - $\cos 30^\circ = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$
   - $\tan 30^\circ = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
   - $\sin 60^\circ = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$
   - $\cos 60^\circ = \frac{5}{10} = \frac{1}{2}$
   - $\tan 60^\circ = \frac{5\sqrt{3}}{5} = \sqrt{3}$
8. If the short leg is x, then the long leg is $x\sqrt{3}$ and the hypotenuse is $2x$. 30° is opposite the short side, or x, and 60° is opposite the long side, or $x\sqrt{3}$.
   - $\sin 30^\circ = \frac{x}{2x} = \frac{1}{2}$
   - $\cos 30^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$
   - $\tan 30^\circ = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
   - $\sin 60^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$
   - $\cos 60^\circ = \frac{x}{2x} = \frac{1}{2}$
   - $\tan 60^\circ = \frac{x\sqrt{3}}{x} = \sqrt{3}$

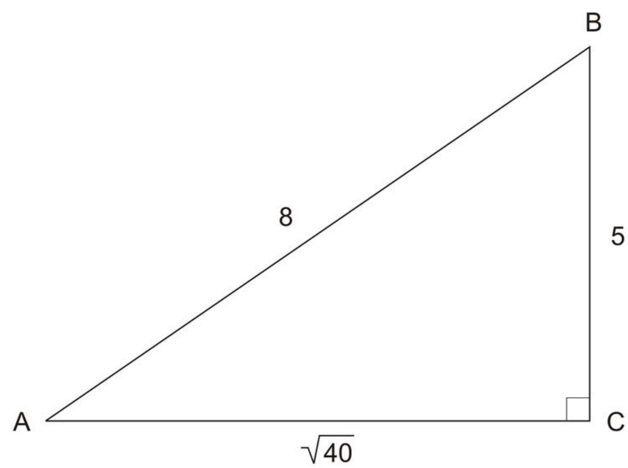
   This tells us that regardless of the length of the sides of a $30-60-90$ triangle, the sine, cosine and tangent of 30° and 60° are always the same. Also, $\sin 30^\circ = \cos 60^\circ$ and $\cos 30^\circ = \sin 60^\circ$.}}
9. If $\sin A = \frac{9}{41}$, then the opposite side is $9x$ (some multiple of 9) and the hypotenuse is $41x$. Therefore, working with the Pythagorean Theorem would give us the length of the other leg. Also, we could notice that this is a Pythagorean Triple and the other leg is $40x$.

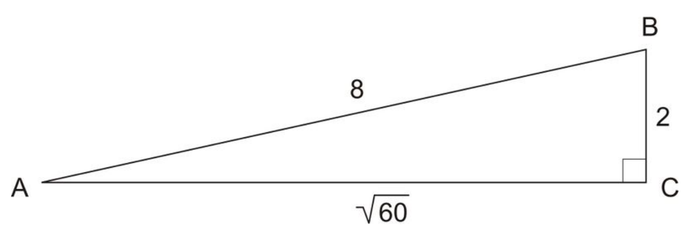
1.4 Solving Right Triangles

1.

$$\angle A = 50^\circ$$
$$b \approx 5.83$$
$$a \approx 9.33$$

2. Anna is correct. There is not enough information to solve the triangle. That is, there are infinitely many right triangles with hypotenuse 8. For example:



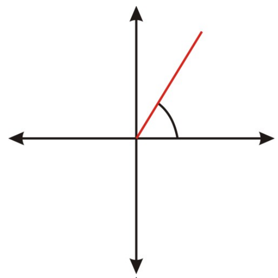


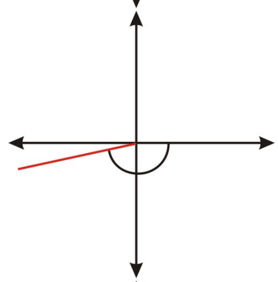
3. $6^2 + 5.03^2 = 36 + 25.3009 = 61.3009 = 7.83^2$.
4. $\angle B \approx 37^\circ$
5. $A = \frac{1}{2} \cdot 10 \cdot 12 \cdot \sin 104^\circ = 58.218$
6. $A = 4 \cdot 9 \cdot \sin 112^\circ = 33.379$
7. About 19.9 feet tall
8. About 120.3 feet
9. The plane has traveled about 203 miles. The two cities are 35 miles apart.
10. About 41.95 feet
11. About 7.44
12.

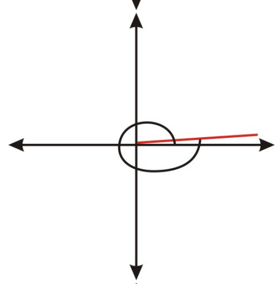
$$\tan\theta = \frac{opposite}{adjacent}$$
$$\tan\theta = 0.625$$
$$\theta = 32^\circ$$

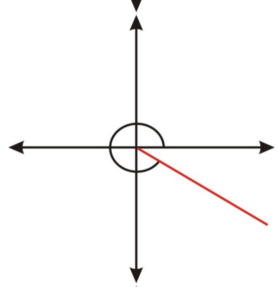
1.5 Measuring Rotation

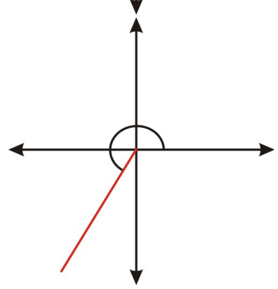
1. Answers:

a. 

b. 

c. 

d. 

e. 

2. Answers will vary. Examples: $450^\circ,\ -270^\circ$
3. Answers:

    a. Answers will vary. Examples: $-240^\circ,\ 480^\circ$
    b. Answers will vary. Examples: $-45^\circ,\ 675^\circ$
    c. Answers will vary. Examples: $210^\circ,\ -510^\circ,\ 570^\circ$

4. The front wheel rotates more because it has a smaller diameter. It rotates 200 revolutions versus 66.67 revolutions for the back wheel, which is a $48,000^\circ$ difference $((200-66.\bar{6})\cdot 360^\circ)$.

1.6 Applying Trig Functions to Angles of Rotation

1. The radius of the circle is 5.

$$\cos\theta = \frac{3}{5} \qquad \sec\theta = \frac{5}{3}$$
$$\sin\theta = \frac{-4}{5} \qquad \csc\theta = \frac{5}{-4}$$
$$\tan\theta = \frac{-4}{3} \qquad \cot\theta = \frac{3}{-4}$$

2. The radius of the circle is 13.

$$\cos\theta = \frac{-5}{13} \qquad \sec\theta = \frac{13}{-5}$$
$$\sin\theta = \frac{-12}{13} \qquad \csc\theta = \frac{13}{-12}$$
$$\tan\theta = \frac{-12}{-5} = \frac{12}{5} \qquad \cot\theta = \frac{-5}{-12} = \frac{5}{12}$$

3. If $\tan\theta = -\frac{2}{3}$, it must be in either Quadrant II or IV. Because $\cos\theta > 0$, we can eliminate Quadrant II. So, this means that the 3 is negative. (All Students Take Calculus) From the Pythagorean Theorem, we find the hypotenuse:

$$2^2 + (-3^2) = c^2$$
$$4 + 9 = c^2$$
$$13 = c^2$$
$$\sqrt{13} = c$$

Because we are in Quadrant IV, the sine is negative. So, $\sin\theta = -\frac{2}{\sqrt{13}}$ or $-\frac{2\sqrt{13}}{13}$ (Rationalize the denominator)

4. If $\csc\theta = -4$, then $\sin\theta = -\frac{1}{4}$, sine is negative, so θ is in either Quadrant III or IV. Because $\tan\theta > 0$, we can eliminate Quadrant IV, therefore θ is in Quadrant III. From the Pythagorean Theorem, we can find the other leg:

$$a^2 + (-1)^2 = 4^2$$
$$a^2 + 1 = 16$$
$$a^2 = 15$$
$$a = \sqrt{15}$$
$$\text{So, } \cos\theta = -\frac{\sqrt{15}}{4}, \sec\theta = -\frac{4}{\sqrt{15}} \text{ or } -\frac{4\sqrt{15}}{15}$$
$$\tan\theta = -\frac{1}{\sqrt{15}} \text{ or } \frac{\sqrt{15}}{15}, \cot\theta = \sqrt{15}$$

5. If the terminal side of θ is on (2, 6) it means θ is in Quadrant I, so sine, cosine and tangent are all positive.

From the Pythagorean Theorem, the hypotenuse is:

$$\begin{aligned} 2^2+6^2 &= c^2 \\ 4+36 &= c^2 \\ 40 &= c^2 \\ \sqrt{40} = 2\sqrt{10} &= c \end{aligned}$$

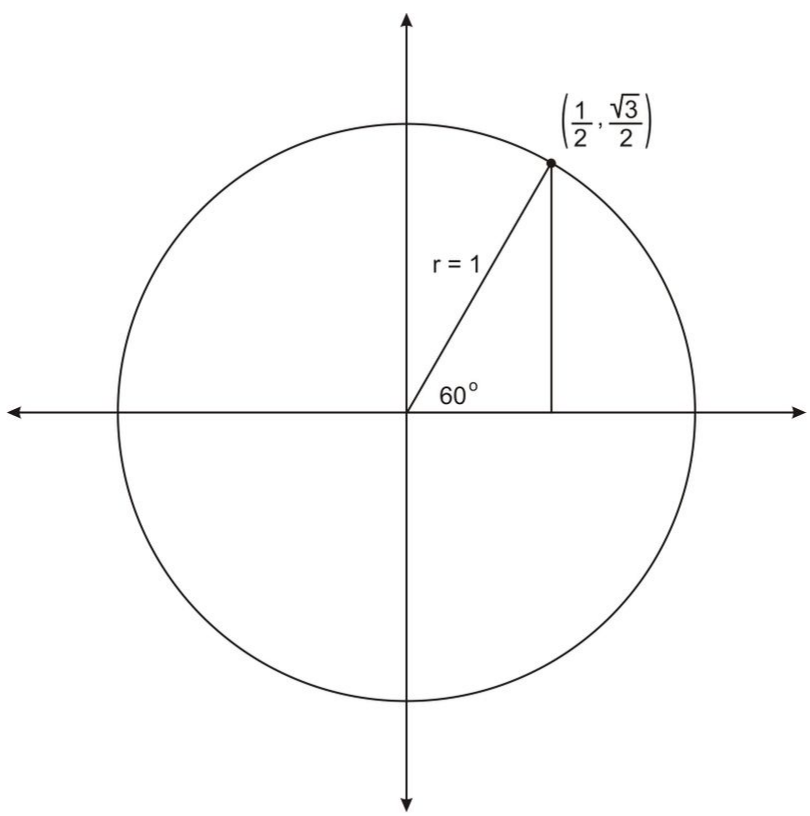
Therefore, $\sin\theta = \frac{6}{2\sqrt{10}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}, \cos\theta = \frac{2}{2\sqrt{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$ and $\tan\theta = \frac{6}{2} = 3$

6.

$\cos 270 = 0$	$\sec 270 = undefined$
$\sin 270 = -1$	$\csc 270 = \frac{1}{-1} = -1$
$\tan 270 = undefined$	$\cot 270 = 0$

7. Answers:

a. The triangle is equiangular because all three angles measure 60 degrees. Angle *DAB* measures 60 degrees because it is the sum of two 30 degree angles.
b. *BD* has length 1 because it is one side of an equiangular, and hence equilateral, triangle.
c. *BC* and *CD* each have length $\frac{1}{2}$, as they are each half of *BD*. This is the case because Triangle *ABC* and *ADC* are congruent.
d. We can use the Pythagorean theorem to show that the length of *AC* is $\frac{\sqrt{3}}{2}$. If we place angle *BAC* as an angle in standard position, then *AC* and *BC* correspond to the x and y coordinates where the terminal side of the angle intersects the unit circle. Therefore the ordered pair is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
e. If we draw the angle 60° in standard position, we will also obtain a $30-60-90$ triangle, but the side lengths will be interchanged. So the ordered pair for 60° is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



8.

$$\begin{aligned}
n^2+n^2 &= 1^2\\
2n^2 &= 1\\
n^2 &= \frac{1}{2}\\
n &= \pm\sqrt{\frac{1}{2}}\\
n &= \pm\frac{1}{\sqrt{2}}\\
n &= \pm\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}
\end{aligned}$$

Because the angle is in the first quadrant, the x and y coordinates are positive.

9. An angle in the first quadrant, as the tangent is the ratio of two positive numbers. And, angle in the third quadrant, as the tangent in the ratio of two negative numbers, which will be positive.
10. The terminal side of the angle is a reflection of the terminal side of 30°. From this, students should see that the ordered pair is $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$.
11. Students should notice that tangent is the ratio of $\frac{\sin}{\cos}$, which is $\frac{y}{x}$, which is also slope.

1.7 Trigonometric Functions of Any Angle

1. Answers:
   a. 10°
   b. 60°
   c. 30°
   d. 45°
2. Answers:
   a. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
   b. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
   c. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
3. Answers:
   a. $-\frac{1}{2}$
   b. 0
   c. $\frac{2}{\sqrt{3}}$
4. Answers:
   a. $\frac{1}{2}$
   b. $-\frac{\sqrt{3}}{2}$
   c. $\sqrt{2}$
5. Answers:
   a. $-\frac{\sqrt{3}}{2}$
   b. -1
   c. $\frac{\sqrt{3}}{2}$
6. Answers:
   a. 0.8828
   b. 1.4281
   c. -0.1736
7. About 11.54 degrees or about 168.46 degrees.
8. This is reasonable because $\tan 45^\circ = 1$ and the $\tan 60^\circ = \sqrt{3} \approx 1.732$, and the $\tan 50^\circ$ should fall between these two values.
9. Conjecture: $\sin a + \sin b \neq \sin(a+b)$
10. Answer:

TABLE 1.1:

a	$(\sin a)^2$	$(\cos a)^2$
0	0	1
25	.1786	.8216
45	$\frac{1}{2}$	$\frac{1}{2}$

TABLE 1.1: (continued)

a	$(\sin a)^2$	$(\cos a)^2$
80	.9698	.0302
90	1	0
120	.75	.25
250	.8830	.1170

Conjecture: $(\sin a)^2 + (\cos a)^2 = 1$.

1.8 Relating Trigonometric Functions

1. Answers:
   a. $\frac{1}{4}$
   b. $\frac{3}{1} = 3$
2. (a)

TABLE 1.2:

Angle	Sin	Csc
10	.1737	5.759
5	.0872	11.4737
1	.0175	57.2987
0.5	.0087	114.5930
0.1	.0018	572.9581
0	0	undefined
-.1	-.0018	-572.9581
-.5	-.0087	-114.5930
-1	-.0175	-57.2987
-5	-.0872	-11.4737
-10	-.1737	-5.759

(b) As the angle gets smaller and smaller, the cosecant values get larger and larger.

(c) The range of the cosecant function does not have a maximum, like the sine function. The values get larger and larger.

(d) Answers will vary. For example, if we looked at values near 90 degrees, we would see the cosecant values get smaller and smaller, approaching 1.

3. The values 90, 270, 450, etc, are excluded because they make the function undefined.
4. Answers:
   a. Quadrant 1; positive
   b. Quadrant 3; negative
   c. Quadrant 4; negative
   d. Quadrant 2; negative
5. $\frac{8}{6} = \frac{4}{3}$
6. The ratio of sine and cosine will be positive in the third quadrant because sine and cosine are both negative in the third quadrant.
7. $\cos\theta \approx .92$
8. $\csc\theta = \sqrt{5}$

9.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$1 + \tan^2\theta = \sec^2\theta$$

10. Using the Pythagorean identities results in a quadratic equation and will have two solutions. Stating that the angle lies in a particular quadrant tells you which solution is the actual value of the expression. In #7, the angle is in the first quadrant, so both sine and cosine must be positive.

Chapter Summary

1. Area 1:

$$(a+b)^2$$
$$(a+b)(a+b)$$
$$a^2 + 2ab + b^2$$

Area 2: Add up 4 triangles and inner square.

$$4 \cdot \frac{1}{2}ab + c^2$$
$$2ab + c^2$$

Set the two equal to each other:

$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$

2. First, find the diagonal of the base. This is a Pythagorean Triple, so the base diagonal is 25 (you could have also done the Pythagorean Theorem if you didn't see this). Now, do the Pythagorean Theorem with the height and the diagonal to get the three-dimensional diagonal.

$$7^2 + 25^2 = d^2$$
$$49 + 225 = d^2$$
$$274 = d^2$$
$$\sqrt{274} = d \approx 16.55$$

3.

$$\angle C = 90^\circ - 23.6^\circ = 66.4^\circ$$

$$\sin 23.6 = \frac{CA}{25} \qquad \cos 23.6 = \frac{AT}{25}$$
$$25 \cdot \sin 23.6 = CA \qquad 25 \cdot \cos 23.6 = AT$$
$$10.01 \approx CA \qquad 22.9 \approx AT$$

4. First do the Pythagorean Theorem to get the third side.

$$\begin{aligned} 7^2+x^2 &= 18^2 \\ 49+x^2 &= 324 \\ x^2 &= 275 \\ x &= \sqrt{275} = 5\sqrt{11} \end{aligned}$$

Second, use one of the inverse functions to find the two missing angles.

$$\begin{aligned} \sin G &= \frac{7}{18} \\ \sin^{-1}\left(\frac{7}{18}\right) &= G \\ G &\approx 22.89^\circ \end{aligned}$$

We can subtract $\angle G$ from 90 to get 67.11°.

5.

$$\begin{aligned} A &= ab\sin C \\ &= 16\cdot 22\cdot \sin 60^\circ \\ &= 352\cdot\frac{\sqrt{3}}{2} \\ &= 176\sqrt{3} \end{aligned}$$

6. Make a right triangle with 165 as the opposite leg and w is the hypotenuse.

$$\begin{aligned} \sin 85^\circ &= \frac{165}{w} \\ w\sin 85^\circ &= 165 \\ w &= \frac{165}{\sin 85^\circ} \\ w &\approx 165.63 \end{aligned}$$

7.

$$\begin{aligned} \cos(90^\circ - x) &= \sin x \\ \sin x &= \frac{2}{7} \end{aligned}$$

8. If $\cos(-x)=\frac{3}{4}$, then $\cos x=\frac{3}{4}$. With $\tan x=\frac{\sqrt{7}}{3}$, we can conclude that $\sin x=\frac{\sqrt{7}}{4}$ and $\sin(-x)=-\frac{\sqrt{7}}{4}$.
9. If $\sin y=\frac{1}{3}$, then we know the opposite side and the hypotenuse. Using the Pythagorean Theorem, we get that the adjacent side is $2\sqrt{2}\left(1^2+b^2=3^2\rightarrow b=\sqrt{9-1}\rightarrow b=\sqrt{8}=2\sqrt{2}\right)$. Thus, $\cos y=\pm\frac{2\sqrt{2}}{3}$ because we don't know if the angle is in the second or third quadrant.
10. $\sin\theta=\frac{1}{3}$, sine is positive in Quadrants I and II. So, there can be two possible answers for the $\cos\theta$. Find the third side, using the Pythagorean Theorem:

$$\begin{aligned} 1^2+b^2 &= 3^2 \\ 1+b^2 &= 9 \\ b^2 &= 8 \\ b &= \sqrt{8} = 2\sqrt{2} \end{aligned}$$

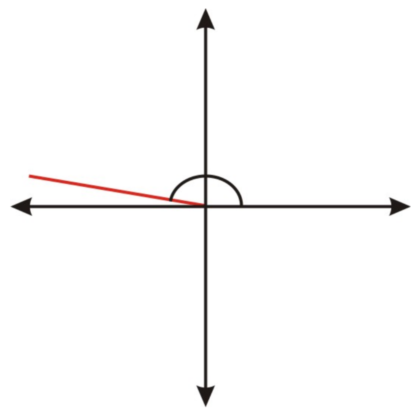
In Quadrant I, $\cos\theta=\frac{2\sqrt{2}}{3}$ In Quadrant II, $\cos\theta=-\frac{2\sqrt{2}}{3}$

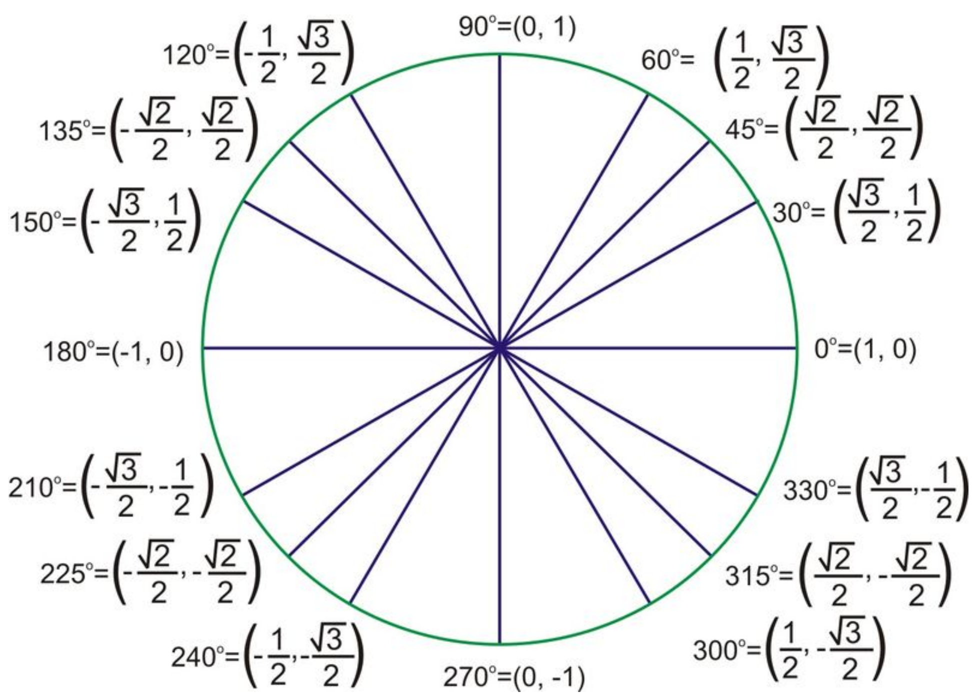
11. $\cos\theta = -\frac{2}{5}$ and is in Quadrant II, so from the Pythagorean Theorem :

$$\begin{aligned} a^2+(-2)^2 &= 5^2 \\ a^2+4 &= 25 \\ a^2 &= 21 \\ a &= \sqrt{21} \end{aligned}$$

So, $\sin\theta = \frac{\sqrt{21}}{5}$ and $\tan\theta = -\frac{\sqrt{21}}{2}$

12. If the terminal side of θ is on (3, -4) means θ is in Quadrant IV, so cosine is the only positive function. Because the two legs are lengths 3 and 4, we know that the hypotenuse is 5. 3, 4, 5 is a Pythagorean Triple (you can do the Pythagorean Theorem to verify). Therefore, $\sin\theta = \frac{3}{5}, \cos\theta = -\frac{4}{5}, \tan\theta = -\frac{4}{3}$
13. Reference angle $= 15°$. Possible coterminal angles $= -195°, 525°$



14. 

CHAPTER 2 Graphing Trigonometric Functions, Solution Key

Chapter Outline

2.1 Radian Measure

1. Answers:

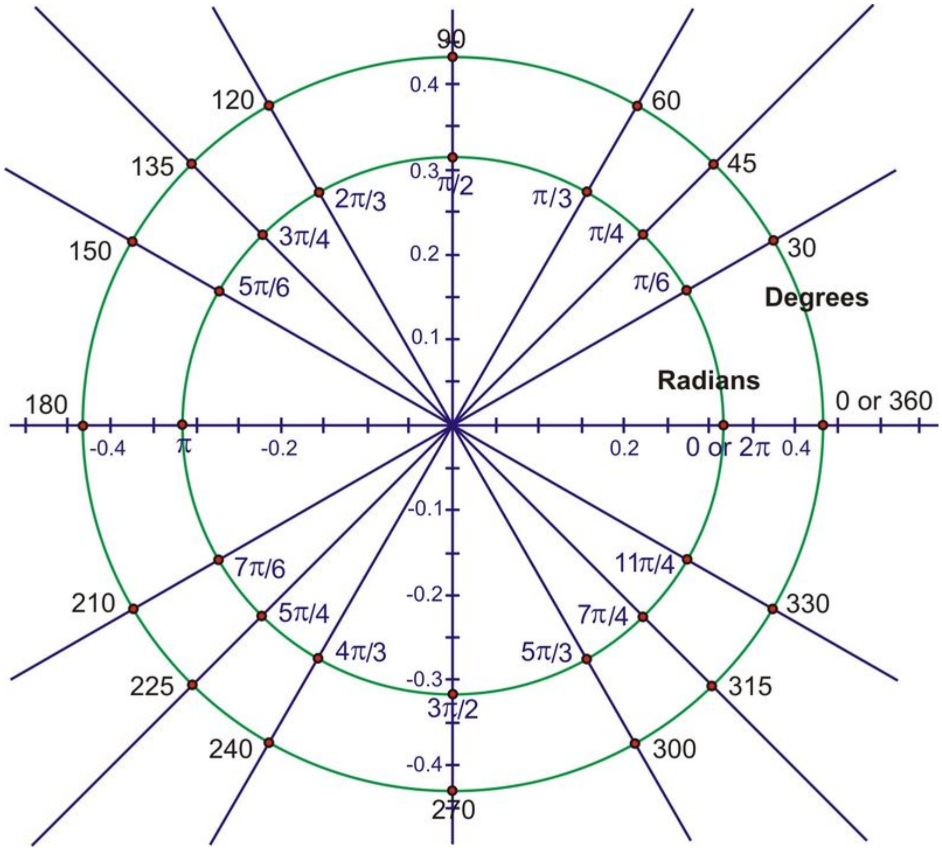
   a. Answer may vary, but 120° seems reasonable.
   b. Based on the answer in part a., the radian measure would be $\frac{2\pi}{3}$
   c. Again, based on part a., $\frac{4\pi}{3}$

2. Answers:

   a. $\frac{4\pi}{3}$
   b. $\frac{3\pi}{2}$
   c. $\frac{7\pi}{4}$
   d. $-\frac{7\pi}{6}$
   e. $\frac{2\pi}{3}$
   f. $\frac{\pi}{12}$
   g. $-\frac{5\pi}{2}$
   h. $\frac{\pi}{5}$
   i. 4π
   j. $\frac{11\pi}{6}$

3. Answers:

   a. 90°
   b. 396°
   c. 120°
   d. 540°
   e. 630°
   f. 54°
   g. 75°
   h. −210°
   i. 1440°
   j. 48°

4. 

5. Answers:
   a. 154.3°
   b. 57.3°
   c. 171.9°
   d. 327.3°
6. Answers:
   a. The correct answer is $-\frac{1}{2}$
   b. Her calculator was is the wrong mode and she calculated the sine of 210 *radians*.
7. Answer:

TABLE 2.1:

x	$\text{Sin}(x)$	$\text{Cos}(x)$	$\text{Tan}(x)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined
$\frac{7\pi}{2}$	-1	0	undefined

2.2 Applications of Radian Measure

1. a. i. $\frac{\pi}{12}$

ii.$\approx$ 0.3 radians
iii.15°
b. i.20°. Answers may vary, anything above 15° and less than 25° is reasonable.
ii.$\frac{\pi}{9}$ Again, answers may vary

2. a. $\frac{\pi}{6}$

b. $\approx 26\ cm$

3. a. $\frac{\pi}{16}$

b. Let's assume, to simplify, that the chord stretches to the center of each of the dots. We need to find the measure of the central angle of the circle that connects those two dots.

Since there are 13 dots, this angle is $\frac{13\pi}{16}$. The length of the chord then is:

$$= 2r\sin\frac{\theta}{2}$$
$$= 2\times 1.2\times \sin(\frac{1}{2}\times\frac{13\pi}{16})$$

The chord is approximately 2.30 m, or 230 cm.

4. Each section is $\frac{\pi}{6}$ radians. The area of one section of the stands is therefore the area of the outer sector minus the area of the inner sector:

$$A = A_{outer} - A_{inner}$$
$$A = \frac{1}{2}(r_{outer})^2\times\frac{\pi}{6} - \frac{1}{2}(r_{inner})^2\times\frac{\pi}{6}$$
$$A = \frac{1}{2}(110)^2\times\frac{\pi}{6} - \frac{1}{2}(55)^2\times\frac{\pi}{6}$$

The area of each section is approximately 2376 ft^2.

a. The students have 4 sections or $\approx 9503\ ft^2$
b. There are 3 general admission sections or $\approx 7127\ ft^2$
c. There is only one press and officials section or $\approx 2376\ ft^2$

5. It is actually easier to calculate the angular velocity first. $\omega = \frac{2\pi}{12} = \frac{\pi}{6}$, so the angular velocity is $\frac{\pi}{6}\ rad$, or 0.524. Because the linear velocity depends on the radius,each girl has her own.

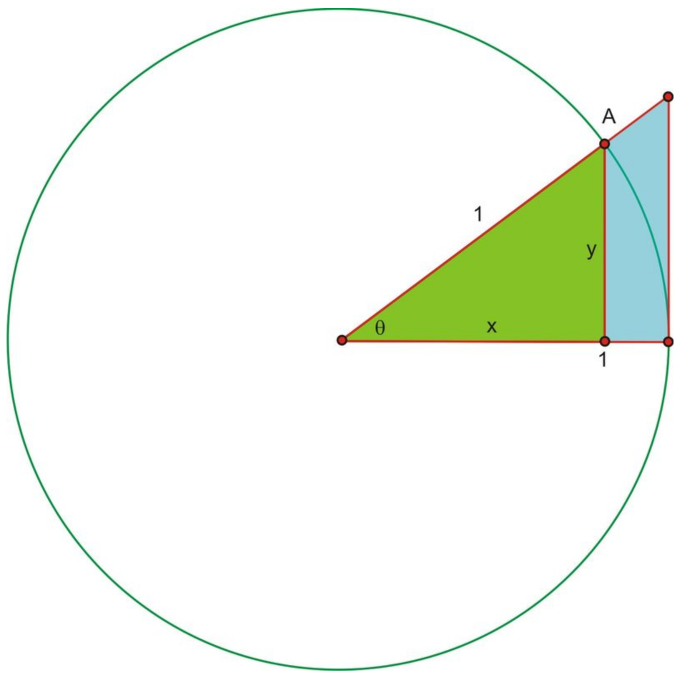
Lois: $v = r\omega = 3 \cdot \frac{\pi}{6} = \frac{\pi}{2}$ or $1.57\ m/sec$
Doris: $v = r\omega = 10 \cdot \frac{\pi}{6} = \frac{5\pi}{3}$ or $5.24\ m/sec$

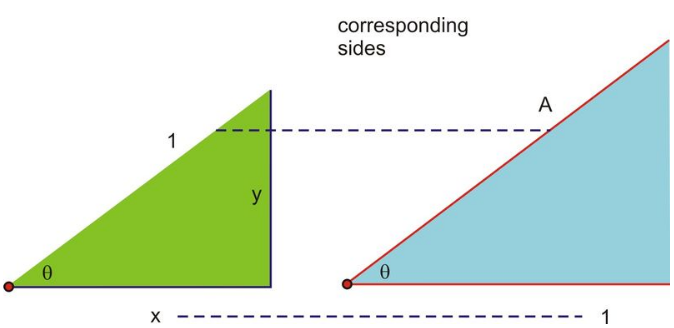
6. a. $v = \frac{d}{t} \rightarrow 3 \times 10^8 = \frac{27{,}000}{t} \rightarrow t = \frac{2.7 \times 10^4}{3 \times 10^8} = 0.9 \times 10^{-4} = 9 \times 10^{-5}$ or 0.00009 seconds.

b. $\omega = \frac{\theta}{t} = \frac{2\pi}{0.00009} \approx 69,813\ rad/sec$
c. The proton rotates around once in 0.00009 seconds. So, in one second it will rotate around the LHC $1 \div 0.00009 = 11,111.11$ times, or just over 11,111 rotations.

2.3 Circular Functions of Real Numbers

1. Use similar triangles:

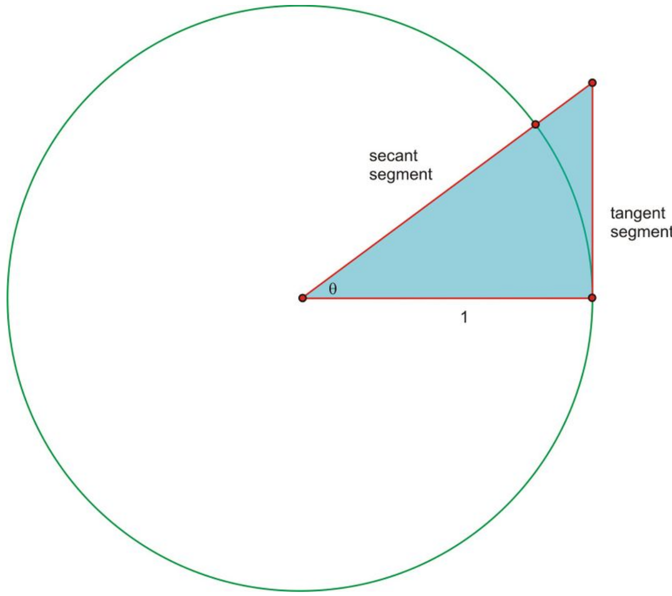


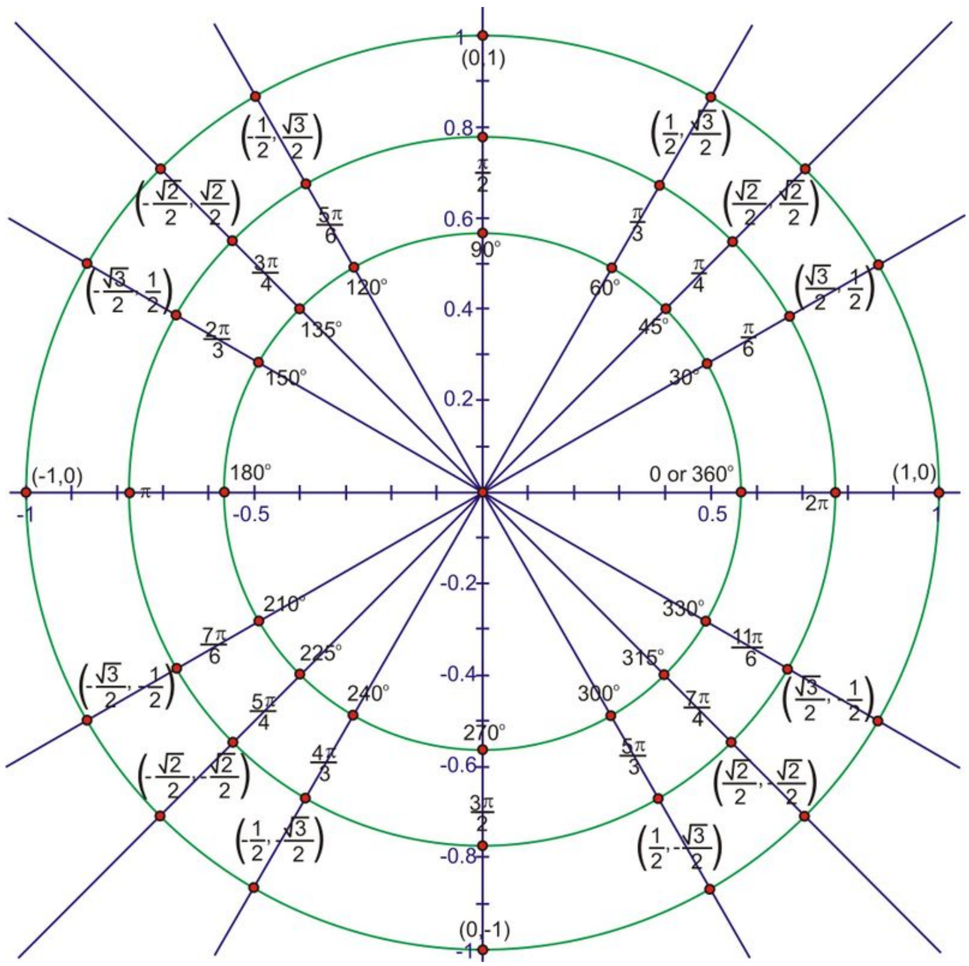


So:

$$\frac{x}{1} = \frac{1}{A} \rightarrow Ax = 1 \rightarrow A = \frac{1}{x}$$
$$\cos\theta = x \rightarrow \frac{1}{\cos\theta} = \frac{1}{x} \rightarrow \frac{1}{\cos\theta} = \sec\theta = \frac{1}{x}$$
$$\therefore \sec\theta = A$$

2. Using the Pythagorean theorem, $\tan^2\theta + 1 = \sec^2\theta$.


secant
segment
tangent
segment
θ
1

3. 
(0,1)
(-1,0)
(1,0)
(0,-1)
0 or 360°
180°
90°
270°
30°
45°
60°
120°
135°
150°
210°
225°
240°
300°
315°
330°
π
2π
0.5
-0.5

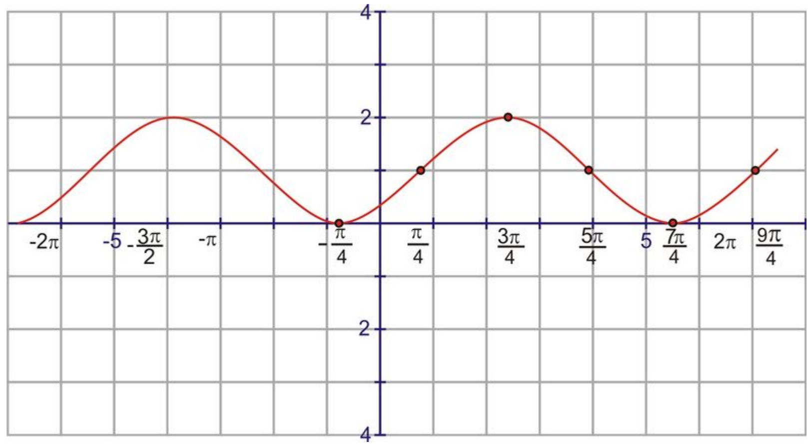
4. b
5. d

2.4 Translating Sine and Cosine Functions

1. B
2. E
3. D
4. C
5. A
6.

$$y = -2 + \sin(x - \pi) \text{ or } y = -2 + \sin(x + \pi)$$
$$y = -2 + \cos\left(x + \frac{\pi}{2}\right) \text{ or } y = -2 + \cos\left(x - \frac{3\pi}{2}\right)$$

Note: this list is *not* exhaustive, there are other possible answers.

7. C
8. D
9. A
10. B
11. 

2.5 Amplitude, Period and Frequency

1.

a. $y = \sec x$: period $= 2\pi$, frequency $= 1$
b. $y = \cot x$: period $= \pi$, frequency $= 2$
c. $y = \csc x$: period $= 2\pi$, frequency $= 1$

Because these are reciprocal functions, the periods are the same as cosine, tangent, and sine, repectively.

2.

a. min: -1, max: 1
b. min: -2, max: 2
c. min: -1, max: 1
d. there is no minimum or maximum, tangent has a range of all real numbers
e. min: $-\frac{1}{2}$, max: $\frac{1}{2}$
f. min: -3, max: 3

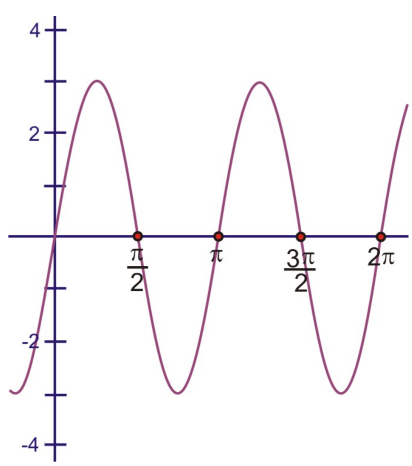
3. d.
4.

a. period: π, amplitude: 1, frequency: 2
b. period: 2π, amplitude: 3, frequency: 1
c. period: 2, amplitude: 2, frequency: π
d. period: $\frac{2\pi}{3}$, amplitude: 2, frequency: 3
e. period: 4π, amplitude: $\frac{1}{2}$, frequency: $\frac{1}{2}$
f. period: 4π, amplitude: 3, frequency: $\frac{1}{2}$

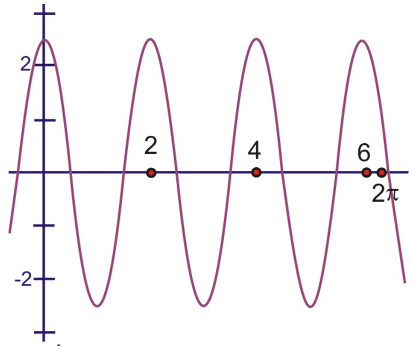
5.

a. period: π, amplitude: 3, frequency: $2, y = 3\cos 2x$
b. period: 4π, amplitude: 2, frequency: $\frac{1}{2}, y = 2\sin\frac{1}{2}x$
c. period: 3, amplitude: 2, frequency: $\frac{2\pi}{3}, y = 2\cos\frac{2\pi}{3}x$
d. period: $\frac{\pi}{3}$, amplitude: $\frac{1}{2}$, frequency: $6, y = \frac{1}{2}\sin 6x$

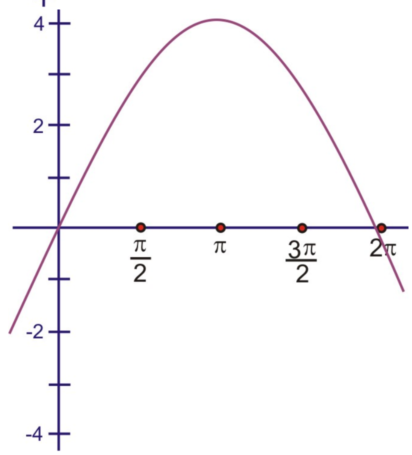
6.



a.



b.



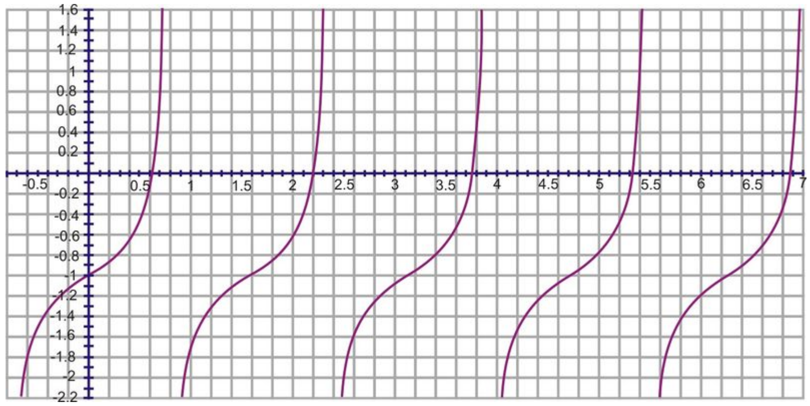
c.

2.6 General Sinusoidal Graphs

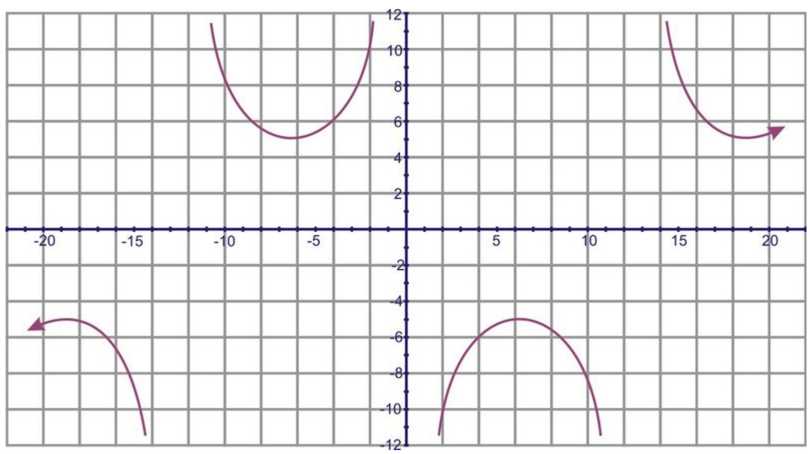
1. This is a sine wave that has been translated 1 unit to the right and 2 units up. The amplitude is 3 and the frequency is 2. The period of the graph is π. The function reaches a maximum point of 5 and a minimum of -1.
2. This is a sine wave that has been translated 1 unit down and $\frac{\pi}{3}$ radians to the left. The amplitude is 1 and the period is 2. The frequency of the graph is π. The function reaches a maximum point of 0 and a minimum of -2.
3. This is a cosine wave that has been translated 5 units up and 120 radians to the right. The amplitude is 1 and the frequency is 40. The period of the graph is $\frac{\pi}{20}$. The function reaches a maximum point of 6 and a minimum of 4.
4. This is a cosine wave that has not been translated vertically. It has been translated $\frac{5\pi}{4}$ radians to the left. The amplitude is 1 and the frequency is $\frac{1}{2}$. The period of the graph is 4π. The function reaches a maximum point of 1 and a minimum of -1. The negative in front of the cosine function does not change the amplitude, it simply reflects the graph across the $x-$axis.
5. This is a cosine wave that has been translate up 3 units and has an amplitude of 2. The frequency is 1 and the period is 2π. There is no horizontal translation. Putting a negative in front of the $x-$value reflects the function across the $y-$axis. A cosine wave that has not been translated horizontally is symmetric to the $y-$axis so this reflection will have no visible effect on the graph. The function reaches a maximum of 5 and a minimum of 1. ***other answers are possible given different horizontal translations of sine/cosine
6. $y = 3 + 2\cos\left(3(x - \frac{\pi}{6})\right)$
7. $y = 2 + \sin x$ or $y = 2 + \cos\left(x - \frac{\pi}{2}\right)$
8. $y = 10 + 20\cos(6(x - 30))$
9. $y = 3 + \frac{3}{4}\cos\left(\frac{1}{2}(x + \pi)\right)$
10. $y = 3 + 7\cos\left(\frac{1}{3}(x - \frac{\pi}{4})\right)$

2.7 Graphing Tangent, Cotangent, Secant, and Cosecant

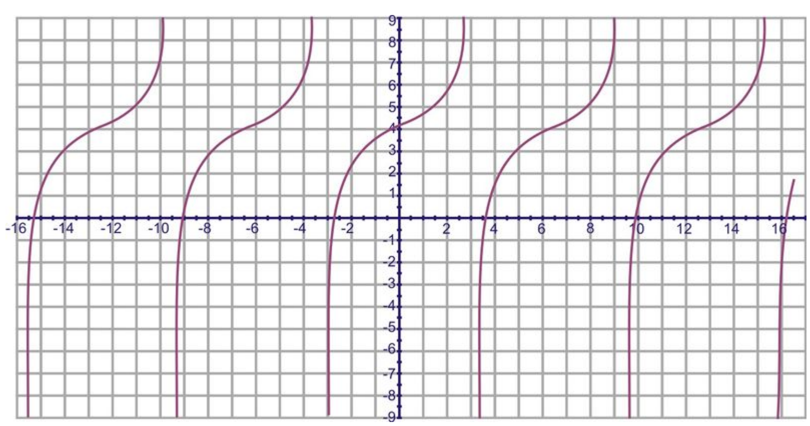
1. $y = -1 + \frac{1}{3} \cot 2x$



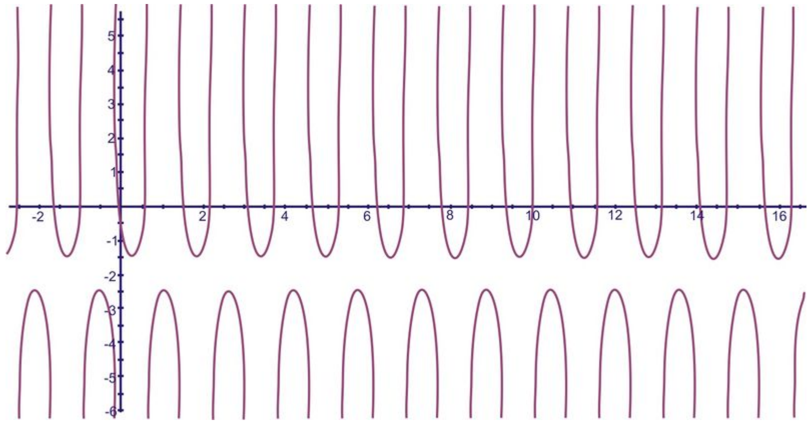
2. $g(x) = 5 \csc \left(\frac{1}{4}(x + \pi) \right)$



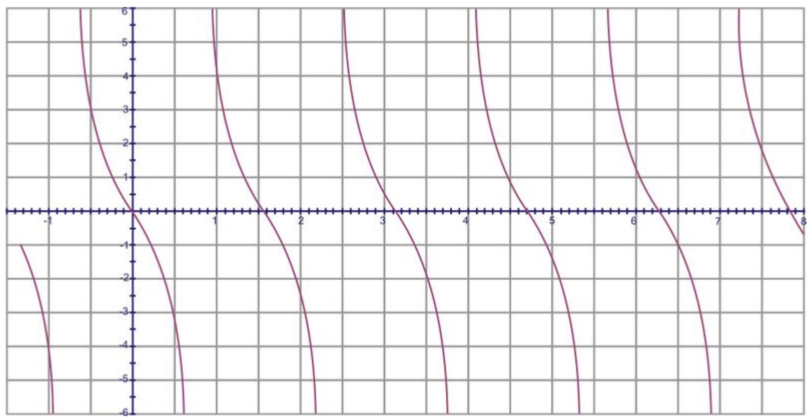
3. $f(x) = 4 + \tan(0.5(x - \pi))$



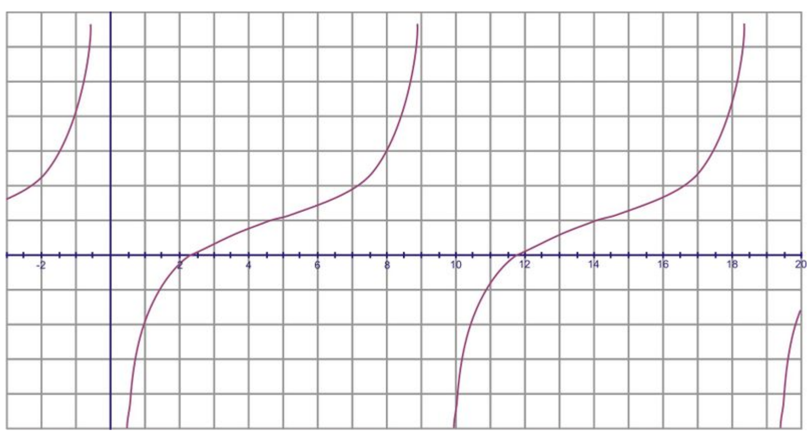
4. $y = -2 + \frac{1}{2} \sec(4(x - 1))$



5. $y = -2\tan 2x$



6. $h(x) = -\cot\left(\frac{1}{3}x\right) + 1$



7. To make cotangent match up with tangent, it is helpful to graph the two on the same set of axis. First, cotangent needs to be flipped, which would make the amplitude of -1. Once cotangent is flipped, it also needs a phase shift of $\frac{\pi}{2}$. So, $\tan x = -\cot\left(x - \frac{\pi}{2}\right)$.
8. This is a tangent graph. From the two points we are given, we can determine the phase shift, vertical shift and frequency. There is no phase shift, the vertical shift is 3 and the frequency is 6. $y = 3 + \tan 6x$
9. This could be either a secant or cosecant function. We will use a cosecant model.

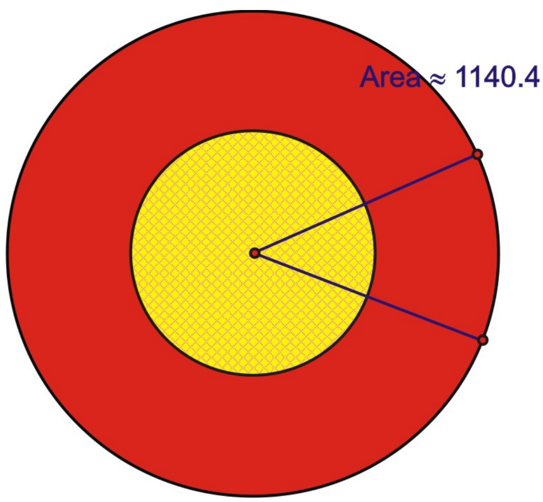
   - First, the vertical shift is -1.
   - The period is the difference between the two given $x-$values, $\frac{7\pi}{4} - \frac{3\pi}{4} = \pi$, so the frequency is $\frac{2\pi}{\pi} = 2$.
   - The horizontal shift incorporates the frequency, so in $y = \csc x$ the corresponding $x-$value to $\left(\frac{3\pi}{4}, 0\right)$ is $\left(\frac{\pi}{2}, 1\right)$.

- The difference between the $x-$values is $\frac{3\pi}{4}-\frac{\pi}{2}=\frac{3\pi}{4}-\frac{2\pi}{4}=\frac{\pi}{4}$ and then multiply it by the frequency, $2\cdot\frac{\pi}{4}=\frac{\pi}{2}$.
- The equation is $y=-1+\csc\left(2(x-\frac{\pi}{2})\right)$.

Chapter Summary

1. $160^\circ\cdot\frac{\pi}{180^\circ}=\frac{16\pi}{18}=\frac{8\pi}{9}$
2. $\frac{11\pi}{12}\cdot\frac{180^\circ}{\pi}=11\cdot 15^\circ=165^\circ$
3. $\cos\frac{3\pi}{4}=\cos 135^\circ=-\frac{\sqrt{2}}{2}$
4. For $\tan\theta=\sqrt{3}$, θ must equal 60° or 240°. In radians, $\frac{\pi}{3}$ or $\frac{4\pi}{3}$.
5. There are many difference approaches to the problem. Here is one possibility: First, calculate the area of the red ring as if it went completely around the circle:

$$A=A_{total}-A_{gold}$$
$$A=\pi\left(\frac{2}{3}\times 66\times\frac{1}{2}\right)^2-\pi\left(\frac{1}{3}\times 66\times\frac{1}{2}\right)^2$$
$$A=\pi\times 22^2-\pi\times 11^2$$
$$A=484\pi-121\pi=363\pi$$
$$A\approx 1140.4\ in^2$$



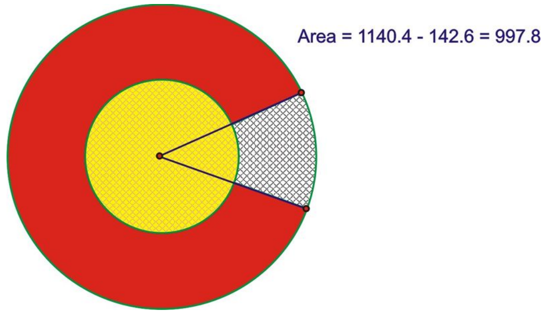
Next, calculate the area of the total sector that would form the opening of the "c"

$$A=\frac{1}{2}r^2\theta$$
$$A=\frac{1}{2}(22)^2\left(\frac{\pi}{4}\right)$$
$$A\approx 190.1\ in^2$$

Then, calculate the area of the yellow sector and subtract it from the previous answer.

$$A=\frac{1}{2}r^2\theta\rightarrow A=\frac{1}{2}(11)^2\left(\frac{\pi}{4}\right)\rightarrow A\approx 47.5\ in^2$$
$$190.1-47.5=142.6\ in^2$$

Finally, subtract this answer from the first area calculated. The area is approximately 998 in^2

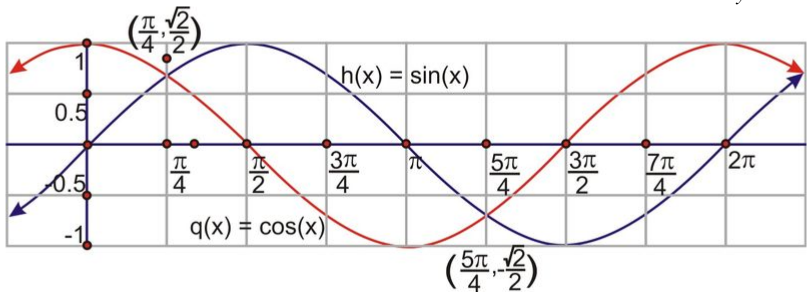


6. Answers:

a. First find the circumference: $2\pi \cdot 7 = 14\pi$. This will be the distance for the linear velocity. $v = dt = 14\pi \cdot 9 = 126\pi \approx 395.84\ cm/sec$

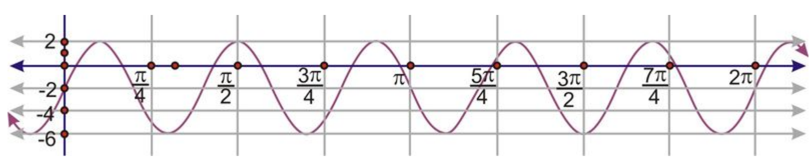
b. $\omega = \frac{\theta}{t} = \frac{2\pi}{9} \approx 0.698\ rad/sec$

7. Given such a quadrilateral, and given that the two transverse angles are identified as equal (i.e., both are marked as θ in the picture), the orange segment must be parallel to the opposite (pink) radius segment, and this quadrilateral would have to be a square. This means that θ must be equal to 45 degrees, and both the tangent and cotangent of 45 degrees are equal to 1. Also, since the radii of the circle are equal to 1 unit, each of the sides of the quadrilateral (including the cotangent segment) are equal to 1 unit. Therefore, since $\cot\theta = \frac{x}{y}$, the number of units that is the length of the cotangent segment must be equal to $\frac{x}{y}$.

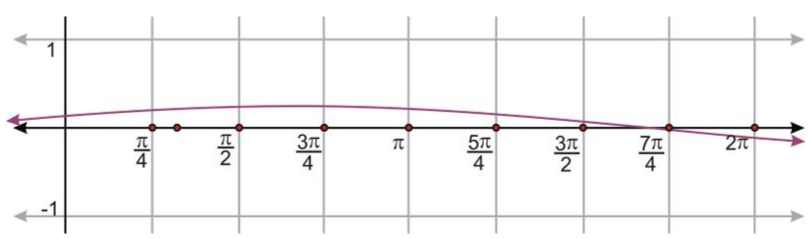
8. 

The intersections are $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$.

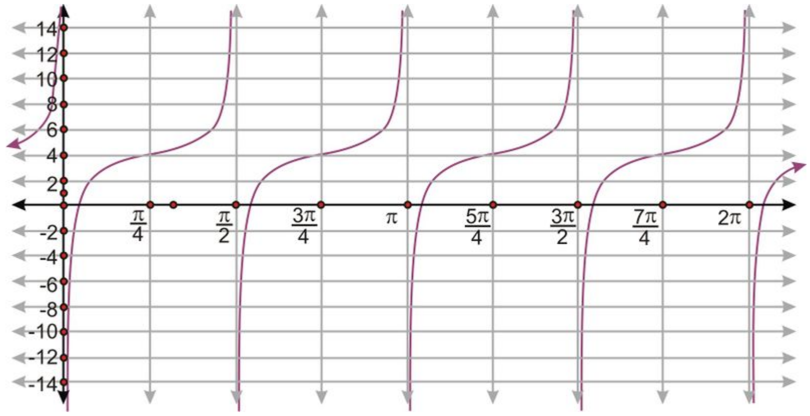
9. $y = -2 + 4\sin 5x, A = 4, B = 5, p = \frac{2\pi}{5}, C = 0, D = -2$



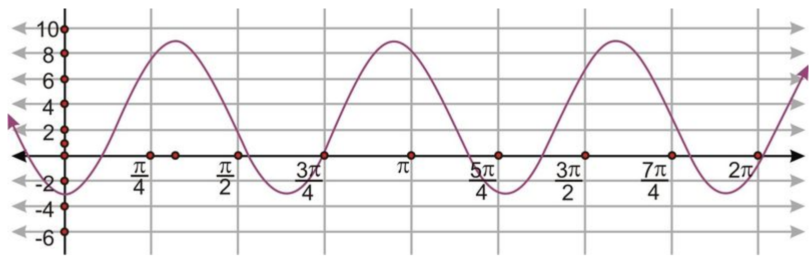
10. $f(x) = \frac{1}{4}\cos\left(\frac{1}{2}(x - \frac{\pi}{3})\right), A = \frac{1}{4}, B = \frac{1}{2}, p = 4\pi, C = \frac{\pi}{3}, D = 0$



11. $g(x) = 4 + \tan\left(2(x + \frac{\pi}{2})\right), A = 1, B = 2, p = \frac{\pi}{2}, C = \frac{-\pi}{2}, D = 4$



12. $h(x) = 3 - 6\cos(\pi x), A = -6, B = \pi, C = 0, D = 3$



13. $y = -1 + \frac{1}{2}\cos 3x$
14. $y = \tan 6x$

CHAPTER 3 Trigonometric Identities and Equations, Solution Key

Chapter Outline

3.1 Fundamental Identities

1. $\tan 270^\circ = \frac{\sin 270^\circ}{\cos 270^\circ} = \frac{-1}{0}$, you cannot divide by zero, therefore $\tan 270^\circ$ is undefined.
2. If $\cos\left(\frac{\pi}{2} - x\right) = \frac{4}{5}$, then, by the cofunction identities, $\sin x = \frac{4}{5}$. Because sine is odd, $\sin(-x) = -\frac{4}{5}$.
3. If $\tan(-x) = -\frac{5}{12}$, then $\tan x = \frac{5}{12}$. Because $\sin x = -\frac{5}{13}$, cosine is also negative, so $\cos x = -\frac{12}{13}$.
4. Use the reciprocal and cofunction identities to simplify

$$\sec x \cos\left(\frac{\pi}{2} - x\right)$$
$$\frac{1}{\cos x} \cdot \sin x$$
$$\frac{\sin x}{\cos x}$$
$$\tan x$$

5. (a) Using the sides 5, 12, and 13 and in the first quadrant, it doesn't really matter which is cosine or sine.
   - So, $\sin^2\theta + \cos^2\theta = 1$ becomes $\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1$. *Simplifying, we get: $\frac{25}{169} + \frac{144}{169} = 1$,
   - Finally $\frac{169}{169} = 1$.

   (b) $\sin^2\theta + \cos^2\theta = 1$ becomes $\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$. Simplifying we get: $\frac{1}{4} + \frac{3}{4} = 1$ and $\frac{4}{4} = 1$.
6. To prove $\tan^2\theta + 1 = \sec^2\theta$, first use $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and change $\sec^2\theta = \frac{1}{\cos^2\theta}$.

$$\tan^2\theta + 1 = \sec^2\theta$$
$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$
$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$

7. If $\csc z = \frac{17}{8}$ and $\cos z = -\frac{15}{17}$, then $\sin z = \frac{8}{17}$ and $\tan z = -\frac{8}{15}$. Therefore $\cot z = -\frac{15}{8}$.
8. (a) Factor $\sin^2\theta - \cos^2\theta$ using the difference of squares.

$$\sin^2\theta - \cos^2\theta = (\sin\theta + \cos\theta)(\sin\theta - \cos\theta)$$

   (b) $\sin^2\theta + 6\ \sin\theta + 8 = (\sin\theta + 4)(\sin\theta + 2)$
9. You will need to factor and use the $\sin^2\theta + \cos^2\theta = 1$ identity.

$$\frac{\sin^4\theta - \cos^4\theta}{\sin^2\theta - \cos^2\theta}$$
$$= \frac{(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)}{\sin^2\theta - \cos^2\theta}$$
$$= \sin^2\theta + \cos^2\theta$$
$$= 1$$

10. To rewrite $\frac{\cos x}{\sec x - 1}$ so it is only in terms of cosine, start with changing secant to cosine.

$$\frac{\cos x}{\sec x - 1} = \frac{\cos x}{\frac{1}{\cos x} - 1}$$

Now, simplify the denominator.

$$\frac{\cos x}{\frac{1}{\cos x} - 1} = \frac{\cos x}{\frac{1 - \cos x}{\cos x}}$$

Multiply by the reciprocal $\frac{\cos x}{\frac{1-\cos x}{\cos x}} = \cos x \div \frac{1-\cos x}{\cos x} = \cos x \cdot \frac{\cos x}{1-\cos x} = \frac{\cos^2 x}{1-\cos x}$

11. The easiest way to prove that tangent is odd to break it down, using the Quotient Identity.

$$\begin{aligned} \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{from this statement, we need to show that } \tan(-x) = -\tan x \\ &= \frac{-\sin x}{\cos x} && \text{because } \sin(-x) = -\sin x \text{ and } \cos(-x) = \cos x \\ &= -\tan x \end{aligned}$$

3.2 Proving Identities

1. Step 1: Change everything into sine and cosine

$$\sin x \tan x + \cos x = \sec x$$
$$\sin x \cdot \frac{\sin x}{\cos x} + \cos x = \frac{1}{\cos x}$$

Step 2: Give everything a common denominator, $\cos x$.

$$\frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} = \frac{1}{\cos x}$$

Step 3: Because the denominators are all the same, we can eliminate them.

$$\sin^2 x + \cos^2 x = 1$$

We know this is true because it is the Trig Pythagorean Theorem

2. Step 1: Pull out a $\cos x$

$$\cos x - \cos x \sin^2 x = \cos^3 x$$
$$\cos x(1 - \sin^2 x) = \cos^3 x$$

Step 2: We know $\sin^2 x + \cos^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$ is also true, therefore $\cos x(\cos^2 x) = \cos^3 x$. This, of course, is true, we are done!

3. Step 1: Change everything in to sine and cosine and find a common denominator for left hand side.

$$\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$$
$$\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = \frac{2}{\sin x} \leftarrow \text{LCD}: \ \sin x(1+\cos x)$$
$$\frac{\sin^2 x + (1+\cos x)^2}{\sin x(1+\cos x)}$$

Step 2: Working with the left side, FOIL and simplify.

$$\frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{\sin x(1+\cos x)} \qquad \rightarrow \text{FOIL } (1+\cos x)^2$$
$$\frac{\sin^2 x + \cos^2 x + 1 + 2\cos x}{\sin x(1+\cos x)} \qquad \rightarrow \text{move } \cos^2 x$$
$$\frac{1+1+2\cos x}{\sin x(1+\cos x)} \qquad \rightarrow \sin^2 x + \cos^2 x = 1$$
$$\frac{2+2\cos x}{\sin x(1+\cos x)} \qquad \rightarrow \text{add}$$
$$\frac{2(1+\cos x)}{\sin x(1+\cos x)} \qquad \rightarrow \text{fator out 2}$$
$$\frac{2}{\sin x} \qquad \rightarrow \text{cancel } (1+\cos x)$$

4. Step 1: Cross-multiply

$$\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$
$$\sin^2 x = (1+\cos x)(1-\cos x)$$

Step 2: Factor and simplify

$$\sin^2 x = 1-\cos^2 x$$
$$\sin^2 x+\cos^2 x = 1$$

5. Step 1: Work with left hand side, find common denominator, FOIL and simplify, using $\sin^2 x+\cos^2 x = 1$.

$$\frac{1}{1+\cos x}+\frac{1}{1-\cos x} = 2+2\cot^2 x$$
$$\frac{1-\cos x+1+\cos x}{(1+\cos x)(1-\cos x)}$$
$$\frac{2}{1-\cos^2 x}$$
$$\frac{2}{\sin^2 x}$$

Step 2: Work with the right hand side, to hopefully end up with $\frac{2}{\sin^2 x}$.

$$= 2+2\cot^2 x$$
$$= 2+2\,\frac{\cos^2 x}{\sin^2 x}$$
$$= 2\left(1+\frac{\cos^2 x}{\sin^2 x}\right) \quad \rightarrow \text{factor out the 2}$$
$$= 2\left(\frac{\sin^2 x+\cos^2 x}{\sin^2 x}\right) \quad \rightarrow \text{common denominator}$$
$$= 2\left(\frac{1}{\sin^2 x}\right) \quad \rightarrow \text{trig pythagorean theorem}$$
$$= \frac{2}{\sin^2 x} \quad \rightarrow \text{simply/multiply}$$

Both sides match up, the identity is true.

6. Step 1: Factor left hand side

$\cos^4 b-\sin^4 b$	$1-2\sin^2 b$
$(\cos^2 b+\sin^2 b)(\cos^2 b-\sin^2 b)$	$1-2\sin^2 b$
$\cos^2 b-\sin^2 b$	$1-2\sin^2 b$

Step 2: Substitute $1-\sin^2 b$ for $\cos^2 b$ because $\sin^2 x+\cos^2 x = 1$.

$(1-\sin^2 b)-\sin^2 b$	$1-2\sin^2 b$
$1-\sin^2 b-\sin^2 b$	$1-2\sin^2 b$
$1-2\ \sin^2 b$	$1-2\sin^2 b$

7. Step 1: Find a common denominator for the left hand side and change right side in terms of sine and cosine.

$$\frac{\sin y+\cos y}{\sin y}-\frac{\cos y-\sin y}{\cos y} = \sec y\csc y$$
$$\frac{\cos y(\sin y+\cos y)-\sin y(\cos y-\sin y)}{\sin y\cos y} = \frac{1}{\sin y\cos y}$$

Step 2: Work with left side, simplify and distribute.

$$\frac{\sin y \cos y + \cos^2 y - \sin y \cos y + \sin^2 y}{\sin y \cos y}$$
$$\frac{\cos^2 y + \sin^2 y}{\sin y \cos y}$$
$$\frac{1}{\sin y \cos y}$$

8. Step 1: Work with left side, change everything into terms of sine and cosine.

$$(\sec x - \tan x)^2 = \frac{1 - \sin x}{1 + \sin x}$$
$$\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)^2$$
$$\left(\frac{1 - \sin x}{\cos x}\right)^2$$
$$\frac{(1 - \sin x)^2}{\cos^2 x}$$

Step 2: Substitute $1 - \sin^2 x$ for $\cos^2 x$ because $\sin^2 x + \cos^2 x = 1$

$$\frac{(1 - \sin x)^2}{1 - \sin^2 x} \rightarrow \text{be careful, these are NOT the same!}$$

Step 3: Factor the denominator and cancel out like terms.

$$\frac{(1 - \sin x)^2}{(1 + \sin x)(1 - \sin x)}$$
$$\frac{1 - \sin x}{1 + \sin x}$$

9. Plug in $\frac{5\pi}{6}$ for x into the formula and simplify.

$$2 \sin x \cos x = \sin 2x$$
$$2 \sin \frac{5\pi}{6} \cos \frac{5\pi}{6} = \sin 2 \cdot \frac{5\pi}{6}$$
$$2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = \sin \frac{5\pi}{3}$$

This is true because $\sin 300^\circ$ is $-\frac{\sqrt{3}}{2}$

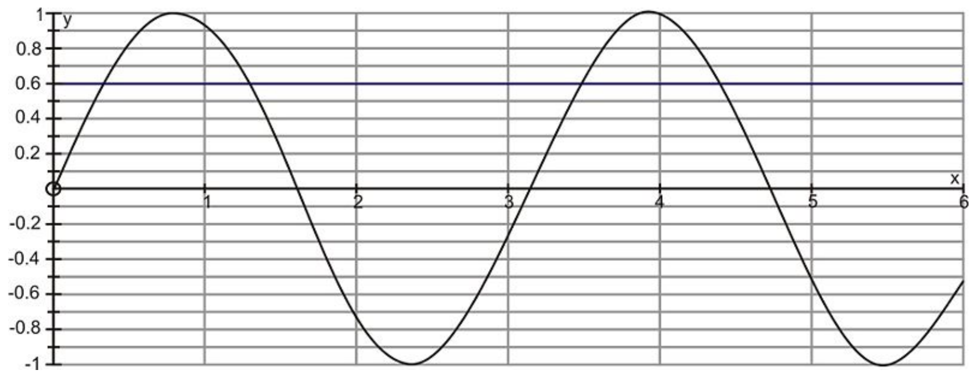
10. Change everything into terms of sine and cosine and simplify.

$$\sec x \cot x = \csc x$$
$$\frac{1}{\cos x} \cdot \frac{\cos x}{\sin x} = \frac{1}{\sin x}$$
$$\frac{1}{\sin x} = \frac{1}{\sin x}$$

3.3 Solving Trigonometric Equations

1. Answer:

- Because the problem deals with 2θ, the domain values must be doubled, making the domain $0 \leq 2\theta < 4\pi$
- The reference angle is $\alpha = \sin^{-1} 0.6 = 0.6435$
- $2\theta = 0.6435, \pi - 0.6435, 2\pi + 0.6435, 3\pi - 0.6435$
- $2\theta = 0.6435, 2.4980, 6.9266, 8.7812$
- The values for θ are needed so the above values must be divided by 2.
- $\theta = 0.3218, 1.2490, 3.4633, 4.3906$
- The results can readily be checked by graphing the function. The four results are reasonable since they are the only results indicated on the graph that satisfy $\sin 2\theta = 0.6$.



2.

$$\cos^2 x = \frac{1}{16}$$
$$\sqrt{\cos^2 x} = \sqrt{\frac{1}{16}}$$
$$\cos x = \pm\frac{1}{4}$$

$$\text{Then } \cos x = \frac{1}{4} \qquad \text{or} \qquad \cos x = -\frac{1}{4}$$
$$\cos^{-1}\frac{1}{4} = x \qquad \cos^{-1} -\frac{1}{4} = x$$
$$x = 1.3181 \textit{ radians} \qquad x = 1.8235 \textit{ radians}$$

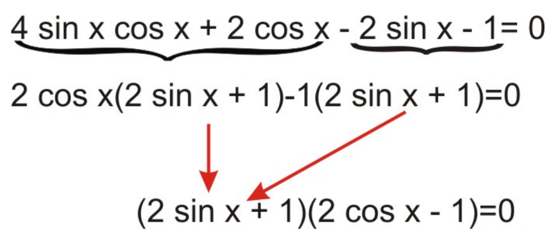
- However, $\cos x$ is also positive in the fourth quadrant, so the other possible solution for $\cos x = \frac{1}{4}$ is $2\pi - 1.3181 = 4.9651$ *radians*
- $\cos x$ is also negative in the third quadrant, so the other possible solution for $\cos x = -\frac{1}{4}$ is $2\pi - 1.8235 = 4.4597$ *radians*

3.

$$\tan^2 x = 1$$
$$\tan x = \pm\sqrt{1}$$
$$\tan x = \pm 1$$

- So, $\tan x = 1$ or $\tan x = -1$.
- Therefore, x is all critical values corresponding with $\frac{\pi}{4}$ within the interval. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

4. Use factoring by grouping.



$$\begin{aligned} 2\sin x + 1 &= 0 \quad \text{or} \quad & 2\cos x - 1 &= 0 \\ 2\sin x &= -1 & 2\cos x &= 1 \\ \sin x &= -\frac{1}{2} & \cos x &= \frac{1}{2} \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6} & x &= \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

5. You can factor this one like a quadratic.

$$\begin{aligned} \sin^2 x - 2\sin x - 3 &= 0 \\ (\sin x - 3)(\sin x + 1) &= 0 \\ \sin x - 3 = 0 & \qquad\qquad \sin x + 1 = 0 \\ \sin x = 3 & \qquad \text{or} \qquad \sin x = -1 \\ x = \sin^{-1}(3) & \qquad\qquad x = \frac{3\pi}{2} \end{aligned}$$

For this problem the only solution is $\frac{3\pi}{2}$ because sine cannot be 3 (it is not in the range).

6.

$$\begin{aligned} \tan^2 x &= 3\tan x \\ \tan^2 x - 3\tan x &= 0 \\ \tan x(\tan x - 3) &= 0 \\ \tan x &= 0 \quad \text{or} \quad \tan x = 3 \\ x &= 0, \pi \qquad\quad x = 1.25 \end{aligned}$$

7. $2\sin^2 \frac{x}{4} - 3\cos \frac{x}{4} = 0$

$$\begin{aligned} 2\left(1-\cos^2 \frac{x}{4}\right) - 3\cos \frac{x}{4} &= 0 \\ 2 - 2\cos^2 \frac{x}{4} - 3\cos \frac{x}{4} &= 0 \\ 2\cos^2 \frac{x}{4} + 3\cos \frac{x}{4} - 2 &= 0 \\ \left(2\cos \frac{x}{4} - 1\right)\left(\cos \frac{x}{4} + 2\right) &= 0 \\ \swarrow \qquad\qquad \searrow & \\ 2\cos \frac{x}{4} - 1 = 0 \quad \text{or} \quad \cos \frac{x}{4} + 2 &= 0 \\ 2\cos \frac{x}{4} = 1 \qquad\qquad \cos \frac{x}{4} &= -2 \\ \cos \frac{x}{4} &= \frac{1}{2} \\ \frac{x}{4} = \frac{\pi}{3} \quad \text{or} \quad & \frac{5\pi}{3} \\ x = \frac{4\pi}{3} \quad \text{or} \quad & \frac{20\pi}{3} \end{aligned}$$

$\frac{20\pi}{3}$ is eliminated as a solution because it is outside of the range and $\cos \frac{x}{4} = -2$ will not generate any solutions because -2 is outside of the range of cosine. Therefore, the only solution is $\frac{4\pi}{3}$.

8.

$$\begin{aligned} 3 - 3\sin^2 x &= 8\sin x \\ 3 - 3\sin^2 x - 8\sin x &= 0 \\ 3\sin^2 x + 8\sin x - 3 &= 0 \\ (3\sin x - 1)(\sin x + 3) &= 0 \\ 3\sin x - 1 = 0 \quad \text{or} \quad \sin x + 3 &= 0 \\ 3\sin x &= 1 \\ \sin x = \frac{1}{3} \qquad \sin x &= -3 \\ x = 0.3398 \textit{ radians} \quad & \text{No solution exists} \\ x = \pi - 0.3398 = 2.8018 \textit{ radians} & \end{aligned}$$

9. $2\sin x \tan x = \tan x + \sec x$

$$\begin{aligned} 2\sin x \cdot \frac{\sin x}{\cos x} &= \frac{\sin x}{\cos x} + \frac{1}{\cos x} \\ \frac{2\sin^2 x}{\cos x} &= \frac{\sin x + 1}{\cos x} \\ 2\sin^2 x &= \sin x + 1 \\ 2\sin^2 x - \sin x - 1 &= 0 \\ (2\sin x + 1)(\sin x - 1) &= 0 \\ 2\sin x + 1 = 0 \quad \text{or} \quad \sin x - 1 &= 0 \\ 2\sin x = -1 \qquad \sin x &= 1 \\ \sin x &= -\frac{1}{2} \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

One of the solutions is not $\frac{\pi}{2}$, because $\tan x$ and $\sec x$ in the original equation are undefined for this value of x.

10.

$$\begin{aligned}
2\cos^2 x+3\sin x-3&=0\\
2(1-\sin^2 x)+3\sin x-3&=0 \text{ Pythagorean Identity}\\
2-2\sin^2 x+3\sin x-3&=0\\
-2\sin^2 x+3\sin x-1&=0 \text{ Multiply by } -1\\
2\sin^2 x-3\sin x+1&=0\\
(2\sin x-1)(\sin x-1)&=0\\
2\sin x-1=0 \qquad &\text{or} \qquad \sin x-1=0\\
2\sin x&=1\\
\sin x=\frac{1}{2} \qquad & \qquad \sin x=1\\
x=\frac{\pi}{6},\frac{5\pi}{6} \qquad & \qquad x=\frac{\pi}{2}
\end{aligned}$$

11. $\tan^2 x+\tan x-2=0$

$$\begin{aligned}
\frac{-1\pm\sqrt{1^2-4(1)(-2)}}{2}&=\tan x\\
\frac{-1\pm\sqrt{1+8}}{2}&=\tan x\\
\frac{-1\pm 3}{2}&=\tan x\\
\tan x&=-2 \quad \text{or} \quad 1
\end{aligned}$$

$\tan x=1$ when $x=\frac{\pi}{4}$, in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$\tan x=-2$ when $x=-1.107$ *rad*

12. $5\cos^2\theta-6\sin\theta=0$ over the interval $[0,2\pi]$.

$$\begin{aligned}
5\left(1-\sin^2 x\right)-6\sin x&=0\\
-5\sin^2 x-6\sin x+5&=0\\
5\sin^2 x+6\sin x-5&=0\\
\frac{-6\pm\sqrt{6^2-4(5)(-5)}}{2(5)}&=\sin x\\
\frac{-6\pm\sqrt{36+100}}{10}&=\sin x\\
\frac{-6\pm\sqrt{136}}{10}&=\sin x\\
\frac{-6\pm 2\sqrt{34}}{10}&=\sin x\\
\frac{-3\pm\sqrt{34}}{5}&=\sin x
\end{aligned}$$

$x=\sin^{-1}\left(\frac{-3+\sqrt{34}}{5}\right)$ or $\sin^{-1}\left(\frac{-3-\sqrt{34}}{5}\right)$$x=0.6018$ *rad* or 2.5398 *rad* from the first expression, the second expression will not yield any answers because it is out the the range of sine.

3.4 Sum and Difference Identities

1. Answers:

a.

$$\cos\frac{5\pi}{12}=\cos\left(\frac{2\pi}{12}+\frac{3\pi}{12}\right)=\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}$$
$$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

b.

$$\cos\frac{7\pi}{12}=\cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right)=\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

c.

$$\sin 345^\circ=\sin(300^\circ+45^\circ)=\sin 300^\circ\cos 45^\circ+\cos 300^\circ\sin 45^\circ$$
$$=-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=-\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

d.

$$\tan 75^\circ=\tan(45^\circ+30^\circ)=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$$
$$=\frac{1+\frac{\sqrt{3}}{3}}{1-1\cdot\frac{\sqrt{3}}{3}}=\frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}=\frac{3+\sqrt{3}}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}}=\frac{9+6\sqrt{3}+3}{9-3}=\frac{12+6\sqrt{3}}{6}=2+\sqrt{3}$$

e.

$$\cos 345^\circ=\cos(315^\circ+30^\circ)=\cos 315^\circ\cos 30^\circ-\sin 315^\circ\sin 30^\circ$$
$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-(-\frac{\sqrt{2}}{2})\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

f.

$$\sin\frac{17\pi}{12}=\sin\left(\frac{9\pi}{12}+\frac{8\pi}{12}\right)=\sin\left(\frac{3\pi}{4}+\frac{2\pi}{3}\right)=\sin\frac{3\pi}{4}\cos\frac{2\pi}{3}+\cos\frac{3\pi}{4}\sin\frac{2\pi}{3}$$
$$=\frac{\sqrt{2}}{2}\cdot(-\frac{1}{2})+-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}=-\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}=\frac{-\sqrt{2}-\sqrt{6}}{4}$$

2. If $\sin y=\frac{12}{13}$ and in Quadrant II, then by the Pythagorean Theorem $\cos y=-\frac{5}{13}(12^2+b^2=13^2)$.
   - And, if $\sin z=\frac{3}{5}$ and in Quadrant I, then by the Pythagorean Theorem $\cos z=\frac{4}{5}(a^2+3^2=5^2)$.
   - $\cos(y-z)=\cos y\cos z+\sin y\sin z$ and $=-\frac{5}{13}\cdot\frac{4}{5}+\frac{12}{13}\cdot\frac{3}{5}=-\frac{20}{65}+\frac{36}{65}=\frac{16}{65}$
3. If $\sin y=-\frac{5}{13}$ and in Quadrant III, then cosine is also negative.
   - By the Pythagorean Theorem, the second leg is $12(5^2+b^2=13^2)$, so $\cos y=-\frac{12}{13}$.
   - If the $\sin z=\frac{4}{5}$ and in Quadrant II, then the cosine is also negative.

- By the Pythagorean Theorem, the second leg is $3(4^2+b^2=5^2)$, so $\cos = -\frac{3}{5}$.
- To find $\sin(y+z)$, plug this information into the sine sum formula.

$$\begin{aligned}\sin(y+z) &= \sin y\cos z+\cos y\sin z\\ &= -\frac{5}{13}\cdot -\frac{3}{5}+-\frac{12}{13}\cdot\frac{4}{5}=\frac{15}{65}-\frac{48}{65}=-\frac{33}{65}\end{aligned}$$

4. Answers:
   a. This is the cosine difference formula, so: $\cos 80^\circ\cos 20^\circ+\sin 80^\circ 20^\circ=\cos(80^\circ-20^\circ)=\cos 60^\circ=\frac{1}{2}$
   b. This is the expanded sine sum formula, so: $\sin 25^\circ\cos 5^\circ+\cos 25^\circ\sin 5^\circ=\sin(25^\circ+5^\circ)=\sin 30^\circ=\frac{1}{2}$
5. Step 1: Expand using the cosine sum formula and change everything into sine and cosine

$$\begin{aligned}\frac{\cos(m-n)}{\sin m\cos n}&=\cot m+\tan n\\ \frac{\cos m\cos n+\sin m\sin n}{\sin m\cos n}&=\frac{\cos m}{\sin m}+\frac{\sin n}{\cos n}\end{aligned}$$

Step 2: Find a common denominator for the right hand side.

$$=\frac{\cos m\cos n+\sin m\sin n}{\sin m\cos n}$$

The two sides are the same, thus they are equal to each other and the identity is true.
6. $\cos(\pi+\theta)=\cos\pi\cos\theta-\sin\pi\sin\theta=-\cos\theta$
7. Step 1: Expand $\sin(a+b)$ and $\sin(a-b)$ using the sine sum and difference formulas. $\sin(a+b)\sin(a-b)=\cos^2 b-\cos^2 a$ $(\sin a\cos b+\cos a\sin b)(\sin a\cos b-\cos a\sin b)$ Step 2: FOIL and simplify.

$$\sin^2 a\cos^2 b-\sin a\cos a\sin b\cos b+\sin a\sin b\cos a\cos b-\cos^2 a\sin^2 b\sin^2 a\cos^2 b-\cos a^2\sin^2 b$$

Step 3: Substitute $(1-\cos^2 a)$ for $\sin^2 a$ and $(1-\cos^2 b)$ for $\sin^2 b$, distribute and simplify.

$$\begin{aligned}&(1-\cos^2 a)\cos^2 b-\cos a^2(1-\cos^2 b)\\ &\cos^2 b-\cos^2 a\cos^2 b-\cos^2 a+\cos^2 a\cos^2 b\\ &\cos^2 b-\cos^2 a\end{aligned}$$

8. $\tan(\pi+\theta)=\frac{\tan\pi+\tan\theta}{1-\tan\pi\tan\theta}=\frac{\tan\theta}{1}=\tan\theta$
9. $\sin\frac{\pi}{2}=\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{4}-\cos\frac{\pi}{4}\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{2}{4}-\frac{2}{4}=0$ This could also be verified by using $60^\circ+30^\circ$
10. Step 1: Expand using the cosine and sine sum formulas.

$$\cos(x+y)\cos y+\sin(x+y)\sin y=(\cos x\cos y-\sin x\sin y)\cos y+(\sin x\cos y+\cos x\sin y)\sin y$$

Step 2: Distribute $\cos y$ and $\sin y$ and simplify.

$$\begin{aligned}&=\cos x\cos^2 y-\sin x\sin y\cos y+\sin x\sin y\cos y+\cos x\sin^2 y\\ &=\cos x\cos^2 y+\cos x\sin^2 y\\ &=\cos x\underbrace{(\cos^2 y+\sin^2 y)}_{1}\\ &=\cos x\end{aligned}$$

11. Step 1: Expand left hand side using the sum and difference formulas

$$\frac{\cos(c+d)}{\cos(c-d)} = \frac{1-\tan c\tan d}{1+\tan c\tan d}$$
$$\frac{\cos c\cos d-\sin c\sin d}{\cos c\cos d+\sin c\sin d} = \frac{1-\tan c\tan d}{1+\tan c\tan d}$$

Step 2: Divide each term on the left side by $\cos c\cos d$ and simplify

$$\frac{\frac{\cos c\cos d}{\cos c\cos d}-\frac{\sin c\sin d}{\cos c\cos d}}{\frac{\cos c\cos d}{\cos c\cos d}+\frac{\sin c\sin d}{\cos c\cos d}} = \frac{1-\tan c\tan d}{1+\tan c\tan d}$$
$$\frac{1-\tan c\tan d}{1+\tan c\tan d} = \frac{1-\tan c\tan d}{1+\tan c\tan d}$$

12. To find all the solutions, between $[0,2\pi)$, we need to expand using the sum formula and isolate the $\cos x$.

$$2\cos^2\left(x+\frac{\pi}{2}\right) = 1$$
$$\cos^2\left(x+\frac{\pi}{2}\right) = \frac{1}{2}$$
$$\cos\left(x+\frac{\pi}{2}\right) = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$
$$\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2} = \pm\frac{\sqrt{2}}{2}$$
$$\cos x\cdot 0-\sin x\cdot 1 = \pm\frac{\sqrt{2}}{2}$$
$$-\sin x = \pm\frac{\sqrt{2}}{2}$$
$$\sin x = \pm\frac{\sqrt{2}}{2}$$

This is true when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$

13. First, solve for $\tan(x+\frac{\pi}{6})$.

$$2\tan^2\left(x+\frac{\pi}{6}\right)+1 = 7$$
$$2\tan^2\left(x+\frac{\pi}{6}\right) = 6$$
$$\tan^2\left(x+\frac{\pi}{6}\right) = 3$$
$$\tan\left(x+\frac{\pi}{6}\right) = \pm\sqrt{3}$$

Now, use the tangent sum formula to expand for when $\tan(x+\frac{\pi}{6}) = \sqrt{3}$.

$$\frac{\tan x+\tan\frac{\pi}{6}}{1-\tan x\tan\frac{\pi}{6}} = \sqrt{3}$$
$$\tan x+\tan\frac{\pi}{6} = \sqrt{3}\left(1-\tan x\tan\frac{\pi}{6}\right)$$
$$\tan x+\frac{\sqrt{3}}{3} = \sqrt{3}-\sqrt{3}\tan x\cdot\frac{\sqrt{3}}{3}$$
$$\tan x+\frac{\sqrt{3}}{3} = \sqrt{3}-\tan x$$
$$2\tan x = \frac{2\sqrt{3}}{3}$$
$$\tan x = \frac{\sqrt{3}}{3}$$

This is true when $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. If the tangent sum formula to expand for when $\tan(x+\frac{\pi}{6}) = -\sqrt{3}$, we get no solution as shown.

$$\begin{aligned}
\frac{\tan x + \tan\frac{\pi}{6}}{1-\tan x \tan\frac{\pi}{6}} &= -\sqrt{3} \\
\tan x + \tan\frac{\pi}{6} &= -\sqrt{3}\left(1-\tan x \tan\frac{\pi}{6}\right) \\
\tan x + \frac{\sqrt{3}}{3} &= -\sqrt{3} + \sqrt{3}\tan x \cdot \frac{\sqrt{3}}{3} \\
\tan x + \frac{\sqrt{3}}{3} &= -\sqrt{3} + \tan x \\
\frac{\sqrt{3}}{3} &= -\sqrt{3}
\end{aligned}$$

Therefore, the tangent sum formula cannot be used in this case. However, since we know that $\tan(x+\frac{\pi}{6}) = -\sqrt{3}$ when $x+\frac{\pi}{6} = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$, we can solve for x as follows.

$$\begin{aligned}
x+\frac{\pi}{6} &= \frac{5\pi}{6} \\
x &= \frac{4\pi}{6} \\
x &= \frac{2\pi}{3}
\end{aligned}$$

$$\begin{aligned}
x+\frac{\pi}{6} &= \frac{11\pi}{6} \\
x &= \frac{10\pi}{6} \\
x &= \frac{5\pi}{3}
\end{aligned}$$

Therefore, all of the solutions are $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

14. To solve, expand each side:

$$\begin{aligned}
\sin\left(x+\frac{\pi}{6}\right) &= \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x \\
\sin\left(x-\frac{\pi}{4}\right) &= \sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x
\end{aligned}$$

Set the two sides equal to each other:

$$\begin{aligned}
\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x &= \frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x\\
\sqrt{3}\sin x+\cos x &= \sqrt{2}\sin x-\sqrt{2}\cos x\\
\sqrt{3}\sin x-\sqrt{2}\sin x &= -\cos x-\sqrt{2}\cos x\\
\sin x\left(\sqrt{3}-\sqrt{2}\right) &= \cos x\left(-1-\sqrt{2}\right)\\
\frac{\sin x}{\cos x} &= \frac{-1-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\\
\tan x &= \frac{-1-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}\\
&= \frac{-\sqrt{3}-\sqrt{2}+\sqrt{6}-2}{3-2}\\
&= -2+\sqrt{6}-\sqrt{3}-\sqrt{2}
\end{aligned}$$

As a decimal, this is -2.69677, so $\tan^{-1}(-2.69677)=x, x=290.35^\circ$ and 110.35°.

3.5 Double Angle Identities

1. If $\sin x = \frac{4}{5}$ and in Quadrant II, then cosine and tangent are negative. Also, by the Pythagorean Theorem, the third side is $3(b = \sqrt{5^2 - 4^2})$. So, $\cos x = -\frac{3}{5}$ and $\tan x = -\frac{4}{3}$. Using this, we can find $\sin 2x, \cos 2x$, and $\tan 2x$.

$$\begin{aligned} \sin 2x &= 2\sin x\cos x \\ &= 2\cdot\frac{4}{5}\cdot-\frac{3}{5} \\ &= -\frac{24}{25} \end{aligned}$$

$$\begin{aligned} \cos 2x &= 1-\sin^2 x \\ &= 1-2\cdot\left(\frac{4}{5}\right)^2 \\ &= 1-2\cdot\frac{16}{25} \\ &= 1-\frac{32}{25} \\ &= -\frac{7}{25} \end{aligned}$$

$$\begin{aligned} \tan 2x &= \frac{2\tan x}{1-\tan^2 x} \\ &= \frac{2\cdot-\frac{4}{3}}{1-\left(-\frac{4}{3}\right)^2} \\ &= \frac{-\frac{8}{3}}{1-\frac{16}{9}} = -\frac{8}{3} \div -\frac{7}{9} \\ &= -\frac{8}{3}\cdot-\frac{9}{7} \\ &= \frac{24}{7} \end{aligned}$$

2. This is one of the forms for $\cos 2x$.

$$\begin{aligned} \cos^2 15^\circ - \sin^2 15^\circ &= \cos(15^\circ\cdot 2) \\ &= \cos 30^\circ \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

3. Step 1: Use the cosine sum formula

$$\begin{aligned} \cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \cos(2\theta+\theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \end{aligned}$$

Step 2: Use double angle formulas for $\cos 2\theta$ and $\sin 2\theta$

$$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$

Step 3: Distribute and simplify.

$$\begin{aligned} &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\ &= -\cos\theta(-2\cos^2\theta + 2\sin^2\theta + 1) \\ &= -\cos\theta[-2\cos^2\theta + 2(1-\cos^2\theta) + 1] && \rightarrow \text{Substitute } 1-\cos^2\theta \text{ for } \sin^2\theta \\ &= -\cos\theta[-2\cos^2\theta + 2 - 2\cos^2\theta + 1] \\ &= -\cos\theta(-4\cos^2\theta + 3) \\ &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

4. Step 1: Expand $\sin 2t$ using the double angle formula.

$$\begin{aligned} \sin 2t - \tan t &= \tan t\cos 2t \\ 2\sin t\cos t - \tan t &= \tan t\cos 2t \end{aligned}$$

Step 2: change $\tan t$ and find a common denominator.

$$\begin{gathered}2\sin t\cos t-\frac{\sin t}{\cos t}\\ \frac{2\sin t\cos^2 t-\sin t}{\cos t}\\ \frac{\sin t(2\cos^2 t-1)}{\cos t}\\ \frac{\sin t}{\cos t}\cdot(2\cos^2 t-1)\\ \tan t\cos 2t\end{gathered}$$

5. If $\sin x=-\frac{9}{41}$ and in Quadrant III, then $\cos x=-\frac{40}{41}$ and $\tan x=\frac{9}{40}$ (Pythagorean Theorem, $b=\sqrt{41^2-(-9)^2}$). So,

$$\begin{aligned}\cos 2x&=2\cos^2 x-1\\ &=2\left(-\frac{40}{41}\right)^2-1\\ &=\frac{3200}{1681}-\frac{1681}{1681}\\ &=\frac{1519}{1681}\end{aligned}$$

$$\begin{aligned}\sin 2x&=2\sin x\cos x\\ &=2\cdot-\frac{9}{41}\cdot-\frac{40}{41}\\ &=\frac{720}{1681}\end{aligned}$$

$$\begin{aligned}\tan 2x&=\frac{\sin 2x}{\cos 2x}\\ &=\frac{\frac{720}{1681}}{\frac{1519}{1681}}\\ &=\frac{720}{1519}\end{aligned}$$

6. Step 1: Expand $\sin 2x$

$$\begin{aligned}\sin 2x+\sin x&=0\\ 2\sin x\cos x+\sin x&=0\\ \sin x(2\cos x+1)&=0\end{aligned}$$

Step 2: Separate and solve each for x.

$$\begin{aligned}\sin x&=0\\ x&=0,\pi\end{aligned}$$

or

$$\begin{aligned}2\cos x+1&=0\\ \cos x&=-\frac{1}{2}\\ x&=\frac{2\pi}{3},\frac{4\pi}{3}\end{aligned}$$

7. Expand $\cos 2x$ and simplify

$$\begin{aligned}\cos^2 x-\cos 2x&=0\\ \cos^2 x-(2\cos^2 x-1)&=0\\ -\cos^2 x+1&=0\\ \cos^2 x&=1\\ \cos x&=\pm 1\end{aligned}$$

$\cos x=1$ when $x=0$, and $\cos x=-1$ when $x=\pi$. Therefore, the solutions are $x=0,\pi$.

8. a. 3.429 b. 0.960 c. 0.280

9. a.

$$
\begin{aligned}
2\csc x\ 2x &= \frac{2}{\sin 2x} \\
2\csc x\ 2x &= \frac{2}{2\sin x\cos x} \\
2\csc x\ 2x &= \frac{1}{\sin x\cos x} \\
2\csc x\ 2x &= \left(\frac{\sin x}{\sin x}\right)\left(\frac{1}{\sin x\cos x}\right) \\
2\csc x\ 2x &= \frac{\sin x}{\sin^2 x\cos x} \\
2\csc x\ 2x &= \frac{1}{\sin^2 x}\cdot\frac{\sin x}{\cos x} \\
2\csc x\ 2x &= \csc^2 x\tan x
\end{aligned}
$$

b.

$$
\begin{aligned}
\cos^4\theta - \sin^4\theta &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \\
\cos^4 - \sin^4\theta &= 1(\cos^2\theta - \sin^2\theta) \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\therefore \cos^4\theta - \sin^4\theta &= \cos 2\theta
\end{aligned}
$$

c.

$$
\begin{aligned}
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x\cos x}{1+(1-2\sin^2 x)} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x\cos x}{2-2\sin^2 x} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x\cos x}{2(1-\sin^2 x)} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x\cos x}{2\cos^2 x} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{\sin x}{\cos x} \\
\frac{\sin 2x}{1+\cos 2x} &= \tan x
\end{aligned}
$$

10. $\cos 2x - 1 = \sin^2 x$

$$
\begin{aligned}
(1-2\sin^2 x) - 1 &= \sin^2 x \\
-2\sin^2 x &= \sin^2 x \\
0 &= 3\sin^2 x \\
0 &= \sin^2 x \\
0 &= \sin x \\
x &= 0, \pi
\end{aligned}
$$

11.

$$\begin{aligned}
\cos 2x &= \cos x \\
2\cos^2 x - 1 &= \cos x \\
2\cos^2 x - \cos x - 1 &= 0 \\
(2\cos x + 1)(\cos x - 1) &= 0 \\
&\searrow \qquad \searrow \\
2\cos x + 1 = 0 \quad &\text{or} \quad \cos x - 1 = 0 \\
2\cos x = -1 \quad & \qquad \cos x = 1 \\
\cos x = -\frac{1}{2} &
\end{aligned}$$

$\cos x = 1$ when $x = 0$ and $\cos x = -\frac{1}{2}$ when $x = \frac{2\pi}{3}$.

12.

$$\begin{aligned}
2\csc 2x \tan x &= \sec^2 x \\
\frac{2}{\sin 2x} \cdot \frac{\sin x}{\cos x} &= \frac{1}{\cos^2 x} \\
\frac{2}{2\sin x \cos x} \cdot \frac{\sin x}{\cos x} &= \frac{1}{\cos^2 x} \\
\frac{1}{\cos^2 x} &= \frac{1}{\cos^2 x}
\end{aligned}$$

13. $\sin 2x - \cos 2x = 1$

$$\begin{aligned}
2\sin x \cos x - (1 - 2\sin^2 x) &= 1 \\
2\sin x \cos x - 1 + 2\sin^2 x &= 1 \\
2\sin x \cos x + 2\sin^2 x &= 2 \\
\sin x \cos x + \sin^2 x &= 1 \\
\sin x \cos x &= 1 - \sin^2 x \\
\sin x \cos x &= \cos^2 x \\
\left(\pm\sqrt{1-\cos^2 x}\right)\cos x &= \cos^2 x \\
\left(1-\cos^2 x\right)\cos^2 x &= \cos^4 x \\
\cos^2 x - \cos^4 x &= \cos^4 x \\
\cos^2 x - 2\cos^4 x &= 0 \\
\cos^2 x(1 - 2\cos^2 x) &= 0 \\
&\swarrow \qquad \searrow \\
& \qquad 1 - 2\cos^2 x = 0 \\
\cos^2 x = 0 \quad & \qquad -2\cos^2 x = -1 \\
\cos x = 0 \quad \text{or} & \qquad \cos^2 x = \frac{1}{2} \\
x = \frac{\pi}{2}, \frac{3\pi}{2} & \qquad \cos x = \pm\frac{\sqrt{2}}{2} \\
& \qquad x = \frac{\pi}{4}, \frac{5\pi}{4}
\end{aligned}$$

Note: If we go back to the equation $\sin x \cos x = \cos^2 x$, we can see that $\sin x \cos x$ must be positive or zero, since $\cos^2 x$ is always positive or zero. For this reason, $\sin x$ and $\cos x$ must have the same sign (or one of them must be zero), which means that x cannot be in the second or fourth quadrants. This is why $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ are not valid solutions.

14. Use the double angle identity for $\cos 2x$.

$$\begin{aligned}
\sin^2 x - 2 &= \cos 2x \\
\sin^2 x - 2 &= \cos 2x \\
\sin^2 x - 2 &= 1 - 2\sin^2 x \\
3\sin^2 x &= 3 \\
\sin^2 x &= 1 \\
\sin x &= \pm 1 \\
x &= \frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}$$

3.6 Half-Angle Identities

1. Answers:

a.

$$\cos 112.5^\circ = \cos \frac{225^\circ}{2} = -\sqrt{\frac{1+\cos 225^\circ}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$$

b.

$$\sin 105^\circ = \sin \frac{210^\circ}{2} = \sqrt{\frac{1-\cos 210^\circ}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

c.

$$\tan \frac{7\pi}{8} = \tan \frac{1}{2} \cdot \frac{7\pi}{4} = \frac{1-\cos \frac{7\pi}{4}}{\sin \frac{7\pi}{4}}$$
$$= \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -\frac{2-\sqrt{2}}{\sqrt{2}} = \frac{-2\sqrt{2}+2}{2} = -\sqrt{2}+1$$

d. $\tan \frac{\pi}{8} = \tan \frac{1}{2} \cdot \frac{\pi}{4} = \frac{1-\cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \sqrt{2}-1$

e. $\sin 67.5^\circ = \sin \frac{135^\circ}{2} = \sqrt{\frac{1-\cos 135^\circ}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

f. $\tan 165^\circ = \tan \frac{330^\circ}{2} = \frac{1-\cos 330^\circ}{\sin 330^\circ} = \frac{1-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\frac{2-\sqrt{3}}{2}}{-\frac{1}{2}} = -\left(2-\sqrt{3}\right) = -2+\sqrt{3}$

2. If $\sin\theta = \frac{7}{25}$, then by the Pythagorean Theorem the third side is 24. Because θ is in the second quadrant, $\cos\theta = -\frac{24}{25}$.

$$\sin \frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1+\frac{24}{25}}{2}} = \sqrt{\frac{49}{50}} = \frac{7}{5\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{10}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1-\frac{24}{25}}{2}} = \sqrt{\frac{1}{50}} = \frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{10}$$

$$\tan \frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1+\frac{24}{25}}{1-\frac{24}{25}}} = \sqrt{\frac{49}{50} \cdot \frac{50}{1}} = \sqrt{49} = 7$$

3. Step 1: Change right side into sine and cosine.

$$\begin{aligned}
\tan\frac{b}{2} &= \frac{\sec b}{\sec b\csc b+\csc b} \\
&= \frac{1}{\cos b} \div \csc b(\sec b+1) \\
&= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1}{\cos b}+1\right) \\
&= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1+\cos b}{\cos b}\right) \\
&= \frac{1}{\cos b} \div \frac{1+\cos b}{\sin b\cos b} \\
&= \frac{1}{\cos b} \cdot \frac{\sin b\cos b}{1+\cos b} \\
&= \frac{\sin b}{1+\cos b}
\end{aligned}$$

Step 2: At the last step above, we have simplified the right side as much as possible, now we simplify the left side, using the half angle formula.

$$\begin{aligned}
\sqrt{\frac{1-\cos b}{1+\cos b}} &= \frac{\sin b}{1+\cos b} \\
\frac{1-\cos b}{1+\cos b} &= \frac{\sin^2 b}{(1+\cos b)^2} \\
(1-\cos b)(1+\cos b)^2 &= \sin^2 b(1+\cos b) \\
(1-\cos b)(1+\cos b) &= \sin^2 b \\
1-\cos^2 b &= \sin^2 b
\end{aligned}$$

4. Step 1: change cotangent to cosine over sine, then cross-multiply.

$$\begin{aligned}
\cot\frac{c}{2} &= \frac{\sin c}{1-\cos c} \\
&= \frac{\cos\frac{c}{2}}{\sin\frac{c}{2}} = \sqrt{\frac{1+\cos c}{1-\cos c}} \\
\sqrt{\frac{1+\cos c}{1-\cos c}} &= \frac{\sin c}{1-\cos c} \\
\frac{1+\cos c}{1-\cos c} &= \frac{\sin^2 c}{(1-\cos c)^2} \\
(1+\cos c)(1-\cos c)^2 &= \sin^2 c(1-\cos c) \\
(1+\cos c)(1-\cos c) &= \sin^2 c \\
1-\cos^2 c &= \sin^2 c
\end{aligned}$$

5.

$$\begin{aligned}
\sin x\tan\frac{x}{2}+2\cos x &= \sin x\left(\frac{1-\cos x}{\sin x}\right)+2\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 1-\cos x+2\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 1+\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 2\cos^2\frac{x}{2}
\end{aligned}$$

6. First, we need to find the third side.

- Using the Pythagorean Theorem, we find that the final side is $\sqrt{105}\left(b=\sqrt{13^2-(-8)^2}\right)$.
- Using this information, we find that $\cos u = \pm\frac{\sqrt{105}}{13}$.
- Plugging this into the half angle formula, we get:

$$\begin{aligned}\cos\frac{u}{2} &= -\sqrt{\frac{1\pm\sqrt{\frac{105}{13}}}{2}}\\ &= -\sqrt{\frac{\frac{13\pm\sqrt{105}}{13}}{2}}\\ &= -\sqrt{\frac{13\pm\sqrt{105}}{26}}\end{aligned}$$

7. To solve $2\cos^2\frac{x}{2}=1$, first we need to isolate cosine, then use the half angle formula.

$$\begin{aligned}2\cos^2\frac{x}{2} &= 1\\ \cos^2\frac{x}{2} &= \frac{1}{2}\\ \frac{1+\cos x}{2} &= \frac{1}{2}\\ 1+\cos x &= 1\\ \cos x &= 0\end{aligned}$$

$\cos x = 0$ when $x=\frac{\pi}{2},\frac{3\pi}{2}$

8. To solve $\tan\frac{a}{2}=4$, first isolate tangent, then use the half angle formula.

$$\begin{aligned}\tan\frac{a}{2} &= 4\\ \sqrt{\frac{1-\cos a}{1+\cos a}} &= 4\\ \frac{1-\cos a}{1+\cos a} &= 16\\ 16+16\cos a &= 1-\cos a\\ 17\cos a &= -15\\ \cos a &= -\frac{15}{17}\end{aligned}$$

Using your graphing calculator, $\cos a = -\frac{15}{17}$ when $a = 152^\circ, 208^\circ$

9.

$$\begin{aligned}
\cos\frac{x}{2} &= 1+\cos x \\
\pm\sqrt{\frac{1+\cos x}{2}} &= 1+\cos x && \text{Half angle identity} \\
\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 &= (1+\cos x)^2 && \text{square both sides} \\
\frac{1+\cos x}{2} &= 1+2\cos x+\cos^2 x \\
2\left(\frac{1+\cos x}{2}\right) &= 2(1+2\cos x+\cos^2 x) \\
1+\cos x &= 2+4\cos x+2\cos^2 x \\
2\cos^2 x+3\cos x+1 &= 0 \\
(2\cos x+1)(\cos x+1) &= 0 \\
\text{Then} \quad 2\cos x+1 &= 0 \\
\frac{2\cos x}{2} &= \frac{-1}{2} \\
x &= \frac{2\pi}{3}, \frac{4\pi}{3} \\
\text{Or} \quad \cos x+1 &= 0 \\
\cos x &= -1 \\
x &= \pi
\end{aligned}$$

10. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$ This is the two formulas for $\tan\frac{x}{2}$. Cross-multiply.

$$\begin{aligned}
\frac{\sin x}{1+\cos x} &= \frac{1-\cos x}{\sin x} \\
(1-\cos x)(1+\cos x) &= \sin^2 x \\
1+\cos x-\cos x-\cos^2 x &= \sin^2 x \\
1-\cos^2 x &= \sin^2 x \\
1 &= \sin^2 x+\cos^2 x
\end{aligned}$$

3.7 Products, Sums, Linear Combinations, and Applications

1. Using the sum-to-product formula:

$$\sin 9x + \sin 5x$$
$$\frac{1}{2}\left(\sin\left(\frac{9x+5x}{2}\right)\cos\left(\frac{9x-5x}{2}\right)\right)$$
$$\frac{1}{2}\sin 7x \cos 2x$$

2. Using the difference-to-product formula:

$$\cos 4y - \cos 3y$$
$$-2\sin\left(\frac{4y+3y}{2}\right)\sin\left(\frac{4y-3y}{2}\right)$$
$$-2\sin\frac{7y}{2}\sin\frac{y}{2}$$

3. Using the difference-to-product formulas:

$$\frac{\cos 3a - \cos 5a}{\sin 3a - \sin 5a} = -\tan 4a$$
$$\frac{-2\sin\left(\frac{3a+5a}{2}\right)\sin\left(\frac{3a-5a}{2}\right)}{2\sin\left(\frac{3a-5a}{2}\right)\cos\left(\frac{3a+5a}{2}\right)}$$
$$-\frac{\sin 4a}{\cos 4a}$$
$$-\tan 4a$$

4. Using the product-to-sum formula:

$$\sin 6\theta \sin 4\theta$$
$$\frac{1}{2}(\cos(6\theta-4\theta)-\cos(6\theta+4\theta))$$
$$\frac{1}{2}(\cos 2\theta - \cos 10\theta)$$

5. (a) If $5\cos x - 5\sin x$, then $A = 5$ and $B = -5$.

   - By the Pythagorean Theorem, $C = 5\sqrt{2}$ and $\cos D = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.
   - So, because B is negative, D is in Quadrant IV.
   - Therefore, $D = \frac{7\pi}{4}$.
   - Our final answer is $5\sqrt{2}\cos\left(x - \frac{7\pi}{4}\right)$.

   (b) If $-15\cos 3x - 8\sin 3x$, then $A = -15$ and $B = -8$.

   - By the Pythagorean Theorem, $C = 17$.
   - Because A and B are both negative, D is in Quadrant III, which means $D = \cos^{-1}\left(\frac{15}{17}\right) = 0.49 + \pi = 3.63$ *rad*.
   - Our final answer is $17\cos 3(x - 3.63)$.

6. Using the sum-to-product formula:

$$\begin{aligned}
\sin 11x - \sin 5x &= 0 \\
2\sin\frac{11x-5x}{2}\cos\frac{11x+5x}{2} &= 0 \\
2\sin 3x\cos 8x &= 0 \\
\sin 3x\cos 8x &= 0
\end{aligned}$$

$\sin 3x = 0$ or $\cos 8x = 0$

$$3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi \qquad 8x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2}$$

$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \qquad x = \frac{\pi}{16}, \frac{3\pi}{16}, \frac{5\pi}{16}, \frac{7\pi}{16}, \frac{9\pi}{16}, \frac{11\pi}{16}, \frac{13\pi}{16}, \frac{15\pi}{16}$$

7. Using the sum-to-product formula:

$$\begin{aligned}
\cos 4x + \cos 2x &= 0 \\
2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2} &= 0 \\
2\cos 3x\cos x &= 0 \\
\cos 3x\cos x &= 0
\end{aligned}$$

So, either $\cos 3x = 0$ or $\cos x = 0$

$$\begin{aligned}
3x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2} \\
x &= \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}
\end{aligned}$$

8. Move $\sin 3x$ over to the other side and use the sum-to-product formula:

$$\begin{aligned}
\sin 5x + \sin x &= \sin 3x \\
\sin 5x - \sin 3x + \sin x &= 0 \\
2\cos\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right) + \sin x &= 0 \\
2\cos 4x\sin x + \sin x &= 0 \\
\sin x(2\cos 4x + 1) &= 0
\end{aligned}$$

So $\sin x = 0$

$$\begin{aligned}
x = 0, \pi \ \text{ or } \ 2\cos 4x &= -1 \\
\cos 4x &= -\frac{1}{2} \\
4x &= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3} \\
&= \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6} \\
x = 0, &= \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}
\end{aligned}$$

9. Using the sum-to-product formula:

$$\begin{aligned}
f(t) &= \sin(200t+\pi)+\sin(200t-\pi) \\
&= 2\sin\left(\frac{(200t+\pi)+(200t-\pi)}{2}\right)\cos\left(\frac{(200t+\pi)-(200t-\pi)}{2}\right) \\
&= 2\sin\left(\frac{400t}{2}\right)\cos\left(\frac{2\pi}{2}\right) \\
&= 2\sin 200t\cos\pi \\
&= 2\sin 200t(-1) \\
&= -2\sin 200t
\end{aligned}$$

10. Derive a formula for $\tan 4x$.

$$\begin{aligned}
\tan 4x &= \tan(2x+2x) \\
&= \frac{\tan 2x+\tan 2x}{1-\tan 2x\tan 2x} \\
&= \frac{2\tan 2x}{1-\tan^2 2x} \\
&= \frac{2\cdot\frac{2\tan x}{1-\tan^2 x}}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2} \\
&= \frac{4\tan x}{1-\tan^2 x}\div\frac{(1-\tan^2 x)^2-4\tan^2 x}{(1-\tan^2 x)^2} \\
&= \frac{4\tan x}{1-\tan^2 x}\div\frac{1-2\tan^2 x+\tan^4 x-4\tan^2 x}{(1-\tan^2 x)^2} \\
&= \frac{4\tan x}{1-\tan^2 x}\cdot\frac{(1-\tan^2 x)^2}{1-6\tan^2 x+\tan^4 x} \\
&= \frac{4\tan x-4\tan^3 x}{1-6\tan^2 x+\tan^4 x}
\end{aligned}$$

11. Let $y=0$.

$$\begin{aligned}
3.50\sin t+1.20\sin 2t &= 0 \\
3.50\sin t+2.40\sin t\cos t &= 0, \text{ Double-Angle Identity} \\
\sin t(3.50+2.40\cos t) &= 0 \\
\sin t = 0 \text{ or } 3.50+2.40\cos t &= 0 \\
2.40\cos t &= -3.50 \\
\cos t &= -1.46 \rightarrow \text{no solution because } -1\le\cos t\le 1. \\
t &= 0,\pi
\end{aligned}$$

Chapter Summary

1. If the terminal side is on $(-8,15)$, then the hypotenuse of this triangle would be 17 (by the Pythagorean Theorem, $c=\sqrt{(-8)^2+15^2}$). Therefore, $\sin x=\frac{15}{17}, \cos x=-\frac{8}{17}$, and $\tan x=-\frac{15}{8}$.
2. If $\sin a=\frac{\sqrt{5}}{3}$ and $\tan a<0$, then a is in Quadrant II. Therefore $\sec a$ is negative. To find the third side, we

need to do the Pythagorean Theorem.

$$\begin{aligned}\left(\sqrt{5}\right)^2+b^2&=3^2\\5+b^2&=9\\b^2&=4\\b&=2\end{aligned}$$

So $\sec a=-\frac{3}{2}$.

3. Factor top, cancel like terms, and use the Pythagorean Theorem Identity. Note that this simplification doesn't hold true for values of x that are $\frac{\pi}{4}+\frac{n\pi}{2}$, where n is a positive integer,, since the original expression is undefined for these values of x.

$$\begin{aligned}&\frac{\cos^4 x-\sin^4 x}{\cos^2 x-\sin^2 x}\\&\frac{(\cos^2 x+\sin^2 x)(\cos^2 x-\sin^2 x)}{\cos^2 x-\sin^2 x}\\&\frac{\cos^2 x+\sin^2 x}{1}\\&\frac{1}{1}\\&1\end{aligned}$$

4. Change secant and cosecant into terms of sine and cosine, then find a common denominator.

$$\begin{aligned}\frac{1+\sin x}{\cos x\sin x}&=\sec x(\csc x+1)\\&=\frac{1}{\cos x}\left(\frac{1}{\sin x}+1\right)\\&=\frac{1}{\cos x}\left(\frac{1+\sin x}{\sin x}\right)\\&=\frac{1+\sin x}{\cos x\sin x}\end{aligned}$$

5.

$$\begin{aligned}\sec\left(x+\frac{\pi}{2}\right)+2&=0\\\sec\left(x+\frac{\pi}{2}\right)&=-2\\\cos\left(x+\frac{\pi}{2}\right)&=-\frac{1}{2}\\x+\frac{\pi}{2}&=\frac{2\pi}{3},\frac{4\pi}{3}\\x&=\frac{2\pi}{3}-\frac{\pi}{2},\frac{4\pi}{3}-\frac{\pi}{2}\\x&=\frac{\pi}{6},\frac{5\pi}{6}\end{aligned}$$

6.

$$\begin{aligned}8\sin\left(\frac{x}{2}\right)-8&=0\\8\sin\frac{x}{2}&=8\\\sin\frac{x}{2}&=1\\\frac{x}{2}&=\frac{x}{2}\\x&=\pi\end{aligned}$$

7.

$$\begin{aligned}2\sin^2 x+\sin 2x&=0\\2\sin^2 x+2\sin x\cos x&=0\\2\sin x(\sin x+\cos x)&=0\\\text{So, }2\sin x&=0 \qquad \text{or} \qquad \sin x+\cos x=0\\2\sin x&=0 \qquad\qquad \sin x+\cos x=0\\\sin x&=0 \qquad\qquad\qquad \sin x=-\cos x\\x&=0,\pi \qquad\qquad\qquad x=\frac{3\pi}{4},\frac{7\pi}{4}\end{aligned}$$

8.

$$\begin{aligned}3\tan^2 2x&=1\\\tan^2 2x&=\frac{1}{3}\\\tan 2x&=\pm\frac{\sqrt{3}}{3}\\2x&=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6},\frac{19\pi}{6},\frac{23\pi}{6}\\x&=\frac{\pi}{12},\frac{5\pi}{12},\frac{7\pi}{12},\frac{11\pi}{12},\frac{13\pi}{12},\frac{17\pi}{12},\frac{19\pi}{12},\frac{23\pi}{12}\end{aligned}$$

9.

$$\begin{aligned}1-\sin x&=\sqrt{3}\sin x\\1&=\sin x+\sqrt{3}\sin x\\1&=\sin x\left(1+\sqrt{3}\right)\\\frac{1}{1+\sqrt{3}}&=\sin x\end{aligned}$$

$\sin^{-1}\left(\frac{1}{1+\sqrt{3}}\right)=x$ or $x=.3747$ radians and $x=2.7669$ radians

10. Because this is $\cos 3x$, you will need to divide by 3 at the very end to get the final answer. This is why we

went beyond the limit of 2π when finding $3x$.

$$\begin{aligned}
2\cos 3x - 1 &= 0 \\
2\cos 3x &= 1 \\
\cos 3x &= \frac{1}{2} \\
3x &= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3} \\
x &= \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}
\end{aligned}$$

11. Rewrite the equation in terms of tan by using the Pythagorean identity, $1 + \tan^2\theta = \sec^2\theta$.

$$\begin{aligned}
2\sec^2 x - \tan^4 x &= 3 \\
2(1 + \tan^2 x) - \tan^4 x &= 3 \\
2 + 2\tan^2 x - \tan^4 x &= 3 \\
\tan^4 x - 2\tan^2 x + 1 &= 0 \\
(\tan^2 x - 1)(\tan^2 x - 1) &= 0
\end{aligned}$$

Because these factors are the same, we only need to solve one for x.

$$\begin{aligned}
\tan^2 x - 1 &= 0 \\
\tan^2 x &= 1 \\
\tan x &= \pm 1 \\
x &= \frac{\pi}{4} + \pi k \text{ and } \frac{3\pi}{4} + \pi k
\end{aligned}$$

Where k is any integer.

12. Use the half angle formula with 315°.

$$\begin{aligned}
\cos 157.5^\circ &= \cos\frac{315^\circ}{2} \\
&= -\sqrt{\frac{1 + \cos 315^\circ}{2}} \\
&= -\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \\
&= -\sqrt{\frac{2 + \sqrt{2}}{4}} \\
&= -\frac{\sqrt{2 + \sqrt{2}}}{2}
\end{aligned}$$

13. Use the sine sum formula.

$$\begin{aligned}
\sin\frac{13\pi}{12} &= \sin\left(\frac{10\pi}{12} + \frac{3\pi}{12}\right) \\
&= \sin\left(\frac{5\pi}{6} + \frac{\pi}{4}\right) \\
&= \sin\frac{5\pi}{6}\cos\frac{\pi}{4} + \cos\frac{5\pi}{6}\sin\frac{\pi}{4} \\
&= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + (-\frac{\sqrt{3}}{2}) \cdot \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2} - \sqrt{6}}{4}
\end{aligned}$$

14.

$$\begin{aligned}4(\cos 5x+\cos 9x) &= 4\left[2\cos\left(\frac{5x+9x}{2}\right)\cos\left(\frac{5x-9x}{2}\right)\right]\\&= 8\cos 7x\cos(-2x)\\&= 8\cos 7x\cos 2x\end{aligned}$$

15.

$$\begin{gathered}\cos(x-y)\cos y-\sin(x-y)\sin y\\\cos y(\cos x\cos y+\sin x\sin y)-\sin y\ (\sin x\cos y-\cos x\sin y)\\\cos x\cos^2 y+\sin x\sin y\cos y-\sin x\sin y\cos y+\cos x\sin^2 y\\\cos x\cos^2 y+\cos x\sin^2 y\\\cos x(\cos^2 y+\sin^2 y)\\\cos x\end{gathered}$$

16. Use the sine and cosine sum formulas.

$$\begin{gathered}\sin\left(\frac{4\pi}{3}-x\right)+\cos\left(x+\frac{5\pi}{6}\right)\\\sin\frac{4\pi}{3}\cos x-\cos\frac{4\pi}{3}\sin x+\cos x\cos\frac{5\pi}{6}-\sin x\sin\frac{5\pi}{6}\\-\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x\\-\sqrt{3}\cos x\end{gathered}$$

17. Use the sine sum formula as well as the double angle formula.

$$\begin{aligned}\sin 6x &= \sin(4x+2x)\\&= \sin 4x\cos 2x+\cos 4x\sin 2x\\&= \sin(2x+2x)\cos 2x+\cos(2x+2x)\sin 2x\\&= \cos 2x\ (\sin 2x\cos 2x+\cos 2x\sin 2x)+\sin 2x(\cos 2x\cos 2x-\sin 2x\ \sin 2x)\\&= 2\sin 2x\cos^2 2x+\sin 2x\cos^2 2x-\sin^3 2x\\&= 3\sin 2x\ \cos^2 2x-\sin^3 2x\\&= \sin 2x(3\cos^2 2x-\sin^2 2x)\\&= 2\sin x\cos x[3(\cos^2 x-\sin^2 x)^2-(2\sin x\cos x)^2\\&= 2\sin x\cos x[3(\cos^4 x-2\sin^2 x\cos^2 x+\sin^4 x)-4\ \sin^2 x\cos^2 x]\\&= 2\sin x\cos x[3\cos^4 x-6\sin^2 x\cos^2 x+3\sin^4 x-4\ \sin^2 x\cos^2 x]\\&= 2\sin x\cos x[3\cos^4 x+3\sin^4 x-10\sin^2 x\cos^2 x]\\&= 6\sin x\cos^5 x+6\sin^5 x\cos x-20\sin^3 x\cos^3 x\end{aligned}$$

18. Using our new formula, $\cos^4 x=\left[\frac{1}{2}(\cos 2x+1)\right]^2$ Now, our final answer needs to be in the first power of cosine, so we need to find a formula for $\cos^2 2x$. For this, we substitute $2x$ everywhere there is an x and the formula translates to $\cos^2 2x=\frac{1}{2}(\cos 4x+1)$. Now we can write $\cos^4 x$ in terms of the first power of cosine as

follows.

$$\begin{aligned}\cos^4 x &= [\frac{1}{2}(\cos 2x+1)]^2\\ &= \frac{1}{4}(\cos^2 2x+2\cos 2x+1)\\ &= \frac{1}{4}(\frac{1}{2}(\cos 4x+1)+2\cos 2x+1)\\ &= \frac{1}{8}\cos 4x+\frac{1}{8}+\frac{1}{2}\cos 2x+\frac{1}{4}\\ &= \frac{1}{8}\cos 4x+\frac{1}{2}\cos 2x+\frac{3}{8}\end{aligned}$$

19. Using our new formula, $\sin^4 x = \left[\frac{1}{2}(1-\cos 2x)\right]^2$ Now, our final answer needs to be in the first power of cosine, so we need to find a formula for $\cos^2 2x$. For this, we substitute $2x$ everywhere there is an x and the formula translates to $\cos^2 2x = \frac{1}{2}(\cos 4x+1)$. Now we can write $\sin^4 x$ in terms of the first power of cosine as follows.

$$\begin{aligned}\sin^4 x &= [\frac{1}{2}(1-\cos 2x)]^2\\ &= \frac{1}{4}(1-2\cos 2x+\cos^2 2x)\\ &= \frac{1}{4}(1-2\cos 2x+\frac{1}{2}(\cos 4x+1))\\ &= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x+\frac{1}{8}\\ &= \frac{1}{8}\cos 4x-\frac{1}{2}\cos 2x+\frac{3}{8}\end{aligned}$$

20. Answers: (a) First, we use both of our new formulas, then simplify:

$$\begin{aligned}\sin^2 x\cos^2 2x &= \frac{1}{2}(1-\cos 2x)\frac{1}{2}(\cos 4x+1)\\ &= \left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)\left(\frac{1}{2}\cos 4x+\frac{1}{2}\right)\\ &= \frac{1}{4}\cos 4x+\frac{1}{4}-\frac{1}{4}\cos 2x\cos 4x-\frac{1}{4}\cos 2x\\ &= \frac{1}{4}(1-\cos 2x+\cos 4x-\cos 2x\cos 4x)\end{aligned}$$

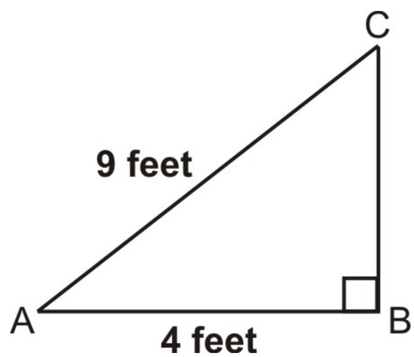
(b) For tangent, we use the identity $\tan x = \frac{\sin x}{\cos x}$ and then substitute in our new formulas. $\tan^4 2x = \frac{\sin^4 2x}{\cos^4 2x} \rightarrow$ Now, use the formulas we derived in #18 and #19.

$$\begin{aligned}\tan^4 2x &= \frac{\sin^4 2x}{\cos^4 2x}\\ &= \frac{\frac{1}{8}\cos 8x-\frac{1}{2}\cos 4x+\frac{3}{8}}{\frac{1}{8}\cos 8x+\frac{1}{2}\cos 4x+\frac{3}{8}}\\ &= \frac{\cos 8x-4\cos 4x+3}{\cos 8x+4\cos 4x+3}\end{aligned}$$

CHAPTER 4 Inverse Trigonometric Functions, Solution Key

Chapter Outline

4.1 Basic Inverse Trigonometric Functions

1. Answers:
   a. $-\frac{1}{2}$
   b. $\sqrt{2}$
   c. $-\sqrt{3}$
2. Answers:
   a. $\frac{\pi}{2}, \frac{3\pi}{2}$
   b. $\frac{2\pi}{3}, \frac{5\pi}{3}$
   c. $\frac{11\pi}{6}, \frac{7\pi}{6}$
3. $\cos\theta = \frac{12}{17} \rightarrow \cos^{-1}\frac{12}{17} = 45.1^\circ$
4. $\sin\theta = \frac{25}{36} \rightarrow \sin^{-1}\frac{31}{36} = 59.44^\circ$
5. This problem uses tangent inverse.
   - $\tan x = \frac{-23}{-14} \rightarrow x = \tan^{-1}\frac{23}{14} = 58.67^\circ$ (value graphing calculator will produce). *However, this is the reference angle.
   - Our angle is in the third quadrant because both the x and y values are negative.
   - The angle is $180^\circ + 58.67^\circ = 238.67^\circ$.
6. 

$$\cos A = \frac{4}{9}$$
$$\cos^{-1}\frac{4}{9} = A$$
$$\angle A = 63.6^\circ$$

7.

$$f(x) = 2x^3 - 5$$
$$y = 2x^3 - 5$$
$$x = 2y^3 - 5$$
$$x + 5 = 2y^3$$
$$\frac{x+5}{2} = y^3$$
$$\sqrt[3]{\frac{x+5}{2}} = y$$

8.

$$\begin{aligned}
y &= \frac{1}{3}\tan^{-1}\left(\frac{3}{4}x-5\right)\\
x &= \frac{1}{3}\tan^{-1}\left(\frac{3}{4}y-5\right)\\
3x &= \tan^{-1}\left(\frac{3}{4}y-5\right)\\
\tan(3x) &= \frac{3}{4}y-5\\
\tan(3x)+5 &= \frac{3}{4}y\\
\frac{4(\tan(3x)+5)}{3} &= y
\end{aligned}$$

9.

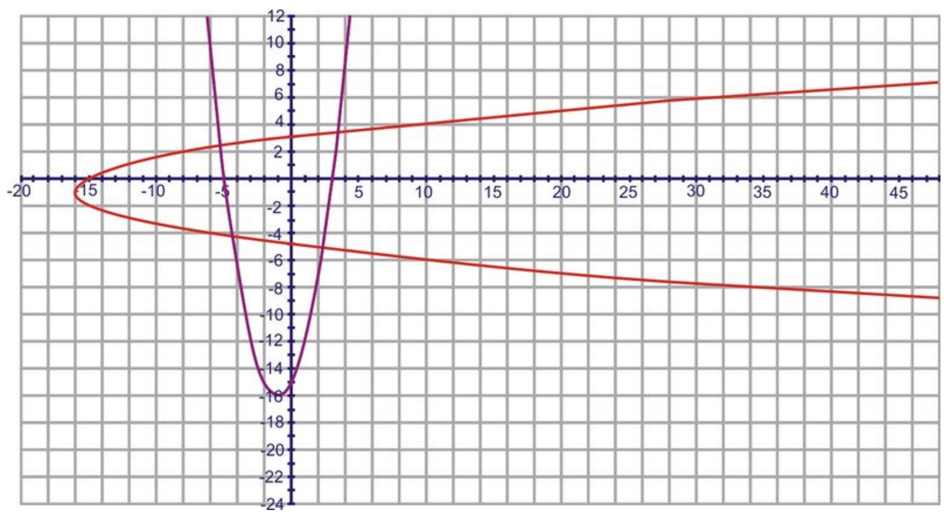
$$\begin{aligned}
g(x) &= 2\sin(x-1)+4\\
y &= 2\sin(x-1)+4\\
x &= 2\sin(y-1)+4\\
x-4 &= 2\sin(y-1)\\
\frac{x-4}{2} &= \sin(y-1)\\
\sin^{-1}\left(\frac{x-4}{2}\right) &= y-1\\
1+\sin^{-1}\left(\frac{x-4}{2}\right) &= y
\end{aligned}$$

10.

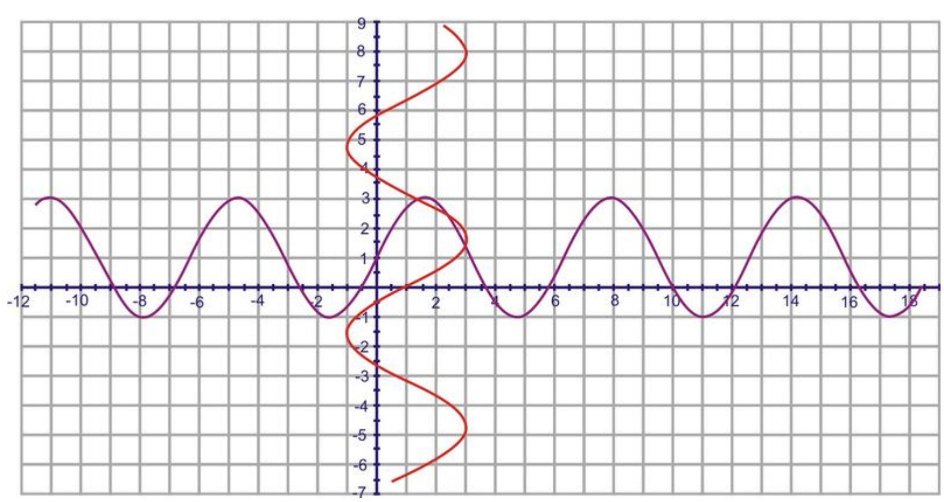
$$\begin{aligned}
h(x) &= 5-\cos^{-1}(2x+3)\\
y &= 5-\cos^{-1}(2x+3)\\
x &= 5-\cos^{-1}(2y+3)\\
x-5 &= -\cos^{-1}(2y+3)\\
5-x &= \cos^{-1}(2y+3)\\
\cos(5-x) &= 2y+3\\
\cos(5-x)-3 &= 2y\\
\frac{\cos(5-x)-3}{2} &= y
\end{aligned}$$

4.2 Graphing Inverse Trigonometric Functions

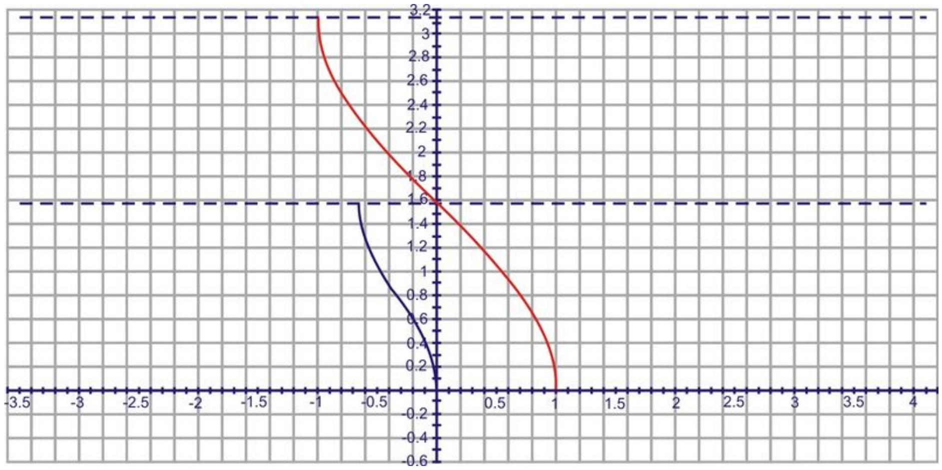
1. The graph represents a one-to-one function. It passes both a vertical and a horizontal line test. The inverse would be a function.
2. The graph represents a function, but is not one-to-one because it does not pass the horizontal line test. Therefore, it does not have an inverse that is a function.
3. The graph does not represent a one-to-one function. It fails a vertical line test. However, its inverse would be a function.
4. By selecting 4-5 points and switching the x and y values, you will get the red graph below.



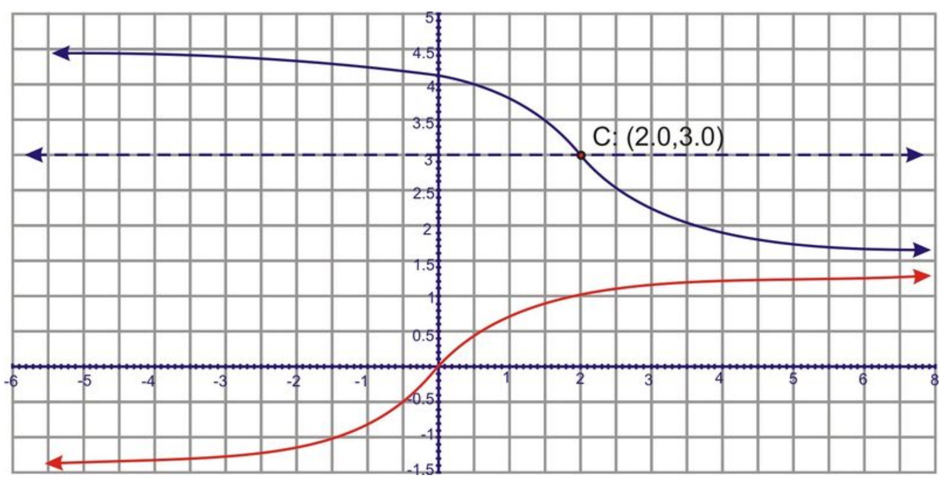
5. By selecting 4-5 points and switching the x and y values, you will get the red graph below.

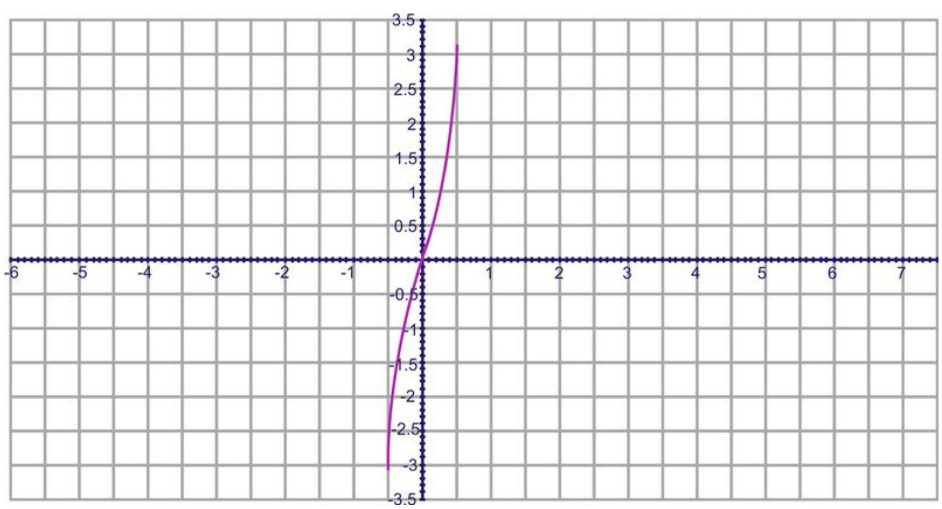


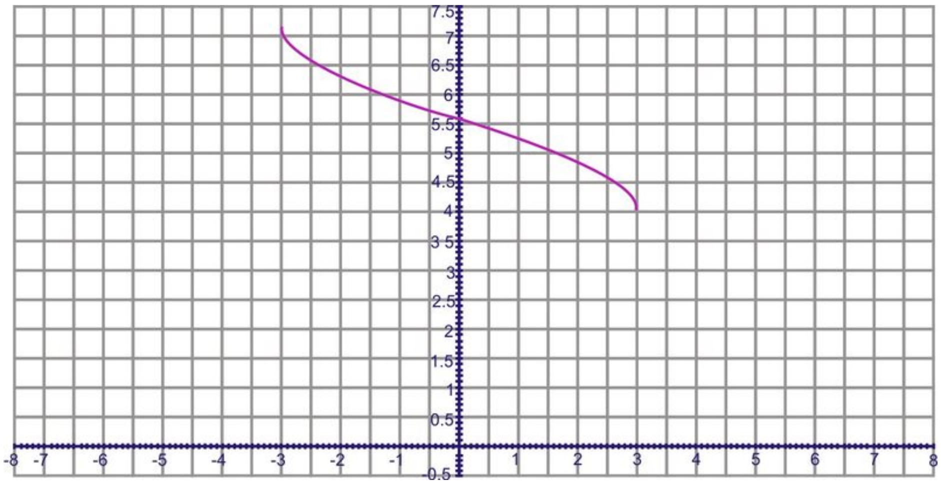
6. $y = \frac{1}{2}\cos^{-1}(3x+1)$ is in blue and $y = \cos^{-1}(x)$ is in red. Notice that $y = \frac{1}{2}\cos^{-1}(3x+1)$ has half the amplitude and is shifted over -1. The 3 seems to narrow the graph.



7. $y = 3 - \tan^{-1}(x-2)$ is in blue and $y = \tan^{-1} x$ is in red. $y = 3 - \tan^{-1}(x-2)$ is shifted up 3 and to the right 2 (as indicated by point C, the "center") and is flipped because of the $-\tan^{-1}$.



8. 

9. 

10.

$$
\begin{aligned}
y &= \cos\left(x - \frac{\pi}{2}\right) \\
x &= \cos\left(y - \frac{\pi}{2}\right) \\
\cos^{-1} x &= y - \frac{\pi}{2} \\
\frac{\pi}{2} + \cos^{-1} x &= y
\end{aligned}
$$

$\sin^{-1} x \neq \frac{\pi}{2} + \cos^{-1} x$, graphing the two equations will illustrate that the two are not the same. This is because of the restricted domain on the inverses. Since the functions are periodic, there is a phase shift of cosine that, when the inverse is found, is equal to sine inverse.

4.3 Inverse Trigonometric Properties

1. Answers:

   a. $\frac{\pi}{6}$
   b. $\frac{\pi}{4}$
   c. $\frac{3\pi}{4}$
   d. $\frac{2\pi}{3}$
   e. $-\frac{\pi}{4}$
   f. $\frac{\pi}{4}$<

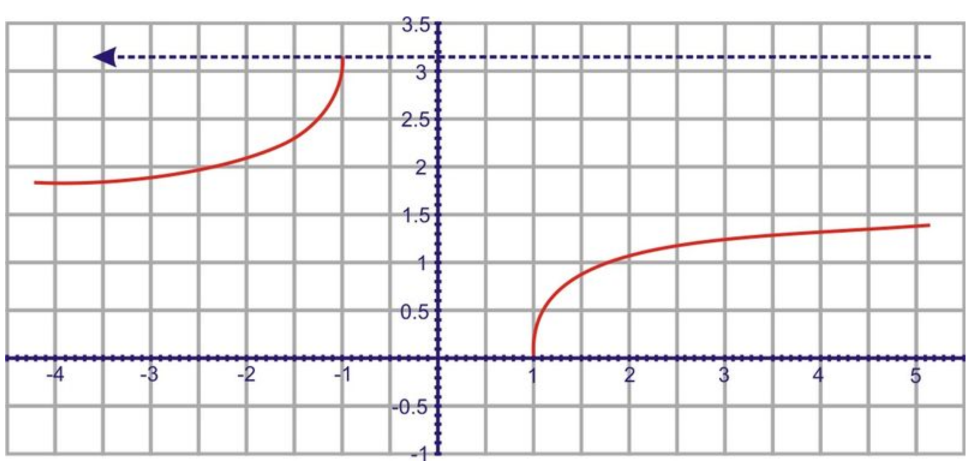
2. Answers:

   a. 2.747
   b. 0.377
   c. 1.397

3. Answers:

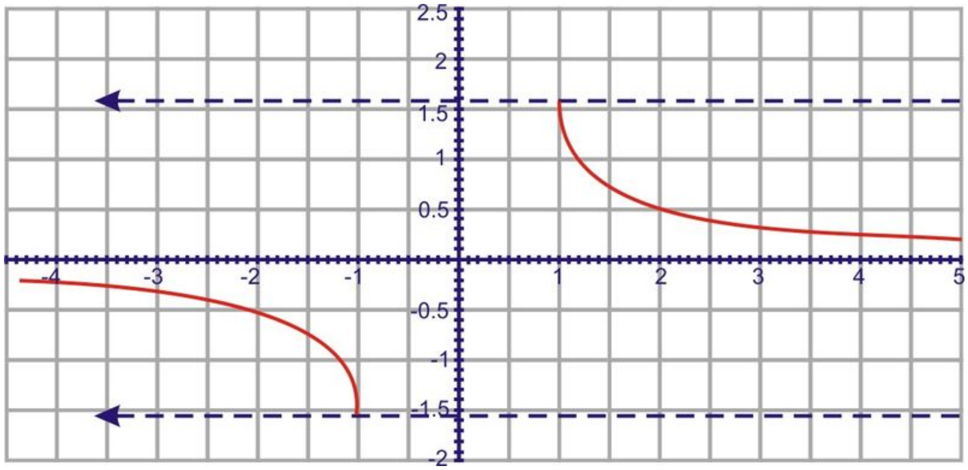
   a. $\csc\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \csc\frac{\pi}{6} = 2$
   b. $\sec^{-1}(\tan(\cot^{-1}1)) = \sec^{-1}\left(\tan\frac{\pi}{4}\right) = \sec^{-1}1 = 0$
   c. $\tan^{-1}\left(\cos\frac{\pi}{2}\right) = \tan^{-1}0 = 0$
   d. $\cot\left(\sec^{-1}\frac{2\sqrt{3}}{3}\right) = \cot\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \cot\frac{\pi}{6} = \frac{1}{\tan\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{3}} = \sqrt{3}$

4. $y = \sec^{-1}x$ when plugged into your graphing calculator is $y = \cos^{-1}\frac{1}{x}$.



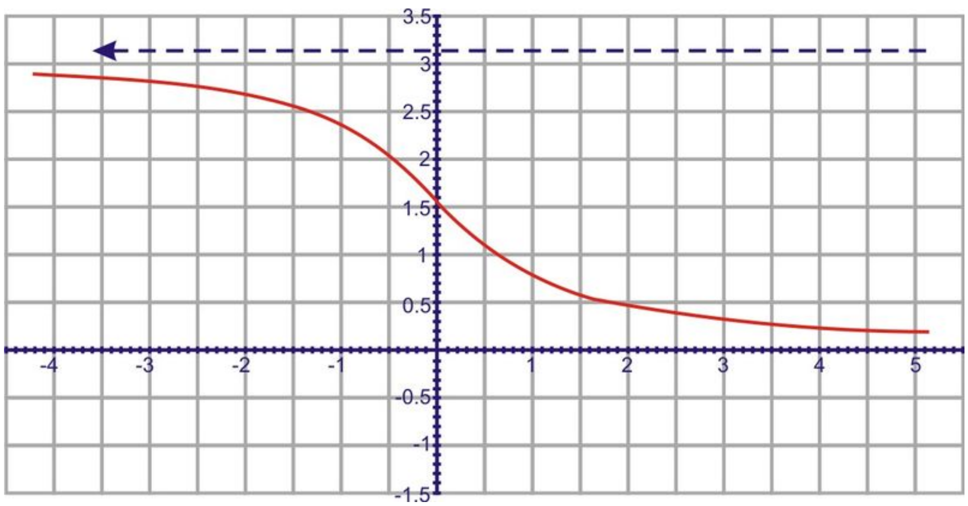
   The domain is all reals, excluding the interval (-1, 1). The range is all reals in the interval $[0,\pi], y \neq \frac{\pi}{2}$. There are no y intercepts and the only x intercept is at 1.

5. $y = \csc^{-1}x$ when plugged into your graphing calculator is $y = \sin^{-1}\frac{1}{x}$.



The domain is all reals, excluding the interval (-1, 1). The range is all reals in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}], y \neq 0$. There are no x or y intercepts.

6. The domain is all real numbers and the range is from $(0, \pi)$. There is an $x-$intercept at $\frac{\pi}{2}$.



7. Answers:

a.

$$\cos\theta = \frac{5}{13}$$
$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin\theta$$
$$\sin\theta = \frac{12}{13}$$

b. $\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right) \to \sin\theta = -\frac{6}{11}$. The third side is $b = \sqrt{121-36} = \sqrt{85}$. $\tan\theta = -\frac{6}{\sqrt{85}} = -\frac{6\sqrt{85}}{85}$

c. $\cos\left(\csc^{-1}\left(\frac{25}{7}\right)\right) \to \csc\theta = \frac{25}{7} \to \sin\theta = \frac{7}{25}$. This two lengths of a Pythagorean Triple, with the third side being 24. $\cos\theta = \frac{24}{25}$

8. Answers:

a. $\frac{1}{x^2+1}$
b. $\frac{1}{x^2}$

9. The adjacent side to θ is $\sqrt{1-x^2}$, so the three trig functions are:

a.

$$\sin(\sin^{-1} x) = \sin\theta = x$$
$$\cos(\sin^{-1} x) = \cos\theta = \sqrt{1-x^2}$$
$$\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

b.

$$\tan(\sin^{-1}(2x^3)) = \frac{2x^3}{\sqrt{1-(2x^3)^2}} = \frac{2x^3}{\sqrt{1-4x^6}}$$

10. The opposite side to θ is $\sqrt{1-x^2}$, so the three trig functions are:

a.

$$\sin(\cos^{-1} x) = \sin\theta = \sqrt{1-x^2}$$
$$\cos(\cos^{-1} x) = \cos\theta = x$$
$$\tan(\cos^{-1} x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$$

b.

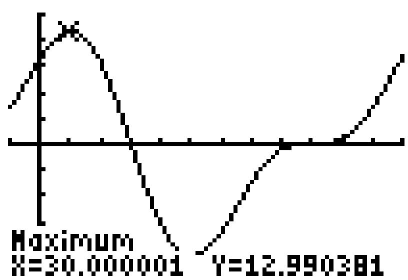
$$\sin^2\left(\cos^{-1}\left(\frac{1}{2}x\right)\right) = \left(\sqrt{1-\left(\frac{1}{2}x\right)^2}\right)^2 = 1-\frac{1}{4}x^2$$

4.4 Applications & Models

1.

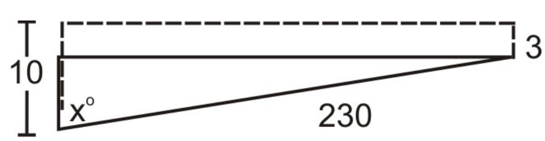
$$
\begin{aligned}
I &= I_0 \sin 2\theta \cos 2\theta \\
\frac{I}{I_0} &= \frac{I_0}{I_0} \sin 2\theta \cos 2\theta \\
\frac{I}{I_0} &= \sin 2\theta \cos 2\theta \\
\frac{2I}{I_0} &= 2\sin 2\theta \cos 2\theta \\
\frac{2I}{I_0} &= \sin 4\theta \\
\sin^{-1} \frac{2I}{I_0} &= 4\theta \\
\frac{1}{4} \sin^{-1} \frac{2I}{I_0} &= \theta
\end{aligned}
$$

2. The volume is 10 feet times the area of the end. The end consists of two congruent right triangles and one rectangle. The area of each right triangle is $\frac{1}{2}(\sin\theta)(\cos\theta)$ and that of the rectangle is $(1)(\cos\theta)$. This means that the volume can be determined by the function $V(\theta) = 10(\cos\theta + \sin\theta\cos\theta)$, and this function can be graphed as follows to find the maximum volume and the angle θ where it occurs.



Therefore, the maximum volume is approximately 13 cubic feet and occurs when θ is about 30°.

3. See the figure below.



$$
\begin{aligned}
\cos x = \frac{7}{230} &\rightarrow x = \cos^{-1} \frac{7}{230} \\
x &= 88.26^\circ
\end{aligned}
$$

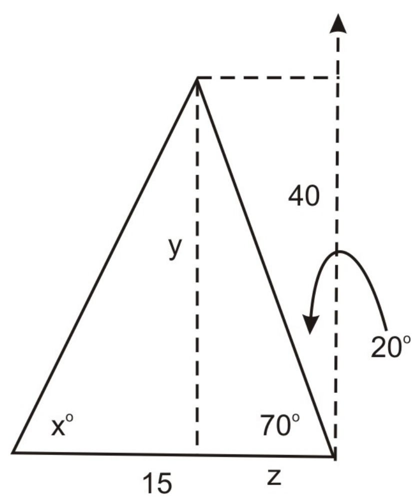
4.

$$\begin{aligned}
i &= I_m[\sin(wt+\alpha)\cos\varphi + \cos(wt+\alpha)\sin\varphi] \\
\frac{i}{I_m} &= \underbrace{\sin(wt+\alpha)\cos\varphi + \cos(wt+\alpha)\sin\varphi}_{\sin(wt+\alpha+\varphi)} \\
\frac{i}{I_m} &= \sin(wt+\alpha+\varphi) \\
\sin^{-1}\frac{i}{I_m} &= wt+\alpha+\varphi \\
\sin^{-1}\frac{i}{I_m} - \alpha - \varphi &= wt \\
\frac{1}{w}\left(\sin^{-1}\frac{i}{I_m} - \alpha - \varphi\right) &= t
\end{aligned}$$

5. Answers:

a. 64° on the 16^{th} of November $= 90^\circ - 64^\circ - 23.5^\circ \cos\left[(320+10)\frac{360}{365}\right] = 6.64^\circ$
b. 15° on the 8^{th} of August $= 90^\circ - 15^\circ - 23.5^\circ \cos\left[(220+10)\frac{360}{365}\right] = 91.07^\circ$

6. We need to find y and z before we can find x°.



$$\sin 70^\circ = \frac{y}{40} \to y = 40\sin 70^\circ = 37.59$$
$$\cos 70^\circ = \frac{z}{40} \to z = 40\cos 70^\circ = 13.68$$

- Using 15-13.68 as the adjacent side for x, we can now find the missing angle.
- $\tan x^\circ = \frac{37.59}{1.32} = 28.48 \to x^\circ = \tan^{-1}(28.48) = 87.99^\circ$.
- Therefore, the bearing from the ship back to the point of departure is $W87.99^\circ S$.

7. The maximum displacement for this equation is simply the amplitude, 4.

8. You can use the same picture from Example 5 for this problem.

$$\begin{aligned}
\tan 18^\circ &= \frac{x}{10,000} \\
x &= 10,000 \tan 18^\circ \\
x &= 3249.2 \\
\tan(18^\circ + y) &= \frac{3249.2 + 500}{10,000} \\
\tan(18^\circ + y) &= \frac{3749.2}{10,000} \\
18^\circ + y &= \tan^{-1} \frac{3749.2}{10,000} \\
18^\circ + y &= 20.55^\circ \\
y &= 2.55^\circ
\end{aligned}$$

So, the towers are 2.6° apart.

Chapter Summary

1. Answers:
   a. $-\frac{\pi}{6}$
   b. $\frac{\pi}{6}$
   c. $-\frac{\pi}{3}$
   d. $\frac{3\pi}{4}$
   e. 0
   f. $\frac{\pi}{4}$
2. Answers:
   a. 0.927
   b. 0.461
   c. 1.446
   d. -1.37
   e. 1.792
   f. 0.586
3. Answers:
   a. $\frac{\sqrt{2}}{2}$
   b. 1
   c. 2
   d. $\frac{5}{13}$
   e. $\frac{5}{2\sqrt{6}}$ or $\frac{5\sqrt{6}}{12}$
   f. $\frac{\pi}{3}$
4. Answers:

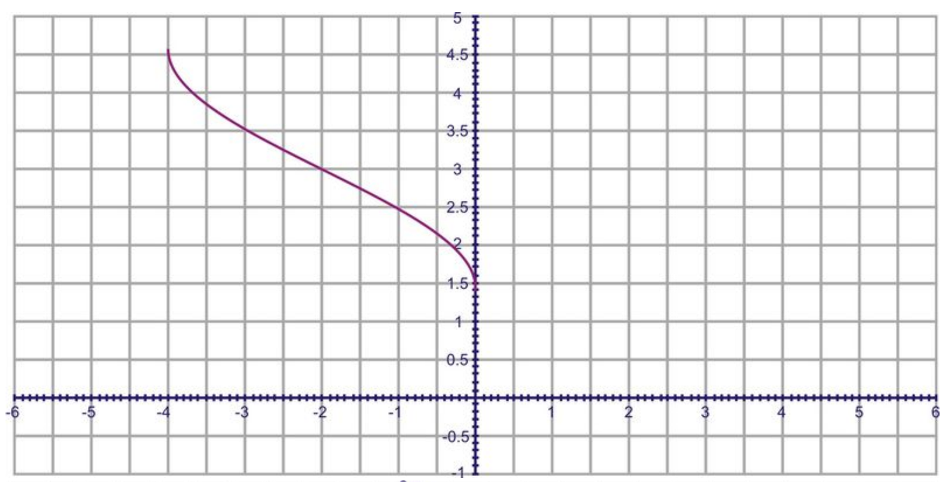
a.

$$\begin{aligned} f(x) &= 5+\cos(2x-1) \\ y &= 5+\cos(2x-1) \\ x &= 5+\cos(2y-1) \\ x-5 &= \cos(2y-1) \\ \cos^{-1}(x-5) &= 2y-1 \\ 1+\cos^{-1}(x-5) &= 2y \\ \frac{1+\cos^{-1}(x-5)}{2} &= y \end{aligned}$$

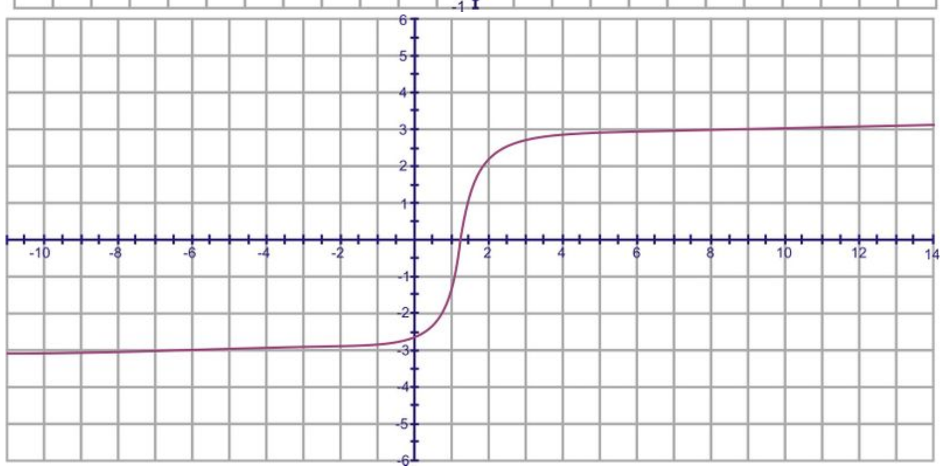
b.

$$\begin{aligned} g(x) &= -4\sin^{-1}(x+3) \\ y &= -4\sin^{-1}(x+3) \\ x &= -4\sin^{-1}(y+3) \\ -\frac{x}{4} &= \sin^{-1}(y+3) \\ \sin\left(-\frac{x}{4}\right) &= y+3 \\ \sin\left(-\frac{x}{4}\right)-3 &= y \end{aligned}$$

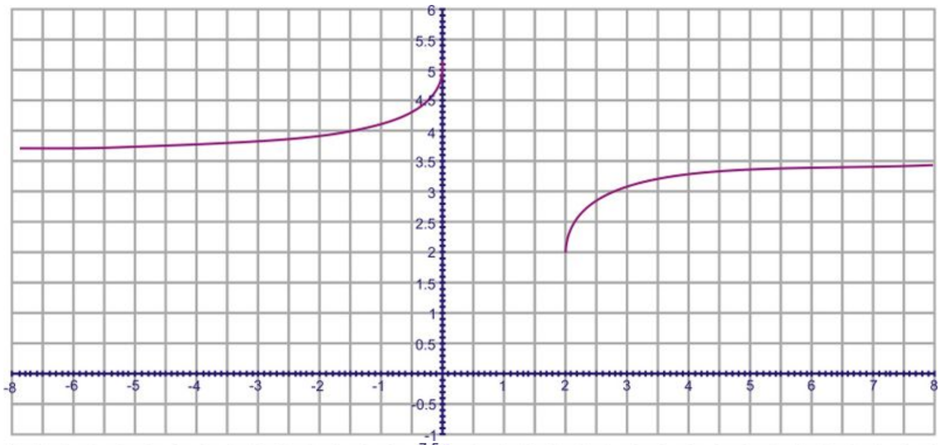
5. Answers:

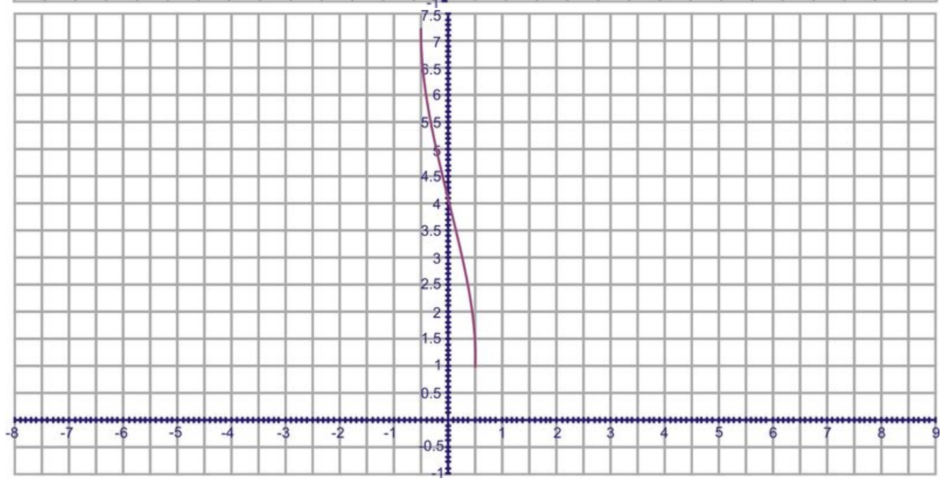


a.



b.

c. 

d. 

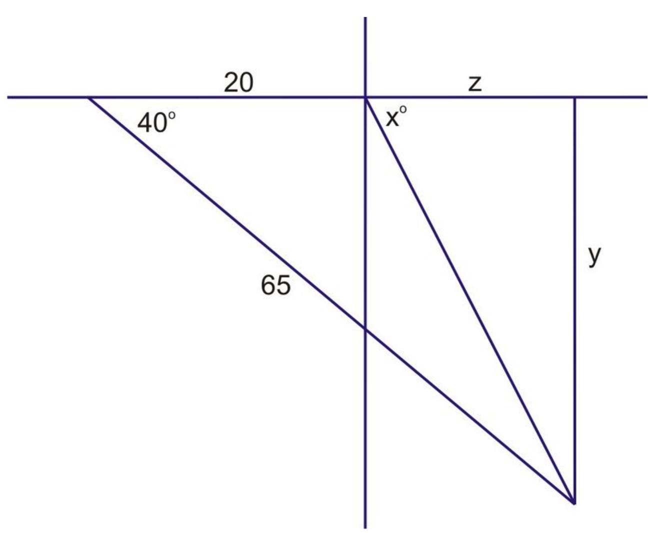
6. Answers:

a. $\sin(\cos^{-1} x^3) = \sqrt{1-(x^3)^2} = \sqrt{1-x^6}$

b. $\tan^2\left(\sin^{-1}\frac{x^2}{3}\right) = \left(\frac{\frac{x^2}{3}}{\sqrt{1-\left(\frac{x^2}{3}\right)^2}}\right)^2 = \frac{\frac{x^4}{9}}{1-\left(\frac{x^4}{9}\right)} = \frac{x^4}{9\left(1-\frac{x^4}{9}\right)} = \frac{x^4}{9-x^4}$

c. $\cos^4(\arctan(2x)^2) = \cos^4(\tan^{-1} 4x^2) = \left(\frac{1}{\sqrt{(4x^2)^2+1}}\right)^4 = \frac{1}{\sqrt{16x^4+1}^4} = \frac{1}{(16x^4+1)^2}$

7. x° can help us find our final answer, but we need to find y and z first.



$$\sin 40^\circ = \frac{y}{65} \rightarrow y = 65\sin 40^\circ = 41.78$$
$$\cos 40^\circ = \frac{20+z}{65} \rightarrow 20+z = 65\cos 40^\circ$$
$$20+z = 49.79 \rightarrow z = 29.79$$
$$\tan x = \frac{41.78}{29.79} \rightarrow x = \tan^{-1}\frac{41.78}{29.79}$$
$$x = 54.51^\circ$$

Now, the bearing from the ship to the point of departure is north, and then $(90-54.51)^\circ$ west, which is written as $N35.49^\circ W$.

8. 36° on the 12^{th} of May $= 90^\circ - 36^\circ - 23.5^\circ \cos\left[(132+10)\frac{360}{365}\right] = 72.02^\circ$
9. $\sin(x \pm y) = \sin x\cos y \pm \cos x \sin y, a = \sin x$ and $b = \sin y \rightarrow x = \sin^{-1} a$ and $y = \sin^{-1} b$

$$\sin(x\pm y) = a\sqrt{1-\sin^2 y} \pm b\sqrt{1-\sin^2 x}$$
$$\sin(x\pm y) = a\sqrt{1-b^2} \pm b\sqrt{1-a^2}$$
$$x \pm y = \sin^{-1}\left(a\sqrt{1-b^2} \pm b\sqrt{1-a^2}\right)$$
$$\sin^{-1} a \pm \sin^{-1} b = \sin^{-1}\left(a\sqrt{1-b^2} \pm b\sqrt{1-a^2}\right)$$

10. $\cos(x \pm y) = \cos x\cos y \mp \sin x \sin y, a = \cos x$ and $b = \cos y \rightarrow x = \cos^{-1} a$ and $y = \cos^{-1} b$

$$\cos(x\pm y) = ab \mp \sqrt{(1-\cos^2 x)(1-\cos^2 y)}$$
$$\cos(x\pm y) = ab \mp \sqrt{(1-a^2)(1-b^2)}$$
$$x\pm y = \cos^{-1}\left(ab \mp \sqrt{(1-a^2)(1-b^2)}\right)$$
$$\cos^{-1} a \pm \cos^{-1} b = \cos^{-1}\left(ab \mp \sqrt{(1-a^2)(1-b^2)}\right)$$

Chapter 5 Triangles and Vectors, Solution Key

Chapter Outline

5.1 The Law of Cosines

1. Answers:

   a. side a
   b. $\angle T, \angle R$, and $\angle I$
   c. side l
   d. $\angle R$ and $\angle D$
   e. side b
   f. $\angle C, \angle D, \angle M$

2. Answers:

   a. $a^2 = 8^2 + 11^2 - 2 \cdot 8 \cdot 11 \cdot \cos 50°, a \approx 8.5$
   b. $11^2 = 6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cdot \cos I, \angle I \approx 115.4°$
   c. $l^2 = 22.4^2 + 13.17^2 - 2 \cdot 22.4 \cdot 13.17 \cdot \cos 79.5°, l \approx 23.8$
   d. $12.8^2 = 17^2 + 18.6^2 - 2 \cdot 17 \cdot 18.6 \cdot \cos D, \angle D \approx 41.8°$
   e. $b^2 = 39^2 + 43^2 - 2 \cdot 39 \cdot 43 \cdot \cos 67.2°, b \approx 45.5$
   f. $11^2 = 9^2 + 13^3 - 2 \cdot 9 \cdot 13 \cdot \cos D, \angle D \approx 56.5°$

3. Answer:

   - $63^2 = 52^2 + 41.9^2 - 2 \cdot 52 \cdot 41.9 \cdot \cos C$
   - $52^2 = 63^2 + 41.9^2 - 2 \cdot 63 \cdot 41.9 \cdot \cos I$
   - $180° - 83.5° - 55.1° = 41.4°$
   - $\angle C \approx 83.5°, \angle I \approx 55.1°, \angle R \approx 41.4°$

4. Answer:

   - First, find AB.
   - $AB^2 = 14.2^2 + 15^2 - 2 \cdot 14.2 \cdot 15 \cdot \cos 37.4° \rightarrow AB = 9.4$
   - $\sin 23.3° = \frac{AD}{9.4} \rightarrow AD = 3.7$.

5. Answer:

   - $\angle HJI = 180° - 96.3° = 83.7°$ (these two angles are a linear pair).
   - $6.7^2 = HJ^2 + 1.9^2 - 2 \cdot HJ \cdot 1.9 \cdot \cos 83.7°$.
   - This simplifies to the quadratic equation $HJ^2 - 0.417HJ - 41.28$.
   - Using the quadratic formula, we can determine that $HJ \approx 6.64$.
   - So, since $HJ + JK = HK, HK \approx 6.64 + 3.6 \approx 10.24$.

6. Answer:

   - To determine this, use the Law of Cosines and solve for d to determine if the picture is accurate.
   - $d^2 = 12^2 + 24^2 - 2 \cdot 12 \cdot 24 \cdot \cos 30°, d = 14.9$, which means d in the picture is off by 1.9.

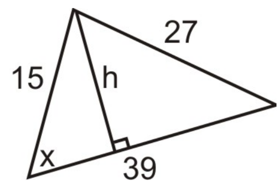
7. Answers:

   - (a) First, find x: $x^2 = 31^2 + 26^2 - 2 \cdot 31 \cdot 26 \cdot \cos 47°, x = 23.187$ *miles*. Dividing the miles by his speed will tell us how long he will have service. $\frac{23.187}{45} = 0.52$ *hr* or 30.9 *min*.
   - (b) $\frac{23.187}{35} = 0.66$ *hr* or 39.7 *min*, so he will have service for 8.8 minutes longer.

8. Answers:

   a. $194.1^2 = 183^2 + 306^2 - 2 \cdot 183 \cdot 306 \cdot \cos a$. The angle formed, a, is 37°.
   b. $207^2 = 183^2 + 329^2 - 2 \cdot 183 \cdot 329 \cdot \cos b$. The new angle, b, will need to be 34.8° rather than 37° or 2.2° less.

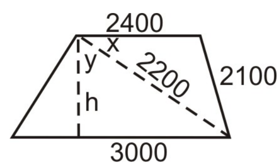
9. $x^2 = 235^2 + 329^2 - 2 \cdot 235 \cdot 329 \cdot \cos 9°$, making the ball 103.6 yards away from the flag.
10. Students answers will vary. The goal is to have each student create their own word problem.
11. Draw a figure as shown below.



We need to find the height in order to get the area.

$$27^2 = 15^2 + 39^2 - 2 \cdot 15 \cdot 39 \cdot \cos x, x = 29.6°$$
$$\sin 29.6° = \frac{h}{15} \rightarrow h = 7.4$$
$$A = \frac{1}{2} \cdot 39 \cdot 7.4 = 144.3$$

12. Draw a figure as shown below.



Recall that the area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$. We need to find the angle x, in order to find y and then h.

$$2100^2 = 2400^2 + 2200^2 - 2 \cdot 2400 \cdot 2200 \cdot \cos x, x = 54.1°.$$
$$90° - 54.1° = 35.9° = y. \cos 35.9° = \frac{h}{2200} \rightarrow h = 1782.1.$$
$$A = \frac{1}{2} 1782.1(2400 + 3000) = 4,811,670 \; sq.ft. \text{ or } 110.5 \; acres.$$

5.2 Area of a Triangle

1. Answers:
   a. $A = \frac{1}{2}\,bh$
   b. Heron's formula
   c. $K = \frac{1}{2}\,bc\sin A$
   d. $A = \frac{1}{2}\,bh$
2. Answers:
   a. $A = 22$
   b. $A = 14.3$
   c. $A = 2514.2$
   d. $A = 144.7$
3. Answers:
   a. $A = \frac{1}{2}\,bh$
   b. $K = \frac{1}{2}\,bc\sin A$
   c. $A = \frac{1}{2}\,bh$
4. Answers:
   a. $h = 89.2$
   b. $\angle A = 67^\circ$
   c. Area of $\triangle ABC = 83.0$
5. Answers: (a) Use Heron's Formula, then multiply your answer by 4, for the 4 sides.
   - $s = \frac{1}{2}(375+375+590) = 670$
   - $A = \sqrt{670(670-375)(670-375)(670-590)} = 68,297.4$
   - $68,297.4$ multiplied by $4 = 273,189.8$ total square feet.

   (b) $\frac{273,189.8}{25} \approx 10,928$ gallons of paint are needed.
6. Using Heron's Formula, s and calculate the area:
   - $s = \frac{1}{2}(8.2+14.6+16.3) = 19.55$
   - $A = \sqrt{19.55(19.55-8.2)(19.55-14.6)(19.55-16.3)} = 59.75\ sq.ft.$
   - He will need 2 bundles $\left(\frac{59.75}{33.3} = 1.8\right)$.
   - The shingles will cost him 2*\$15.45 = \$30.90
   - $6.92 square feet$ will go to waste $(66.67 - 59.75 = 6.92)$.
7. Answers:
   a. Use $K = \frac{1}{2}\,bc\sin A, K = \frac{1}{2}(186)(205)\sin 148^\circ$. So, the area that needs to be replaced is 10102.9 square yards.
   b. $K = \frac{1}{2}(186)(288)\sin 148^\circ = 14193.4$, the area has increased by 4090.5 yards.
8. You need to use the $K = \frac{1}{2}\,bc\sin A$ formula to find DE and GF.

$$56.5 = \frac{1}{2}(13.6)DE\sin 39^\circ \rightarrow DE = 13.2 \qquad 84.7 = \frac{1}{2}(13.6)EF\sin 60^\circ \rightarrow EF = 14.4$$

Second, you need to find sides DG and GF using the Law of Cosines.

$$DG^2 = 13.2^2 + 13.6^2 - 2\cdot 13.2\cdot 13.6\cdot\cos 39^\circ \rightarrow DG = 8.95$$
$$GF^2 = 14.4^2 + 13.6^2 - 2\cdot 14.4\cdot 13.6\cdot\cos 60^\circ \rightarrow GF = 14.0$$

The perimeter of the quadrilateral is 50.55.

9. Answer:

- First, find BD by using the Pythagorean Theorem. $BD = \sqrt{16.2^2 - 14.4^2} = 7.42$.
- Then, using the area and formula $(A = \frac{1}{2}bh)$, you can find AC. $232.96 = \frac{1}{2}(7.42)AC$
- $AC = 62.78$.
- $DC = 62.78 - 14.4 = 48.38$.

10. Answer:

$$\begin{aligned} d^2 &= e^2 + f^2 - 2ef\cos D \\ e^2 &= d^2 + f^2 - 2df\cos E \qquad \text{All three versions of the Law of Cosines} \\ f^2 &= d^2 + e^2 - 2de\cos F \end{aligned}$$

Add the three formulas together, we get:

$$\begin{aligned} d^2 + e^2 + f^2 &= e^2 + f^2 - 2ef\cos D + d^2 + f^2 - 2df\cos E + d^2 + e^2 - 2de\cos F \\ d^2 + e^2 + f^2 &= 2(d^2 + e^2 + f^2) - 2(ef\cos D + df\cos E + de\cos F) \\ -(d^2 + e^2 + f^2) &= -2(ef\cos D + df\cos E + de\cos F) \\ d^2 + e^2 + f^2 &= 2(ef\cos D + df\cos E + de\cos F) \end{aligned}$$

5.3 The Law of Sines

1. Answers:
   a. ASA
   b. AAS
   c. neither
   d. ASA
   e. AAS
   f. AAS
2. Student answers will vary but they should notice that in both cases you know or can find an angle and the side across from it.
3. Answers:
   a. $\frac{\sin 11.7^\circ}{a} = \frac{\sin 144.5^\circ}{16}, a = 5.6$
   b. $\frac{\sin 41.3^\circ}{214.9} = \frac{\sin 39.7^\circ}{d}, d = 208.0$
   c. not enough information
   d. $\frac{\sin 40.3^\circ}{l} = \frac{\sin 123.5^\circ}{6.3}, l = 4.9$
   e. $\frac{\sin 9^\circ}{o} = \frac{\sin 31^\circ}{15}, o = 4.6$
   f. $\frac{\sin 127^\circ}{q} = \frac{\sin 21.8^\circ}{3.62}, q = 7.8$
4. $\angle G = 180^\circ - 62.1^\circ - 21.3^\circ = 96.6^\circ$

$$\frac{\sin 96.6^\circ}{g} = \frac{\sin 21.3^\circ}{108}, g = 295.3 \qquad \frac{\sin 62.1^\circ}{h} = \frac{\sin 21.3^\circ}{108}, h = 262.8$$

5.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of Sines}$$
$$a(\sin B) = b(\sin A) \qquad \text{Cross multiply}$$
$$\frac{a}{b} = \frac{\sin A}{\sin B} \qquad \text{Divide by } b(\sin B)$$

6. Answers:
   a. $\tan 54^\circ = \frac{h}{7.15} \rightarrow h = 9.8, \cos 67^\circ = \frac{9.8}{x} \rightarrow x = 25.2$
   b. The angle we are finding is the one at the far left side of the triangle.

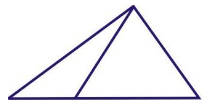
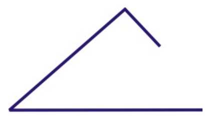
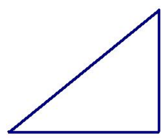
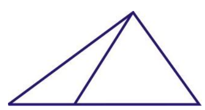
$$8.9^2 = 11.2^2 + 12.6^2 - 2 \cdot 11.2 \cdot 12.6 \cos A \rightarrow A = 43.4^\circ$$
$$\frac{\sin 43.4^\circ}{x} = \frac{\sin 31}{11.2} \rightarrow x = 14.9.$$

7. First we need to find the other two sides in the triangle.
   - $\frac{\sin 64^\circ}{218} = \frac{\sin 11^\circ}{x} = \frac{\sin 105^\circ}{y}, x = 46.3, y = 234.3$, where y is the length of the original fight plan.
   - The modified flight plan is $218 + 46.3 = 264.3$.
   - Dividing both by 495 mi/hr, we get 32 min (modified) and 28.4 min (original).
   - Therefore, the modified flight plan is 3.6 minutes longer.
8. First, we need to find the distance between Stop B (B) and Stop C (C).

- $\frac{\sin 36^\circ}{12.3} = \frac{\sin 41^\circ}{B} = \frac{\sin 103^\circ}{C} B = 13.7, C = 20.4.$
- The total length of her route is $1.1 + 12.3 + 13.7 + 20.4 + 1.1 = 48.6$ *miles*.
- Dividing this by 45 mi/hr, we get that it will take her 1.08 hours, or 64.8 minutes, of actual driving time.
- In addition to the driving time, it will take her 6 minutes (three stops at 2 minutes per stop) to deliver the three packages, for a total roundtrip time of 70.8 minutes.
- Subtracting this 70.8 minutes from 10:00 am, she will need to leave by 8:49 am.

5.4 The Ambiguous Case

TABLE 5.1:

$a >, =,$ **or** $< b\sin A$	**Diagram**	**Number of solutions**
a. $26 > 19.9$		2
b. $16 < 17.5$		0
c. $13.48 = 13.48$		1
d. $3.4 > 3.3$		2

2. Answers:
   a. $\frac{\sin 32.5^\circ}{26} = \frac{\sin B}{37} \rightarrow B = 49.9^\circ$ or $180^\circ - 49.9^\circ = 130.1^\circ$
   b. no solution
   c. $\frac{\sin 47.8^\circ}{13.48} = \frac{\sin B}{18.2} \rightarrow B = 90^\circ$
   d. $\frac{\sin 51.5^\circ}{3.4} = \frac{\sin B}{4.2} \rightarrow B = 75.2^\circ$ or $180^\circ - 75.2^\circ = 104.8^\circ$
3.

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin C}{c} \\ c\sin A &= a\sin C \\ c\sin A - c\sin C &= a\sin C - c\sin C \\ c(\sin A - \sin C) &= \sin C(a - c) \\ \frac{\sin A - \sin C}{\sin C} &= \frac{a - c}{c} \end{aligned}$$

4. Student answers will vary. Student should mention using $a > b\sin A$ in their explanation.
5. The answers can be determined as follows:
   a. $a < b\sin A \rightarrow \frac{a}{b} < \sin A \rightarrow \frac{22}{31} < \sin A \rightarrow A > 45.2^\circ$

b. $a = b\sin A \rightarrow \frac{a}{b} = \sin A \rightarrow \frac{22}{31} = \sin A \rightarrow A = 45.2^\circ$
c. $a > b\sin A \rightarrow \frac{a}{b} > \sin A \rightarrow \frac{22}{31} > \sin A \rightarrow A < 45.2^\circ$

6. This problem can be done entirely with right triangle trig, but there are several different ways to solve this particular problem.

$$BD = \sqrt{13.7^2 - 9.8^2} = 9.6 \qquad \tan 42.6^\circ = \frac{9.8}{DC} \rightarrow DC = 10.7$$
$$\sin 42.6^\circ = \frac{9.8}{AC} \rightarrow AC = 14.5 \qquad BC = 9.6 + 10.7 = 20.3$$
$$\sin B = \frac{9.8}{13.7} \rightarrow \angle B = 45.7^\circ \qquad \angle A = 180^\circ - 45.7^\circ - 42.6^\circ = 91.7^\circ$$

7. Answers:

a. $\angle EBD \Rightarrow 7.6^2 = 9.9^2 + 10.2^2 - 2 \cdot 9.9 \cdot 10.2 \cos EBD \Rightarrow 44.4^\circ$
b. $\angle BDE \Rightarrow \frac{\sin BDE}{9.9} = \frac{\sin 44.4^\circ}{7.6} \Rightarrow 65.7^\circ$
c. $\angle DEB \Rightarrow 180^\circ - 65.7^\circ - 44.4^\circ \Rightarrow 69.9^\circ$
d. $\angle BDC \Rightarrow 180^\circ - 65.7^\circ \Rightarrow 114.3^\circ$
e. $\angle BEA \Rightarrow 180^\circ - 69.9^\circ \Rightarrow 110.1^\circ$
f. $\angle DBC \Rightarrow 180^\circ - 114.3^\circ - 21.8^\circ \Rightarrow 43.9^\circ$
g. $\angle ABE \Rightarrow 109.6^\circ - 43.9^\circ - 44.4^\circ \Rightarrow 21.3^\circ$
h. $\angle BAE \Rightarrow 180^\circ - 21.3^\circ - 110.1^\circ \Rightarrow 48.6^\circ$
i. $BC \Rightarrow \frac{\sin 114.3^\circ}{BC} = \frac{\sin 21.8^\circ}{10.2} \Rightarrow 25.0$
j. $AB \Rightarrow \frac{\sin 110.1^\circ}{AB} = \frac{\sin 48.6^\circ}{9.9} \Rightarrow 12.4$
k. $AE \Rightarrow \frac{\sin 21.3^\circ}{AE} = \frac{\sin 48.6^\circ}{9.9} \Rightarrow 4.8$
l. $DC \Rightarrow \frac{\sin 43.9^\circ}{DC} = \frac{\sin 21.8^\circ}{9.9} \Rightarrow 19.0$
m. $AC = 19 + 4.8 + 7.6 = 31.4$

8. We need to find the distance between sensors 2 and 3. If it is less than 6000 ft, then the sensor will be able to detect all motion between the two. First, $4500 > 4000$, so there is going to be one solution.

- $\frac{\sin S2}{4000} = \frac{\sin 56^\circ}{4500} \rightarrow S2 = 47.47^\circ$
- $S1 + 47.47^\circ + 56^\circ = 180^\circ \rightarrow S1 = 76.53^\circ$
- $\frac{\sin 76.53^\circ}{x} = \frac{\sin 56^\circ}{4500} \rightarrow x = 5279$
- Since $x < 6000$ ft., Sensor 3 will be able to pick up all movement from its location to the location of Sensor 2.

9. The length from $S3$ to $S4$ is x and from $S4$ to $S2$ is y. $180^\circ - 36^\circ - 49^\circ = 95^\circ$, which is the angle at $S4$.

$$\frac{\sin 95^\circ}{5279} = \frac{\sin 36^\circ}{x} = \frac{\sin 49^\circ}{y} \qquad x = 3114.8, y = 3999.3$$

5.5 General Solutions of Triangles

1. Answers:

    a. AAS, Law of Sines, one solution
    b. SAS, Law of Cosines, one solution
    c. SSS, Law of Cosines, one solution
    d. SSA, Law of Sines, no solution $(15 < 25\sin 58^\circ)$
    e. SSA, Law of Sines, two solutions $(45 > 60\sin 47^\circ)$

2. Answers:

    a. $\frac{\sin 69^\circ}{22.3} = \frac{\sin 12^\circ}{b}, b = 4.97$
    b. $c^2 = 1.4^2 + 2.3^2 - 2(1.4)(2.3)\cos 58^\circ, c = 2.0$
    c. $3.3^2 = 6.1^2 + 4.8^2 - 2(6.1)(4.8)\cos A, A = 32.6^\circ$
    d. No solution
    e. $\frac{\sin 47^\circ}{45} = \frac{\sin B}{60}, B = 77.2^\circ$ or $180^\circ - 77.2^\circ = 102.8^\circ$

3. (a) need angle C and side c.

$$C = 180^\circ - 69^\circ - 12^\circ = 99^\circ$$
$$\frac{\sin 99^\circ}{c} = \frac{\sin 69^\circ}{22.3}, c = 23.6$$

(b) need angle A and angle B.

$$\frac{\sin 58^\circ}{2} = \frac{\sin A}{1.4}, A = 36.4^\circ$$
$$180^\circ - 36.4^\circ - 58^\circ, B = 85.6^\circ$$

(c) need angle B and angle C.

$$\frac{\sin 32.6^\circ}{3.3} = \frac{\sin B}{6.1}, B = 84.8^\circ$$
$$180^\circ - 32.6^\circ - 84.8^\circ, C = 62.6^\circ$$

(d) No solution (e) Both cases need angle C and side c.

$$\text{Case 1}: C = 180^\circ - 47^\circ - 77.2^\circ = 55.8^\circ, \frac{\sin 55.8^\circ}{c} = \frac{\sin 47^\circ}{45}, c = 50.9$$
$$\text{Case 2}: C = 180^\circ - 47^\circ - 102.8^\circ = 30.2^\circ, \frac{\sin 30.2^\circ}{c} = \frac{\sin 47^\circ}{45}, c = 30.95$$

4. To find the area of the rhombus, use the formula $K = \frac{1}{2}\, bc\sin A$ and then multiply that by 2. We first need to find one of the angles that are opposite the given diagonal (they are both the same measurement). We will call it angle A. $21.5^2 = 12^2 + 12^2 - 2(12)(12)\sin A, A = 127.2^\circ$, which means the other two angles are both $52.8^\circ(360^\circ - 127.2^\circ - 127.2^\circ$ and then divide by 2).

$$K = 2\left(\frac{1}{2}(12)(12)\sin 127.2^\circ\right) = 114.7$$

5. Divide the pentagon into three triangles, drawing segments from $\angle 2$ to $\angle 5$, called x below, and $\angle 2$ to $\angle 4$, called y below. With these three triangles, only the middle triangle needs us to find two sides and the angle between them (called $\angle Z$ below) to use $K = \frac{1}{2}\, bc \sin A$ (the outer two triangles already have two sides and an angle that fit this criteria).

$$x^2 = 192^2 + 190.5^2 - 2(192)(190.5)\cos 81^\circ \rightarrow x = 248.4$$
$$y^2 = 146^2 + 173.8^2 - 2(146)(173.8)\cos 73^\circ \rightarrow x = 191.5$$
$$118^2 = 248.4^2 + 191.5^2 - 2(248.4)(191.5)\cos Z \rightarrow \angle Z = 27.4^\circ$$

Areas: $K = \frac{1}{2}(190.5)(192)\sin 81^\circ = 18062.8$

$$k = \frac{1}{2}(248.4)(191.5)\sin 27.4^\circ = 10945.5$$
$$K = \frac{1}{2}(173.8)(146)\sin 73^\circ = 12133.0 \qquad \text{Total Area}: \ 41141.3$$

6. altitude, x: $\sin 56.8^\circ = \frac{x}{38} \rightarrow x = 31.8$

$$GT = \sqrt{38^2 - 31.8^2} = 20.8 \qquad GI = 88 - 20.8 = 67.2$$
$$A_{small} = \frac{1}{2}(20.8)(31.8) = 330.8 \qquad A_{big} = \frac{1}{2}(67.2)(31.8) = 1068.5$$
$$RI = \sqrt{67.2^2 + 31.8^2} = 74.3 \qquad \angle I \rightarrow \sin I = \frac{31.8}{74.3} \rightarrow 25.3^\circ$$

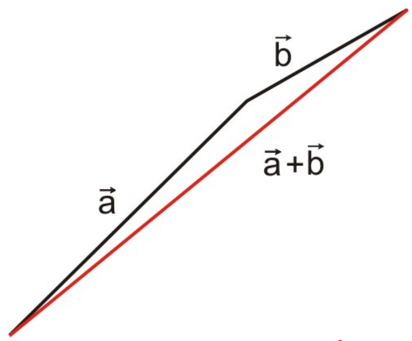
7. The headings would be as follows:
   a. $W 28.2^\circ N$
   b. $S 28.8^\circ W$
   c. $N 90^\circ E$
8. Answers:
   a. $a^2 = 187^2 + 218^2 - 2(187)(218)\cos 115^\circ$, he would need to hit the ball 342.0 yards.
   b. $\frac{\sin 115^\circ}{342} = \frac{\sin B}{187}$, he would have to hit the ball at a 29.7° angle.
9. Answers:
   a. $180^\circ - 14.2^\circ - 162.2^\circ = 3.6^\circ$
   b. $\frac{\sin 14.2^\circ}{b} = \frac{\sin 162.2^\circ}{320}, 256.8\ yards$
   c. $\frac{\sin 162.2^\circ}{320} = \frac{\sin 3.6^\circ}{c}, 65.7\ yards$

5.6 Vectors

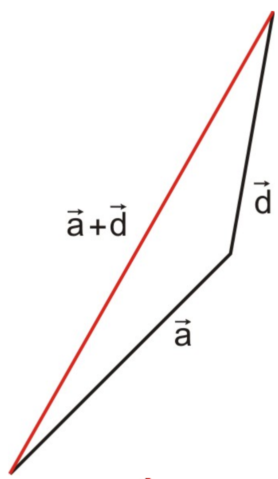
1. For each problem below, use the Pythagorean Theorem to find the magnitude and $\tan\theta = \frac{|\vec{n}|}{|\vec{m}|}$

    a. magnitude $= 48.1$, direction $= 51.7^\circ$
    b. magnitude $= 6.1$, direction $= 62.6^\circ$
    c. magnitude $= 15.2$, direction $= 38.3^\circ$

2. Answers:



a.



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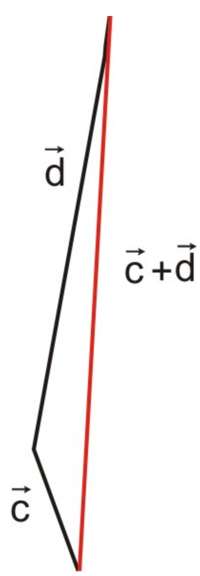

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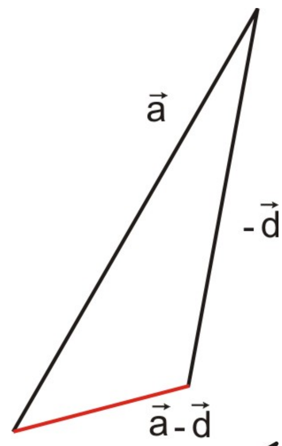
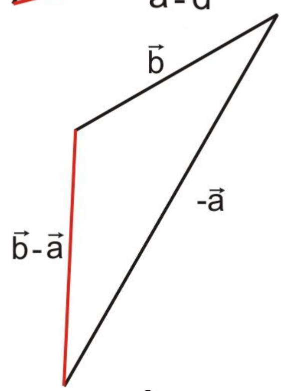
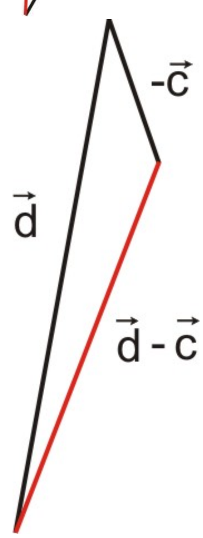


d.



e.



f.

3. When two vectors are summed, the magnitude of the resulting vector is almost always different than the sum of the magnitudes of the two initial vectors. The only times that $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}|$ would be true is when 1) the magnitude of at least one of the two vectors to be added is zero, or 2) both of the vectors to be added have the same direction.
4. Speed (magnitude): $\sqrt{18^2+225^2}=225.7$ and its direction is $\tan\theta=\frac{18}{225}=N4.6^\circ W$.
5. The magnitude is $\sqrt{330^2+410^2}=526.3$ Newtons and the direction is $\tan^{-1}\left(\frac{410}{330}\right)=E51.2^\circ N$.
6. Answers:
   a. $|\vec{a}|=\sqrt{12^2+18^2}=21.6$, direction $=\tan^{-1}\left(\frac{18}{12}\right)=56.3^\circ$.
   b. $|\vec{a}|=\sqrt{(-3)^2+6^2}=6.7$, direction $=\tan^{-1}\left(\frac{6}{-3}\right)=116.6^\circ$.
7. Answers:
   a. $|\vec{a}|=\sqrt{(2-8)^2+(4-6)^2}=6.3$, direction $=\tan^{-1}\left(\frac{4-6}{2-8}\right)=18.4^\circ$.
   b. $|\vec{a}|=\sqrt{(5-3)^2+(-2-1)^2}=3.6$, direction $=\tan^{-1}\left(\frac{-2-1}{5-3}\right)=123.7^\circ$. Note that when you use your

calculator to solve for $\tan^{-1}(\frac{-2-1}{5-3})$, you will get -56.3°. The calculator produces this answer because the range of the calculator's $y = \tan^{-1} x$ function is limited to $-90^\circ < y < 90^\circ$. You need to sketch a draft of the vector to see that its direction when placed in standard position is into the second quadrant (and not the fourth quadrant), and so the correct angle is calculated by moving the angle into the second quadrant through the equation $-56.3^\circ + 180^\circ = 123.7^\circ$.

8. In both *a* and *b*, we have the SAS case, so you can do the Law of Cosines, followed by the Law of Sines.

    a. $(\vec{a}+\vec{b})^2 = 31^2 + 31^2 - 2(31)(31)\cos 132, \vec{a}+\vec{b} = 56.6, \frac{\sin 132}{56.6} = \frac{\sin x}{31}, x = 24^\circ$
    b. $(\vec{a}+\vec{b})^2 = 29^2 + 44^2 - 2(29)(44)\cos 26, \vec{a}+\vec{b} = 22, \frac{\sin x}{44} = \frac{\sin 26}{22}, x = 61.3^\circ$

5.7 Component Vectors

1. Answers:

   a. $2\vec{b} = 2\langle 5,4\rangle = \langle 10,8\rangle = 10\hat{\imath} + 8\hat{\jmath}$
   b. $-\frac{1}{2}\vec{c} = -\frac{1}{2}\langle -3,7\rangle = \langle 1.5,-3.5\rangle = 1.5\hat{\imath} - 3.5\hat{\jmath}$
   c. $0.6\vec{b} = 0.6\langle 5,4\rangle = \langle 3,2.4\rangle = 3\hat{\imath} + 2.4\hat{\jmath}$
   d. $-3\vec{b} = -3\langle 5,4\rangle = \langle -15,-12\rangle = -15\hat{\imath} - 12\hat{\jmath}$

2. All of these need to be translated to (0,0). Also, recall that magnitudes are always positive.

   - (a) $(-3,8) + (3,-8) = (0,0)$ $(2,-1) + (3,-8) = (5,-9)$

   horizontal $= 5$, vertical $= 9$

   - (b) $(7,13) + (-7,-13) = (0,0)$ $(11,19) + (-7,-13) = (4,6)$

   horizontal $= 4$, vertical $= 6$

   - (c) $(4.2,-6.8) + (-4.2,6.8) = (0,0)$ $(-1.3,-9.4) + (-4.2,6.8) = (-5.5,-2.6)$

   horizontal $= 5.5$, vertical $= 2.6$
3. Answers:

   a. $\cos 35^\circ = \frac{x}{75}, \sin 35^\circ = \frac{y}{75}, x = 61.4, y = 43$
   b. $\cos 162^\circ = \frac{x}{3.4}, \sin 162^\circ = \frac{y}{3.4}, x = 3.2, y = 1.1$
   c. $\cos 12^\circ = \frac{x}{15.9}, \sin 12^\circ = \frac{y}{15.9}, x = 15.6, y = 3.3$

4. magnitude $= 33.2$, direction $= 75.2^\circ$ from the smaller force
5. magnitude $= 304, 18.3^\circ$ between resultant and larger force
6. $y = 12\sin 28.2^\circ = 5.7, x = 12\cos 28.2^\circ = 10.6$
7. Answer:

   - Recall that headings and angles in triangles are complementary.
   - So, an 83° heading translates to 7° from the horizontal.
   - Adding that to 35° (270° from 305°) we get 42° for two of the angles in the parallelogram.
   - So, the other angles in the parallelogram measure 138° each, $\frac{360-2(42)}{2}$.
   - Using 138° in the Law of Cosines, we can find the diagonal or resultant, $x^2 = 42^2 + 155^2 - 2(42)(155)\cos 138$, so $x = 188.3$.
   - We then need to find the angle between the resultant and the speed using the Law of Sines. $\frac{\sin a}{42} = \frac{\sin 138}{188.3}$, so $a = 8.6^\circ$.
   - To find the actual heading, this number needs to be added to 83°, getting 91.6°.

8. The heading is just $\tan\theta = \frac{2}{10}$, or 11.3° against the current.
9. BA is the same vector as AB, but because it starts with B it is in the opposite direction. Therefore, when you add the two together, you will get (0,0).

Chapter Summary

1. Use Law of Cosines to solve for angle B. $17.6^2 = 15^2 + 20.9^2 - 2(15)(20.9)\cos B, B = 55.8^\circ$, which means the picture was drawn accurately.

2.

$$CD^2 = 32.6^2 + 51.4^2 - 2(32.6)(51.4)\cos 27, CD = 26.8$$
$$\angle C,\ \angle C = 33.5^\circ$$
$$AB^2 = 64.1^2 + 51.4^2 - 2(64.1)(51.4)\cos 33.5, AB = 35.4$$

This means that the artist's drawing is off by 1.1 foot.

3. The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$. In the problem, we were given both bases, a side and a diagonal. So, we need to find the height. In order to do this, we first need to find the angle between 2200 and 2400(x), use that to find it's complement, then we can take the cosine to find the height.

$$2100^2 = 2400^2 + 2200^2 - 2(2400)(2200)\cos x$$
$$x = 54.1^\circ \rightarrow 90^\circ - 54.1^\circ = 35.9^\circ$$
$$\cos 35.9^\circ = \frac{h}{2200} \rightarrow h = 1782.1$$
$$A = \frac{1}{2}(1782.1)(2400 + 3000) = 4,811,670\ sq.\ ft.$$

4. $AB^2 = 4^2 + 21^2 - 2(4)(21)\cos 120, AB = 23.3$ *minus* 5, new AB is 18.3 new $\angle E$ is $18.3^2 = 4^2 + 21^2 - 2(4)(21)\cos E, \angle E = 42.6^\circ$
5. Answers:
   a. $x^2 = 13.3^2 + 17.6^2 - 2(13.3)(17.6)\cos 132, x = 28.3$
   b. $\cos 52 = \frac{x}{19.3}, x = 11.9$
   c. $\frac{\sin 42}{18.6} = \frac{\sin x}{15.9}, x = 34.9^\circ$
   d. $\frac{\sin 46}{92.7} = \frac{\sin 12}{x}, x = 26.8$
6. Answers:
   a. We will call the distance across the canyon h. $\frac{\sin 32}{12} = \frac{\sin 87}{x}, x = 22.6.\sin 61 = \frac{h}{22.6}$, 19.8 km.
   b. We will call the distance between the two boulders y. $y = a + b$, where a is the distance from boulder 1 to the first stop (that the surveyor took) and b is the distance from that spot to boulder 2.

   •

   $$\tan 3^\circ = \frac{a}{19.8} \qquad \tan 37^\circ = \frac{b}{19.8}$$

   • So, $a + b = 19.8(\tan 3^\circ + \tan 37^\circ) = 16.0$ km.
7. Company A coverage: $x^2 = 38^2 + 47^2 - 2(38)(47)\cos 72.8, x = 51$ Company B coverage: $\frac{\sin x}{58} = \frac{\sin 12}{59}, x = 11.8^\circ \rightarrow \frac{\sin 12}{59} = \frac{\sin 156.2}{b}, b = 114.52$ Overlap: $\frac{\sin a}{38} = \frac{\sin 72.8}{51}, a = 45.4^\circ \rightarrow \frac{\sin 12}{15} = \frac{\sin 122.6}{o}, o = 60.8$
8. Answers:
   a. magnitude $= \sqrt{48.3^2 + 47.6^2} = 67.8$, direction $= \tan\theta = \frac{47.6}{48.3} \rightarrow 44.6^\circ$
   b. magnitude $= \sqrt{18.6^2 + 17.5^2} = 25.5$, direction $= \tan\theta = \frac{47.6}{48.3} \rightarrow 43.3^\circ$
9. Answers:
   a. magnitude $= \sqrt{(19-1)^2 + (-4-12)^2} = 24.1$, direction $= \tan\theta = \frac{-18}{16} \rightarrow -48.4^\circ$
   b. magnitude $= \sqrt{(-21+11)^2 + (11-21)^2} = 14.1$, direction $= \tan\theta = \frac{-10}{-10} \rightarrow 45^\circ$
10. This is the SSA or ambiguous case. Because $137 > 425\sin 16^\circ$, we will have two solutions or two different lengths that the golfer could have hit the ball. $\frac{\sin 16}{137} = \frac{\sin x}{425}, x = 58.8^\circ$ or 121.2° (this is the angle at the ball) Case 1: Use 58.8°, the angle at the pin is $105.2^\circ \frac{\sin 105.2}{d} = \frac{\sin 16}{137}$, distance is 479.6 yards. Case 2: Use 121.2°, the angle at the pin is $42.8^\circ \frac{\sin 42.8}{d} = \frac{\sin 16}{137}$, distance is 337.7 yards. It is safe to say that the golfer did not hit the ball 479.6 yards, considering that Tiger Woods longest recorded drive is 425 yards. So, we can logically rule out Case 1 and our answer is that the golfer's drive was 337.7 yards.

11. $\cos 36^\circ = \frac{x}{0.5}, x = 0.4\ miles$
12. The third angle in the triangle is 35°, from the Triangle Sum Theorem.

$$\frac{\sin 35}{127.3} = \frac{\sin 127}{x}, x = 177.2$$

Therefore, the ball was hit 177.2 feet. The distance between second base and the ball can be calculated as follows:

$$\frac{\sin 35}{127.3} = \frac{\sin 18}{x}, x = 68.6$$

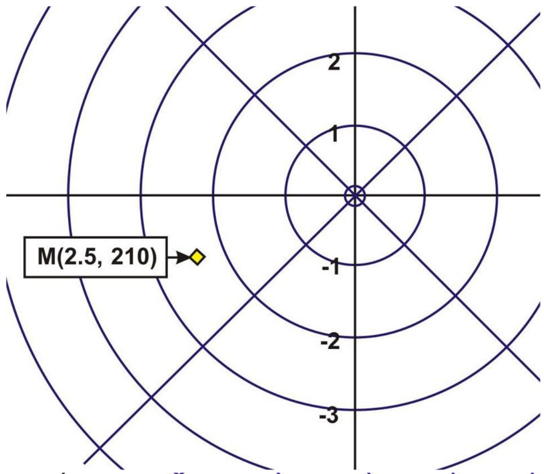
13. Therefore, the distance from second base to the ball is 68.6 feet.
14. The distance between the tower and target 2 is: $x^2 = 37^2 + 18^2 - 2(37)(18)\cos 67.2^\circ, 34.3\ miles$. This means that the second target is out of range by 4.3 miles.
15. Answer:
    - We need to find area, use Heron's Formula. $s = \frac{1}{2}(587 + 247 + 396) = 615$
    - $A = \sqrt{615(615-587)(615-247)(615-396)} = 37,253.1$, times $5.2 \times 10^{13} = 1.9 \times 10^{18}$ bacteria.

CHAPTER 6 The Polar System, Solution Key

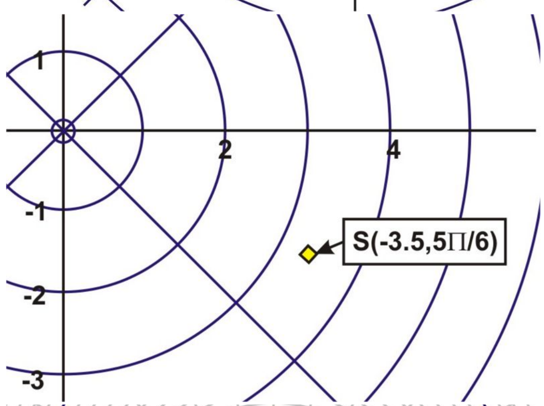
Chapter Outline

6.1 Polar Coordinates

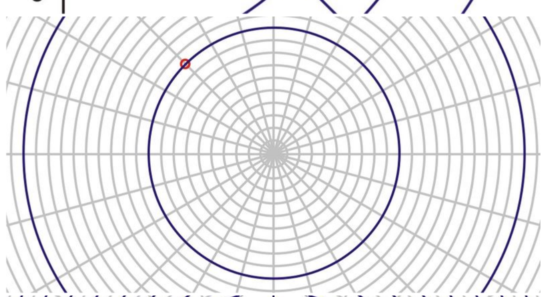
1. Answers:



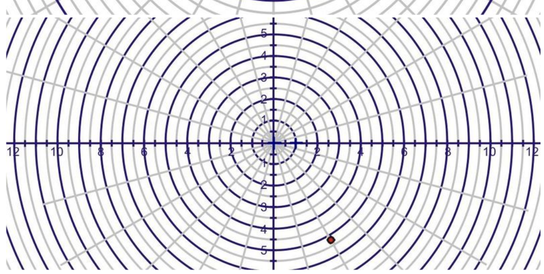
a.



b.



c.



d.

2.

$\left(-4, \frac{\pi}{4}\right)$	all positive $\rightarrow$	$\left(4, \frac{5\pi}{4}\right)$
	both negative $\rightarrow$	$\left(-4, \frac{-7\pi}{4}\right)$
	negative angle $\rightarrow$	$\left(4, \frac{-3\pi}{4}\right)$

3.

$(2, 120^\circ)$	negative angle only $\rightarrow$	$(2, -240^\circ)$
	negative radius only $\rightarrow$	$(-2, 300^\circ)$
	both negative $\rightarrow$	$(-2, -60^\circ)$

4. Use $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$.

a.

$$P_1P_2 = \sqrt{1^2 + 6^2 - 2(1)(6)\cos(135^\circ - 30^\circ)}$$
$$P_1P_2 \approx 6.33 \ units$$

b.

$$P_1P_2 = \sqrt{2^2 + 9^2 - 2(2)(9)\cos 150^\circ}$$
$$= 10.78$$

c.

$$P_1P_2 = \sqrt{3^2 + 10^2 - 2(3)(10)\cos(322 - 272)^\circ}$$
$$= 8.39$$

d.

$$P_1P_2 = \sqrt{5^2 + 16^2 - 2(5)(16)\cos(200 - 25)^\circ}$$
$$= 20.99$$

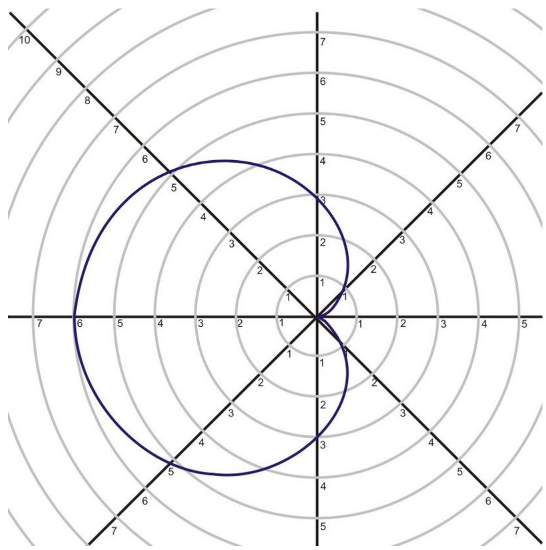
6.2 Graphing Basic Polar Equations

1. Answers:

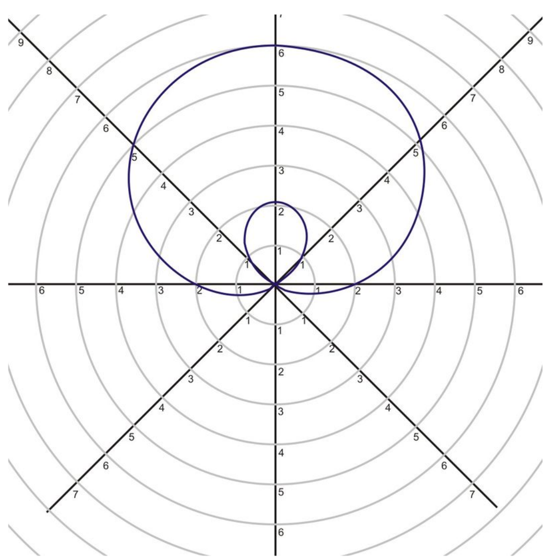
   a. a limaçon with an innerloop. $r = 2 - 3\sin\theta$
   b. a cardioid $r = 2 + 2\sin\theta$
   c. a dimpled limaçon $r = 5.5 + 4.5\sin\theta$

2. Answers:

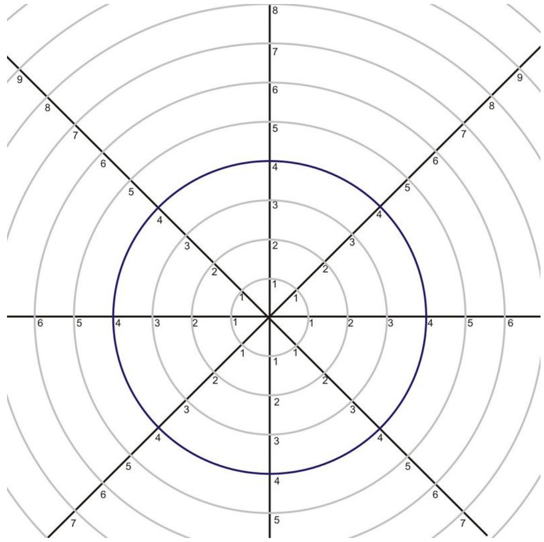
   a. $r = -3 - 3\cos\theta$



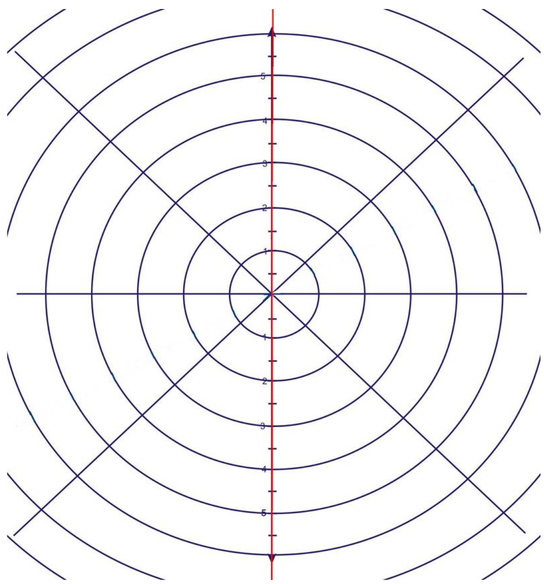
   b. $r = 2 + 4\sin\theta$



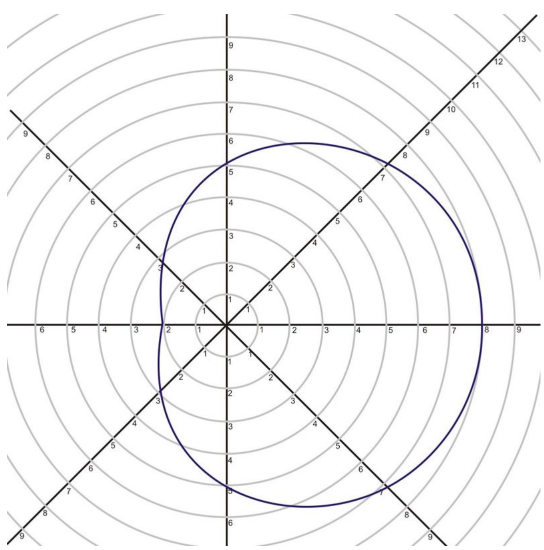
   c. $r = 4$



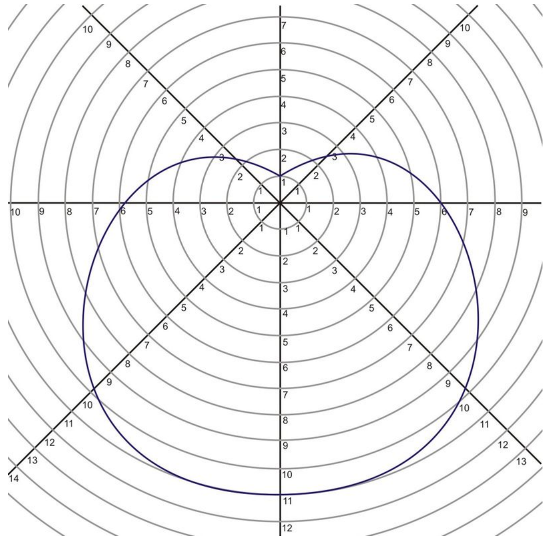
d. $\theta = \frac{\pi}{2}$



e. $r = 5 + 3\cos\theta$



f. $r = -6 - 5\sin\theta$



To determine the equation of these curves, first notice that cosine curves are along the horizontal axis and sine curves are along the vertical axis. Second, where the curve passes through the axis on the non-dimpled side is a in $r = a + b\sin\theta$. b is a little harder to see, but it is the average of the two intercepts where the curve crosses the axis on the dimpled axis. If there is an inner loop, use the innermost value of the loop.

3. Answer:

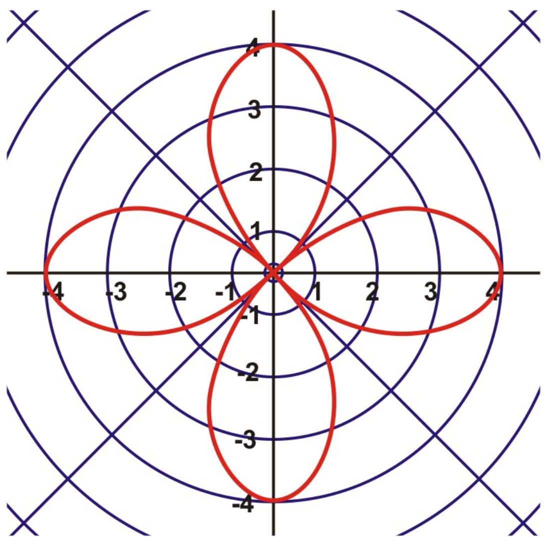


TABLE 6.1:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$4\cos 2\theta$	4	2	-2	-4	-2	2	4	2	-2	-4	-2	2	4

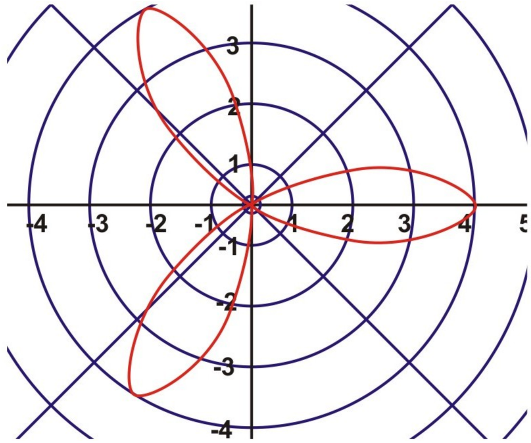


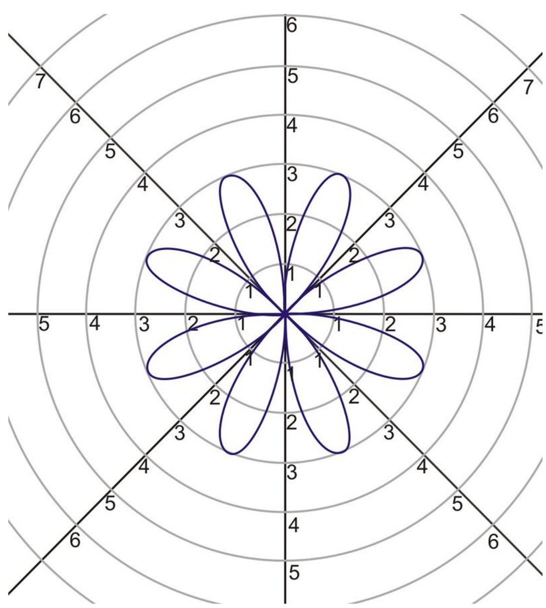
TABLE 6.2:

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$4\cos 3\theta$	4	0	-4	0	4	0	-4	0	4	0	-4	0	4

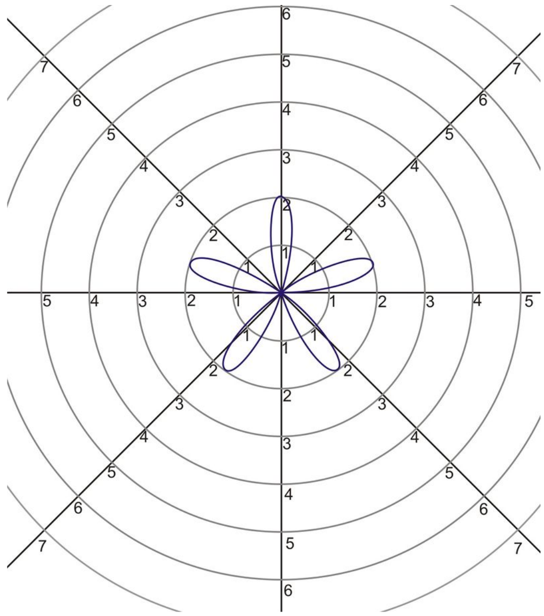
In the graph of $r = 4\cos 2\theta$, the rose has four petals on it but the graph of $r = 4\cos 3\theta$ has only three petals. It appears, that if n is an even positive integer, the rose will have an even number of petals and if n is an odd positive integer, the rose will have an odd number of petals.

4. Answers:

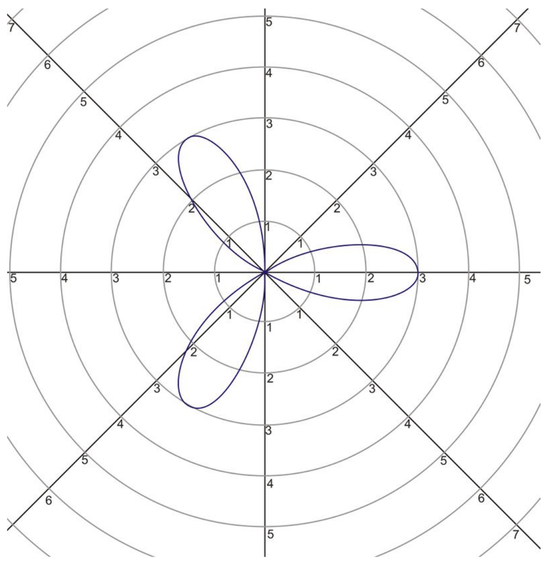
a. $r = 3\sin 4\theta$



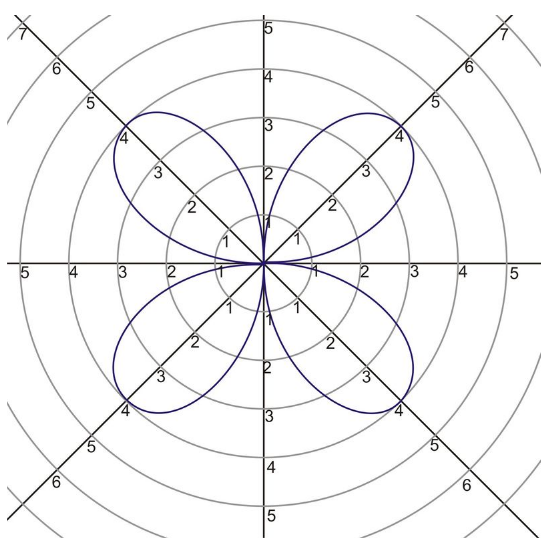
b. $r = 2\sin 5\theta$



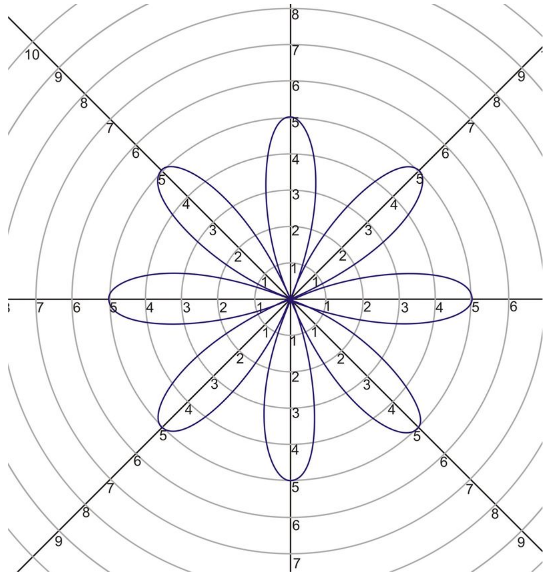
c. $r = 3\cos 3\theta$



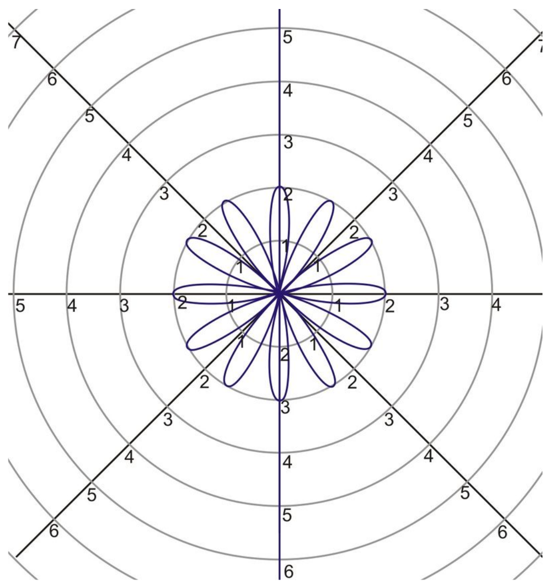
d. $r = -4\sin 2\theta$



e. $r = 5\cos 4\theta$



f. $r = -2\cos 6\theta$



For roses, the general equation is $r = a\sin n\theta$ or $r = a\cos n\theta$. a indicates how long each petal is, and depending on if n is even or odd indicates the number of pedals. If n is ood, there are n pedals and if n is even there are $2n$ pedals.

6.3 Converting Between Systems

1. For $A, r = -4$ and $\theta = \frac{5\pi}{4}$

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$x = -4\cos\frac{5\pi}{4} \qquad y = -4\sin\frac{5\pi}{4}$$
$$x = -4\left(-\frac{\sqrt{2}}{2}\right) \qquad y = -4\left(-\frac{\sqrt{2}}{2}\right)$$
$$x = 2\sqrt{2} \qquad y = 2\sqrt{2}$$

For $B, r = -3$ and $\theta = 135^\circ$

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$x = -3\cos 135^\circ \qquad y = -3\sin 135^\circ$$
$$x = -3 - \frac{\sqrt{2}}{2} \qquad y = -3\frac{\sqrt{2}}{2}$$
$$x = \frac{3\sqrt{2}}{2} \qquad y = \frac{-3\sqrt{2}}{2}$$

For C, $r = 5$ and $\theta = \left(\frac{2\pi}{3}\right)$

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$x = 5\cos\frac{2\pi}{3} \qquad y = 5\sin\frac{2\pi}{3}$$
$$x = 5\left(-\frac{1}{2}\right) \qquad y = 5\left(\frac{\sqrt{3}}{2}\right)$$
$$x = -2.5 \qquad y = \frac{5\sqrt{3}}{2}$$

2. Answers:

a.

$$\begin{aligned} r &= 6\cos\theta \\ r^2 &= 6r\cos\theta \\ x^2 + y^2 &= 6x \\ x^2 - 6x + y^2 &= 0 \\ x^2 - 6x + 9 + y^2 &= 9 \\ (x-3)^2 + y^2 &= 9 \end{aligned}$$

b.

$$\begin{aligned} r\sin\theta &= -3 \\ y &= -3 \end{aligned}$$

c.

$$\begin{aligned} r &= 2\sin\theta \\ r^2 &= 2r\sin\theta \\ x^2+y^2 &= 2y \\ y^2-2y &= -x^2 \\ y^2-2y+1 &= -x^2+1 \\ (y-1)^2 &= -x^2+1 \\ x^2+(y-1)^2 &= 1 \end{aligned}$$

d.

$$\begin{aligned} r\sin^2\theta &= 3\cos\theta \\ r^2\sin^2\theta &= 3r\cos\theta \\ y^2 &= 3x \end{aligned}$$

3. Answers:
   a. For $A(-2,5) x = -2$ and $y = 5$. The point is located in the second quadrant and $x < 0$.

   $$r = \sqrt{(-2)^2+(5)^2} = \sqrt{29} \approx 5.39,\ \theta = Arc\ \tan\frac{5}{-2}+\pi = 1.95.$$

   The polar coordinates for the rectangular coordinates $A(-2,5)$ are $A(5.39,1.95)$
   b. For $B(5,-4) x = 5$ and $y = -4$. The point is located in the fourth quadrant and $x > 0$.

   $$r = \sqrt{(5)^2+(-4)^2} = \sqrt{41} \approx 6.4,\ \theta = \tan^{-1}\left(\frac{-4}{5}\right) \approx -0.67$$

   The polar coordinates for the rectangular coordinates $B(5,-4)$ are $A(6.40,-0.67)$
   c. $C(1,9)$ is located in the first quadrant.

   $$r = \sqrt{1^2+9^2} = \sqrt{82} \approx 9.06,\ \theta = \tan^{-1}\frac{9}{1} \approx 83.66^\circ.$$

   d. $D(-12,-5)$ is located in the third quadrant and $x < 0$.

   $$r = \sqrt{(-12)^2+(-5)^2} = \sqrt{169} = 13,\ \theta = \tan^{-1}\frac{5}{12}+180^\circ \approx 202.6^\circ.$$

4. Answers:
   a.

$$\begin{aligned} (x-4)^2+(y-3)^2 &= 25 \\ x^2-8x+16+y^2-6y+9 &= 25 \\ x^2-8x+y^2-6y+25 &= 25 \\ x^2-8x+y^2-6y &= 0 \\ x^2+y^2-8x-6y &= 0 \\ r^2-8(r\cos\theta)-6(r\sin\theta) &= 0 \\ r^2-8r\cos\theta-6r\sin\theta &= 0 \\ r(r-8\cos\theta-6\sin\theta) &= 0 \\ r = 0 \text{ or } r-8\cos\theta-6\sin\theta &= 0 \\ r = 0 \text{ or } r = 8\cos\theta+6\sin\theta & \end{aligned}$$

From graphing $r-8\cos\theta-6\sin\theta = 0$, we see that the additional solutions are 0 and 8.

b.

$$\begin{aligned} 3x - 2y &= 1 \\ 3r\cos\theta - 2r\sin\theta &= 1 \\ r(3\cos\theta - 2\sin\theta) &= 1 \\ r &= \frac{1}{3\cos\theta - 2\sin\theta} \end{aligned}$$

c.

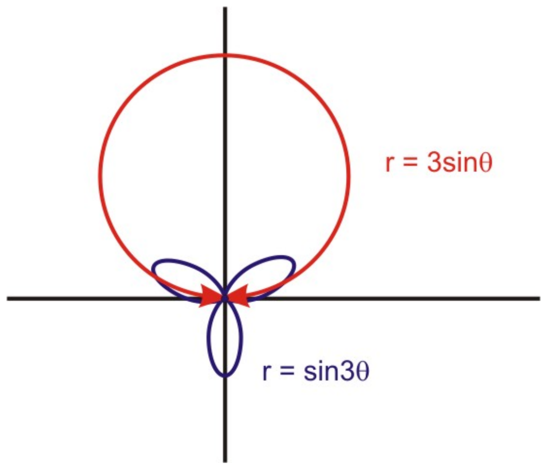
$$\begin{aligned} x^2 + y^2 - 4x + 2y &= 0 \\ r^2\cos^2\theta + r^2\sin^2\theta - 4r\cos\theta + 2r\sin\theta &= 0 \\ r^2(\sin^2\theta + \cos^2\theta) - 4r\cos\theta + 2r\sin\theta &= 0 \\ r(r - 4\cos\theta + 2\sin\theta) &= 0 \\ r = 0 \text{ or } r - 4\cos\theta + 2\sin\theta &= 0 \\ r = 0 \text{ or } r &= 4\cos\theta - 2\sin\theta \end{aligned}$$

d.

$$\begin{aligned} x^3 &= 4y^2 \\ (r\cos\theta)^3 &= 4(r\sin\theta)^2 \\ r^3\cos^3\theta &= 4r^2\sin^2\theta \\ \frac{4r^2\sin^2\theta}{r^3\cos^3\theta} &= 1 \\ \frac{4\tan^2\theta\sec\theta}{r} &= 1 \\ 4\tan^2\theta\sec\theta &= r \end{aligned}$$

6.4 More with Polar Curves

1. Answer:

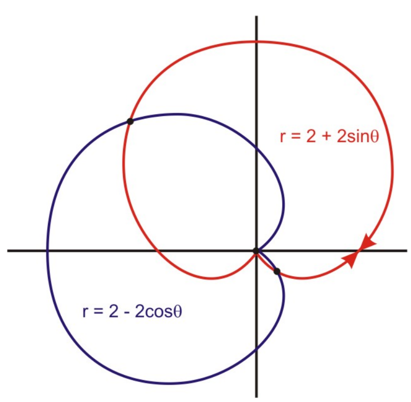


There appears to be one point of intersection.

$$\begin{aligned} r &= \sin 3\theta \\ 0 &= \sin 3\theta \\ 0 &= \theta \end{aligned} \qquad \begin{aligned} r &= 3\sin\theta \\ 0 &= 3\sin\theta \\ 0 &= \sin\theta \\ 0 &= \theta \end{aligned}$$

The point of intersection is (0, 0)

2. Answer:



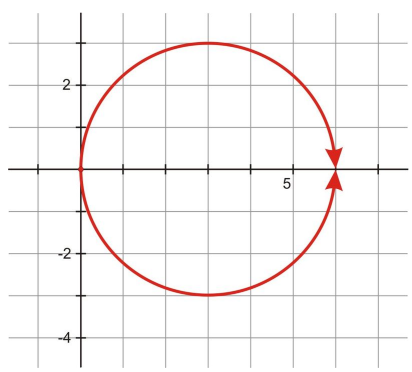
$$\begin{aligned} r &= 2+2\sin\theta & r &= 2+2\sin\theta \\ r &= 2+2\sin\left(\frac{3\pi}{4}\right) & r &= 2+2\sin\frac{7\pi}{4} \\ r &\approx 3.4 & r &\approx 0.59 \\ r &= 2+2\sin\theta & r &= 2-2\cos\theta \\ 0 &= 2+2\sin\theta & 0 &= 2-2\cos\theta \\ -1 &= \sin\theta & 1 &= \cos\theta \\ \theta &= \frac{3\pi}{2} & \theta &= 0 \end{aligned}$$

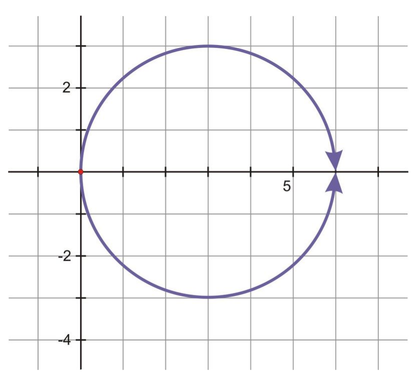
Since both equations have a solution at $r = 0$, that is (0, $\frac{3\pi}{2}$) and (0, 0), respectively, and these two points are equivalent, the two equations will intersect at (0, 0).

$$\begin{aligned} r &= 2+2\sin\theta \\ r &= 2-2\cos\theta \\ 2+2\sin\theta &= 2-2\cos\theta \\ 2\sin\theta &= -2\cos\theta \\ \frac{2\sin\theta}{2\cos\theta} &= -\frac{2\cos\theta}{2\cos\theta} \\ \frac{\sin\theta}{\cos\theta} &= -1 \\ \tan\theta &= -1 \\ \theta &= \frac{3\pi}{4} \text{ and } \theta = \frac{7\pi}{4} \end{aligned}$$

The points of intersection are $\left(3.4, \frac{3\pi}{4}\right), \left(0.59, \frac{7\pi}{4}\right)$ and $(0,0)$.

3. Answer:

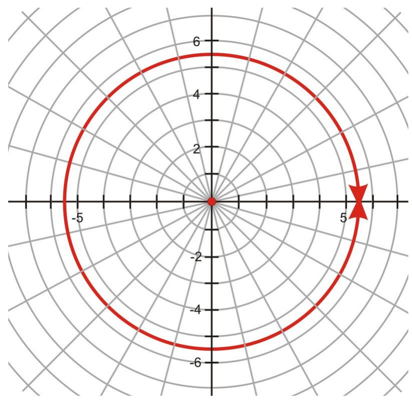




$$\begin{aligned} x^2+y^2 &= 6x \\ r^2 &= 6(r\cos\theta) && r^2 = x^2+y^2 \quad \text{and} \quad x = y\cos\theta \\ r &= 6\cos\theta && \text{divide by } r \end{aligned}$$

Both equations produced a circle with center $(3,0)$ and a radius of 3.

4. $r=-2+\sin\theta$ and $r=2-\sin\theta$ are not equivalent because the sine has the opposite sign. $r=-2+\sin\theta$ will be primarily above the horizontal axis and $r=2-\sin\theta$ will be mostly below. However, the two do have the same pole axis intercepts.
5. $r=-3+4\cos(-\pi)$ and $r=3+4\cos\pi$ are equivalent because the sign of a does not matter, nor does the sign of θ.
6. Yes, the equations produced the same graph so they are equivalent.



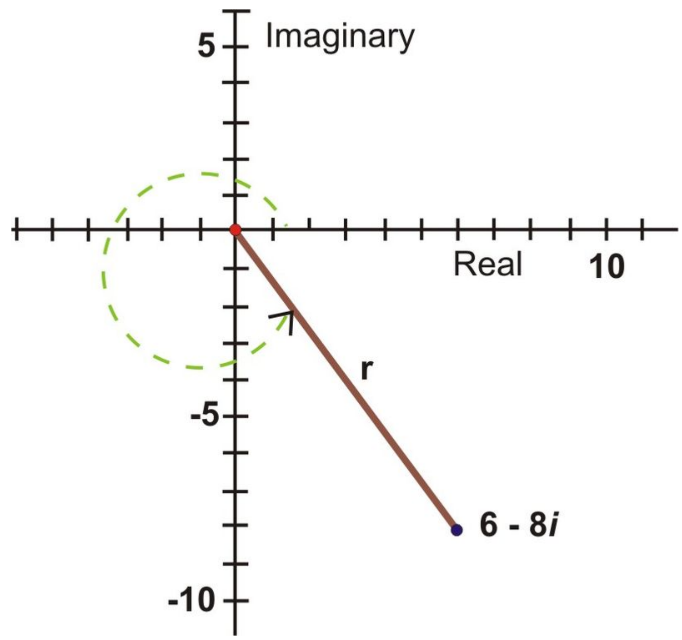
7. Students answers will vary, but they need to include that b must be the same sign. They should also mention that the sign of a does not matter, nor does the sign of θ.
8. There are several answers here. The most obvious are any two pairs of circles, for example $r=3$ and $r=9$.

6.5 The Trigonometric Form of Complex Numbers

1. Answers:

    a. $5\ cis\frac{\pi}{6} = 5\angle\frac{\pi}{6} = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
    b. $3\angle 135^\circ = 3cis135^\circ = 3(\cos 135^\circ + i\sin 135^\circ)$
    c. $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2cis\frac{2\pi}{3} = 2\angle\frac{2\pi}{3}$

2. $6-8i$



$$6-8i$$

$$x = 6 \text{ and } y = -8 \qquad \tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{-8}{6}$$

$$r = \sqrt{(6)^2+(-8)^2} \qquad \theta = -53.1^\circ$$

$$r = 10$$

Since θ is in the fourth quadrant then $\theta = -53.1^\circ + 360^\circ = 306.9^\circ$ Expressed in polar form $6-8i$ is $10(\cos 306.9^\circ + i\sin 306.9^\circ)$ or $10\angle 306.9^\circ$

3. Answers:

    a. $4+3i \to x = 4, y = 3$

$$r = \sqrt{4^2+3^2} = 5, \tan\theta = \frac{3}{4} \to \theta = 36.87^\circ \to 5(\cos 36.87^\circ + i\sin 36.87^\circ)$$

    b. $-2+9i \to x = -2, y = 9$

$$r = \sqrt{(-2)^2+9^2} = \sqrt{85} \approx 9.22, \tan\theta = -\frac{9}{2} \to \theta = 102.53^\circ \to 9.22(\cos 102.53^\circ + i\sin 102.53^\circ)$$

c. $7-i \rightarrow x=7, y=-1$

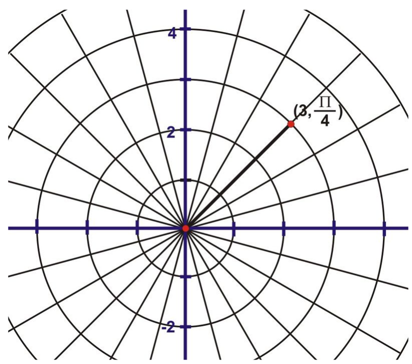
$$r=\sqrt{7^2+1^2}=\sqrt{50}\approx 7.07, \tan\theta=-\frac{1}{7}\rightarrow\theta=351.87^\circ\rightarrow 7.07(\cos 351.87^\circ+i\sin 351.87^\circ)$$

d. $-5-2i \rightarrow x=-5, y=-2$

$$r=\sqrt{(-5)^2+(-2)^2}=\sqrt{29}\approx 5.39, \tan\theta=\frac{2}{5}\rightarrow\theta=201.8^\circ\rightarrow 5.39(\cos 201.8^\circ+i\sin 201.8^\circ)$$

- Note: The range of a graphing calculator's $\tan^{-1}$ function is limited to Quadrants I and IV, and for points located in the other quadrants, such as $-2+9i$ in part b (in Quadrant II), you must add 180° to get the correct angle θ for numbers given in polar form.

4. Answer:



$$3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$
$$r=3$$
$$x=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$
$$y=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

The standard form of the polar complex number $3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ is $\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i$.

5. Answers:

a. $2cis\frac{\pi}{2}\rightarrow x=\cos\frac{\pi}{2}=0, y=\sin\frac{\pi}{2}=1\rightarrow 2(0)+2(1i)=2i$

b. $4\angle\frac{5\pi}{6}\rightarrow x=\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}, y=\sin\frac{5\pi}{6}=\frac{1}{2}\rightarrow 4\left(-\frac{\sqrt{3}}{2}\right)+4\left(i\frac{1}{2}\right)=-2\sqrt{3}+2i$

c. $8\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)\rightarrow x=\cos\left(-\frac{\pi}{3}\right)=\frac{1}{2}, y=\sin\left(-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}\rightarrow 8\left(\frac{1}{2}\right)+8\left(-\frac{\sqrt{3}}{2}i\right)=4-4i\sqrt{3}$

6.6 The Product & Quotient Theorems

1. Answers:

 a. $2\angle 56°, 7\angle 113° = (2)(7)\angle(56° + 113°) = 14\angle 169°$
 b. $3(\cos\pi + i\sin\pi), 10\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = (3)(10)cis\left(\pi + \frac{5\pi}{3}\right) = 30cis\frac{8\pi}{3} = 30cis\frac{2\pi}{3}$
 c. $2+3i, -5+11i \rightarrow$ change to polar

$$x = 2, y = 3 \qquad x = -5, y = 11$$
$$r = \sqrt{2^2+3^2} = \sqrt{13} \approx 3.61 \qquad r = \sqrt{(-5)^2+11^2} = \sqrt{146} \approx 12.08$$
$$\tan\theta = \frac{3}{2} \rightarrow \theta = 56.31° \qquad \tan\theta = -\frac{11}{5} \rightarrow \theta = 114.44°$$

$$(3.61)(12.08)\angle(56.31° + 114.44°) = 43.61\angle 170.75°$$

 d. $6-i, -20i \rightarrow$ change to polar

$$x = 6, y = -1 \qquad x = 0, y = -20$$
$$r = \sqrt{6^2+(-1)^2} = \sqrt{37} \approx 6.08 \qquad r = \sqrt{0^2+(-20)^2} = \sqrt{40} = 20$$
$$\tan\theta = -\frac{1}{6} \rightarrow \theta = 350.54° \qquad \tan\theta = \frac{-20}{0} = und \rightarrow \theta = 270°$$

$$(6.08)(20)\angle(350.54° + 270°) = 121.6\angle 620.54° = 121.6\angle 260.54°$$

2. Without changing complex numbers to polar form, you mulitply by FOIL-ing.

$$(2+3i)(-5+11i) = -10+22i-15i+33i^2 = -10-33+7i = -43+7i$$

 The answer is student opinion, but they seem about equal in the degree of difficulty.

3. Answer:

$$\begin{aligned}4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^2 &= 4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\cdot 4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\\ &= 16\left(\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\right)\\ &= 16\left(\cos\frac{\pi}{2} i\sin\frac{\pi}{2}\right)\end{aligned}$$

4. Answer:

$$P = (6.80)(7.05)\angle(56.3° - 15.8°), P = 47.9\angle 40.5° watts$$

5. Answers:

 a. $\frac{2\angle 56°}{7\angle 113°} = \frac{2}{7}\angle(56° - 113°) = \frac{2}{7}\angle -57°$
 b. $\frac{10\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)}{5(\cos\pi+i\sin\pi)} = 2\left(\cos\left(\frac{5\pi}{3}-\pi\right)+i\sin\left(\frac{5\pi}{3}-\pi\right)\right) = 2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

c. $\frac{2+3i}{-5+11i} \rightarrow$ change each to polar.

$$x = 2, y = 3 \qquad x = -5, y = 11$$
$$r = \sqrt{2^2+3^2} = \sqrt{13} \approx 3.61 \qquad r = \sqrt{(-5)^2+11^2} = \sqrt{146} \approx 12.08$$
$$\tan\theta = \frac{3}{2} \rightarrow \theta = 56.31^\circ \qquad \tan\theta = -\frac{11}{5} \rightarrow \theta = 114.44^\circ$$

$\frac{3.61}{12.08}\angle(56.31^\circ - 114.44^\circ) = 0.30\angle-58.13^\circ$

d. $\frac{6-i}{1-20i} \rightarrow$ change both to polar

$$x = 6, y = -1 \qquad x = 1, y = -20$$
$$r = \sqrt{6^2+(-1)^2} = \sqrt{37} \approx 6.08 \qquad r = \sqrt{1^2+(-20)^2} = \sqrt{401} = 20.02$$
$$\tan\theta = -\frac{1}{6} \rightarrow \theta = 350.54^\circ \qquad \tan\theta = \frac{-20}{1} \rightarrow \theta = 272.68^\circ$$

$\frac{6.08}{20.02}\angle(350.54^\circ - 272.86^\circ) = 0.304\angle 77.68^\circ$

6. Answer:

$$\frac{2+3i}{-5+11i} = \frac{2+3i}{-5+11i}\cdot\frac{-5-11i}{-5-11i} = \frac{-10-22i-15i+33}{25+121} = \frac{23-37i}{146}$$

Again, this is opinion, but in general, using the polar form is "easier."

7. Answer:

$$[4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)]^3 = [4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)]^2\cdot 4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right).$$

From #3, $[4\left(\cos\frac{\pi}{4}i\sin\frac{\pi}{4}\right)]^2 = 16\left(\cos\frac{\pi}{2}i\sin\frac{\pi}{2}\right)$. So,

$$\begin{aligned}[4\left(\cos\frac{\pi}{4}i\sin\frac{\pi}{4}\right)]^3 &= 16\left(\cos\frac{\pi}{2}i\sin\frac{\pi}{2}\right)\cdot 4\left(\cos\frac{\pi}{4}i\sin\frac{\pi}{4}\right)\\ &= 64\left(\cos\frac{3\pi}{4}i\sin\frac{3\pi}{4}\right)\end{aligned}$$

8. Even though 1 is not a complex number, we can still change it to polar form.

$$1 \rightarrow x = 1, y = 0$$

$$r = \sqrt{1^2+0^2} = 1$$
$$\tan\theta = \frac{0}{1} = 0 \rightarrow \theta = 0^\circ$$

$$\text{So, } \frac{1}{4cis\frac{\pi}{6}} = \frac{1cis0}{4cis\frac{\pi}{6}} = \frac{1}{4}cis\left(0-\frac{\pi}{6}\right) = \frac{1}{4}cis\left(-\frac{\pi}{6}\right).$$

6.7 De Moivre's and the nth Root Theorems

1. Express z in polar form:

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$r = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$
$$\theta = \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) = 120^\circ$$

The polar form is $z = 1(\cos 120^\circ + i\sin 120^\circ)$

$$z^n = [r(\cos\theta + i\sin\theta)^n = r(\cos n\theta + i\sin n\theta)$$
$$z^3 = 1^3[\cos 3(120^\circ) + i\sin(120^\circ)]$$
$$z^3 = 1(\cos 360^\circ + i\sin 360^\circ)$$
$$z^3 = 1(1+0i)$$
$$z^3 = 1$$

2. Answers:

a. $\left[\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^8 = \left(\frac{\sqrt{2}}{2}\right)^8\left(\cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4}\right) = \frac{1}{16}\cos 2\pi + \frac{i}{16}\sin 2\pi = \frac{1}{16}$

b.

$$\left[3(\sqrt{3} - i\sqrt{3})\right]^4 = (3\sqrt{3} - 3i\sqrt{3})^4$$
$$r = \sqrt{(3\sqrt{3})^2 + (3\sqrt{3})^2} = 3\sqrt{6}, \tan\theta = \frac{3\sqrt{3}}{3\sqrt{3}} = 1 \to 45^\circ$$
$$= \left(3\sqrt{6}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^4 = (3\sqrt{6})^4\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right)$$
$$= 81(36)[-1+i(0)] = -2936$$

c.

$$(\sqrt{5} - i)^7 \to r = \sqrt{(\sqrt{5})^2 + (-1)^2} = \sqrt{6}, \tan\theta = -\frac{1}{\sqrt{5}} \to \theta = 335.9^\circ$$
$$\sqrt{6}(\cos 335.9^\circ + i\sin 335.9^\circ)^7 = (\sqrt{6})^7(\cos(7\cdot 335.9^\circ) + i\sin(7\cdot 335.9^\circ))$$
$$= 216\sqrt{6}(\cos 2351.3^\circ + i\sin 2351.3^\circ)$$
$$= 216\sqrt{6}(-0.981 - 0.196i)$$
$$= -519.04 - 103.7i$$

d. $\left[3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{12} = 3^{12}(\cos 2\pi + i\sin 2\pi) = 531,441$

3. Answer:

$$r = 2 \text{ and } \theta = 315^\circ \text{ or } \frac{7\pi}{4}.$$
$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$z^3 = 2^3\left[(\cos 3\left(\frac{7\pi}{4}\right) + i\sin 3\left(\frac{7\pi}{4}\right)\right]$$
$$z^3 = 8\left(\cos\frac{21\pi}{4} + i\sin\frac{21\pi}{4}\right)$$
$$z^3 = 8\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$
$$z^3 = -4\sqrt{2} - 4i\sqrt{2}$$

$\frac{21\pi}{4}$ is in the third quadrant so both are negative.

4. Answer:

$$a = 0 \textit{ and } b = 27$$
$$\sqrt[3]{27i} = (0 + 27i)^{\frac{1}{3}} \qquad x = 0 \textit{ and } y = 27$$
$$\text{Polar Form} \qquad r = \sqrt{x^2 + y^2} \qquad \theta = \frac{\pi}{2}$$
$$r = \sqrt{(0)^2 + (27)^2}$$
$$r = 27$$
$$\sqrt[3]{27i} = \left[27\left(\cos(\frac{\pi}{2} + 2\pi k) + i\sin(\frac{\pi}{2} + 2\pi k)\right)\right]^{\frac{1}{3}} \text{ for } k = 0, 1, 2$$
$$\sqrt[3]{27i} = \sqrt[3]{27}\left[\cos\left(\frac{1}{3}\right)\left(\frac{\pi}{2} + 2\pi k\right) + i\sin\left(\frac{1}{3}\right)\left(\frac{\pi}{2} + 2\pi k\right)\right] \text{ for } k = 0, 1, 2$$
$$\sqrt[3]{27i} = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \text{ for } k = 0$$
$$\sqrt[3]{27i} = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \text{ for } k = 1$$
$$\sqrt[3]{27i} = 3\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right) \text{ for } k = 2$$
$$\sqrt[3]{27i} = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), 3\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right), -3i$$

5. Answer:

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\sqrt{2}}{2} \qquad \text{Polar Form} = \sqrt{2}cis\frac{\pi}{4}$$
$$r = \sqrt{(1)^2 + (1)^2}$$
$$r = \sqrt{2}$$

$$(1+i)^{\frac{1}{5}} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{\frac{1}{5}}$$
$$(1+i)^{\frac{1}{5}} = \sqrt{2}^{\frac{1}{5}}\left[\cos\left(\frac{1}{5}\right)\left(\frac{\pi}{4}\right) + i\sin\left(\frac{1}{5}\right)\left(\frac{\pi}{4}\right)\right]$$
$$(1+i)^{\frac{1}{5}} = \sqrt[10]{2}\left(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right)$$

In standard form $(1+i)^{\frac{1}{5}} = (1.06 + 1.06i)$ and this is the principal root of $(1+i)^{\frac{1}{5}}$.

6. $81i$ in polar form is:

$$r = \sqrt{0^2+81^2} = 81, \tan\theta = \frac{81}{0} = und \rightarrow \theta = \frac{\pi}{2} \quad 81\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$
$$\left[81\left(\cos\left(\frac{\pi}{2}+2\pi k\right)+i\sin\left(\frac{\pi}{2}+2\pi k\right)\right)\right]^{\frac{1}{4}}$$
$$3\left(\cos\left(\frac{\frac{\pi}{2}+2\pi k}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\pi k}{4}\right)\right)$$
$$3\left(\cos\left(\frac{\pi}{8}+\frac{\pi k}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{\pi k}{2}\right)\right)$$
$$z_1 = 3\left(\cos\left(\frac{\pi}{8}+\frac{0\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{0\pi}{2}\right)\right) = 3\cos\frac{\pi}{8}+3i\sin\frac{\pi}{8} = 2.77+1.15i$$
$$z_2 = 3\left(\cos\left(\frac{\pi}{8}+\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{\pi}{2}\right)\right) = 3\cos\frac{5\pi}{8}+3i\sin\frac{5\pi}{8} = -1.15+2.77i$$
$$z_3 = 3\left(\cos\left(\frac{\pi}{8}+\frac{2\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{2\pi}{2}\right)\right) = 3\cos\frac{9\pi}{8}+3i\sin\frac{9\pi}{8} = -2.77-1.15i$$
$$z_4 = 3\left(\cos\left(\frac{\pi}{8}+\frac{3\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{3\pi}{2}\right)\right) = 3\cos\frac{13\pi}{8}+3i\sin\frac{13\pi}{8} = 1.15-2.77i$$

7. Answer:

$$x^4+1=0 \qquad r=\sqrt{x^2+y^2}$$
$$x^4=-1 \qquad r=\sqrt{(-1)^2+(0)^2}$$
$$x^4=-1+0i \qquad r=1$$
$$\theta = \tan^{-1}\left(\frac{0}{-1}\right)+\pi=\pi$$

Write an expression for determining the fourth roots of $x^4=-1+0i$

$$(-1+0i)^{\frac{1}{4}} = [1(\cos(\pi+2\pi k)+i\sin(\pi+2\pi k))]^{\frac{1}{4}}$$
$$(-1+0i)^{\frac{1}{4}} = 1^{\frac{1}{4}}\left(\cos\frac{\pi+2\pi k}{4}+i\sin\frac{\pi+2\pi k}{4}\right)$$
$$x_1 = 1\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} \qquad \text{for } k=0$$
$$x_2 = 1\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} \qquad \text{for } k=1$$
$$x_3 = 1\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2} \qquad \text{for } k=2$$
$$x_4 = 1\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2} \qquad \text{for } k=3$$

If a line segment is drawn from each root on the polar plane to its adjacent roots, the four roots will form the corners of a square.

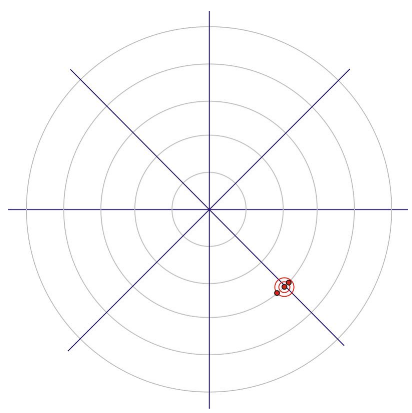
8. Answer:

$$x^3-64=0 \rightarrow x^3=64+0i$$
$$64+0i=64(\cos(0+2\pi k)+i\sin(0+2\pi k))$$
$$x=(x^3)^{\frac{1}{3}}=(64+0i)^{\frac{1}{3}}=\sqrt[3]{64}\left(\cos\left(\frac{0+2\pi k}{3}\right)+i\sin\left(\frac{0+2\pi k}{3}\right)\right)$$
$$z_1=4\left(\cos\left(\frac{0+2\pi 0}{3}\right)+i\sin\left(\frac{0+2\pi 0}{3}\right)\right)=4\cos 0+4i\sin 0=4 \text{ for } k=0$$
$$z_2=4\left(\cos\left(\frac{0+2\pi}{3}\right)+i\sin\left(\frac{0+2\pi}{3}\right)\right)=4\cos\frac{2\pi}{3}+4i\sin\frac{2\pi}{3}=-2+2i\sqrt{3} \text{ for } k=1$$
$$z_3=4\left(\cos\left(\frac{0+4\pi}{3}\right)+i\sin\left(\frac{0+4\pi}{3}\right)\right)=4\cos\frac{4\pi}{3}+4i\sin\frac{4\pi}{3}=-2-2i\sqrt{3} \text{ for } k=2$$

If a line segment is drawn from each root on the polar plane to its adjacent roots, the three roots will form the vertices of an equilateral triangle.

Chapter Summary

1. $A\left(-3,\frac{3\pi}{4}\right)$



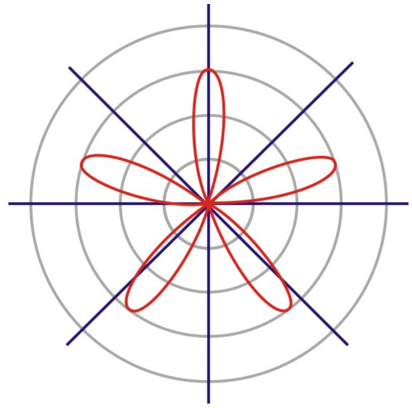
three equivalent coordinates $\rightarrow \left(3,-\frac{\pi}{4}\right),\left(3,\frac{7\pi}{4}\right),\left(-3,-\frac{5\pi}{4}\right)$.

2. $(2,94^\circ)$ and $(7,-73^\circ)$

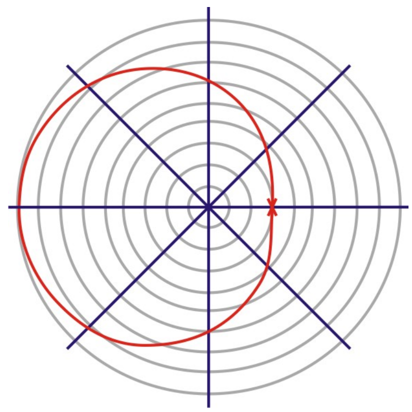
$$\begin{aligned} d &= \sqrt{2^2+7^2-2(2)(7)\cos(94^\circ-(-73^\circ))} \\ &= \sqrt{4+49-28\cos 167^\circ} \\ &= \sqrt{80.28}\approx 8.96 \end{aligned}$$

3. Answers:

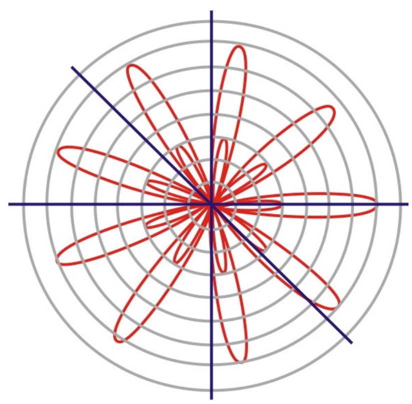
a. $r=3\sin 5\theta$



b. $r = 6 - 3\cos\theta$



c. $r = 2 + 5\cos 9\theta$



4. Answers:

a. $r = 2 - 6\cos\theta$
b. $r = 7 + 3\sin\theta$

5. Answers:

a. $A(-6, 11) \rightarrow r = \sqrt{36 + 121} \approx 12.59, \tan\theta = -\frac{11}{6}, \theta = 118.6^\circ \rightarrow (12.59, 118.6^\circ)$
b. $B(15, -8) \rightarrow r = \sqrt{225 + 64} = 17, \tan\theta = -\frac{8}{15}, \theta = -28.1^\circ \rightarrow (17, -28.1^\circ)$
c. $C(9, 40) \rightarrow r = \sqrt{91 + 1600} = 41, \tan\theta = \frac{40}{9}, \theta = 77.3^\circ \rightarrow (41, 77.3^\circ)$

d.

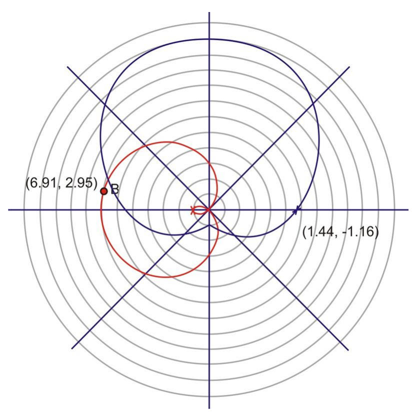
$$\begin{aligned} x^2+(y-6)^2 &= 36 \\ r^2\cos^2\theta+(r\sin\theta-6)^2 &= 36 \\ r^2\cos^2\theta+r^2\sin^2\theta-12r\sin\theta+36 &= 36 \\ r^2-12r\sin\theta &= 0 \text{ or} \\ r^2 &= 12r\sin\theta \\ r &= 12\sin\theta \end{aligned}$$

6. Answers:

a. $D\left(4,-\frac{\pi}{3}\right) \to x=4\cos\left(-\frac{\pi}{3}\right)=2, y=4\sin\left(-\frac{\pi}{3}\right)=-2\sqrt{3} \to (2,-2\sqrt{3})$
b. $E(-2,135^\circ) \to x=-2\cos 135^\circ=\sqrt{2}, y=-2\sin 135^\circ=-\sqrt{2} \to (\sqrt{2},-\sqrt{2})$
c. $r=7 \to r^2=49 \to x^2+y^2=49$
d.

$$\begin{aligned} r &= 8\sin\theta \\ r^2 &= 8r\sin\theta \\ x^2+y^2 &= 8y \\ y^2-8y &= -x^2 \\ y^2-8y+16 &= 16-x^2 \\ (y-4)^2 &= 16-x^2 \\ x^2+(y-4)^2 &= 16 \end{aligned}$$

7. $r=6+5\sin\theta$ and $r=3-4\cos\theta$



- angle measures in the graph are in radians
- Note: The two determined points of intersection [$(6.91, 2.95)$ and $(1.44, -1.16)$] were estimated from the trace function on a graphing calculator and are not precise solutions for either equation.

8. Answer:

- $-3+8i, x=-3, y=8 \to r=\sqrt{(-3)^2+8^2} \approx 8.54$
- $\tan\theta=-\frac{8}{3} \to \theta=110.56^\circ$
- $8.54(\cos 110.56^\circ + i\sin 110.56^\circ)$

9. Answer:

- $15\angle 240^\circ, r=15, \theta=240^\circ$
- $x=15\cos 240^\circ=-7.5, y=15\sin 240^\circ=-\frac{15\sqrt{3}}{2}=-7.5\sqrt{3}$

- So, $15\angle 240^\circ = -7.5 - 7.5i\sqrt{3}$.

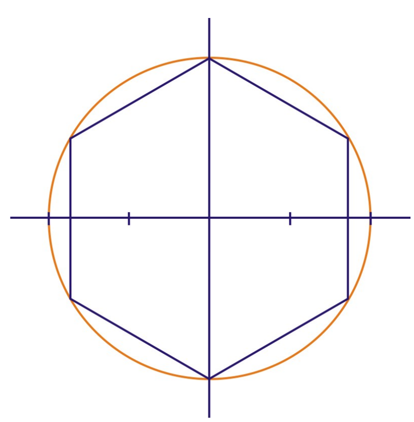
10. Answers:

a. $\left(7cis\frac{7\pi}{4}\right)\cdot\left(3cis\frac{\pi}{3}\right) = 21cis\left(\frac{7\pi}{4}+\frac{\pi}{3}\right) = 21cis\frac{25\pi}{12}$
b. $\frac{8\angle 80^\circ}{2\angle -155^\circ} = 4\angle(80^\circ - (-155^\circ)) = 4\angle 235^\circ$

11. $\left[4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^6 = 4^6\left(\cos\frac{6\pi}{4}+i\sin\frac{6\pi}{4}\right) = 4096\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$
12. -64 in polar form is $64(\cos\pi - i\sin\pi)$

$$[64(\cos(\pi+2\pi k)+i\sin(\pi+2\pi k))]^{\frac{1}{6}}$$
$$2\left(\cos\left(\frac{\pi+2\pi k}{6}\right)+i\sin\left(\frac{\pi+2\pi k}{6}\right)\right)$$
$$2\left(\cos\left(\frac{\pi}{6}+\frac{\pi k}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{\pi k}{3}\right)\right)$$
$$z_1 = 2\left(\cos\left(\frac{\pi}{6}+\frac{0\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{0\pi}{3}\right)\right) = 2\cos\frac{\pi}{6}+2i\sin\frac{\pi}{6} = \frac{2\sqrt{3}}{2}+\frac{2i}{2} = \sqrt{3}+i$$
$$z_2 = 2\left(\cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right)\right) = 2\cos\frac{\pi}{2}+2i\sin\frac{\pi}{2} = 2i$$
$$z_3 = 2\left(\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)\right) = 2\cos\frac{5\pi}{6}+2i\sin\frac{5\pi}{6} = -\frac{2\sqrt{3}}{2}+\frac{2i}{2} = -\sqrt{3}+i$$
$$z_4 = 2\left(\cos\left(\frac{\pi}{6}+\pi\right)+i\sin\left(\frac{\pi}{6}+\pi\right)\right) = 2\cos\frac{7\pi}{6}+2i\sin\frac{7\pi}{6} = -\frac{2\sqrt{3}}{2}-\frac{2i}{2} = -\sqrt{3}-i$$
$$z_5 = 2\left(\cos\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)\right) = 2\cos\frac{3\pi}{2}+2i\sin\frac{3\pi}{2} = -2i$$
$$z_6 = 2\left(\cos\left(\frac{\pi}{6}+\frac{5\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{5\pi}{3}\right)\right) = 2\cos\frac{11\pi}{6}+2i\sin\frac{11\pi}{6} = \frac{2\sqrt{3}}{2}-\frac{2i}{2} = \sqrt{3}-i$$

Graph of the solutions:



13. Answer:

$$x^4+32=0 \rightarrow x^4=-32+0i=-32(\cos\pi+i\sin\pi)$$

$$[32(\cos(\pi+2\pi k)+i\sin(\pi+2\pi k))]^{\frac{1}{4}}$$

$$2\sqrt[4]{2}\left(\cos\left(\frac{\pi+2\pi k}{4}\right)+i\sin\left(\frac{\pi+2\pi k}{4}\right)\right)$$

$$2\sqrt[4]{2}\left(\cos\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)\right)$$

$$z_1=2\sqrt[4]{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)=2\sqrt[4]{2}\cos\frac{\pi}{4}+2i\sqrt[4]{2}\sin\frac{\pi}{4}=\frac{2\sqrt[4]{2}\sqrt{2}}{2}+\frac{2i\sqrt[4]{2}\sqrt{2}}{2}$$

$$=\sqrt[4]{2}^3+i\sqrt[4]{2}^3$$

$$z_2=2\sqrt[4]{2}\left(\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right)=2\sqrt[4]{2}\cos\frac{3\pi}{4}+2i\sqrt[4]{2}\sin\frac{3\pi}{4}=-\frac{2\sqrt[4]{2}\sqrt{2}}{2}+\frac{2i\sqrt[4]{2}\sqrt{2}}{2}$$

$$=-\sqrt[4]{2}^3+i\sqrt[4]{2}^3$$

$$z_3=2\sqrt[4]{2}\left(\cos\left(\frac{\pi}{4}+\pi\right)+i\sin\left(\frac{\pi}{4}+\pi\right)\right)=2\sqrt[4]{2}\cos\frac{5\pi}{4}+2i\sqrt[4]{2}\sin\frac{5\pi}{4}=-\frac{2\sqrt[4]{2}\sqrt{2}}{2}-\frac{2i\sqrt[4]{2}\sqrt{2}}{2}$$

$$=-\sqrt[4]{2}^3-i\sqrt[4]{2}^3$$

$$z_4=2\sqrt[4]{2}\left(\cos\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)\right)=2\sqrt[4]{2}\cos\frac{7\pi}{4}+2i\sqrt[4]{2}\sin\frac{7\pi}{4}=\frac{2\sqrt[4]{2}\sqrt{2}}{2}-\frac{2i\sqrt[4]{2}\sqrt{2}}{2}$$

$$=\sqrt[4]{2}^3-i\sqrt[4]{2}^3$$