LECTURE 12: HOPF ALGEBRA $U_q(\mathfrak{sl}_2)$

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Introduction

In this lecture we start to study quantum groups $U_q(\mathfrak{g})$, certain deformations of the universal enveloping algebras $U(\mathfrak{g})$. The algebras $U_q(\mathfrak{g})$ are *Hopf algebras* that basically means that we can take tensor products and duals of their representations. In Section 1 we define Hopf algebras.

In Section 2 we start discussing quantum groups themselves concentrating mostly on the simplest case, $U_q(\mathfrak{sl}_2)$. An important feature here is that the tensor product is not commutative in a naive sense. This is a feature and not a bug, this is one of the main reasons why the quantum groups were introduced.

1. Hopf algebras

1.1. **Tensor products and duals.** Recall that for a group G and two G-modules V_1, V_2 we can define G-module structures on $V_1 \otimes V_2$ and V_1^* by

$$g.(v_1 \otimes v_2) := gv_1 \otimes gv_2, \langle g.\alpha, v_1\rangle := \langle \alpha, g^{-1}v_1\rangle.$$

We also have the trivial one-dimensional module $\mathbb{C}$, where $g \in G$ acts by 1.

Similarly, for a Lie algebra $\mathfrak{g}$ and two $\mathfrak{g}$-modules V_1, V_2, we can define $\mathfrak{g}$-module structures on $V_1 \otimes V_2$ and V_1^* by

$$x.(v_1 \otimes v_2) = (x.v_1) \otimes v_2 + v_1 \otimes (x.v_2), \langle x.\alpha, v_1\rangle = -\langle \alpha, x.v_1\rangle.$$

And we have the trivial one-dimensional module $\mathbb{C}$, where $x \in \mathfrak{g}$ acts by 0.

Recall also that a G-module (resp., $\mathfrak{g}$-module) is the same thing as a module over the group algebra $\mathbb{C}G$ (resp., over the universal enveloping algebra $U(\mathfrak{g})$). Both $\mathbb{C}G, U(\mathfrak{g})$ are associative algebras. Note, however, that if A is an associative algebra, then we do not have natural A-module structures on $V_1 \otimes V_2, V_1^*, \mathbb{C}$ (where V_1, V_2 are A-modules). Indeed, $V_1 \otimes V_2$ carries a natural structure of $A \otimes A$-module by $(a \otimes b).(v_1 \otimes v_2) = (av_1) \otimes (bv_2)$. The dual space V_1^* is naturally a module over the opposite algebra A^{op}, which is the same vector space as A but with opposite multiplication: $a \cdot b := ba$. An A^{op}-module is the same thing as a right A-module, and we set $(\alpha a)(v_1) := \alpha(av_1)$. Finally, $\mathbb{C}$ is naturally a $\mathbb{C}$-module. We could equip $V_1 \otimes V_2$ with an A-module structure if we have a distinguished algebra homomorphism $\Delta : A \to A \otimes A$ (then we just pull the $A \otimes A$-module structure back to A). This homomorphism Δ is called a *coproduct*. Similarly, to equip V_1^* and $\mathbb{C}$ with A-module structures we need algebra homomorphisms $S : A \to A^{op}$ (antipode) and $\eta : A \to \mathbb{C}$ (counit).

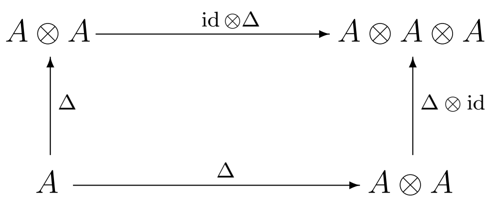
Let us construct these homomorphisms for $A = \mathbb{C}G$ and $A = U(\mathfrak{g})$.

Example 1.1. For $A = \mathbb{C}G$, we have $\Delta(g) := g \otimes g, S(g) = g^{-1}, \eta(g) = 1$ for $g \in G$.

Example 1.2. Let $A = U(\mathfrak{g})$. Since Δ, S, η are supposed to be algebra homomorphisms, it is enough to define them on $\mathfrak{g}$. We set $\Delta(x) = x \otimes 1 + 1 \otimes x, S(x) = -x, \eta(x) = 1$, where $x \in \mathfrak{g}$.

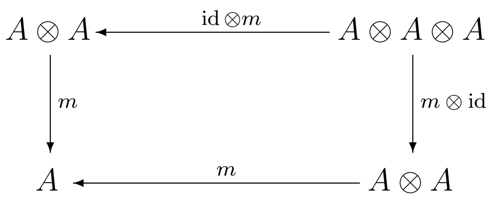
1.2. **Coassociativity.** We need some additional assumptions on Δ, S, ϵ in order to guarantee some natural properties of tensor products such as associativity. Axiomatizing these properties, we arrive at the definition of a *Hopf algebra*.

First, let us examine the associativity of the tensor product. We have a natural isomorphism $(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3), (v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$. We want this isomorphism to be A-linear. We have two homomorphisms $A \to A^{\otimes 3}$ produced from Δ. First, we have $(\Delta \otimes \mathrm{id}) \circ \Delta$. The algebra A acts on $(V_1 \otimes V_2) \otimes V_3$ via this homomorphism $A \to A^{\otimes 3}$. Indeed, if $\Delta(a) = \sum_{i=1}^k a_i^1 \otimes a_i^2$, then $a.((v_1 \otimes v_2) \otimes v_3) = \sum_{i=1}^k a_i^1.(v_1 \otimes v_2) \otimes a_i^2 v_3 = \sum_{i=1}^k \Delta(a_i^1)(v_1 \otimes v_2) \otimes a_i^2 v_3$, and $(\Delta \otimes \mathrm{id}) \circ \Delta(a) = \sum_{i=1}^k \Delta(a_i^1) \otimes a_i^2$. Similarly, A acts on $V_1 \otimes (V_2 \otimes V_3)$ via $(\mathrm{id} \otimes \Delta) \circ \Delta : A \to A^{\otimes 3}$. So, if we want to the isomorphism $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$ to be A-linear, it is natural to require that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. In other words, we want the following diagram to be commutative.



If this holds, then we say that Δ is coassociative.

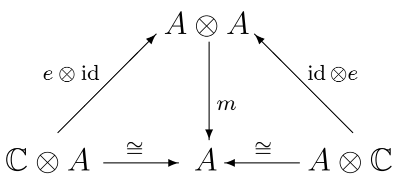
Let us motivate the terminology ("coproduct" and "coassociative"). Let A be a finite dimensional algebra. Let us write $m : A \otimes A \to A$ for the product. Then m is associative (i.e., $m(m(a \otimes b) \otimes c) = m(a \otimes m(b \otimes c))$) if and only if the following diagram is commutative.



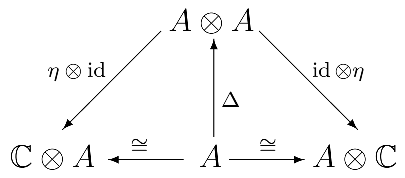
Now let us dualize. We get the space A^* together with the map $m^* : A^* \to A^* \otimes A^*$ that is natural to call a coproduct. Clearly, m is associative if and only if m^* is coassociative.

1.3. **Axioms of Hopf algebras.** We need to axiomatically describe two more maps: the counit $\eta : A \to \mathbb{C}$ and the antipode $S : A \to A^{op}$.

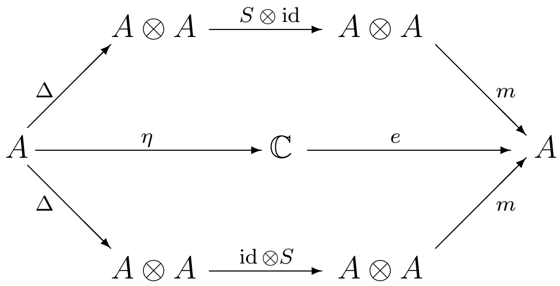
An axiom of a counit should be dual to that of the unit, $e : \mathbb{C} \to A, z \mapsto z \cdot 1$. The element $e(1)$ is a unit if and only if the following diagram is commutative.



Dualizing this diagram we get the counit axiom: the following diagram is commutative.



Finally, the antipode axiom is the commutativity of the following diagram.



Let us illustrate this axiom in the example of $A = \mathbb{C}G$, where $S(g) = g^{-1}$. There $\Delta(g) = g \otimes g, S \otimes \mathrm{id}(g \otimes g) = g^{-1} \otimes g$, $m(g^{-1} \otimes g) = 1 = e \circ \eta(g)$.

Definition 1.3. By a Hopf algebra we mean a $\mathbb{C}$-vector space A with five maps (m, e, Δ, η, S), where $m : A \otimes A \to A, e : \mathbb{C} \to A, \Delta : A \to A \otimes A, \eta : A \to \mathbb{C}, S : A \to A$ such that:

(1) (A, m, e) is an associative unital algebra.
(2) $\Delta : A \to A \otimes A, S : A \to A^{op}, \eta : A \to \mathbb{C}$ are algebra homomorphisms.
(3) Δ is coassociative, and η satisfies the counit axiom.
(4) S satisfies the antipode axiom.

Remark 1.4. In fact, once m, e, Δ are specified, S and η are recovered in at most one way.

It is straightforward to check that $\mathbb{C}G$ and $U(\mathfrak{g})$ are Hopf algebras.

1.4. **Duality of Hopf algebras.** Now let $(A, m, e, \Delta, \eta, S)$ be a finite dimensional Hopf algebra. One can show that $(A^*, \Delta^*, \eta^*, m^*, e^*, S^*)$ is a Hopf algebra as well.

Example 1.5. Let us describe $(\mathbb{C}G)^*$. As a vector space, $(\mathbb{C}G)^*$ is the algebra of functions on G, to be denoted by $\mathbb{C}[G]$. The map $\Delta : \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G$ sends g to $g \otimes g$. So $\Delta^*(\alpha \otimes \beta)(g) = \alpha \otimes \beta(g \otimes g) = \alpha(g)\beta(g)$ is the usual multiplication of functions. Similarly, η^* sends 1 to the identity function. The map $m^* : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G] = \mathbb{C}[G \times G]$ sends $\alpha \in \mathbb{C}[G]$ to $m^*(\alpha)(g, h) := \alpha(gh)$. The map $e^* : \mathbb{C}[G] \to \mathbb{C}$ maps α to $\alpha(1)$. Finally, we have $(S^*\alpha)(g) = \alpha(g^{-1})$.

1.5. **Cocommutativity.** In the cases of $A = U(\mathfrak{g}), \mathbb{C}G$ the isomorphism $V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$ is that of A-modules. The reason for this is that the *opposite coproduct* $\Delta^{op} := \sigma \circ \Delta$, where $\sigma : A^{\otimes 2} \to A^{\otimes 2}, a \otimes b \mapsto b \otimes a$, coincides with Δ. The Hopf algebras with $\Delta = \Delta^{op}$ are called *cocommutative*. However, there are Hopf algebras that are not cocommutative, e.g. $\mathbb{C}[G]$.

The Hopf algebras we have encountered so far are commutative as algebras ($\mathbb{C}[G]$) or cocommutative ($\mathbb{C}G, U(\mathfrak{g})$). Of course, one can cook a Hopf algebra that is neither commutative nor cocommutative: the tensor product of two Hopf algebras carries a natural Hopf algebra structure and we can take the tensor product of a non-commutative Hopf algebra with a non-cocommutative one. But this is very boring. In the next section, we will study a far more interesting example.

2. $U_q(\mathfrak{sl}_2)$

2.1. $U_q(\mathfrak{sl}_2)$ **as a Hopf algebra.** We will define the "quantum $\mathfrak{sl}_2$" by generators and relations (as an algebra) and then define Δ, η, S on the generators.

Let $q \in \mathbb{C} \setminus \{0, \pm 1\}$ (we can also take q to be an independent variable in the field of rational functions $\mathbb{C}(q)$). We define the algebra $U_q(\mathfrak{sl}_2)$ generated by E, F, K, K^{-1} subject to the following relations:

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Note that the algebra $U := U_q(\mathfrak{sl}_2)$ is spanned by the monomials $F^k K^\ell E^m$, where $k, m \in \mathbb{Z}_{\geqslant 0}$, and $\ell \in \mathbb{Z}$. In fact, these monomials are linearly independent (the PBW theorem).

Now let us define the Hopf algebra structure. We set

$$\begin{aligned} &\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K, \\ &\eta(E) = \eta(F) = 0, \quad \eta(K) = 1, \\ &S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1}. \end{aligned} \tag{2.1}$$

Proposition 2.1. Δ, η, S *extend to required algebra homomorphisms. Moreover,* U *becomes a Hopf algebra.*

Proof. This is a mighty tedious check... What we need to verify is that Δ, S, η respect the relations in U and that the axioms (3),(4) in the definition of a Hopf algebra hold on the generators E, K, F. Let us check that $\Delta([E,F]) = [\Delta(E), \Delta(F)]$, which is the hardest relation to check. We have

$$\Delta([E,F]) = \Delta\left(\frac{K - K^{-1}}{q - q^{-1}}\right) = \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}}$$

On the other hand,

$$\begin{aligned} &[\Delta(E), \Delta(F)] = [E \otimes 1 + K \otimes E, F \otimes K^{-1} + 1 \otimes F] = [E,F] \otimes K^{-1} + K \otimes [E,F] + \\ &+ [K \otimes E, F \otimes K^{-1}] = \frac{(K - K^{-1}) \otimes K^{-1}}{q - q^{-1}} + \frac{K \otimes (K - K^{-1})}{q - q^{-1}} + KF \otimes EK^{-1} - \\ &- FK \otimes K^{-1}E = \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}} + KF \otimes EK^{-1} - (q^2 KF) \otimes (q^{-2}EK^{-1}) = \\ &= \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}}. \end{aligned}$$

□

We note that $\Delta \neq \Delta^{op}$. In particular, the map $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ does not give an isomorphism $V_1 \otimes V_2 \to V_2 \otimes V_1$, in general. However, in the next lecture we will find an element $R \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ (this is a slight lie, we need a certain completion) with $R^{-1}\Delta(u)R = \Delta^{op}(u)$. This element, called the universal R-matrix, is extremely important. In particular, it will allow us to construct link invariants, such as the Jones polynomial.

2.2. $U_q(\mathfrak{sl}_2)$ **vs** $U(\mathfrak{sl}_2)$**.** The algebra $U_q(\mathfrak{sl}_2)$ should be thought as a deformation of $U(\mathfrak{sl}_2)$ (the latter corresponds to $q = 1$). This however requires some care, we cannot put $q = 1$ in the definition of $U_q(\mathfrak{sl}_2)$. In order to make the claim about the deformation more precise, we will need to consider the formal version of $U_q(\mathfrak{sl}_2)$, we will call it $U_\hbar(\mathfrak{sl}_2)$. This will be an algebra over $\mathbb{C}[[\hbar]]$.

By definition, as an algebra, $U_\hbar(\mathfrak{sl}_2)$ is the quotient of $T(\mathfrak{sl}_2)[[\hbar]]$ by the relations

$$[h,e]=2e, [h,f]=-2f, [e,f]=\frac{\exp(\hbar h)-\exp(-\hbar h)}{\exp(\hbar)-\exp(-\hbar)}.$$

Note that $\frac{\exp(\hbar h)-\exp(-\hbar h)}{\exp(\hbar)-\exp(-\hbar)}$ is a formal power series in $\hbar$, modulo $\hbar$ it equals h. It follows that $U_\hbar(\mathfrak{sl}_2)/(\hbar)=U(\mathfrak{sl}_2)$.

One can show that $\hbar$ is not a zero divisor in $U_\hbar(\mathfrak{sl}_2)$. Note that $E=e, F=f, K=\exp(\hbar h), q=\exp(\hbar)$ satisfy the relations of $U_q(\mathfrak{sl}_2)$. Indeed, for example, we get

$$\exp(\hbar h)e\exp(-\hbar h)=\exp(\hbar\,\mathrm{ad}(h))e=\exp(2\hbar)e.$$

One can introduce the Hopf algebra structure on $U_\hbar(\mathfrak{sl}_2)$ but one needs to extend the definition to allow Δ to be a homomorphism $U_\hbar(\mathfrak{sl}_2)\to U_\hbar(\mathfrak{sl}_2)\widehat{\otimes}_{\mathbb{C}[[\hbar]]}U_\hbar(\mathfrak{sl}_2)$. Here $\widehat{\otimes}$ denotes the *completed tensor product*. While the usual tensor product consists of all finite sums of decomposable tensors, the completed product consists of all converging (in the $\hbar$-adic topology) infinite sums.

2.3. **Algebras $U_q(\mathfrak{g})$.** We can define quantum groups $U_q(\mathfrak{g})$ for any semisimple Lie algebra $\mathfrak{g}$ (or, more generally, any Kac-Moody algebra $\mathfrak{g}(A)$ for a symmetrizable Cartan matrix A). Let us start with $\mathfrak{g}=\mathfrak{sl}_{n+1}$.

Recall that the usual universal enveloping algebra $U(\mathfrak{sl}_{n+1})$ is defined by the generators $e_i,h_i,f_i, i=1,\ldots,n$, and the following relations:

(i) $[h_i,e_i]=2e_i, [h_i,f_i]=-2f_i, [e_i,f_i]=h_i$.
(ii) $[h_i,h_j]=0$.
(iii) $[h_i,e_j]=a_{ij}e_j, [h_i,f_j]=-a_{ij}f_j$.
(iv) $e_if_j=f_je_i, i\neq j$.
(v) $e_ie_j=e_je_i$, if $a_{ij}=0$, and $e_i^2e_j-2e_ie_je_i+e_je_i^2=0$, if $a_{ij}=-1$.
(vi) $f_if_j=f_jf_i$, if $a_{ij}=0$, and $f_i^2f_j-2f_if_jf_i+f_jf_i^2=0$, if $a_{ij}=-1$.

Recall that here $a_{ij}=-1$ if $|i-j|=1$ and $a_{ij}=0$ if $|i-j|>1$.

The quantum group $U_q(\mathfrak{sl}_{n+1})$ is defined by the generators $E_i,K_i^{\pm1},F_i, i=1,\ldots,n$, with relations

(i_q) $K_iE_iK_i^{-1}=q^2E_i, K_iF_iK_i^{-1}=q^{-2}F_i, [E_i,F_i]=\frac{K_i-K_i^{-1}}{q-q^{-1}}$.
(ii_q) $[K_i,K_j]=0$.
(iii_q) $K_iE_jK_i^{-1}=q^{a_{ij}}E_j, K_iF_jK_i^{-1}=q^{-a_{ij}}F_j$.
(iv_q) $E_iF_j=F_jE_i, i\neq j$.
(v_q) $E_iE_j=E_jE_i$ if $a_{ij}=0$ and $E_i^2E_j-[2]_qE_iE_jE_i+E_jE_i^2=0$ if $a_{ij}=-1$.
(vi_q) $F_iF_j=F_jF_i$ if $a_{ij}=0$ and $F_i^2F_j-[2]_qF_iF_jF_i+F_jF_i^2=0$ if $a_{ij}=-1$.

Here $[2]_q$ denotes the "quantum 2", i.e., $q+q^{-1}$.

The similar definition will work for any simply laced Cartan matrix A (meaning that $a_{ij}\in\{0,-1\}$ if $i\neq j$). When A is not simply laced (e.g., of type B_n,C_n,F_4,G_2), the definition is more technical, one needs to use different q's for the "$\mathfrak{sl}_2$-subalgebras" of $U_q(\mathfrak{g})$ according the length of the corresponding root. Namely, when $\mathfrak{g}$ is finite dimensional, we define $d_i\in\{1,2,3\}$ as $(\alpha_i,\alpha_i)/2$, where $(\cdot,\cdot)$ is a W-invariant form on $\mathfrak{h}^*$ normalized in such a way that $(\alpha,\alpha)=2$ for the short roots (we have two different root lengthes). This can be generalized to an arbitrary symmetrizable Kac-Moody algebra but we are not going to explain that.

Now set $q_i := q^{d_i}$ (so that $q_1 = q$). We also define the quantum integer $[n]_{q_i} = q_i^{n-1} + q_i^{n-2} + \ldots + q_i^{1-n}$, and the quantum factorial $[n]_{q_i}! = [1]_{q_i} \ldots [n]_{q_i}$. We set

$$\binom{n}{k}_{q_i} = \frac{[n]_{q_i}!}{[k]_{q_i}![n-k]_{q_i}!}.$$

Now we define $U_q(\mathfrak{g})$ as the algebra generated by E_i, K_i, F_i subject to the relations

(i$_q$) $K_i E_i K_i^{-1} = q_i^2 E_i, K_i F_i K_i^{-1} = q_i^{-2} F_i, [E_i, F_i] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$.
(ii$_q$) $[K_i, K_j] = 0$.
(iii$_q$) $K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$.
(iv$_q$) $E_i F_j = F_j E_i, i \neq j$.
(v$_q$) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0$.
(vi$_q$) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0$.

Note that they are obtained from the relations for $U(\mathfrak{g})$ in the same fashion as the relations for $U_q(\mathfrak{sl}_{n+1})$ are obtained from those for $U(\mathfrak{sl}_{n+1})$.

The Hopf algebra structure on $U_q(\mathfrak{g})$ is introduced as follows: we just define Δ, S, η on E_i, F_i, K_i as in $U_{q_i}(\mathfrak{sl}_2)$.