

## Math 1330 – Conic Sections

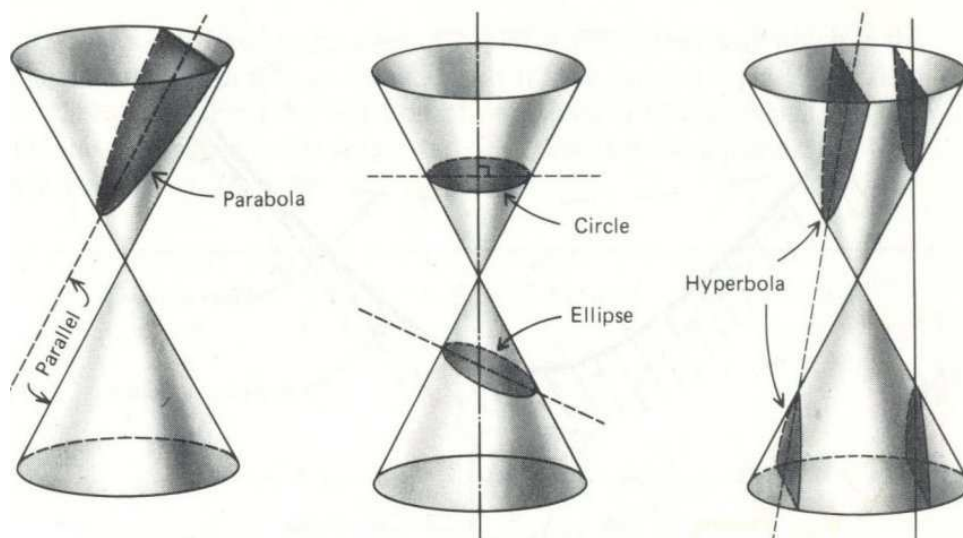
In this chapter, we will study conic sections (or conics). It is helpful to know exactly what a conic section is. This topic is covered in Chapter 8 of the online text.

We start by looking at a double cone. Think of this as two “pointy” ice cream cones that are connected at the small tips:

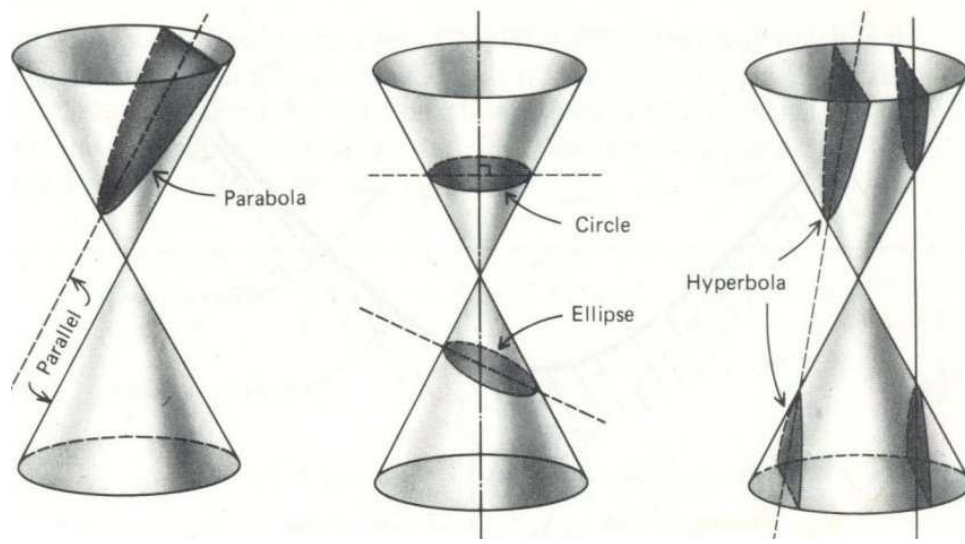


To form a conic section, we'll take this double cone and slice it with a plane. When we do this, we'll get one of several different results.

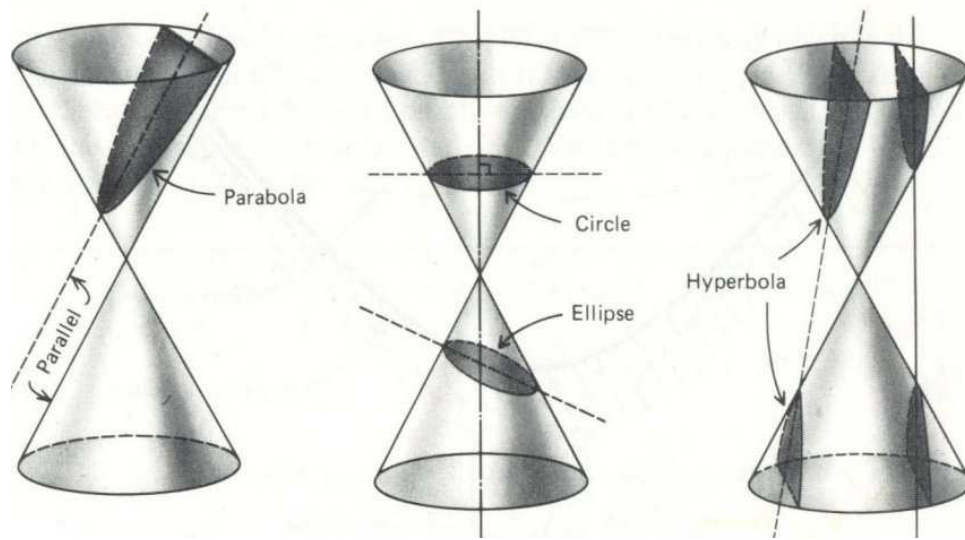
### 1. Parabola



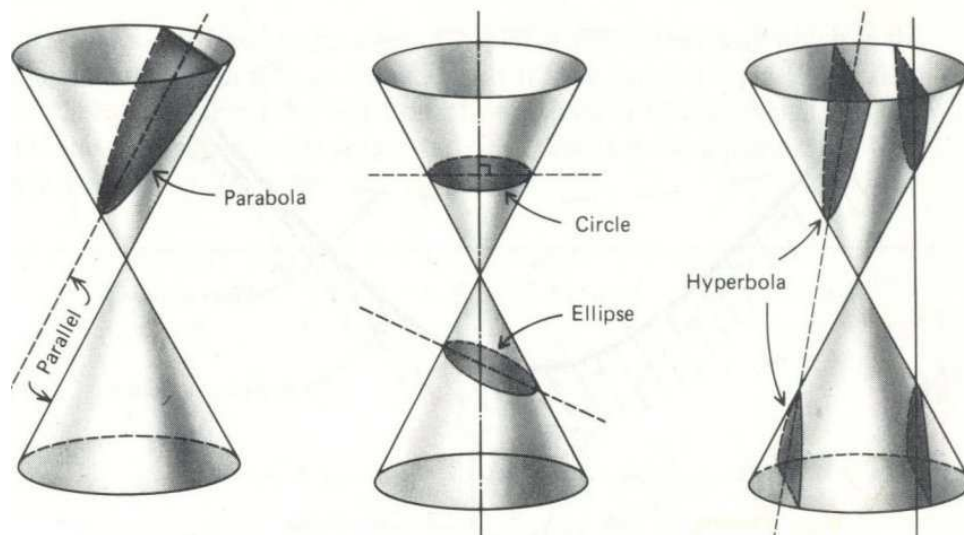
## 2. Ellipse



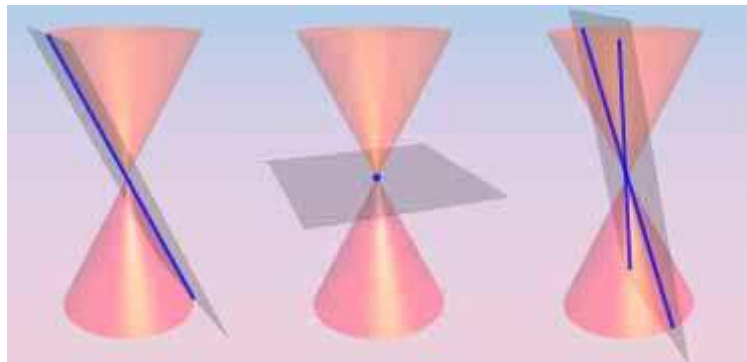
## 3. Circle



#### 4. Hyperbola



#### 5. Degenerate conic sections



As we study conic sections, we will be looking at special cases of the general second-degree equation:  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ .

## Math 1330 – Section 8.1

### Parabolas

Next, we'll look at parabolas. We previously studied parabolas as the graphs of quadratic functions. Now we will look at them as conic sections. There are a few differences. For example, when we studied quadratic functions, we saw that the graphs of the functions could open up or down. As we look at conic sections, we'll see that the graphs of these second degree equations can also open left or right. So, not every parabola we'll look at in this section will be a function.

We already know that the graph of a quadratic function  $f(x) = ax^2 + bx + c$  is a parabola. But there is more to be learned about parabolas.

**Definition:** A *parabola* is the set of all points equally distant from a fixed line and a fixed point not on the line. The fixed line is called the *directrix*. The fixed point is called the *focus*.

The *axis*, or *axis of symmetry*, runs through the focus and is perpendicular to the directrix.

The *vertex* is the point **halfway between** the focus and the directrix.

We won't be working with slanted parabolas, just with "horizontal" and "vertical" parabolas.

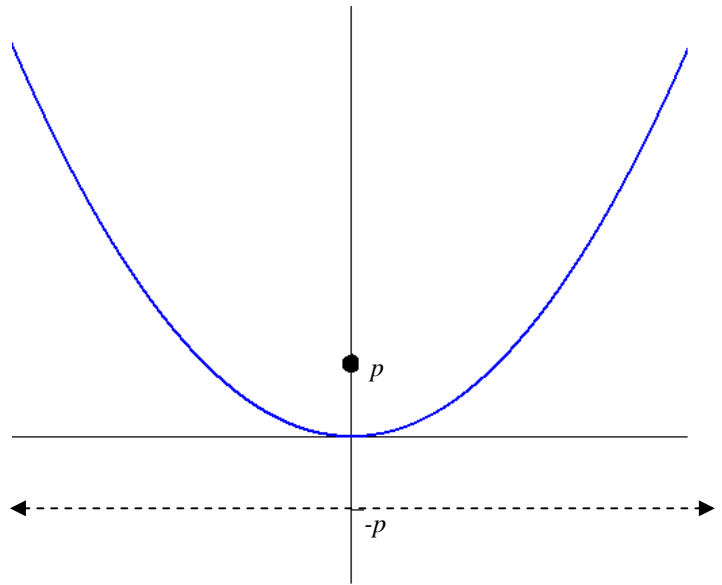
### Basic “Vertical” Parabola:

Equation:  $x^2 = 4py$

Focus:  $(0, p)$

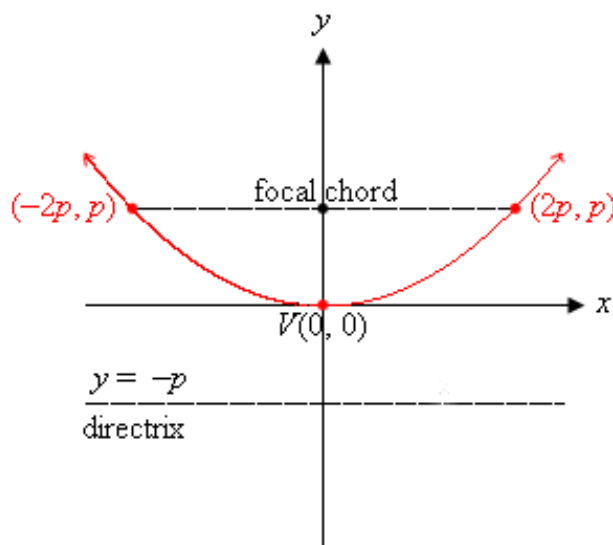
Directrix:  $y = -p$

Focal Width:  $|4p|$



*Note: This can be written as  $y = f(x) = \frac{x^2}{4p}$ . It is a function (passes vertical line test).*

The line segment that passes through the focus and perpendicular to the axis with endpoints on the parabola is called the focal chord. Its length (called the focal width) is  $4p$ .



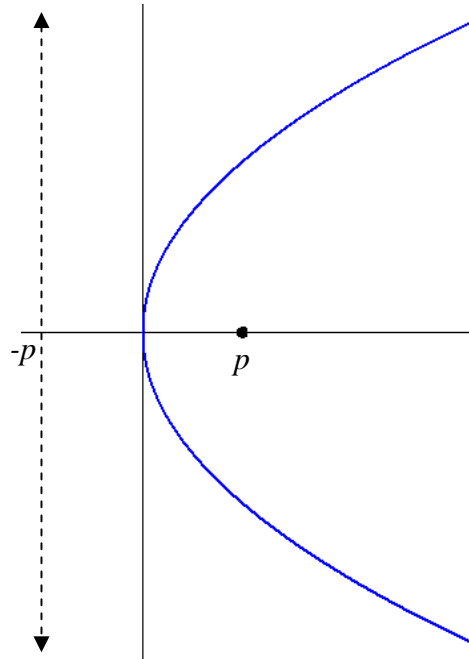
**Basic “Horizontal” Parabola:**

Equation:  $y^2 = 4px$

Focus:  $(p, 0)$

Directrix:  $x = -p$

Focal Width:  $|4p|$



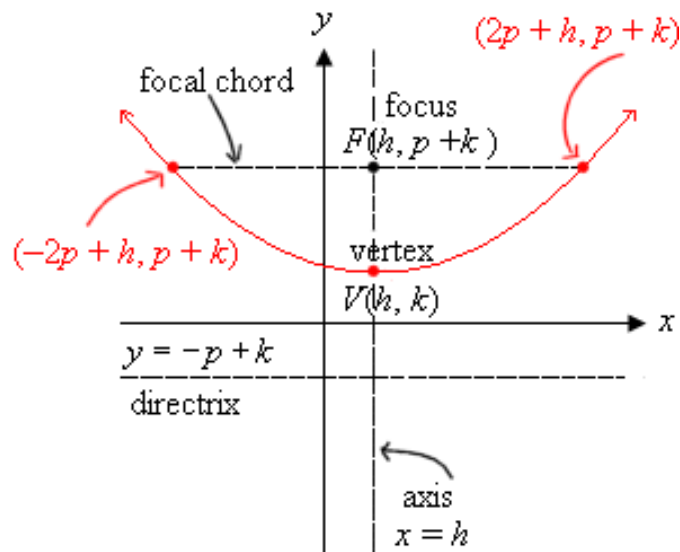
*Note: This is not a function (fails vertical line test). However, the top half  $y = \sqrt{x}$  is a function and the bottom half  $y = -\sqrt{x}$  is also a function.*

### Graphing parabolas with vertex at the origin:

- When you have an equation, look for  $x^2$  or  $y^2$
- If it has  $x^2$ , it's a "vertical" parabola. If it has  $y^2$ , it's a "horizontal" parabola.
- Rearrange to look like  $y^2 = 4px$  or  $x^2 = 4py$ . In other words, isolate the squared variable.
- Determine  $p$ .
- Determine the direction it opens.
  - If  $p$  is positive, it opens right or up.
  - If  $p$  is negative, it opens left or down.
- Starting at the origin, place the focus  $p$  units to the inside of the parabola. Place the directrix  $p$  units to the outside of the parabola.
- Use the focal width  $4p$  ( $2p$  on each side) to make the parabola the correct width at the focus.

### Graphing parabolas with vertex not at the origin:

- Rearrange (complete the square) to look like  $(y-k)^2 = 4p(x-h)$  or  $(x-h)^2 = 4p(y-k)$ .
- Vertex is  $(h,k)$ . Draw it the same way, except start at this vertex.



Graph of the parabola  $(x-h)^2 = 4p(y-k)$ .



**Example 1:** Write  $y^2 - 20x = 0$  in standard form and graph it.

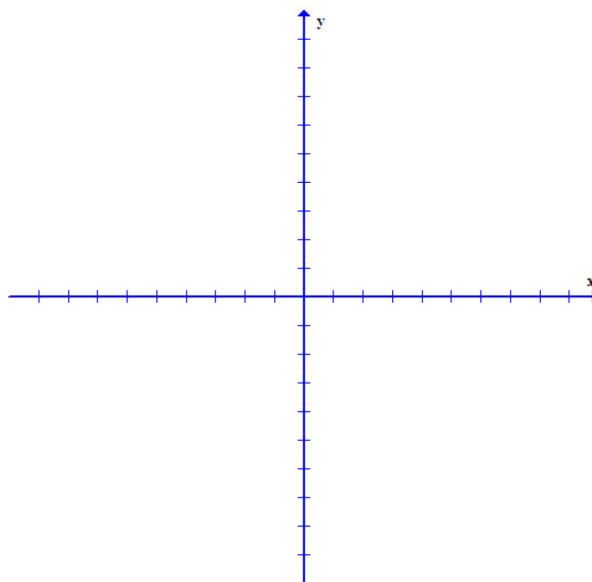
Vertex:

Focus:

Directrix:

Focal width:

Endpoints of focal chord:



**Example 2:** Write  $6x^2 + 24y = 0$  in standard form and graph it.

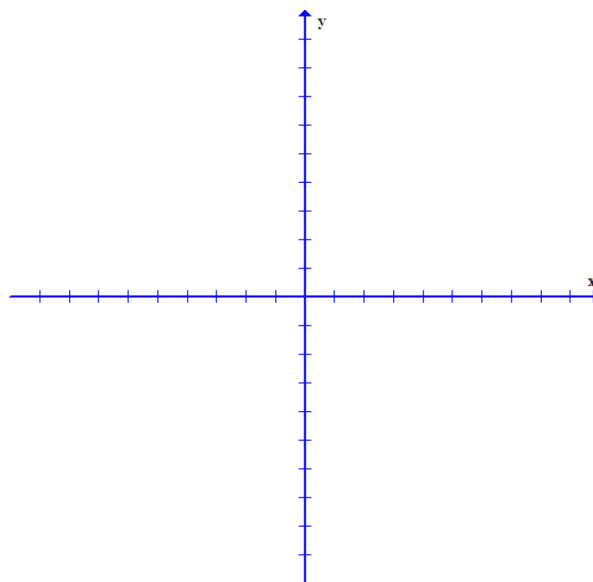
Vertex:

Focus:

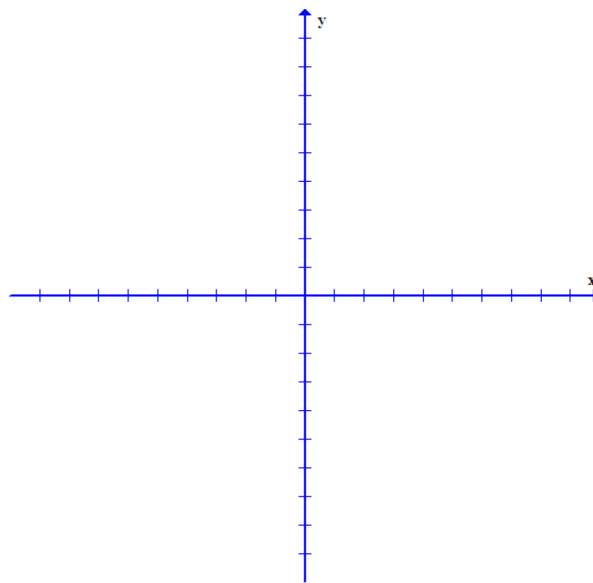
Directrix:

Focal width:

Endpoints of focal chord:



**Example 3:** Write  $y^2 - 6y = 8x + 7$  in standard form and graph it.



Vertex:

Focus:

Directrix:

Focal width:

Endpoints of focal chord:

**Example 4:** Suppose you know that the vertex of a parabola is at  $(-3, 5)$  and its focus is at  $(1, 5)$ . Write an equation for the parabola in standard form.

**Example 5:** Suppose you know that the focus of a parabola is  $(-1, 3)$  and the directrix is the line  $y = -1$ . Write an equation for the parabola in standard form.

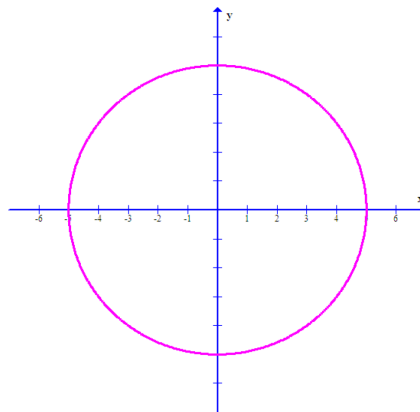
## Section 8.2a: The Circle

**Definition:** A circle is the set of all points that are equidistant from a fixed point. The fixed point is called the *center* and the distance from the center to any point on the circle is called the *radius*.

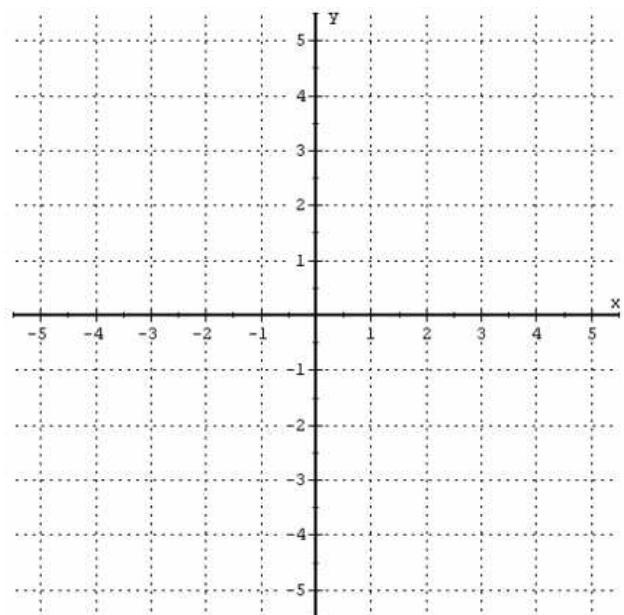
An equation of a circle whose center is at the origin will be  $x^2 + y^2 = r^2$ , where  $r$  is the radius of the circle.

So  $x^2 + y^2 = 25$  is an equation of a circle with center  $(0, 0)$  and radius 5.

Here's the graph of this circle:



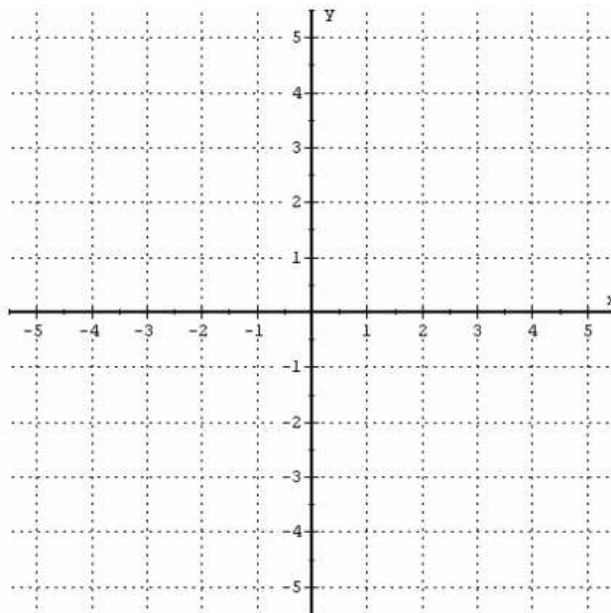
**Example 1:** State the center and the radius of the circle and then graph it:  $x^2 + y^2 - 16 = 0$ .



The standard form of the equation of a circle is  $(x - h)^2 + (y - k)^2 = r^2$ , where the center of the circle is the point  $(h, k)$  and the radius is  $r$ . Notice that if the center of the circle is  $(0, 0)$  you'll get the equation we saw earlier.

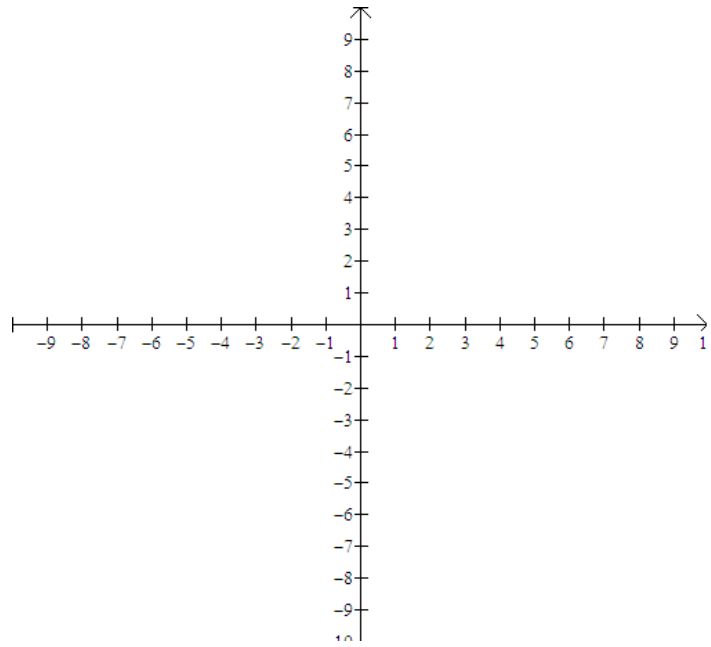
**Example 2:** State the center and the radius of the circle and then graph it:

$$(x - 2)^2 + (y + 3)^2 = 4$$



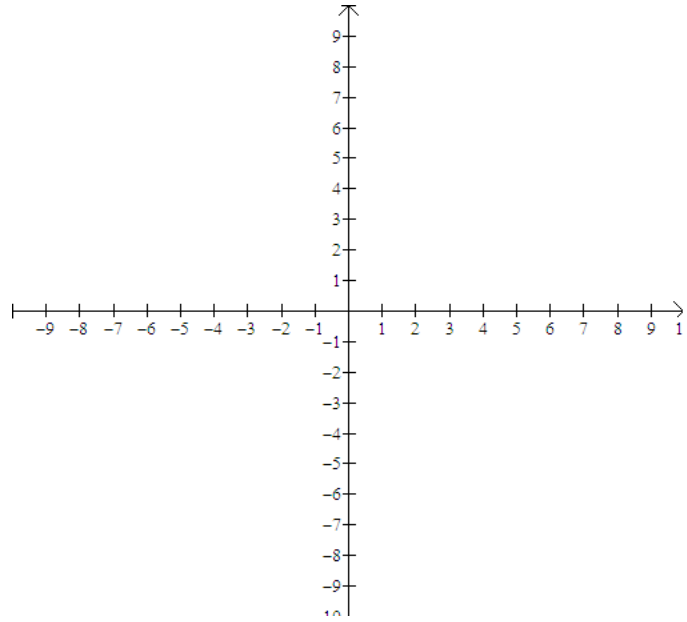
Sometimes the equation will be given in the general form, and your first step will be to rewrite the equation in the standard form. You'll need to **complete the square** to do this.

**Example 3:** Write the equation in standard form, find the center and the radius and then graph the circle:  $x^2 + y^2 + 6x - 10y + 44 = 26$



**Example 4:** Write the equation in standard form, find the center and the radius and then graph the circle:

$$5x^2 + 5y^2 - 20x + 10y = 20$$





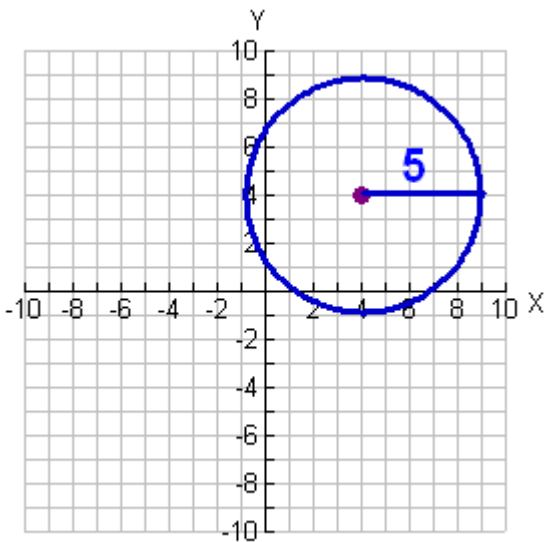
We can also write the equation of a circle, given appropriate information.

**Example 5:** Write the equation of a circle with center  $(2, 5)$  and radius  $2\sqrt{5}$ .

**Example 6:** Write an equation of a circle with center  $(-1, 3)$  which passes through the point  $(4, -7)$ .

**Example 7:** Write an equation of a circle if the endpoints of the diameter of the circle are  $(6, -3)$  and  $(-4, 7)$ .

**Example 8:** What is the equation of the given circle?



## Math 1330 – Section 8.2

### Ellipses

**Definition:** An *ellipse* is the set of all points, the sum of whose distances from two fixed points is constant. Each fixed point is called a *focus* (plural = *foci*).

#### Basic ellipses (centered at origin):

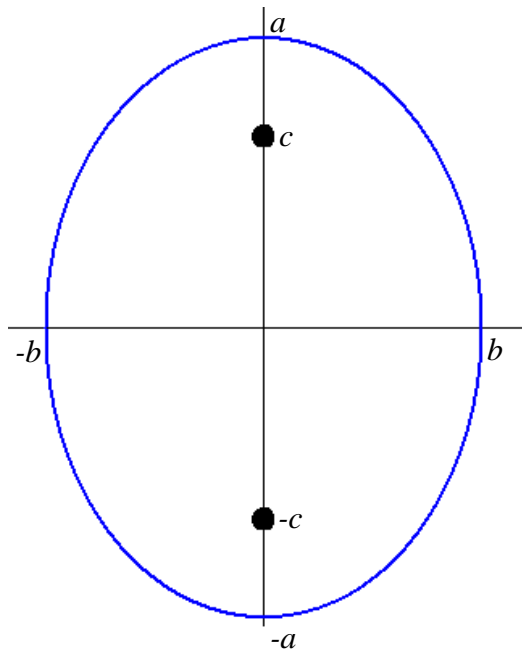
Basic “vertical” ellipse:

Equation:  $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$

Foci:  $(0, \pm c)$ , where  $c^2 = a^2 - b^2$

Vertices:  $(0, \pm a)$

Eccentricity:  $e = \frac{c}{a}$



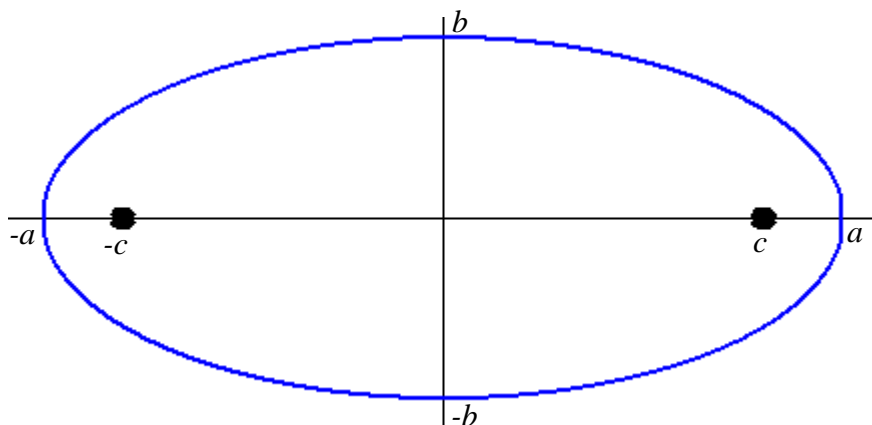
Basic “horizontal” ellipse:

Equation:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$

Foci:  $(\pm c, 0)$ , where  $c^2 = a^2 - b^2$

Vertices:  $(\pm a, 0)$

Eccentricity:  $e = \frac{c}{a}$

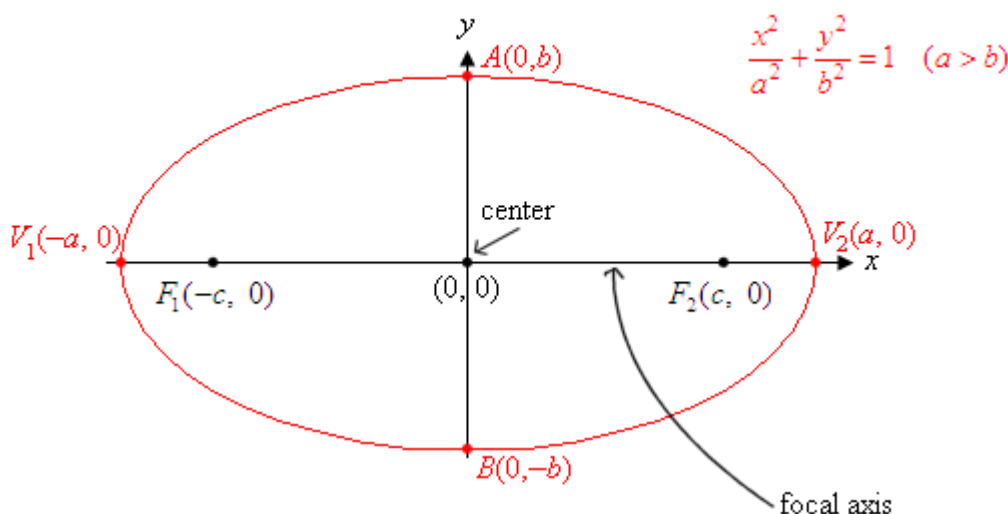


The eccentricity provides a measure on how much the ellipse deviates from being a circle. The *eccentricity*  $e$  is a number between 0 and 1.

- small  $e$ : graph resembles a circle (foci close together)
- large  $e$ : flatter, more elongated (foci far apart)
- if the foci are the same, it's a circle!

**Graphing ellipses:** To graph an ellipse with **center at the origin**:

- Rearrange into the form  $\frac{x^2}{\text{number}} + \frac{y^2}{\text{number}} = 1$ .
- Decide if it's a "horizontal" or "vertical" ellipse.
  - if the bigger number is under  $x^2$ , it's horizontal (longer in  $x$ -direction).
  - if the bigger number is under  $y^2$ , it's vertical (longer in  $y$ -direction).
- Use the square root of the number under  $x^2$  to determine how far to measure in  $x$ -direction.
- Use the square root of the number under  $y^2$  to determine how far to measure in  $y$ -direction.
- Draw the ellipse with these measurements. Be sure it is smooth with no sharp corners
- $c^2 = a^2 - b^2$  where  $a^2$  and  $b^2$  are the denominators. So  $c = \sqrt{\text{big denom} - \text{small denom}}$
- The foci are located  $c$  units from the center on the long axis.



Center:  $(0, 0)$

Foci:  $F_1(-c, 0)$  and  $F_2(c, 0)$ , where  $c^2 = a^2 - b^2$

Vertices:  $V_1(-a, 0)$  and  $V_2(a, 0)$

Major Axis:  $\overline{V_1V_2}$       Length of Major Axis:  $2a$

Minor Axis:  $\overline{AB}$       Length of Minor Axis:  $2b$

To graph an ellipse with **center not at the origin**:

- Rearrange (complete the square if necessary) to look like  $\frac{(x-h)^2}{\text{number}} + \frac{(y-k)^2}{\text{number}} = 1$ .
- Start at the center  $(h, k)$  and then graph it as before.

The following list reflects the changes in translating the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  to the

ellipse  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$  ( $a > b$ ):

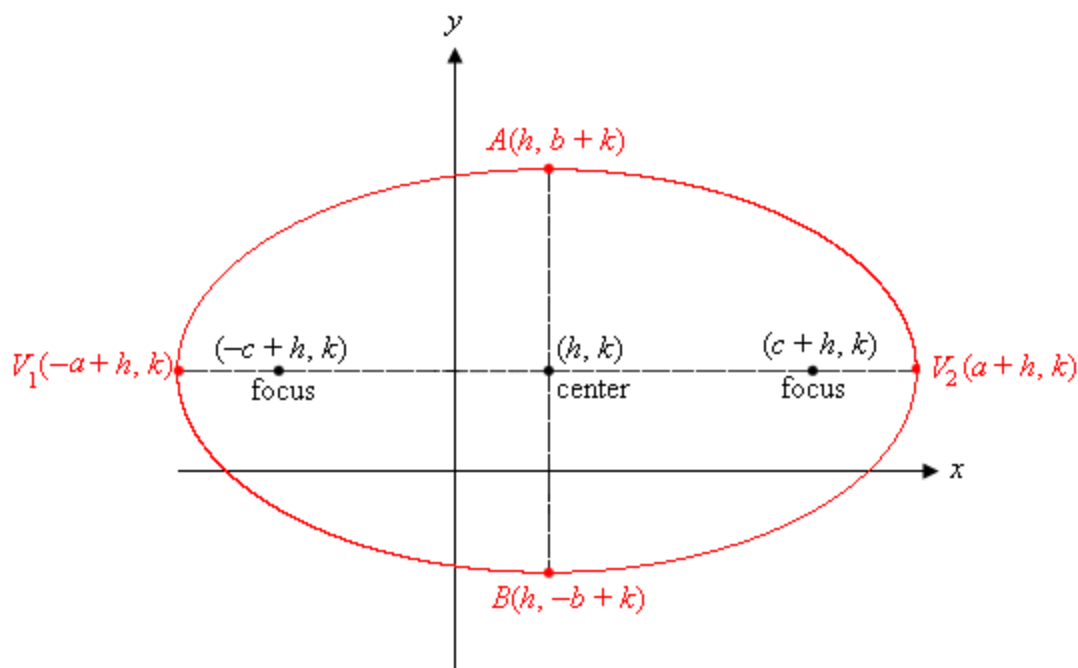
Center: The point  $(0, 0)$  changes to the point  $(h, k)$ .

Foci: The foci change from the points  $(-c, 0)$  and  $(c, 0)$  to the points  $(-c + h, k)$  and  $(c + h, k)$ , where  $c^2 = a^2 - b^2$ .

Vertices: The vertices change from the points  $(-a, 0)$  and  $(a, 0)$  to the points  $(-a + h, k)$  and  $(a + h, k)$ .

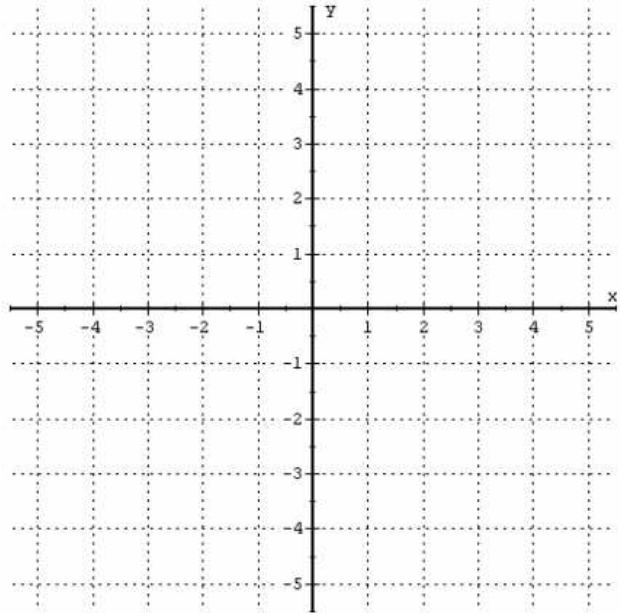
Major Axis:  $\overline{V_1V_2}$  Length of Major Axis:  $2a$

Minor Axis:  $\overline{AB}$  Length of Minor Axis:  $2b$



When graphing, you will need to find the orientation, center, values for  $a$ ,  $b$  and  $c$ , vertices, foci, lengths of the major and minor axes and eccentricity.

**Example 1:** Find all relevant information and graph  $\frac{x^2}{16} + \frac{y^2}{9} = 1$ .



Orientation:

Center:

Vertices:

Foci:

Length of major axis:

Length of minor axis:

Coordinates of the major axis:

Coordinates of the minor axis:

Eccentricity:

**Example 2:** Find all relevant information and graph  $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{25} = 1$ .

Orientation:

Center:

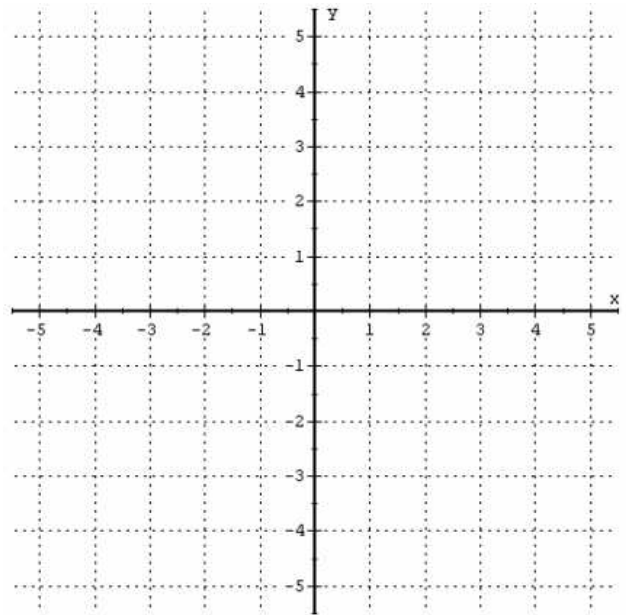
Vertices:

Foci:

Length of major axis:

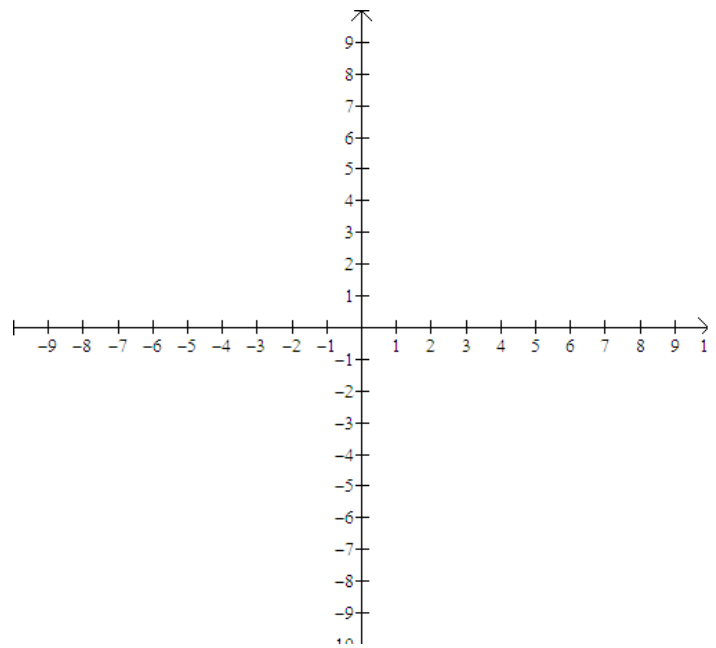
Length of minor axis:

Eccentricity:



**Example 3:** Write the equation in standard form. Find all relevant information and graph:

$$4x^2 - 8x + 9y^2 - 54y = -49.$$





**Example 4:** Find the equation for the ellipse satisfying the given conditions.

Foci  $(\pm 3, 0)$ , vertices  $(\pm 5, 0)$

**Example 5:** Write an equation of the ellipse with vertices  $(5, 9)$  and  $(5, 1)$  if one of the foci is  $(5, 7)$ .

**(Extra) Example:** Write an equation of the ellipse with center  $(2,1)$  if  
The length of major axis is: 10  
The length of minor axis is: 2  
Vertical major axis.

Sometimes, you'll need to be able to manipulate an equation of a circle:

**(Extra) Example 9:** Suppose  $(x - 2)^2 + (y + 1)^2 = 9$ . Solve the equation for  $x$ . Then solve the equation for  $y$ .