Diversity and no arbitrage

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A stock market is called diverse if no stock can dominate the market in terms of relative capitalization. On one hand, this natural property leads to arbitrage in diffusion models under mild assumptions. On the other hand, it is also easy to construct diffusion models which are both diverse and free of arbitrage. Can one tell whether an observed diverse market admits arbitrage?

In the present paper we argue that this may well be impossible by proving that the known examples of diverse markets in the literature (which do admit arbitrage) can be approximated uniformly (on the logarithmic scale) by models which are both diverse and arbitrage-free.

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1. Introduction

Stochastic portfolio theory is a relatively new branch of mathematical finance. It was introduced and studied by Fernholz [2, 3], and then further developed by Fernholz, Karatzas and Kardaras [4]. It provides a framework for analysing portfolio performance under an angle which is different from the usual one.

One of the most important notions here is *diversity* of a market. In short, diversity means that no single stock is ever allowed to dominate the market. Diversity was proposed based on empirical grounds and is conform with intuition.

Absence of arbitrage (riskless profit) is the cornerstone of modern mathematical finance. At the technical level, there are various formulations of arbitrage but basic economic considerations forbid that such opportunities persist in a liquid market.

If the log-prices follow an Itô process with uniformly non-degenerate voltility matrix, diversity of a market implies the existence of arbitrage opportunities relative to the market portfolio (see section 7 of Fernholz and Karatzas [5]), and thus the non-existence

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of equivalent martingale measure also follows (Proposition 6.2 of Fernholz and Karatzas [5]).

This situation may seem dramatic at first sight: the common sense notion of diversity contradicting the most fundamental principle of asset pricing. There must be a way out: indeed, relaxing the hypothesis of uniformly nondegenerate volatility one may easily construct models where both diversity and absence of arbitrage hold true. However, a much harder question immediately arises: can we tell whether the price processes seen in today's market (which clearly satisfy the diversity assumption) are arbitrage-free or not?

In this paper we derive the somewhat unsettling conclusion that possibly there is no way to answer this question based on statistical analysis. Our conclusions parallel those of [7].

We are looking at the diverse market models of Fernholz, Karatzas and Kardaras [4] and Osterrieder and Rheinländer [9]. We prove that under an arbitrarily small model misspecification diversity is retained but relative arbitrage is not. More precisely, we show that to these diverse market models (admitting relative arbitrage) there are models arbitrarily close on the logarithmic scale that no longer admit arbitrage (though they are still diverse).

The rest of the paper is organized as follows. Section 2 introduces the notion of diversity presenting examples with and without relative arbitrage opportunities. Section 3 contains the result on consistent price system that is needed in subsection 2.3. Section 4 provides a new result on conditional full support in higher dimensions, an extension of the work of Pakkanen [10].

2. Diversity

Let T > 0 be a fixed time horizon. We consider a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbf{P})$, where the filtration is assumed to satisfy the usual conditions with \mathcal{F}_0 being trivial and all events belong to \mathcal{F}_T .

Fernholz, Karatzas and Kardaras [4], see also Fernholz and Karatzas [5], call a market diverse "if no single stock is ever allowed to dominate the entire market in terms of relative capitalization". To give a formulation of this requirement let the positive processes S_i , i = 1, ..., n, denote the capitalization of the i^{th} company. The market weights of the companies are defined by

$$\mu_i(t) = \frac{S_i(t)}{\sum_{j=1}^n S_j(t)}$$

and we let $\mu_{(1)}(t) = \max_j \mu_j(t)$ the largest market weight.

A market is called *diverse* on the time-horizon [0,T] if there exists $\delta \in (0,1)$ such that

$$\mu_{(1)}(t) < 1 - \delta$$
, almost surely for all $t \in [0, T]$.

Similarly a market is called weakly diverse on the time-horizon [0,T] if for some $\delta \in (0,1)$

$$\frac{1}{T} \int_0^T \mu_{(1)}(t) dt < 1 - \delta$$
, almost surely.

A portfolio process $\pi'(t) = (\pi_1(t), \dots, \pi_n(t))$ describes the proportion of wealth invested in the stocks. It is required that π is progressively measurable and $\pi_i(t) \geq 0$ for $t \in [0,T]$, $i=1,\dots,n$ and $\sum_i \pi_i(t) = 1$ for all $t \in [0,T]$. An example of a portfolio process is the *market portfolio* defined by the market weights μ .

Fernholz and Karatzas [5] consider markets where the evolution of the prices are Itô processes, written on the logarithmic scale as

$$d\log S_i(t) = \gamma_i(t)dt + \sum_{\nu=1}^d \sigma_{i\nu}(t)dW_{\nu}(t), \quad i = 1, \dots, n,$$
(1)

where W is a d-dimensional Brownian motion in the filtration \mathcal{F} and the coefficients γ, σ are progressively measurable and satisfy the integrability condition $\int_0^T |\gamma(t)| + \|\sigma(t)\|^2 dt < \infty$.

The value $V^{z,\pi}$ of a portfolio π with initial value z is given by

$$\frac{dV^{z,\pi}(t)}{V^{z,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dS_i(t)}{S_i(t)}, \quad V^{z,\pi}(0) = z.$$

Given two portfolios π and ρ , we say that π represents an arbitrage opportunity relative to ρ over the time-horizon [0,T] if we have $V^{\pi}(0) = V^{\rho}(0) > 0$ and

$$\mathbf{P}(V^{\pi}(T) \ge V^{\rho}(T)) = 1$$
 and $\mathbf{P}(V^{\pi}(T) > V^{\rho}(T)) > 0$.

It is an interesting property of diverse market models that there exists arbitrage relative to the market portfolio μ provided that there exist $\varepsilon, M>0$ finite constants such that

$$\varepsilon |\xi|^2 \le |\sigma'(t)\xi|^2 \le M|\xi|^2$$
, a.s. for all $t \in [0,T]$ and $\xi \in \mathbb{R}^n$. (2)

Roughly speaking, $a(t) = \sigma(t)\sigma'(t)$ is bounded and non-degenerate uniformly in $(t, \omega) \in [0, T] \times \Omega$.

For the proof of this claim, we refer the reader to [4, 5]. Note that the existence of relative arbitrage opportunity excludes the possibility of the existence of an equivalent martingale measure, although equivalent local martingale measure may exist.

2.1. Examples of diverse market

We recall in this subsection two examples of diverse markets. The first one is due to Fernholz, Karatzas and Kardaras [4, Theorem 6.1]. In this type of example the drift is positive for all but the largest company. The drift of the largest price has a log-pole-type singularity that prevents its market weights to reach $1 - \delta$.

In the simplest such example, that Fernholz, Karatzas and Kardaras [4] present, the evolution of the price is written as in (1). The volatility matrix σ satisfies (2). The crucial assumption is about the drift γ . They fix a vector $g = (g_1, \ldots, g_n)$ of positive numbers. Then γ is expressed as

$$\gamma_i(t) = \mathbb{1}_{(\mu(t) \notin \mathcal{O}_i)} g_i + \mathbb{1}_{(\mu(t) \in \mathcal{O}_i)} \frac{M}{\delta(\log(\mu_{(1)}(t)) - \log(1 - \delta))},\tag{3}$$

where $\mathcal{O}_i = \{x \in \mathbb{R}^n : \max_{j < i} x_j < x_i, \max_{j > i} x_j \leq x_i\}$. Then γ_i is g_i except when the *i*-th company has the largest market weight. In the latter case γ_i is negative and decreases to $-\infty$ as $\mu_{(1)}(t)$ approchaes $1 - \delta$. This negative drift is strong enough to make the market diverse, that is, it keeps the process S(t) in the open set

$$0 = 0(\delta) = \left\{ x \in (0, \infty)^n : \max_j \frac{x_j}{x_1 + \dots + x_n} < 1 - \delta \right\}.$$
 (4)

Osterrieder and Rheinländer [9] concerns arbitrage possibilities of diverse markets, in the usual sense, i.e. in the sense of Delbaen and Schachermayer [1]. They give a general construction of diverse markets by conditioning the price process to stay in \mathcal{O} for the entire time-horizon [0,T]. They use a condition called \mathbf{ND} , staying for non-degeneracy, that ensures that arbitrage possibilities exist in the diverse market constructed.

To be more precise and concrete, one can start with a pre-model under some probability \mathbf{P}_0 . We may assume, as Osterrieder and Rheinländer [9] do, that under the probability \mathbf{P}_0 the price processes are positive continuous local martingales, that is,

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \mathrm{d}M_i(t), \quad 1 \le i \le n, \quad t \ge 0.$$

where M is a continuous local martingale under \mathbf{P}_0 .

Then **P** is obtained by conditioning

$$\mathbf{P}(A) = \mathbf{P}_0(A \mid \forall t \in [0, T], S(t) \in \mathcal{O}), \text{ for } A \in \mathcal{F}_T.$$

We apply this construction with a special form of M, namely with

$$dM_i(t) = \sum_{\nu=1}^d \sigma_{i\nu}(t) dW_{\nu}(t),$$

where the volatility matrix σ satisfies (2) under \mathbf{P}_0 . This condition implies that \mathbf{ND} holds on sufficiently small time-horizons so arbitrage in the usual sense exists under \mathbf{P} . Also, there is arbitrage relative to the market portfolio as (2) holds under \mathbf{P} .

2.2. Diverse market models without relative arbitrage

The market with two assets

$$S_1(t) = \exp\{W_1(t)\}\$$
and $S_2(t) = \exp\{W_1(t) + \arctan(W_2(t))\}\$

(driven by the 2-dimensional Brownian motion W) is clearly diverse and admits an equivalent martingale measure at the same time. Note, however, that the volatility of $(\log S_1, \log S_2)'$ is *not* uniformly non-degenerate, which was an important hypothesis for showing the existence of relative arbitrage.

Our goal now is to show that in many cases, especially in the examples recalled in the previous subsection, even though diverse markets present relative arbitrage opportunities, small model misspecifications or proportional transaction costs lead to diverse models that no longer admit arbitrage.

To state the main theorem of the paper we need the following variant of the notion of conditional full support.

Definition 1. Let $0 \subset \mathbb{R}^n$ be open set and $(S(t))_{t \in [0,T]}$ be a continuous adapted process taking values in 0. We say that S has conditional full support in 0 if for all $t \in [0,T]$ and open set $G \subset C([0,T],0)$

$$\mathbf{P}(S \in G \mid \mathcal{F}_t) > 0$$
, a.s. on the event $S|_{[0,t]} \in \{g|_{[0,t]} : g \in G\}$. (5)

We will also say that S has full support in \mathcal{O} , or simply full support when $\mathcal{O} = \mathbb{R}^n$, if (5) holds for t = 0 and for all open subset of $C([0, T], \mathcal{O})$.

Recall also, the notion of consistent price system.

Definition 2. Let $\varepsilon > 0$. An ε -consistent price system to S is a pair (\tilde{S}, \mathbf{Q}) , where \mathbf{Q} is a probability measure equivalent to \mathbf{P} and \tilde{S} is a \mathbf{Q} -martingale in the filtration \mathcal{F} , such that

$$\frac{1}{1+\varepsilon} \leq \frac{\tilde{S}_i(t)}{S_i(t)} \leq 1+\varepsilon, \quad \text{almost surely for all } t \in [0,T] \text{ and } i=1,\dots,n.$$

Note, that \tilde{S} is a martingale under \mathbf{Q} , hence we may assume that it is càdlàg, but it is not required in the definition that \tilde{S} is continuous.

Theorem 1. Let $0 \subset (0,\infty)^n$ be the open set defined by (4) and assume that the price process takes values and has conditional full support in 0.

Then for any $\varepsilon > 0$ there is an ε -consistent price system (\tilde{S}, \mathbf{Q}) such that \tilde{S} takes values in \mathbb{O} .

The proof of Theorem 1 is given in Section 3. In the rest of this subsection we show that the examples recalled in subsection 2.1 have conditional full support in \mathcal{O} . Then Theorem 1 applies and we can conclude that for any $\varepsilon > 0$ there is a price process \tilde{S} , uniformly ε -close to S on the logarithmic scale, such that no arbitrage (absolute or relative to the market portfolio) possibilities exist for the price \tilde{S} .

To check the condition of Theorem 1 we apply the next Theorem whose proof is given in section 4. To compare it with existing results we mention that it seems to be new in the sense, that we do not assume that our process solves a stochastic differential equation as it is done in Stroock and Varadhan [12] and it is not only for one dimensional processes as it is in Pakkanen [10].

Theorem 2. Let X be a n-dimensional Itô process on [0,T], such that

$$dX_i(t) = \mu_i(t)dt + \sum_{\nu=1}^n \sigma_{i\nu}(t)dW_{\nu}(t)$$

Assume that $|\mu|$ is bounded and σ satisfies (2). Then X has conditional full support.

Consider first the example of diverse market due to Fernholz, Karatzas and Kardaras [4], see also the review paper [5], recalled in subsection 2.1. So fix a $\delta \in (0,1)$ such that $\mathcal{O}(\delta)$ is not empty, and take the coefficients γ defined in (3), with M taken from (2).

Then we take the open sets $\mathcal{O}_k = \mathcal{O}(\delta + 1/k)$ for $k \geq 1$ and note that for any open set $G \subset C([0,T],\mathcal{O})$ we have

$$G = \bigcup_{k=1}^{\infty} G_k,$$

where $G_k = G \cap C([0,T], \mathcal{O}_k)$. Hence it is enough to show that for $t \in [0,T]$

$$\mathbf{P}(S \in G \mid \mathcal{F}_t) > 0, \quad \text{on } S|_{[0,t]} \in \{g|_{[0,t]} : g \in G_k\}.$$
(6)

Let $\tau_k = \inf\{t \in [0,T] : S(t) \notin \mathcal{O}_k\}$. Then $\tau_k = \infty$ exactly when $S \in C([0,T],\mathcal{O}_k)$ while $\tau_k = 0$ when $S(0) \notin \mathcal{O}_k$.

For $k = 1, 2, \ldots$ we define the process $S^{(k)}$ with the equation

$$dS^{(k)}(t) = \gamma^{(k)}(t)dt + \sigma(t)dW, \quad S^{(k)}(0) = S(0),$$

where

$$\gamma^{(k)}(t) = \gamma(t \wedge \tau_k) \mathbb{1}_{(S(0) \in \mathcal{O}_k)}.$$

Note that $S^{(k)}$ satisfies the conditions of Theorem 2, hence $S^{(k)}$ has conditional full support (in \mathbb{R}^n), and $S = S^{(k)}$ on the event $\tau_k = \infty$.

The conditional full support property of $S^{(k)}$ gives, for the open set $G_k = G \cap C([0,T], \mathcal{O}_k)$, that for $t \in [0,T]$

$$\mathbf{P}\Big(S^{(k)} \in G_k \,|\, \mathfrak{F}_t\Big) > 0, \text{a.s. on } S^{(k)}|_{[0,t]} \in \big\{g|_{[0,t]} \,:\, g \in G_k\big\}.$$

To obtain (6) one has to add only that $\{S^{(k)} \in G_k\} = \{S \in G_k\} \subset \{S \in G\}$. This proves that Theorem 1 applies to the diverse market constructed by Fernholz, Karatzas and Kardaras [4].

Next we turn to diverse market model attributed to Osterrieder and Rheinländer in subsection 2.1. By Theorem 2 the process S has conditional full support under \mathbf{P}_0 . For an open set $G \subset C([0,T], \mathcal{O})$ we have by Bayes formula

$$\mathbf{P}(S \in G \mid \mathcal{F}_t) = \frac{\mathbf{E}_0 \left(\mathbb{1}_{(S \in G)} \frac{d\mathbf{P}}{d\mathbf{P}_0} \mid \mathcal{F}_t \right)}{\mathbf{E}_0 \left(\frac{d\mathbf{P}}{d\mathbf{P}_0} \mid \mathcal{F}_t \right)} = \frac{\mathbf{P}_0(S \in G \mid \mathcal{F}_t)}{\mathbf{P}_0(S \in C([0, T], 0) \mid \mathcal{F}_t)}.$$

Since S has conditional full support under \mathbf{P}_0 both the numerator and the denominator are positive on the event $S|_{[0,t]} \in \{g|_{[0,t]} : g \in G\}$. Then

$$\mathbf{P}(S \in G \mid \mathcal{F}_t) > 0$$
, on $S|_{[0,t]} \in \{g|_{[0,t]} : g \in G\}$.

So S has conditional full support in ${\mathfrak O}$ under the measure ${\mathbf P}$ and Theorem 1 applies to this type of examples as well.

3. Consistent Price System and Conditional Full support

The aim of this section is to prove Theorem 1. It will follow from the following reinforcement of a result due to Guasoni, Rásonyi and Schachermayer [6].

Theorem 3. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and $(S(t))_{t \in [0,T]}$ be an \mathcal{O} -valued, continuous adapted process having conditional full support in \mathcal{O} .

Besides, let $(\varepsilon_t)_{t\in[0,T]}$ be a continuous positive process, that satisfies

$$|\varepsilon_t - \varepsilon_s| \le L_s \sup_{s \le u \le t} |S(u) - S(s)|, \quad \text{for all } 0 \le s \le t \le T$$
 (7)

with some progressively measurable finite valued $(L_s)_{s \in [0,T]}$.

Then S admits an ε -consistent price system in the sense that, there is an equivalent probability \mathbf{Q} on \mathfrak{F}_T , a process $(\tilde{S}(t))_{t\in[0,T]}$ taking values in \mathfrak{O} , such that \tilde{S} is a \mathbf{Q} martingale, bounded in $L^2(\mathbf{Q})$ and finally $|S(t)-\tilde{S}(t)| \leq \varepsilon_t$ almost surely for all $t\in[0,T]$.

The main theorem of [6], covers the case when $\mathfrak{O} = (0, \infty)^n$ and $-\eta S_i(t)/(1+\eta) \le \tilde{S}_i(t) - S_i(t) \le \eta S_i(t)$ for $i = 1, \ldots, n$, with some positive constant $\eta > 0$. That is, we get their result by the choice

$$\varepsilon_t = \frac{\eta}{1+\eta} \min_i S_i(t) \tag{8}$$

and (7) holds with $L_s = \eta$. So Theorem 3 contains the result of Guasoni, Rásonyi and Schachermayer [6] as a special case. We also have to mention the recent paper of Maris and Sayit [8]. They prove a similar statement with $\varepsilon_t = \varepsilon$ constant.

Our Theorem 1 also follows easily from Theorem 3; the choice of ε_t given in (8) yields an η -consistent price system evolving in O.

Proof of Theorem 3. To keep the process \tilde{S} inside 0 we decrease ε_t , if necessary, such that

$$0 < \varepsilon_t < \inf\{|S_t - x| : x \notin \emptyset\}, \quad \text{holds for all } t \in [0, T]. \tag{9}$$

Indeed, taking $\bar{\varepsilon}_t = \varepsilon_t \wedge \frac{1}{2} \inf\{|S_t - x| : x \notin \mathcal{O}\}\$ the process $\bar{\varepsilon}$ is positive and fulfills (7) with $\bar{L} = L \vee (1/2)$. So in what follows we assume that (9) holds also.

The proof is based on two steps. First, similarly to the proof in [6], a random walk with retirement is constructed. The properties of this random walk are collected in the next Lemma.

Lemma 4. Under the assumption of Theorem 3 there is a sequence of stopping times $(\tau_k)_{k\geq 1}$, a sequence of random variables $(X_k)_{k\geq 0}$ and an equivalent probability \mathbf{Q} such that

- (i) $\tau_0 = 0$, (τ_k) is increasing and $\cup_k \{\tau_k = T\}$ has full probability,
- (ii) $(X_k)_{k\geq 0}$ is a \mathbf{Q} martingale in the discrete time filtration $(\mathfrak{G}_k = \mathfrak{F}_{\tau_k})_{k\geq 0}$, bounded in $L^2(\mathbf{Q})$.
- (iii) if $\tau_k \leq t \leq \tau_{k+1}$ then $|S_t X_{k+1}| \leq \varepsilon_t$.

The second step of the argument is to take $\tilde{S}_t = \mathbf{E}_{\mathbf{Q}}(X \mid \mathcal{F}_t)$, where $X = \lim_{k \to \infty} X_k$. Then \tilde{S} is a martingale under \mathbf{Q} bounded in $L^2(\mathbf{Q})$ since the variable X is in $L^2(\mathbf{Q})$.

It remains to show that $|\tilde{S}_t - S_t| \leq \varepsilon_t$ for $t \in [0, T]$. By (9) this ensures also that $\tilde{S}_t \in \mathcal{O}$. By the right continuity of \tilde{S} and S it is enough to deal with t > 0. For $t \in (0, T]$ introduce the random index $\nu = \nu(t) = \inf\{k : \tau_k \geq t\}$. Note that ν is almost surely finite by (i) of Lemma 4. Clearly ν is a \mathcal{G} stopping time, and τ_{ν} is a stopping time in the filtration \mathcal{F} . For the stopped σ -fields we have $\mathcal{G}_{\nu} = \mathcal{F}_{\tau_{\nu}}$.

Then $\tau_{\nu-1} \leq t \leq \tau_{\nu}$ by the definition of ν . As $(X_k)_{k\geq 0}$ is a martingale, by property (ii), in the filtration \mathcal{G} we have $X_{\nu} = \mathbf{E}_{\mathbf{Q}}(X \mid \mathcal{G}_{\nu}) = \mathbf{E}_{\mathbf{Q}}(X \mid \mathcal{F}_{\tau_{\nu}})$. By property (iii) $|S_t - X_{\nu}| \leq \varepsilon_t$. Putting all these together, we get

$$|S_t - \tilde{S}_t| = |S_t - \mathbf{E}_{\mathbf{Q}}(X \mid \mathcal{F}_t)| = |S_t - \mathbf{E}_{\mathbf{Q}}(X_{\nu(t)} \mid \mathcal{F}_t)| \le \mathbf{E}_{\mathbf{Q}}(|X_{\nu(t)} - S_t| \mid \mathcal{F}_t) \le \varepsilon_t. \quad \Box$$

We use the next corollary of the conditional full support property, which also justifies the name. It is related to the strong conditional full support in the terminology of [6]. We give at the end of this section a direct proof instead of referring to the indirect proof using measurable selection of [6].

Corollary 5. Assume that the continuous adapted process S evolving in $\mathfrak O$ has conditional full support in $\mathfrak O$. Let τ be a stopping time and denote by $Q_{S|\mathcal F_{\tau}}$ the regular version of the conditional distribution of S given $\mathcal F_{\tau}$.

Then the support of the random measure $Q_{S|\mathcal{F}_{\tau}}$ is

$$\operatorname{supp} Q_{S|\mathcal{F}_{\sigma}} = \{ g \in C([0,T], 0) : g|_{[0,\tau]} = S|_{[0,\tau]} \}, \quad almost \ surely.$$

Proof of Lemma 4. Without loss of generality we may assume that ε is decreasing. Indeed, by taking $\bar{\varepsilon}_t = \min_{s \leq t} \varepsilon_s$, we have $0 < \bar{\varepsilon}_t \leq \varepsilon_t$ and for $s \leq t$

$$|\bar{\varepsilon}_t - \bar{\varepsilon}_s| \le \sup_{s \le u \le t} |\varepsilon_u - \varepsilon_s| \le L_s \sup_{s \le u \le t} |S_u - S_s|$$

So the condition (7) holds for $\bar{\varepsilon}$ as well.

So in what follows we assume that $(\varepsilon_t)_{t\in[0,T]}$ is decreasing and (9) holds.

The definition of $(\tau_k, X_k)_{n\geq 0}$ is then straightforward. We take $\tau_0 = 0$ and $X_0 = S_0$. If (τ_k, X_k) are already defined then we take

$$\tau_{k+1} = T \wedge \inf\{t > \tau_k : |S_t - S_{\tau_k}| > \varepsilon_t/2\},$$

$$X_{k+1} = X_k \mathbb{1}_{(\tau_{k+1} = T)} + S_{\tau_{k+1}} \mathbb{1}_{(\tau_{k+1} < T)}.$$

Now, it is easily seen from the definition that $|X_{k+1} - S_{\tau_k}| \le \varepsilon_{\tau_{k+1}}/2$. Indeed, there are three cases

- (1) $\tau_{k+1} < T$, then $X_{\tau_{k+1}} = S_{\tau_{k+1}}$ and the estimation follows by the choice of τ_{k+1} .
- (2) $\tau_k < T = \tau_{k+1}$, then $X_{k+1} = X_k = S_{\tau_k}$ and the estimation is obvious.
- (3) $\tau_k = \tau_{k+1} = T$. Then there is $k_0 < k$ such that $\tau_{k_0} < \tau_{k_0+1} = T$, and $X_{k+1} = X_k = \cdots = X_{k_0} = S_{\tau_{k_0}}$ and $S_{\tau_k} = S_T = S_{\tau_{k_0+1}}$. By the choice of τ_{k_0+1} we have that $|X_{k+1} S_{\tau_k}| = |S_{\tau_{k_0+1}} S_{\tau_{k_0}}| \le \varepsilon_T/2 = \varepsilon_{\tau_{k+1}}/2$.

Then for $\tau_k \leq t \leq \tau_{k+1}$

$$|X_{k+1} - S_t| \le |X_{k+1} - S_{\tau_k}| + |S_t - S_{\tau_k}| \le \frac{1}{2} (\varepsilon_{\tau_{k+1}} + \varepsilon_t) \le \varepsilon_t$$

as ε is decreasing. Hence Property (iii) holds.

Property (i) follows easily from the continuity of the sample path of S on [0, T]. Indeed, assume that for a given ω , we have $\tau_k(\omega) < T$ for all k. Then at $\tau(\omega) = \sup_k \tau_k(\omega)$ the sample path $S(\omega)$ could not be continuous, as $|S_{\tau_{k+1}}(\omega) - S_{\tau_k}(\omega)| \ge \varepsilon_T(\omega)/2$.

To construct the probability measure \mathbf{Q} and prove Property (ii) we apply the argument of Guasoni, Rásonyi and Schachermayer [6]. With the notation $\Delta_{k+1} = X_{k+1} - X_k$ they showed that if

$$0 \in \operatorname{int conv supp} Q_{\Delta_{k+1}|\mathcal{F}_{\tau_k}}, \quad \text{almost surely on } \tau_{k+1} < T$$
 (10)

$$\mathbf{P}(\tau_{k+1} = T \mid \mathfrak{F}_{\tau_k}) > 0. \tag{11}$$

then there exists an equivalent probability \mathbf{Q} satisfying the requirements of the statement. Roughly speaking, (10) implies the existence of Z_k such that $\mathbf{E}(Z_k\Delta_k | \mathcal{F}_{\tau_{k-1}}) = 0$ and $\mathbf{E}(Z_k | \mathcal{F}_{\tau_{k-1}}) = 1$. One can define Z_k in such a way that it charges most of the mass to the events $\{\tau_k = T\}$. With this it is possible to achieve that

$$\mathbf{E}\left(Z_k|\Delta_k|^2\,\middle|\,\mathcal{F}_{\tau_{k-1}}\right) \le 2^{-k},\tag{12}$$

and that the partial products $L_k = \prod_{\ell=1}^k Z_\ell$ are convergent in L^1 . Then with $L = \prod Z_k$ and $d\mathbf{Q} = Ld\mathbf{P}$, using (12) one can show that $X \in L^2(\mathbf{Q})$. For details we refer the reader to the proof of Theorem 1.2 in [6, pages 508-510].

So to finish the proof we have to show (10) and (11). Note, that it is enough to elaborate the proof for k=0 and \mathcal{F}_0 being trivial, as by conditioning on \mathcal{F}_{τ_k} and linearly relabelling the time interval $[\tau_k, T]$ into [0, T] we can reduce the general case to this special case. Indeed all our arguments are based on full support of the conditional law of

S given \mathcal{F}_{τ_k} , the properties of (S, ε) given in (7), (9) and the non-increase of ε . Each of these hold under the regular version of the conditional law of (S, ε) given \mathcal{F}_{τ_k} and they are not sensitive to a continuous time-change. Even though the time-change is random it depends only on τ_k , which is measurable with respect to \mathcal{F}_{τ_k} .

For (11) it is enough to show that $\mathbf{P}(\tau_1 = T) > 0$. Given L_0, ε_0 we define $\eta = \varepsilon_0/(3(L_0 \vee 1))$. Then

$$\mathbf{P}(\forall t \in [0, T], |S_t - S_0| < \eta) > 0, \tag{13}$$

by Corollary 5. Using condition (7) we have

$$\varepsilon_t \geq \varepsilon_0 - L_0 \eta \geq \frac{2}{3} \varepsilon_0 \geq 2\eta > 2|S_t - S_0|, \quad \text{for all } t \in [0,T] \text{ when } \sup_{t \in [0,T]} |S_t - S_0| < \eta \ .$$

That is

$$\left\{ \sup_{t \in [0,T]} |S_t - S_0| < \eta \right\} \subset \{ \tau_1 = T \}.$$

Hence (11) follows by (13) in the special case k = 0 and \mathcal{F}_{τ_0} being trivial, and also in the general case as we have already remarked in the previous paragraph.

Next we turn to (10). For the special case k = 0 and \mathcal{F}_{τ_0} trivial, it simplifies to (by a slight abuse of notation)

$$0 \in \operatorname{int} \operatorname{conv} \operatorname{supp}(X_1 - X_0). \tag{14}$$

So we prove (14), from this the general case follows.

Let us denote by π_r the projection onto the ball with center 0 and radius r, that is

$$\pi_r(y) = \begin{cases} \frac{r}{|y|} y & |y| \ge r, \\ y & \text{otherwise.} \end{cases}$$

We show below that there is a positive δ such that $\pi_{\delta}(\operatorname{supp}(X_1 - X_0))$ contains the entire sphere $\{y : |y| = \delta\}$. This clearly implies (14).

Let v be a unit vector in \mathbb{R}^n and $\eta > 0$. Define

$$G_{\eta} = \{ g \in C([0, T], 0) : |g(t) - (S_0 + v\varepsilon_0 t/T)| < \eta, t \in [0, T] \}.$$

By (9) ε_0 is smaller than the distance of S_0 from the complement of \mathcal{O} , hence G_{η} is a non-empty open subset of $C([0,T],\mathcal{O})$. By the full support property $\mathbf{P}(S \in G_{\eta}) > 0$ for all $\eta > 0$.

When $S \in G_{\eta}$ and η is smaller than $\varepsilon_0/2$, then S exits the ball with center S_0 and radius $\varepsilon_0/2$ hence $\tau_1 < T$. At τ_1 we have that $|X_1 - X_0| = |S_{\tau_1} - S_0| = \varepsilon_{\tau_1}/2$ and by (7)

$$\varepsilon_{\tau_1} \ge \varepsilon_0 - L_0 \sup_{0 \le u \le \tau_1} |S_u - S_0|$$

Define $f(t) = S_0 + v\varepsilon_0 t/T$. Since $S \in G_\eta$

$$\sup_{0 \le u \le \tau_1} |S_u - S_0| \le \eta + \sup_{0 \le u \le \tau_1} |f(u) - S_0|$$

$$= \eta + |f(\tau_1) - S_0| \le 2\eta + |S_{\tau_1} - S_0|$$

$$= 2\eta + \frac{1}{2}\varepsilon_{\tau_1},$$

that is

$$\varepsilon_{\tau_1} \ge \varepsilon_0 - L_0(2\eta + \frac{1}{2}\varepsilon_{\tau_1}), \quad \varepsilon_{\tau_1} \ge \frac{\varepsilon_0 - L_02\eta}{1 + L_0/2}$$

and

$$|X_1 - X_0| = \frac{\varepsilon_{\tau_1}}{2} \ge \frac{\varepsilon_0 - L_0 2\eta}{2 + L_0}.$$

Now, taking η so small that $2L_0\eta < \varepsilon_0/2$ and $\delta = \varepsilon_0/(4+2L_0)$ we obtain that the closed set $\pi_\delta(\sup(X_1 - X_0))$ intersects the set $\{y : |y| = \delta, |y - \delta v| \le \eta\}$. Since this is true for all η small enough and unit vector v (14) follows and the proof is complete.

Proof of Corollary 5. For a fixed open $G \subset C([0,T], 0)$ the process $M_t = \mathbf{P}(S \in G \mid \mathcal{F}_t)$ is a non-negative martingale. We may take the càdlàg version of this martingale. Let $A = \{M_\tau = 0\}$. Then by the martingale property $(M_t - M_{t \wedge \tau})\mathbb{1}_A = \int_0^t \mathbb{1}_A \mathbb{1}_{(\tau < s)} dM_s$ is a non-negative martingale starting from zero, hence

$$\mathbf{P}(\forall t \in [0, T], (M_t - M_{t \wedge \tau}) \mathbb{1}_A = 0) = 1.$$
(15)

We use the notation $G_t = \{g|_{[0,t]} : g \in G\}$ and note that as S has conditional full support in the sense of Definition 1 we have that $M_t > 0$ on the event $S|_{[0,t]} \in G_t$.

Next we approximate τ by stopping times $\tau_k = 2^{-k}([2^k\tau] + 1)$ and note that by (15)

$$A \cap \{\tau_k = \ell 2^{-k}\} \subset \{M_{\ell 2^{-k}} = 0\},\$$

holds up to a null set. Thus

$$A^{c} \cap \{\tau_{k} = \ell 2^{-k}\} \supset \{M_{\ell 2^{-k}} > 0\} \cap \{\tau_{k} = \ell 2^{-k}\} \supset \{S|_{[0,\ell 2^{-k}]} \in G_{\ell 2^{-k}}\} \cap \{\tau_{k} = \ell 2^{-k}\}$$

Taking union we obtain

$$A^c \supset \{S|_{[0,\tau_k]} \in G_{\tau_k}\},$$

and

$$A^c \supset \bigcup_{k=1}^{\infty} \{ S|_{[0,\tau_k]} \in G_{\tau_k} \} = \{ S|_{[0,\tau]} \in G_{\tau} \},$$

where in the last step we used that τ_k approaches τ from the right and G is open. On the other hand $\{S|_{[0,\tau]} \notin G_{\tau}\} \subset A$ is obvious so we can conclude that the events

$$A^{c} = \{ M_{\tau} = \mathbf{P}(S \in G \mid \mathcal{F}_{\tau}) > 0 \}, \text{ and } \{ S|_{[0,\tau]} \in G_{\tau} \}$$

are equal up to a negligible event.

Since the space C([0,T], 0) is second countable, its topology has a countable base and we may conclude that there is Ω' of full probability such that on Ω' for all open $G \subset C([0,T], 0)$

$$Q_{S|\mathcal{F}_{\tau}}(G) > 0$$
, exactly when $S|_{[0,\tau]} \in G_{\tau}$.

But then for $\omega \in \Omega'$ the support of the random Borel measure $Q_{S|\mathcal{F}_{\tau}}(.,\omega)$ on $C([0,T],\mathcal{O})$ is the (random) closed set $\{g:g|_{[0,\tau(\omega)]}=S(\omega)|_{[0,\tau(\omega)]}\}$ as stated.

4. Conditional full support; extension of a result of Pakkanen

In this section we prove Theorem 2. That is, we give a sufficient condition for a multidimensional continuous semimartingale to have full support. It also gives the conditional full support of the process. As we already remarked it seems to be new in the sense, that we do not assume that our process solves a stochastic differential equation as it is done in [12] and it is not only for one dimensional processes as it is in [10]. We use comparison with a squared Bessel process of suitably chosen dimension.

Theorem 6. Let X be a d-dimensional Itô process, such that

$$dX_t = \mu_t dt + \sigma_t dW_t$$

Assume that $|\mu|$, ||a|| and $||a^{-1}||$ are bounded processes, where $a_t = \sigma_t \sigma_t'$. Then X has full support.

The conditional full support of X, that is Theorem 2, follows from the observation that $(X_u)_{u \in [s,T]}$ under the regular version of its conditional law given \mathcal{F}_s is an Itô process on the time interval [s,T] satisfying the assumptions of Theorem 6.

As it is observed in [10], under the assumption of Theorem 6, it is enough to consider the case when $\mu=0,\,X_0=0$ and G is an open ball around the identically zero function. This is the content of Proposition 7 below. When $\mu=0$ and $X_0=0$ the process X is a martingale starting from zero, whose quadratic variation process $\langle X \rangle_t = \int_0^t a_t \mathrm{d}t$ satisfies

$$cI_d \le a_t = \frac{\mathrm{d}\langle X \rangle_t}{\mathrm{d}t} \le CI_d, \quad \text{for all } t \ge 0.$$
 (16)

Here I_d is the identity matrix of dimension d.

We can also assume that X is defined on $[0, \infty)$ although in Theorem 6 it is defined only on [0, T]. We can simply extend it using an independent d dimensional Brownian motion B, by the formula

$$\tilde{X}_t = \begin{cases} X_t & \text{if } t \le T, \\ c' B_{t-T} + X_T & \text{if } t > T \end{cases}$$

where $c' \in [c, C]$ is arbitrary.

Proposition 7. Assume that for the d-dimensional continuous martingale $(X_t, \mathcal{F}_t)_{t\geq 0}$ with $X_0 = 0$ (16) holds. Then for all T > 0 and $\varepsilon > 0$

$$\mathbf{P}\left(\sup_{t\leq T}|X_t|\leq\varepsilon\right)>0.$$

Proof. Let $R_t = |X_t|^2 = \sum_{i=1}^d (X_t^{(i)})^2$. Then

$$R_t = 2 \int_0^t \sqrt{R_s} \frac{X_s^T dX_s}{|X_s|} + \text{Tr}(\langle X \rangle_t)$$

By (16)

$$N_t = \int_0^t \frac{X_s^T \mathrm{d} X_s}{|X_s|}$$

is a one dimensional martingale with $c \leq \frac{\mathrm{d}\langle N \rangle_t}{\mathrm{d}t} \leq C$. Let $(\eta(t))_{t\geq 0}$ be the time change making N a one-dimensional Brownian motion $\beta_t = N_{\eta(t)}$. For the time changed process $\tilde{R}_t = R_{\eta(t)}$ we have $R_t = \tilde{R}_{\langle N \rangle_t}$ and

$$\tilde{R}_t = R_{\eta(t)} = 2 \int_0^t \sqrt{\tilde{R}_s} d\beta_s + \int_0^t b(s) ds$$

Here $b(s) = \text{Tr}(a_{\eta(s)}) \frac{d\eta(s)}{ds}$ and $0 \le b(s) \le Cd/c$.

Now we compare \tilde{R} to the solution of

$$dZ_t = 2\sqrt{Z_t}d\beta_t + \delta dt, \quad Z_0 = 0.$$

where $\delta \geq dC/c$. Note that the drift of Z is bigger than that of \tilde{R} .

Then by a standard comparison result of the solutions of SDEs, $\tilde{R}_t \leq Z_t$ for all $t \geq 0$, see [11, chapter IX, (3.7) Theorem on page 394.]. This argument is based on two simple observation. First, $\tilde{R}-Z$ can not accumulate local time at level zero, and then by Tanaka formula $|\tilde{R}-Z|_+$ is a non-negative continuous semimartingale starting from zero having non-positive drift. This is only possible if $|\tilde{R}-Z|_+$ is identically zero.

Now, Z is a squared Bessel process of dimension δ , so it stays below ε^2 on [0, s] with positive probability for any $\varepsilon > 0$ and $s \ge 0$. Since, $R_t = \tilde{R}_{\langle N \rangle_t}$ and $\langle N \rangle_t \le CT$ for $t \le T$ we have that

$$\mathbf{P}\left(\sup_{s \le T} |X_s| < \varepsilon\right) = \mathbf{P}\left(\sup_{s \le T} R_s < \varepsilon^2\right)$$

$$\geq \mathbf{P}\left(\sup_{s < CT} \tilde{R}_s < \varepsilon^2\right) \geq \mathbf{P}\left(\sup_{s < CT} Z_s < \varepsilon^2\right) > 0. \quad \Box$$

References

- [1] Delbaen, F. and Schachermayer, W. (2006). The mathematics of arbitrage. Springer Finance. Springer-Verlag, Berlin. MR2200584 (2007a:91001)
- [2] FERNHOLZ, R. (1999). On the diversity of equity markets. *J. Math. Econom.* **31** 393–417. doi: 10.1016/S0304-4068(97)00018-9. MR1684221 (2000e:91072)
- [3] FERNHOLZ, R. (2001). Equity portfolios generated by functions of ranked market weights. Finance Stoch. 5 469–486. doi: 10.1007/s007800100044. MR1861997 (2002h:91052)
- [4] FERNHOLZ, R., KARATZAS, I. and KARDARAS, C. (2005). Diversity and relative arbitrage in equity markets. Finance Stoch. 9 1–27. doi: 10.1007/s00780-004-0129-4. MR2210925 (2006k:60123)
- [5] FERNHOLZ, R. and KARATZAS, I. (2009). Stochastic Portfolio Theory: an Overview. In Handbook of Numerical Analysis, (P. G. Ciarlet, ed.). Handbook of Numerical Analysis 15 89 - 167. Elsevier. doi: 10.1016/S1570-8659(08)00003-3
- [6] GUASONI, P., RÁSONYI, M. and SCHACHERMAYER, W. (2008). Consistent price systems and face-lifting pricing under transaction costs. Ann. Appl. Probab. 18 491– 520. doi: 10.1214/07-AAP461. MR2398764 (2009a:91050)
- [7] GUASONI, P. and RÁSONYI, M. (2012). Fragility of Arbitrage and Bubbles in Diffusion Models. doi: 10.2139/ssrn.1856223
- [8] Maris, F. and Sayit, H. (2012). Consistent Price Systems in Multiasset Markets. *International Journal of Stochastic Analysis* 2012 14 pages. Article ID 687376. doi: 10.1155/2012/687376
- OSTERRIEDER, J. R. and RHEINLÄNDER, T. (2006). Arbitrage opportunities in diverse markets via a non-equivalent measure change. Annals of Finance 2 287–301. doi: 10.1007/s10436-006-0037-z
- [10] PAKKANEN, M. S. (2010). Stochastic integrals and conditional full support. *Journal of Applied Probability* 47 650–667. doi: 10.1239/jap/1285335401. MR2731340
- [11] REVUZ, D. and YOR, M. (1991). Continuous martingales and Brownian motion. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 293. Springer-Verlag, Berlin. MR1083357 (92d:60053)
- [12] STROOCK, D. W. and VARADHAN, S. R. S. (2006). Multidimensional diffusion processes. Classics in Mathematics. Springer-Verlag, Berlin. Reprint of the 1997 edition. MR2190038 (2006f:60005)