Appendix A

The Riemann Zeta Function

We now give the analytic continuation and the functional equation of the Riemann zeta function, which is based on the functional equation of the theta series. First we need the gamma function:

For $\mathrm{Re}(s) > 0$ the integral

$$\Gamma(s) \;=\; \int_0^\infty t^{s-1} e^{-t} dt$$

converges and gives a holomorphic function in that range. We integrate by parts to get for $\mathrm{Re}(s) > 0$,

$$\Gamma(s+1) \;=\; \int_0^\infty t^s e^{-t} dt \;=\; \int_0^\infty s t^{s-1} e^{-t} dt \;=\; s\Gamma(s),$$

i.e.,

$$\Gamma(s) \;=\; \frac{\Gamma(s+1)}{s}.$$

In the last equation the right-hand-side gives a meromorphic function on $\mathrm{Re}(s) > -1$, and thus $\Gamma(s)$ extends meromorphically to that range. But again the very same equation extends $\Gamma(s)$ to $\mathrm{Re}(s) > -2$, and so on. We find that $\Gamma(s)$ extends to a meromorphic function on the entire plane that is holomorphic except for simple poles at $s = 0, -1, -2, \dots$.

Recall from Section 3.6 the theta series

$$\Theta(t) \;=\; \sum_{k \in \mathbb{Z}} e^{-t\pi k^2}, \text{ for } t > 0,$$

which satisfies

$$\Theta(t) \;=\; \frac{1}{\sqrt{t}}\Theta\left(\frac{1}{t}\right),$$

as was shown in Theorem 3.7.1. We now introduce the Riemann zeta function:

Lemma A.1 *For* $\mathrm{Re}(s) > 1$ *the series*

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely and defines a holomorphic function there. This function is called the Riemann zeta function.

Proof: Since the summands $1/n^s$ are entire functions, it needs to be shown only that the series $\sum_{n=1}^{\infty} |n^{-s}|$ converges locally uniformly in $\mathrm{Re}(s) > 1$. In that range we compute

$$\begin{aligned}
\frac{1}{\mathrm{Re}(s)-1} \;&=\; \left.\frac{x^{-\mathrm{Re}(s)+1}}{1-\mathrm{Re}(s)}\right|_1^{\infty} \;=\; \int_1^{\infty} x^{-\mathrm{Re}(s)}dx \\
&=\; \int_2^{\infty} (x-1)^{-\mathrm{Re}(s)}dx \;\geq\; \int_2^{\infty} [x]^{-\mathrm{Re}(s)}dx \\
&=\; \sum_{n=2}^{\infty} n^{-\mathrm{Re}(s)} \;=\; \sum_{n=2}^{\infty} \left|\frac{1}{n^s}\right|,
\end{aligned}$$

where for $x \in R$ the number $[x]$ is the largest integer k that satisfies $k \leq x$. The lemma follows. $\square$

Theorem A.2 *(The functional equation of the Riemann zeta function)*

The Riemann zeta function $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$, holomorphic up to a simple pole at $s = 1$, and the function

$$\xi(s) \;=\; \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

satisfies

$$\xi(s) \;=\; \xi(1-s)$$

for every $s \in \mathbb{C}$.

Proof: Note that the expression dt/t is invariant under the substitution $t \mapsto ct$ for $c > 0$ and up to sign under $t \mapsto 1/t$. Using these facts, we compute for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\xi(s) &= \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} = \sum_{n=1}^{\infty}\int_0^{\infty} n^{-s}t^{s/2}\pi^{-s/2}e^{-t}\frac{dt}{t} \\
&= \sum_{n=1}^{\infty}\int_0^{\infty}\left(\frac{t}{n^2\pi}\right)^{s/2} e^{-t}\frac{dt}{t} = \sum_{n=1}^{\infty}\int_0^{\infty} t^{s/2}e^{-n^2\pi t}\frac{dt}{t} \\
&= \int_0^{\infty} t^{s/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t}.
\end{aligned}$$

We split this integral into a sum of an integral over $(0,1)$ and an integral over $(1,\infty)$. The latter one,

$$\int_1^{\infty} t^{s/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t},$$

is an entire function, since the function $t \mapsto \Theta(t) - 1$ is rapidly decreasing at ∞. The other summand is

$$\begin{aligned}
\int_0^1 t^{s/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t} &= \int_1^{\infty} t^{-s/2}\frac{1}{2}\left(\Theta\left(\frac{1}{t}\right)-1\right)\frac{dt}{t} \\
&= \int_1^{\infty} t^{-s/2}\frac{1}{2}\left(\sqrt{t}\Theta(t)-1\right)\frac{dt}{t} \\
&= \int_1^{\infty} t^{-s/2}\frac{1}{2}\left(\sqrt{t}(\Theta(t)-1)+\sqrt{t}-1\right)\frac{dt}{t},
\end{aligned}$$

which equals the sum of the entire function

$$\int_1^{\infty} t^{(1-s)/2}\frac{1}{2}(\Theta(t)-1)\frac{dt}{t}$$

and

$$\frac{1}{2}\int_1^{\infty} t^{(1-s)/2}\frac{dt}{t} - \frac{1}{2}\int_1^{\infty} t^{-s/2}\frac{dt}{t} = \frac{1}{s-1} - \frac{1}{s}.$$

Summarizing, we get

$$\xi(s) = \int_1^{\infty}\left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right)\frac{1}{2}(\Theta(t)-1)\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}.$$

$\square$

Using the functional equation and knowing the locations of the poles of the Γ-function, we can see that the Riemann zeta function has zeros at the even negative integers $-2, -4, -6, \dots$, called the *trivial zeros*. It can be shown that all other zeros are in the strip $0 < \mathrm{Re}(s) < 1$. The up to now unproven *Riemann hypothesis* states that all nontrivial zeros should be in the set $\mathrm{Re}(s) = \frac{1}{2}$. This would have deep consequences about the distribution of primes through the prime number theorem [13].

This technique for constructing the analytic continuation of the zeta function dates back to Riemann, and can be applied to other Dirichlet series as well.

Appendix B

Haar Integration

Let G be an LC group. We here give the proof of the existence of a Haar integral.

Theorem B.1 *There exists a non-zero invariant integral I of G. If I' is a second non-zero invariant integral, then there is a number $c > 0$ such that $I' = cI$.*

For the uniqueness part of the theorem we say that the invariant integral is *unique up to scaling.*

The idea of the proof resembles the construction of the Riemann integral on $\mathbb{R}$. To construct the Riemann integral of a positive function one finds a step function that dominates the given function and adds the lengths of the intervals needed multiplied by the values of the dominating function. Instead of characteristic functions of intervals one could also use translates of a given continuous function with compact support, and this is exactly what is done in the general situation.

Proof of the Theorem: For the existence part, let $C_c^+(G)$ be the set of all $f \in C_c(G)$ with $f \geq 0$. For $f, g \in C_c(G)$ with $g \neq 0$ there are $c_j > 0$ and $s_j \in G$ such that

$$f(x) \;\leq\; \sum_{j=1}^{n} c_j g(s_j^{-1}x).$$

Let $(f:g)$ denote

$$\inf\left\{\sum_{j=1}^n c_j \;\middle|\; \begin{array}{c} c_1,\dots,c_n>0 \text{ and there are } s_1,\dots,s_n\in G \\ \text{such that } f(x)\le\sum_{j=1}^n c_j g(s_j x)\end{array}\right\}.$$

Lemma B.2 *For $f,g,h\in C_{\mathrm{c}}^+(G)$ with $g\neq 0$ we have*

(a) $(L_s f:g)=(f:g)$ *for every* $s\in G$,

(b) $(f+h:g)\le(f:g)+(h:g)$,

(c) $(\lambda f:g)=\lambda(f,g)$ *for* $\lambda\ge 0$,

(d) $f\le h \;\Rightarrow\; (f:g)\le(h:g)$,

(e) $(f:h)\le(f:g)(g:h)$ *if* $h\neq 0$, *and*

(f) $(f:g)\ge\frac{\max f}{\max g}$, *where* $\max f=\max\{f(x)|x\in G\}$.

Proof: The items (a) to (d) are trivial. For item (e) let $f(x)\le\sum_j c_j g(s_j x)$ and $g(y)\le\sum_k d_k h(t_k y)$; then

$$f(x)\;\le\;\sum_{j,k} c_j d_k h(t_k s_j x),$$

so that $(f:h)\le\sum_j c_j\sum_k d_k$.

For (f) choose $x\in G$ with $\max f=f(x)$; then

$$\max f\;=\;f(x)\;\le\;\sum_j c_j g(s_j x)\;\le\;\sum_j c_j\max g.$$

$\square$

Fix some $f_0\in C_{\mathrm{c}}^+(G)$, $f_0\neq 0$. For $f,\varphi\in C_{\mathrm{c}}^+(G)$ with $\varphi\neq 0$ let

$$J(f,\varphi)\;=\;J_{f_0}(f,\varphi)\;=\;\frac{(f:\varphi)}{(f_0:\varphi)}.$$

Lemma B.3 *For $f,h,\varphi\in C_{\mathrm{c}}^+(G)$ with $f,\varphi\neq 0$ we have*

(a) $\frac{1}{(f_0:f)}\le J(f,\varphi)\le(f:f_0)$,

(b) $J(L_s f, \varphi) = J(f, \varphi)$ *for every* $s \in G$,

(c) $J(f + h, \varphi) \le J(f, \varphi) + J(h, \varphi)$, *and*

(d) $J(\lambda f, \varphi) = \lambda J(f, \varphi)$ *for every* $\lambda \ge 0$.

Proof: This follows from the last lemma. $\square$

The function $f \mapsto J(f, \varphi)$ does not give an integral, since it is not additive but only subadditive. However, as the support of φ shrinks it will become asymptotically additive, as the next lemma shows.

Lemma B.4 *Given* $f_1, f_2 \in C_c^+(G)$ *and* $\varepsilon > 0$ *there is a neighborhood* V *of the unit in* G *such that*

$$J(f_1, \varphi) + J(f_2, \varphi) \ \le \ J(f_1 + f_2, \varphi)(1 + \varepsilon)$$

holds for every $\varphi \in C_c^+(G)$, $\varphi \ne 0$ *with support contained in* V.

Proof: Choose $f' \in C_c^+(G)$ such that f' is identically equal to 1 on the support of $f_1 + f_2$. For the existence of such a function see Exercise 8.2. Let $\delta, \varepsilon > 0$ be arbitrary and set

$$f \ = \ f_1 + f_2 + \delta f', \quad h_1 \ = \ \frac{f_1}{f}, \quad h_2 \ = \ \frac{f_2}{f},$$

where it is understood that $h_j = 0$ where $f = 0$. It follows that $h_j \in C_c^+(G)$.

Choose a neighborhood V of the unit such that $|h_j(x) - h_j(y)| < \varepsilon/2$ whenever $x^{-1}y \in V$. If $\mathrm{supp}(\varphi) \subset V$ and $f(x) \le \sum_k c_k \varphi(s_k x)$, then $\varphi(s_k x) \ne 0$ implies

$$|h_j(x) - h_j(s_k^{-1})| < \frac{\varepsilon}{2},$$

and

$$\begin{aligned} f_j(x) \ &= \ f(x) h_j(x) \ \le \ \sum_k c_k \varphi(s_k x) h_j(x) \\ &\le \ \sum_k c_k \varphi(s_k x) \left(h_j(s_k^{-1}) + \frac{\varepsilon}{2} \right), \end{aligned}$$

so that

$$(f_j : \varphi) \ \le \ \sum_k c_k \left(h_j(s_k^{-1}) + \frac{\varepsilon}{2} \right),$$

and so

$$(f_1 : \varphi) + (f_2 : \varphi) \ \leq \ \sum_k c_k(1+\varepsilon).$$

This implies

$$\begin{aligned} J(f_1,\varphi) + J(f_2,\varphi) &\leq J(f,\varphi)(1+\varepsilon) \\ &\leq (J(f_1+f_2,\varphi) + \delta J(f',\varphi))(1+\varepsilon). \end{aligned}$$

Letting δ tend to zero gives the claim. $\square$

Let F be a countable subset of $C_c^+(G)$, and let V_F be the complex vector space spanned by all translates $L_s f$, where $s \in G$ and $f \in F$. A linear functional $I : V_F \to \mathbb{C}$ is called an *invariant integral* on V_F if $I(L_s f) = I(f)$ holds for every $s \in G$ and every $f \in V_F$ and

$$f \in F \quad \Rightarrow \quad I(f) \geq 0.$$

An invariant integral I_F on V_F is called *extensible* if for every countable set $F' \subset C_c^+(G)$ that contains F there is an invariant integral $I_{F'}$ on $V_{F'}$ extending I_F.

Lemma B.5 *For every countable set $F \subset C_c^+(G)$ there exists an extensible invariant integral I_F that is unique up to scaling.*

Proof: Fix a metric on G. For $n \in \mathbb{N}$ let $\varphi_n \in C_c^+(G)$ be nonzero with support in the open ball of radius $1/n$ around the unit. Suppose that $\varphi_n(x) = \varphi_n(x^{-1})$ for every $x \in G$.

Let $F = \{f_1, f_2, \dots\}$. Since the sequence $J(f_1, \varphi_n)$ lies in the compact interval $[1/(f_0 : f_1), (f_1 : f_0)]$ there is a subsequence φ_n^1 of φ_n such that $J(f_1, \varphi_n^1)$ converges. Next there is a subsequence φ_n^2 of φ_n^1 such that $J(f_2, \varphi_n^2)$ also converges. Iterating this gives a sequence (φ_n^j) of subsequences. Let $\psi_n = \varphi_n^n$ be the diagonal sequence. Then for every $j \in \mathbb{N}$ the sequence $(J(f_j, \psi_n))$ converges, so that the definition

$$I_{f_0,(\psi_n)_{n\in\mathbb{N}}}(f_j) \ = \ \lim_{n\to\infty} J(f_j, \psi_n)$$

makes sense. By Lemma B.4, the map I indeed extends to a linear functional on V_F that clearly is a nonzero invariant integral.

This integral is extensible, since for every countable $F' \supset F$ in $C_c^+(G)$ one can iterate the process and go over to a subsequence of ψ_n.

This does not alter I_{f_0,ψ_n}, since every subsequence of a convergent sequence converges to the same limit.

We shall now establish the uniqueness. Let $I_F = I_{f_0,\psi_n}$ be the invariant integral just constructed. Let I be another extensible invariant integral on V_F. Let $f \in F$, $f \neq 0$; then we will show that

$$I_{f_0,\psi_n}(f) \;=\; \frac{I(f)}{I(f_0)}.$$

The assumption of extensibility will enter our proof in that we will freely enlarge F in the course of the proof. Now let the notation be as in the lemma. Let $\varphi \in F$ and suppose $f(x) \leq \sum_{j=1}^m d_j\varphi(s_jx)$ for some positive constants d_j and some elements s_j of G. Then

$$I(f) \;\leq\; \sum_{j=1}^m d_j I(\varphi),$$

and therefore

$$\frac{I(f)}{I(\varphi)} \;\leq\; (f:\varphi).$$

Let $\varepsilon > 0$. Since f is uniformly continuous, there is a neighborhood V of the unit such that for $x, s \in G$ we have $x \in sV \;\Rightarrow\; |f(x) - f(s)| < \varepsilon$. Let $\varphi \in C_c^+$ be zero outside V and suppose $\varphi(x) = \varphi(x^{-1})$. Let C be a countable dense set in G. The existence of such a set is clear by Lemma 6.3.1. Now suppose that for every $x \in C$ the function $s \mapsto f(s)\varphi(s^{-1}x)$ lies in F. For $x \in C$ consider

$$\int_G f(s)\varphi(s^{-1}x)ds \;=\; I(f(.)\varphi(.^{-1}x)).$$

Now, $\varphi(s^{-1}x)$ is zero unless $x \in sV$, so

$$\begin{aligned}\int_G f(s)\varphi(s^{-1}x)ds \;&>\; (f(x)-\varepsilon)\int_G \varphi(s^{-1}x)ds \\ &=\; (f(x)-\varepsilon)\int_G \varphi(x^{-1}s)ds \\ &=\; (f(x)-\varepsilon)I(\varphi).\end{aligned}$$

Therefore,

$$(f(x)-\varepsilon) \;<\; \frac{1}{I(\varphi)}\int_G f(s)\varphi(s^{-1}x)ds.$$

Let $\eta > 0$, and let W be a neighborhood of the unit such that

$$x, y \in G, \quad x \in Wy \;\;\Rightarrow\;\; |\varphi(x) - \varphi(y)| \;<\; \eta.$$

There are finitely many $s_j \in G$ and $h_j \in C_c^+(G)$ such that the support of h_j is contained in s_jW and

$$\sum_{j=1}^{m} h_j \;\equiv\; 1 \quad \text{on } \operatorname{supp}(f).$$

Such functions can be constructed using the metric (see Exercise 8.2). We assume that for each j the function $s \mapsto f(s)h_j(s)\varphi(s^{-1}x)$ lies in F for every $x \in C$. Then it follows that

$$\int_G f(s)\varphi(s^{-1}x)ds \;=\; \sum_{j=1}^{m} \int_G f(s)h_j(s)\varphi(s^{-1}x)ds.$$

Now $h_j(s) \neq 0$ implies $s \in s_jW$, and this implies

$$\varphi(s^{-1}x) \leq \varphi(s_j^{-1}x) + \eta.$$

Assuming that the fh_j lie in F, we conclude that

$$\int_G f(s)\varphi(s^{-1}x)ds \;\leq\; \sum_{j=1}^{m} I(fh_j)(\varphi(s_j^{-1}x) + \eta).$$

Let $c_j = I(h_jf)/I(\varphi)$; then $\sum_j c_j = I(f)/I(\varphi)$ and

$$f(x) \;\leq\; \varepsilon + \eta\sum_{j=1}^{m} c_j + \sum_{j=1}^{m} c_j\varphi(s_j^{-1}x).$$

Let $\chi \in C_c^+(G)$ be such that $\chi \equiv 1$ on $\operatorname{supp}(f)$. Then

$$f(x) \;\leq\; \left(\varepsilon + \eta\sum_{j=1}^{m} c_j\right)\chi(x) + \sum_{j=1}^{m} c_j\varphi(s_j^{-1}x).$$

This result is valid for $x \in C$ in the first instance, but the denseness of C implies it for all $x \in G$. As $\eta \to 0$ it follows that

$$(f : \varphi) \;\leq\; \varepsilon(\chi : \varphi) + \frac{I(f)}{I(\varphi)}.$$

Therefore,

$$\frac{(f : \varphi)}{(f_0 : \varphi)} \;\leq\; \varepsilon\frac{(\chi : \varphi)}{(f_0 : \varphi)} + \frac{I(f)}{I(\varphi)(f_0 : \varphi)} \;\leq\; \varepsilon\frac{(\chi : \varphi)}{(f_0 : \varphi)} + \frac{I(f)}{I(f_0)}.$$

So, as $\varepsilon \to 0$ and as φ runs through the ψ_n, we get

$$I_{f_0,\psi_n}(f) \;\le\; \frac{I(f)}{I(f_0)}.$$

Applying the same argument with the roles of f and f_0 interchanged gives

$$I_{f,\psi_n}(f_0) \;\le\; \frac{I(f_0)}{I(f)}.$$

Now note that both sides of these inequalities are antisymmetric in f and f_0, so that the second inequality gives

$$I_{f_0,\psi_n}(f) \;=\; I_{f,\psi_n}(f_0)^{-1} \;\ge\; \left(\frac{I(f_0)}{I(f)}\right)^{-1} \;=\; \frac{I(f)}{I(f_0)}.$$

Thus it follows that $I_{f_0,\psi_n}(f) = I(f)/I(f_0)$ and the lemma is proven. $\square$

Finally, the proof of the theorem proceeds as follows. For every countable set $F \subset C_c^+(C)$ with $f_0 \in F$, let I_F be the unique extensible invariant integral on V_F with $I_F(f_0) = 1$. We define an invariant integral on all $C_c(G)$ as follows: For $f \in C_c^+(G)$ let

$$I(f) \;=\; I_{\{f_0,f\}}(f).$$

Then I is additive, since for $f, g \in C_c^+(G)$,

$$\begin{aligned} I(f+g) &= I_{\{f_0,f+g\}}(f+g) \;=\; I_{\{f_0,f,g\}}(f+g) \\ &= I_{\{f_0,f,g\}}(f) + I_{\{f_0,f,g\}}(g) \;=\; I_{\{f_0,f\}}(f) + I_{\{f_0,g\}}(g) \\ &= I(f) + I(g). \end{aligned}$$

Thus I extends to an invariant integral on $C_c(G)$, with the invariance being clear from Lemma B.5. $\square$

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