From triangles to a concept: a phenomenographic study of A-level students' development of the concept of trigonometry.

By Michele Challenger

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University of Warwick
Department of Education

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Dedication

This study is dedicated to the late Jim Pattison, an exemplary mathematics teacher universally respected by both his colleagues and his students.

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Declaration

I, the author, declare that this thesis is my own work and has not been submitted for a degree at any other institution.

Abstract

This thesis describes an investigation of the trigonometry schemas developed by a group of 16-18 year old English students during their study of A-level mathematics. It is concerned with identifying differences in the schemas of students who are successful with solving trigonometric problems to those who are less successful.

The study is guided by the theoretical frameworks of mathematical schema development proposed by Sfard (1991) and Dubinsky's (1991) APOS theory that describe how operational knowledge of one or more procedures develops into an understanding that is conceptual. The benefits of a conceptual understanding are greater flexibility in problem solving and greater cognitive economy. The study of trigonometry prior to starting the A-level course is predominantly concerned with problems relating to triangles either right-angled or scalene and it is during the A-level course that trigonometry broadens into the study of the properties of function. Experience as a mathematics tutor suggests however that not all students finish the A-level course with a conceptual understanding of trigonometric functions that is a coherent entity. Some students have little more than a collection of arbitrary facts and procedures that they struggle to use cohesively. Traditionally trigonometry is taught by mediating the core ideas through a mixture of spatial-visual images and algebraic identities that together provide the basis for function properties and behaviour. This study examines student perceptions of these mediating representations through a phenomenological investigation based on concept maps, interviews, classroom observations and observed problem-solving by selected students.

The results of the study suggest that different students focus upon different aspects of the mediating representations. Students schemas as evidenced by the concept maps varied between those that were predominantly composed of algebraic representations for instance formulae, to those that were composed of a mixture of algebraic and specific spatial visual representations such as graphs, the unit circle and special angles triangles, to those that portrayed trigonometry through a series of overlaid graphs that signified the essence of function behaviour. The students whose schemas included spatial-visual components were more successful in problem solving and assessments than those whose schemas were focused on algebraic aspects. The study also supported documented research by Gray, Pinto, Pinner & Tall (1999) that spatial-visual imagery has a qualitative aspect and by Delice & Monaghan (2005) that teaching style plays a considerable part in the students' development of schema. A significant aspect to the development of a flexible schema is the teacher's philosophy of trigonometry and approach to the construction of sub concepts. Finally the study considers the merits of the two main theoretical frameworks of mathematical development proposed by Sfard and Dubinsky's APOS theory from a teaching perspective and concludes that the empirical findings of this study are better described by Sfard's explanation of the dual nature of mathematical conceptions whereby a process schema has the potential to develop into a flexible, stable object conception through interiorisation, condensation and reification.

Chapter 1
Overview of the Study

This thesis is focused on the trigonometric schemas that a group of English students in years 12 and 13 developed during their study of AS/A2 mathematics. This study seeks to discover how the understanding of trigonometry is transformed into a concept of trigonometric functions by a group of students studying A-level mathematics. The perspective taken is that knowledge is an individual's construction and thus the focus of the study is the cognitive organisation of new information by students into a coherent structure of links, axioms and procedures that together form an abstract concept of trigonometric function.

The study is concerned with the qualitative nature of these students' individual schemas and with identifying differences in their understanding of the syllabus material and the consequential effects of these differences, if any, in assessments. [§1.2]

The development of A-level mathematics since its inception has seen a change in purpose, content, format and assessment procedure. Initially its purpose was to provide a route to the study of mathematics-based courses at university. The syllabus was determined by Higher Education Institutions and the examinations were set and assessed under their control. The grades awarded to students reflected their achievement in comparison to that of the other students, in that year's cohort.

During the 1980's Education reform became one of the key areas on the political agenda of the Conservative government and radical reforms of the secondary education system for both pre-16 students [§2.2.2.1] and 16-18 year olds [§2.2.2.2] within England and Wales were introduced. During the 1990's the pace of change quickened [§2.2.3] but reports from the mathematics community expressing deep concern about the ill preparation of A-level students for degree courses [§2.3] and the perception that standards of assessment were declining [§2.4]. The topics most frequently cited by universities as showing a decline in understanding and process are algebra and geometric subjects such as trigonometry.

It is a feature of this study that given the consistent emphasis on these weaknesses and the changes that a variety of studies and reports have recommended that this study seeks to consider student understanding and perception of one part of the syllabus, trigonometry, in the context of the quality of their learning and the students' perception of what it is they should focus upon.

Of central importance in this context is Curriculum 2000 [§2.9] which partitioned the A-level into two one-year courses: the first year was now classified as the AS course and the second year the A2 course [§2.10].

Trigonometry is usually introduced to year 9 students and then returned to in years 10 and 11. It involves learning the ratios of sin, cos and tan and a complex coordination between diagrams of right angled triangles and algebraic manipulation of the ratios in order to find required lengths or angles. For those doing the Higher papers at GCSE the subject is broadened to include the sine rule and cosine rule and the values of sin, cos and tan for angles in the range 0 to 360 degrees. Skemp [§3.1.2] theorised that there are two types of mathematical understanding: instrumental which is knowledge of a procedure or a set of procedures; and relational understanding which is knowledge of why instrumental procedures are appropriate, their limitations and alternatives. The sine rule and cosine rule are formulae that are used instrumentally to find required lengths or angles of scalene triangles. The graphs, which, to an expert, describe the nature of the sin, cos and tan function between 0 and 360 degrees and by extension from negative infinity to positive infinity, are in my experience frequently conceived by students as an instrumental means to a solution rather than a representation of the function. This experience leads me to surmise that at the end of the GCSE course the majority of the students who have studied trigonometry under the National Curriculum have a predominantly instrumental understanding of trigonometry. The development of students' understanding of trigonometry from a set of procedures to an entity that has a multiplicity of representations both algebraic (as in the identities) and spatial therefore takes place (if at all) during the study of the A-level course. This study set out to observe this transformation in the light of two theoretical frameworks for mathematical development proposed by Sfard (1991) and Dubinsky (1991) respectively.

Sfard [§3.1.4] theorises that mathematical development is characterised by the progression from operational understanding (that is knowledge of a procedure) to conceptual understanding (that is a static object that links together facts, processes, properties, algebraic and spatial representations simultaneously) through interiorisation, condensation and reification. A process or representation is interiorised by an individual then condensed mentally. Reification is a cognitive reorganisation that enables the individual to link the process to other facts and processes creating an object conception. This static object helps inform decisions on suitable procedures for subsequent problem solving and is itself informed by new processes and representations thus a duality of understanding is constructed with the operational feeding the reified object which in turn feeds operational know how. Sfard states that reification is not an automatic development and some students never move from an operational understanding to the mental object construction but when it does happen the result is greater cognitive flexibility and efficiency.

The APOS theory [§3.1.5] first proposed by Dubinsky (1991) is an acronym for the words Action-Process-Object-Schema. An *Action* is a physical action that includes application of a procedure or use of a known fact to facilitate an answer. A *Process* denotes recognition of an Action as having a beginning and an end and the individual is in conscious control of each step able to reverse the procedure if required or an use alternative procedure or known fact. An *Object* conception the result of the encapsulation of a process and is recognisable by the individual's use of one process upon another. For example once the graph of sin x is encapsulated transformations such as stretches and translations may be applied to it. A *Schema* is the sum of all the encapsulated objects that link together.

One feature of schema development appears to be the focus of students' attention when they are studying trigonometry. The second feature is a difference of perception of the everyday mathematical terms and visual symbols used to communicate ideas within the classroom. Gray & Tall (1994) found that mathematical symbols, (including words) have different meanings to different individuals [§3.1.6]. Some individuals perceive the meaning as an instruction to undertake an action, whilst others perceive the symbol or word as simultaneously indicating the undertaking of the action and the result of that action. Gray & Tall defined symbols that can be interpreted both operationally and conceptually as 'procepts'. Within trigonometry there are numerous procepts; indeed the word trigonometry itself is a procept that may be perceived as a set of processes or as a code word for an amalgam of visual images, identities and processes that together form an abstracted mathematical object. The procepts that predicate an Object concept of trigonometry are numerous and include the terms sine, cosine and tangent. In order to determine the nature of a students' trigonometry schema therefore it is necessary to understand the meaning given by the student to the terms and sub-concepts in common discourse. The main ideas of trigonometry are mediated through a mixture of algebraic representations such as ratios and formulae, and geometric representations such as triangles, circles and graphs. The use of imagery is ubiquitous by students and teachers however spatial images have been found to have a qualitative dimension (Gray, Pinto, Pitta & Tall, 1999) which requires investigation. The research literature that documents the extent to which spatial visual imagery increases mental flexibility by providing a complementary representation is described and the observations of other documented research on the study of trigonometry are also noted.

The focus of this study was the observation of students' development in understanding trigonometry and whether there is empirical evidence to support the theoretical frameworks. In particular it was designed to investigate changes in students perceptions of trigonometry, the development (or not) of a flexible schema [§3.1.3] the means by which this was promoted and whether key stages of development can be identified. The ultimate aim is for a more informed and beneficial teaching strategy.

Students who study mathematics at AS/A-level have to have done well at the higher GCSE paper, usually getting at least an A grade, and like the subject enough to want to choose to study it further. So it is reasonable to assume that at the beginning of the course all the students are not only good at mathematics but also optimistic about their future prospects.

As a mathematics tutor I have the opportunity to teach students from different schools who have been exposed to different teaching styles. The advantage of teaching on a one to one basis is the opportunity it affords to identify types of understanding. Some students clearly build knowledge personally whilst others prefer to learn procedures and formulae for the purpose of problem solving. The focus of these students is the memorisation of key facts and instrumental processes for application to a given type of problem. When I asked students who have done well at GCSE through memorising facts and procedures why they have chosen to do A-level mathematics their answers were typically:

"I find it easy to remember it"

"I usually do well at it in exams"

"I find it easier than the other subjects; you just have to learn it"

However, as the A-level course progresses many of these students are completely at a loss with trigonometry in particular. Their confusion leads to a diffidence that further incapacitates them as evidenced by comments such as:

"Are we talking about triangle trigonometry or circle trigonometry here?"

"I used to understand it when it was just triangles but now I don't know where to start".

Such comments contrast starkly with comments from other students such as:

"In the beginning I hated trigonometry but now I really like it"

"It was weird at first having to label opposite, adjacent and hypotenuse on all the triangles but at A-level you spend a lot of time on it and it starts to make sense - I love solving trig equations now".

This observed phenomenon led me to enquire about the experiences of other teachers of A-level trigonometry. Here are some of their comments:

"Students find trigonometry one of the hardest subjects on the A-level. They have difficulty with it as it becomes more abstract"

"Trigonometry at A-level is the topic that sorts the sheep from the goats. It is a good indicator of true mathematical ability and if they can master it then it indicates that they have the potential to become a real mathematician."

"There is a big difference between the simple trigonometry that we teach at GCSE and what we do at A-level. There are loads more formulas for a start and the graph transformations. I never really totally got to grips with graph transformations and prefer to concentrate on the algebra but it is hard for the students to try and remember it all."

My perception that trigonometry is one of the key areas of A-level mathematics that students (and some teachers) struggle with is one that is acknowledged by others in the teaching profession.

This then is an exploratory study with two central themes:

How do A-level students develop their understanding of trigonometry? What distinguishes the thinking of more able students from the less able?

Can we as educators gain any insight into the sorts of cognitive constructions that are more beneficial?

To gain an insight into the meaning of trigonometry to students and any change in meaning over time it was decided to frame this study on phenomological principles [§4.3]. The aim of the study was to investigate differences in the perception of trigonometry between students in the group and longitudinally between the start and the end of the course for a set of selected students. The means for the investigation was primarily students' concept maps [§4.5.2] drawn freely without teacher or observer intervention or suggestion at the start of the course and at the end of each session of lessons on a trigonometry component of the syllabus. The analysis of the concept maps was supported by informal student interviews [§4.5.3], lesson observations [§4.5.4], observations of students as they attempted routine and later, integrated questions [§7.9.2] and assessment results [§7.11].

A pilot study was undertaken with a year 12 class from the same school as the main study [Chapter 5]. All of the students had followed a course in trigonometry prior to the start of their A-level component. During the pilot study lesson observations were made and student interviews undertaken to investigate how the new material was being incorporated into the students' trigonometry schemas and if any reconstruction was taking place. The development of 5 out of 6 students' schemas in trigonometry was observed up to the AS assessment. This pilot study indicated that a refinement for the main study should be the inclusion of a more detailed indication of students' operational ability at the start of the A-level course so prior to the main study the new cohort of students' initial knowledge of trigonometry was investigated [Chapter 6]. Chapter 7 details the main study which followed the students through the full 2 year A-level study of trigonometry. The main study group of 17 students was much larger than the pilot study group and further refinements were required to deal with this. The group's teacher was asked to indicate students that he thought were at the top, middle and bottom end of the ability range within the group. 4 students were then chosen from his selection one from the top, one from the bottom and two randomly selected from the middle range. The teaching style of the pilot study teacher and the main study teacher differed considerably. The pilot study teacher focussed strongly on the forthcoming assessment and frequently urged the students to remember significant images or results [§5.6]. He confessed a preference for algebraic processes over geometric ones. The main study teacher stated that he considered trigonometry to be essentially a geometric subject and frequently linked algebraic representations to spatial representations during the course of his exposition [§7.5]. This difference had a significant impact on the resultant schemas of their students. The four students selected at the start of the main study were closely observed and asked to answer integrated questions talking through their thinking as they did so [§7.9]. The student identified as coming from the bottom of the ability range dropped mathematics after one year and there is a section on his reflections of the course [§7.8]. The research found evidence to support the theoretical frameworks of a difference in meaning assigned to terms such as sine, cosine and tangent as described by Gray and Tall (1994). It also found evidence that one group of students remained focused on processes whilst another was starting to build a cognitive construct of related images, formulae and procedures which gave them greater mental flexibility when problem solving [§7.9.3]. In addition there was evidence that those students who always used algebraic representations of sub-concept were at a disadvantage to those who could switch easily between algebraic and geometric representations. The research found that there was a qualitative difference in students own spatial visual representations and the quality of the spatial visual imagery [§7.9.1] provided good indications of differences in schema between the students [§7.9.2]. The indications of these differences of schema were provided by the concept maps [§7.10] and supported by the students' attempts at the integrated questions and their assessment results [§7.11].

Students' own reflections on their learning of trigonometry at A-level were then considered. The mastery (or lack of it) of the subject as perceived by the 3 remaining selected students and other students in the group is described here [§8.2]. The assessments and the pattern of study necessary for regular assessments were specifically considered in this study. The students were in favour of unit assessments on balance especially since it afforded them the opportunity to resit a particular unit assessment if they wanted a better grade [§8.3]. However the students also identified that the pattern of the course required two types of learning: one for assessment and one for depth. The more successful students identified the teacher's pedagogic style as the most significant factor in their understanding and problem solving capabilities [§8.3.3].

Chapter 9 analyses the differences in students' schemas and the specific influences of the teacher in their development. Different trends of schema development are identified and these appear to have an influence on problem-solving and assessment results. Those students whose schema alternated flexibly between algebraic and geometric representations had an advantage over those who focused exclusively on algebraic representations and processes. Students who focused predominantly on geometric representations also had an advantage over those who focused on algebraic processes. The key features of the teacher's pedagogic delivery and the effect this has on his students' understanding is shown. This study found that students were dependent on the teacher for the opportunity to interiorise sub-concepts and hence network them into a cohesive structural Object construction. They were also dependent on the teacher for clarity of meaning for the vocabulary commonly used in classroom discourse and for experience of problems that tested their understanding and problem solving techniques with respect to economy.

Finally the merits from a teaching/observational perspective of the theoretical frameworks described in chapter 3 are discussed. The conclusion was that the stages of development described by Sfard's description of concept formation through interiorisation, condensation and reification could be more easily identified in this empirical research than the developmental stages described by APOS or network theory. This gives it the advantage from a teaching perspective.

Chapter 2

The Development of A-level Mathematics

2.1 Introduction

This chapter starts with a brief outline of the history of A-level mathematics in England and attempts to describe the changes that have taken place in both its content and its format, [§2.2]. During the 1980's Education reform became one of the key areas on the political agenda of the Conservative government and radical reforms of the secondary education system for both pre-16 students [§ 2.2.2.1] and 16-18 year olds [§ 2.2.2.2] within England and Wales were introduced. During the 1990's the pace of change quickened [§2.2.3] but reports from the mathematics community expressing deep concern about the ill preparation of A-level students for degree courses [§2.3] and the perception that standards of assessment were declining [§2.4]. In addition the poor take up of mathematics and science courses by students and the decline in the number of specialist mathematics teachers was regarded as very serious [§2.5]. As the situation continued there was open acknowledgement of a 'Crisis in Mathematics' and the first of a series of independent consultations under Sir Ron Dearing was held to review the A-level curriculum [§2.6]. In 1997 the Labour party were elected into government with the pledge that education reform would be their first priority. In the decade between 1997 and 2007 there were 11 new Education Acts passed through government which cumulatively transformed the structure of education in general and A-level mathematics in particular. The most radical of these was Curriculum 2000 [§2.9] which partitioned the A-level into two one-year courses: the first year was now classified as the AS course and the second year the A2 course [§2.10].

2.2 A- level Mathematics.

The content and format of the A-level underwent little change during the 1960's and 1970's although the change that did take place was to have significant ramifications for Institutions of Higher Education.

2.2.1 The 1950's, 1960's and 1970's.

A-levels were first introduced in 1951 and were assigned 3 grades of achievement, Pass, competence at O- level standard, and Fail. In 1953 a further classification was added for the top performers called Distinction. In 1963, due to pressure from higher education institutions who claimed that grades were too broad, a norm- referenced grading system was introduced for grades A to E. This meant that the assignment of a particular grade for assessment was not awarded on the student having attained a specified mark but on how he or she compared with the other students of that years cohort. So the top 10% were credited a grade A, the next 15% a grade B etc regardless of the marks obtained.

The A-level syllabus, set by the universities and used by them as an entrance exam, was tailored to their needs [Savage, Kitchen, Sutherland & Porkiss, (2007), Gordon (2005) Sutherland & Pozzi (1995) etc]. The syllabus consisted of mainly Pure Mathematics and Mechanics. Its assessment consisted of two 3-hour examination papers taken at the end of a 2 year course. In the mid 1970's Statistics became an alternative option of study to Mechanics: Statistics and Mechanics, henceforth termed the 'Applied component of A-level' were allocated approximately 50% study time. Kitchen et al (2007) say that Universities were able to cope with the variation in students Applied background because:

> "Pure Mathematics remained solid and students continued to be generally well prepared with regard to study skills, problem solving skills and basic mathematical capabilities" (p2)

However because students with A-level mathematics applying to study mathematics at university did not necessarily have mechanics, some universities (Warwick, Newcastle etc) introduced additional support in mechanics and diagnostic testing.

2.2.2 The 1980's

A paradigm shift in secondary education occurred during the 1980's when a National Curriculum was introduced for all state schools in England and Wales for students aged up to 16 years old. This was complemented by new assessments to replace the O-levels and CSE examinations [§2.2.2.1]. Post 16, a core curriculum was designed, universities lost control of the A-level syllabus and assessment as new qualifications-awarding bodies were created and the content of the A-level was broadened. The A-level grading system was changed from a proportion scale to a mark scale and modular assessment was introduced [§2.2.2.2].

2.2.2.1 Changes to Pre-16 Secondary Education.

The National Curriculum was introduced into English schools in 1988 for students up to the age of 16 years and the General Certificate of Education (GCE) Ordinary level (O-level), originally designed for Grammar school students, was combined with the less academic Certificate of School Education (CSE) course and was henceforth to be called the GCSE. The assessment was designed to take into account all abilities by having 9 grades of achievement awarded (A* to G) instead of the previous 6 (A-E or 1-6 according to the awarding university body) for the O-level and 6 for the CSE. Candidates would study the mathematics GCSE at one of three levels: Higher, Intermediate or Foundation according to ability. The Higher papers provided for grades A*, A, B, C or U (Ungraded). The Intermediate papers provided for grades B, C, D, E and U and the Foundation papers provided for grades D, E, F and G. This meant that unlike other subjects those students studying GCSE at Foundation level had no possibility of gaining a C grade, which became widely regarded as a pass in the workplace and the outside world [§2.14] making it difficult for teachers to motivate the students taking this tier.

Traditionally O-level Additional Mathematics had been viewed by schools as a necessary preparation for most students intending to study the General Certificate of Education at Advanced level (A-level) mathematics. In the new National Curriculum Additional Mathematics, which had included the study of calculus from first principles, logarithm theory, the factor theorem and other algebraic techniques and trigonometry for angles in radians, was abolished. The knowledge and skills necessary to be awarded the new A and A* (“A star”, the maximum grade available) grades were now regarded as sufficient preparation for post-16 A-level study. Many teachers contested this claim and Sutherland and Pozzi (1995) compared the syllabuses of 1993 mathematics GCSE with those of 1983 O-level courses. They found that there had been an overall reduction in content in the move from O-level to GCSE and that this was particularly marked in the areas of trigonometry and algebra.

2.2.2.2 Changes to A-level Mathematics.

In 1983 the apparent loss of uniformity of A-level courses was addressed by the introduction of a common core for A-level subjects including mathematics to provide some commonality between A-Level syllabuses, ensure comparable standards and enable higher education and employers to have an idea of the scope and content of Advanced level studies [Easingwood, 1997]

In 1984, the Secondary Examinations Council advised that grade boundaries should be based on the partition of the mark scale rather than on proportions of candidates, in a move towards a criterion-referenced system. This meant that whereas previously the top 10% of students were awarded a grade A regardless of their actual examination mark the new system would award all students who achieved above a specified percentage an A grade. Examiner judgment was to be the basis for the award of grades B and E, with the remaining grades determined by dividing the mark range between these two points into equal intervals. This system was introduced in 1987 and remained in force until the introduction of the new curriculum in 2000 [§2.9].

From the mid 1980's there were increasing concerns about the steady decline in the number of students choosing to take A-level mathematics [§2.5] and the root problem was thought to be the A-level's traditional aim to be the foundation of Higher Education study of mathematics or mathematics-based courses such as engineering, computer studies or the sciences. The problem that resulted from this aim was that the content of the syllabus was considered too narrowly defined and was thus unappealing to students who might otherwise have chosen to study mathematics at A-level. In response to this the A-level was re-focused to be the final, and most advanced part, of the study of mathematics in school as well as an entrance examination for the universities. The main thrust of this argument as indicated by Reid (1991) was that policies on access for 16+ year olds should not be dictated by the ideals of Higher Education Institutions and England and Wales but must follow other countries in treating this stage of education as an integral part of the system as a whole and apply rules for transfer to university which are in the interests of all parties and not just teachers in Higher Education [§ 2.13].

The examinations were no longer to be set by the universities but by the new qualification awarding bodies of, respectively: Secondary Examinations Council (SEC), School Examinations and Assessment Council (SEAC), School Curriculum and Assessment Authority (SCAA) and currently the Qualification and Curriculum Authority (QCA). The new rationale was to broaden the scope of the A-level content to widen its future application and interest to students.

New modular syllabuses were designed and trialled. In 1989 the Advanced Subsidiary (AS) level GCE was first examined. It was designed to have restricted content i.e. not all the A-level topics would be included but this content would be studied to the same depth as those studying the full A-level, a vertical structure. This gave the Higher Education Institutions a problem determining how much the new AS was worth. Clearly it was worth more than a GCSE and less than a full A-level but its true value was unclear. The new AS also gave schools and colleges a problem in that separate classes, in addition to the A-level mathematics classes, had to be run to accommodate it. This put extra pressure on teaching resources and classroom accommodation.

2.2.3 The 1990's.

In 1993 the mathematics core was rewritten to accommodate the new AS. The syllabuses based on this reconfigured core were examined for the first time in 1996. However reports and research studies started to be published that signalled two particular problems. The first was the increasing perception from Higher Education tutors that students with A-level mathematics, even those with the best grades, had unsatisfactory knowledge or understanding of the subject. The second was the worrying decline in the uptake of A-level mathematics; a phenomenon which became known as 'The Crisis in Maths'.

2.3 Problems for Higher Education

In 1994 Fitzgibbon & Vincent published research that showed that mathematics and the sciences were considered difficult by students when compared to other subjects.

In 1995 the London Mathematical Society in conjunction with the Institute of Mathematics and its Applications and the Royal Statistical Society published '*Tackling the Mathematics Problem'*. This indicated the concerns of Higher Education admission tutors that many students lacked an essential mathematical facility, were unable to cope with simple problems requiring more than one step to solution, and had limited regard to the essential place of precision and proof in mathematics. Students who had good grades at A-level, they claimed, were unprepared for mathematics–based degree programmes and this was leading to high failure rates at the end of the first year of study. Mathematics, Science and Engineering departments 'appeared unanimous' in their perception of a qualitative change in the mathematical preparedness of incoming students, even the very best. [§ 2.16]

At the same time, the report states, the proportion of the A-level cohort who had opted for mathematics, science and engineering over the past decade was in decline and Higher Education departments were having difficulty recruiting sufficient numbers of students to fill the increase in spaces created by the expansion of the Higher Education sector. [§ 2.5 and §2.8]

The same year Sutherland & Pozzi (1995) published *The Changing Mathematical Background of Engineers* which stated that students were now being accepted on engineering degree courses with relatively low mathematics qualifications in comparison with ten years previously and, they also claimed, too many graduate engineers were perceived to be deficient in mathematical concepts and fluency. The report also highlighted the many pressures to reduce and simplify the mathematical content.

In 1996 Reynolds & Farrell published the book *Worlds apart* which was a review of international surveys of educational achievement involving England in which they described a crisis in maths under English State Education. Also that year The Standing Conference on School Science and technology, Society of Education Officers & the Engineering council published *Mathematics matters* (1996) which reported that Higher Education lecturers perceived that students were having difficulty forming mathematical models for engineering problems. UCAS (1996) published a report stating that only one third of students with A-level mathematics now went on to read mathematics, science or engineering and only 40% of engineering students had any A-level mathematics.

There was some evidence that the perceived change in calibre of students on entry to Higher Education courses was having an impact on the courses the institutions were able to provide. Kahn and Hoyles (1997) in a case study of single honours mathematics in England and Wales found that the range of mathematics had broadened away from traditional pure mathematics, the advanced content had been reduced and assessment had changed with more structured questions and more calculation at the expense of proof. They concluded that these changes had led to a reduction in the rigour and depth of degrees. This view appears to be supported by evidence from subsequent studies; Sutherland and Dewhurst (1999) considered the mathematical knowledge of undergraduates as they enter university across various mathematics-based disciplines such as physics, computer science, engineering etc, and concluded that the school curriculum did not adequately prepare students for a mathematics degree or for the mathematical components of other higher degree courses. The study pointed out that many universities are dealing with this by a variety of measures that included diagnostic testing, setting course entrance examinations such as Advanced Extension Awards (AEA) or Sixth Term Examination paper (STEP), redesigning first year mathematics courses, providing remedial centres of help, drop in work-shops and computer based mathematics learning centres. However, they noted that the effectiveness of diagnostic testing and computer based learning had not been systematically evaluated. [§2.16] Gordon, (2005) suggested that students seem weak in the fundamental concepts such as algebra manipulation.

2.4 Grade Inflation

A further cause for concern was the increase year on year since 1982 in the number of students gaining passes at A-level and in particular gaining the high grades that Higher Education Institutions sought. Studies associated with identifying the calibre of the students entering into university are frequently associated with evidence drawn from The A-level Information System (ALIS). This began in 1983 as a system for helping schools compare progress between their students and the students of other schools. Currently 1400 schools and colleges participate in the project which processes about half of the A-levels taken in the UK. An optional part of the scheme is the Test for Developmental Ability (TDA) which is offered free of charge to participants of ALIS.

A report for the Sunday Times by Tymms, Coe and Merrell (2005) found that grades had improved but there had been a decline in the TDA scores of the candidates between 1988 and 2001 which was most noticeable for Mathematics.

> From 1988 until 2004 the achievement levels have risen by about 1½ grades across all subjects on average. Exceptionally, from 1988 the rise appears to be about 3 grades for Mathematics. This could be due to this severely graded subject being brought more into line with other subjects.
>
> (Tymms, Coe & Merrell. pp14-15)

Their report concluded that A-levels have generally become more leniently graded through a combination of syllabus change, modularisation and alterations to the exam formats. In many ways, they suggest this has been a good thing since it allowed an increasing number of candidates to access education at higher levels, but it has meant that the very top levels of attainment have been removed from A-level. [§ 2.14]

The topics most frequently cited by universities as showing a decline in understanding and process are algebra and geometric subjects such as trigonometry.

2.5 Decline in the Student Uptake of A-level Mathematics; the Crisis in Mathematics

Since 1985 there has been an overall increase in the number of students studying A-levels. In 1985 20% were studying A-levels, in 1994 35% (Bell, Bramley & Raikes, 1997) and in the 3rd quarter of 2005, the Labour Force Survey of the Department of Trade and Industry (DTI), reported that 73% of 16 to 18 year olds were in full time education, most studying A-levels. There has not been the corresponding increase in the number of students studying A-level mathematics however, which has declined significantly. The pattern of decline is evidenced by the statistics: In 1989, 84,744 studied mathematics at A-level, in 1995 there were 62,188 candidates and in 2004 there were 52,788. (Gordon, 2005). The percentage of A-level applicants who opt to do mathematics has fallen from approximately 9% during the years 1993-2000 to 6.9% in 2004/5. Of these, up to 40% go on to take non-mathematics based subjects at university so the low number of students entering Higher Education to study mathematics-based subjects is causing great concern. Many mathematics departments have suffered a drop in funding to the extent that they have had to merge their mathematics departments with other associated disciplines; for example Nottingham Trent combined mathematics with computing and Bangor combined mathematics with engineering to form the new subject of ‘informatics’. Some have had to consider a severe reduction or, as in the case of the Universities of Hull and Essex, a complete deletion of their mathematics departments. A survey undertaken by the London Mathematical Society in 1995 led by James Smith stated that 25% of mathematics departments were under threat. A further survey by Middleton in 2001 indicated similar findings with departments reporting the loss of service teaching, the cessation of single honours mathematics and down sizing or embedding into larger groupings.

A postnote to a paper by the Parliamentary Office of Science and Technology called *Strategic Science* (2007) reported that since 1999, 5 mathematics departments had closed and many of the remaining 46 were under pressure. This means that not only is there a significant reduction in the number of people taking mathematics degrees, but there is a decline in the quantity of service teaching which mathematics departments offer to other disciplines. A further consequence is that it leads to fewer mathematics graduates and hence fewer mathematics graduates becoming teachers which has serious implications for the quality of learning for future A- level candidates.

In addition, many universities had introduced mathematics entrance examinations [§2.3 and § 2.16]. The growing concerns about standards at A-level and the decline in its attractiveness to candidates led to a review of 16-19 education by the School Curriculum and Assessment Authority (SCAA) and The Office of Standards in Education (OFSTED), headed by Ron Dearing in 1996.

2.6 The Dearing Review

The Dearing review in 1996 acknowledged the AS was not as successful as had been hoped. Take up was low and was further declining especially in mathematics. Because it studied topics to the full depth of A-level, it had been found to be relatively too demanding for many learners and the judgement was that it was failing to achieve its main purpose of increasing breadth of study in post-16 education. The report suggested a reformulated AS. Instead of covering half the A-level syllabus to full depth the new AS would cover the syllabus breadth and content appropriate to one years study post GCSE; a horizontal structure. The AS and A-level could hence be co-teachable. The AS assessment should be graded on an A - E scale like the A-level and, for the purposes of Higher Education recruiters, the AS component should be weighted as 40% of the total marks of the A-level. The A-level should be reviewed again to ensure progression from AS to A-level and the content of the new core should be specified in greater detail to give more guidance to examining boards of what has to be included in their syllabuses and to give Higher Education tutors a clear indication of the work covered by all students. It also recommended a continuing move from linear to modular examinations.

The report acknowledged that mathematics and science were more difficult than other subjects and recommended the other subjects be levelled up.

2.7 An Interim Response to the Dearing Report

In February 1997 a new core was agreed by the Secretary of State and examination boards were asked to submit first drafts of syllabuses and specimen examination papers by early June as an interim response to the Dearing report. The AS was changed to 50% of the total content whilst a The Pure mathematics component was to be 50% of the AS and A-level. At least 25% of the total assessment had to be without the aid of a calculator. This was a response to the charge that students were over-dependent on calculators but led to fears that this would result in less satisfactory examination questions in topics such as trigonometry and numerical methods. There was more Pure material than previously. Sections on proof (for AS and A-level) and Vectors (for A-level only) were included and the Mathematics of Uncertainty was removed as it did not fit easily with the other pure components and overlapped with material in the statistics modules. For the first time a list of required background knowledge was specified to candidates.

There followed a series of consultations with teachers, lecturers and the mathematics community called Q*ualifying for Success* which were held over the 2 years following the Dearing report. However, pressure continued to mount over a perception of grade inflation in A-level mathematics and the continued decline in standards.

2.8 Further Reports on the Crisis in Mathematics

In September 1997 Bell, Bramley & Raikes from The University of Cambridge Local Examinations Syndicate presented a paper at the British Educational Research Association Annual conference entitled *Standards in A-level 1986-1996.* It was the result of an investigation into the ways in which the A-level had changed and an examination of a claim that grades were being awarded differently. Their conclusions were that change within the syllabus is natural as the relative importance of component skills and knowledge change over time. With regard to grade inflation they concluded that there was no change for the grading for the A and B boundaries and a small increase in grading standard at the grade E boundary. This, they concluded, meant that the improvement of the grade distribution must be explained by an improvement in the candidates. They further noted that there was considerable evidence that the standard of mathematics attained by English 18 year olds who specialise in mathematics was higher than in other countries such as Germany and Japan.

In contrast a report *Teaching and Learning Algebra pre-19* published by the Royal Society and the Joint Mathematics Council of the United Kingdom (1997) concluded that the vast majority of non-mathematics pre-university students in Germany and France are expected to be competent with manipulative algebraic skills equivalent to those experienced by students studying a single A-level mathematics within England and Wales. They also concluded that what had been algebraic content within an O-level course was now being taught at A-level and consequently they recommended a bridging course between GCSE and A-level mathematics. Although they recognised the difficulty in comparing A-level papers with those of the past due to the variation in question papers from different boards, they noted that whilst universities had adjusted their courses to use the first year as a levelling course to take account of the differing backgrounds of their students, many had initiated four year degrees.

In July 1998 the Royal Society published *Mathematics Education pre 19* which noted the difficulty of recruitment to teacher training courses in mathematics and science and the lack of highly qualified specialists entering the profession. It recommended the matter be tackled with utmost urgency at a national level. It also recommended increasing the focus on key mathematical concepts by reducing the breadth of content in A-level syllabus specifications for mechanics and statistics. The report concluded by noting that any revision of the syllabuses should involve Higher Education bodies since they are the major users of A-level mathematics students.

In 1998 the new interim syllabuses agreed after the Dearing report was introduced by schools and colleges for assessment in the summer of 2000.

In 1999 The Engineering Council together with the Learning and Teaching Support Network (LTSN), The Institute of Mathematics and its Applications, and The London Mathematical Society published *Measuring the Mathematics Problem,* which reported on the serious decline in students' mastery of basic mathematical skills. The first two recommendations of the report were that students embarking on mathematics-based degree courses should have a diagnostic test on entry and that prompt and effective support should be available to students whose mathematical background is found wanting by the tests. [§ 2.16].

The result of the Q*ualifying for Success* consultations was the development and implementation of Curriculum 2000 (NC2000).

2.9 Curriculum 2000

One of the conclusions of the Q*ualifying for Success* consultations was that the post-16 curriculum in England was too narrow and inflexible and that it had to be adapted to enable 16 and 17 year olds to compete with their peers in other European countries. It was concerned that most of our European competitors offered their young people broader programmes with a much more demanding schedule. The reforms introduced in September 2000 were intended to encourage young people to study more subjects over two years than had been the case previously, while also helping them to combine academic and vocational study [QCA 1999]. The traditional 2 year course would now be split into AS for the first year and A2 for the second year. Units would be designated as AS modules or A2 modules Students were henceforth to study 4 AS levels in the first year of the sixth form and in the second year they should continue 3 of their chosen subjects on to A2 and possibly study a further AS level course.

2.10 Changes in the Structure of Assessment

The reforms included plans to provide world class tests to stretch the most able students and give a clearer indication of their abilities. In addition, it was intended that students should develop their competence in the key skills of communication, application of number and information technology (IT) which could accrue UCAS points, and Problem Solving, Working with Others and Improving Own Learning and Performance which could not. Major structural changes to the curriculum in England and Wales were implemented which introduced assessments of key skills at levels 1, 2 and 3. In the new national qualifications framework a GCSE grade of D, E, F or G would indicate level 1 key skills. A GCSE grade or A*, A, B or C would show level 2 key skills. Mathematics beyond GCSE, but pre-university level, was classified as a level 3 qualification.

The A-levels would be modular and each A-level would be comprised of 6 modules in total. Candidates would be required to sit 3 AS modules, 2 of which must be Pure, to qualify for the AS and 3 more A2 level modules, including 2 further Pure units, to qualify for the A-level. The content of the A-level was largely unchanged but there was more emphasis than previously on correct notation, algebraic manipulation, logical deduction and proof. There were to be more multi-step problems and fewer structured questions. At least 25% of the examination would require the use of a simple (i.e. non graphic, and non-programmable) scientific calculator only [Abramsky (2001)]

Within England there were 5 approved syllabus specifications for AS/A-level, and 2 additional ones for Wales and Northern Ireland. The assessment, by standard examination, was designed to be 8-9 hours for the A-level and 4-4.5 hours for the AS.

2.11 Curriculum 2000 Results.

After Curriculum 2000 reports from schools and colleges were negative. Phrases such as “sweat shop sixth forms” were commonly used and teachers complained that they were moving away from meaningful education under pressure of continual assessment. In addition the 2001 results for the new AS qualification were deeply disappointing when it was revealed that almost one third of students had failed the new examination. The perception of students that mathematics is harder than other subjects was reinforced and the take up rate amongst candidates fell further the following year. Curriculum 2000 was heavily criticised by teaching professionals who claimed that the work load for the AS was too demanding and the overall effect had been to damage mathematics post-16 (Hodgson, & Spours, ,2002;Porkiss,2005;Gordon 2005).

The first A2 results under Curriculum 2000 were released in August 2002. The number of students finishing the full A-level had fallen by 12,000 and university applications had fallen by 10%. QCA were immediately asked to revise the syllabus and in October 2002 new syllabuses were published to be taught from September 2004.

2.12 QCA 2004 Syllabus.

The 2 AS Pure modules, P1 and P2, were reorganised into 3 Core modules, C1, C2 and C3. The award of AS mathematics would be given after successful completion of units: C1, C2 and a third Applied module: either Mechanics (M1), Statistics (S1) or Decision (D1) and the A2 award would be given on assessment of the C3 and C4 modules with in addition, a second module chosen from: Mechanics 2 (M2), Statistics 2 (S2) or Decision 2 (D2) or some combination of the applied modules such as S1, M1 or D1, S1 etc. So the ratio of Pure mathematics to Applied mathematics changed from $^1/_2$:$^1/_2$ to $^2/_3$:$^1/_3$ with no increase in the amount of Pure content. The use of calculators was further reduced.

2.13 Assessment Objectives.

Curriculum 2000 had specified *key skills* that would be incorporated throughout the study of core mathematics for students. These included developing understanding, coherence and mathematical progression; developing abilities to reason logically, and recognise incorrect reasoning and to take increasing responsibility for their own learning and the evaluation of their own mathematical development.

These were retained in the new syllabus [QCA 2004] with the addition of a specified percentage assigned to each skill for the purpose of the assessment.

The new 2004 syllabus started being taught for AS assessment in 2005 and A2 in 2006 with the approval of teachers who thought that covering the work would be more realistic in the available time frame (Porkiss, 2005).

During this research the pilot study was carried out according to the 2000 syllabus whilst the main study was carried out during the introduction and implementation of the 2004 syllabus.

2.14 The Tomlinson Report and the Smith Report.

Since 2004 further changes have been implemented. In September 2002 another independent inquiry was commissioned by the government chaired by Mike Tomlinson to consider the effects of Curriculum 2000 and make recommendations for reforms. This was published in 2004 under the title *14-19 Curriculum and Qualifications Reform* and recommended a radical overhaul of the 14 -19 curriculum and assessment structure and the introduction of diplomas to incorporate A-levels and vocational qualifications. It further recommended that material be included in the course and assessment to stretch students and two further grades of achievement A* and A** should be available for the most able.

At the same time another inquiry headed by Adrian Smith was commissioned to specifically consider the situation in mathematics. In 2004 the results of the Smith inquiry were published as *Making Mathematics Count* in which it identified 3 major areas of concern: the shortage of specialist mathematics teachers, the failure of the current curriculum, assessment and qualifications framework to meet the needs of students or to satisfy the expectations of employers and Higher Education Institutions, and the lack of resources to support mathematics teachers. It attributed the possible factors of the decline in student uptake to be: the perceived poor quality of the teaching and learning experience, the perceived relative difficulty of the subject, and the failure of the curriculum to excite interest and provide motivation. The report recommended a two tier GCSE rather than the current three tier one so that all students have the possibility of attaining a C grade [§2.2.2.1]. In addition extra courses should be provided for the best mathematical talent at GCSE and A-level. With respect to A-level the report stated that Curriculum 2000 has been a disaster for post-16 study and the AS/A2 split has not worked. Students could not cope with the material within the laid down timetable and the pass rate had dropped from 90% to under 70% which has had a detrimental effect on the image of mathematics and further decline in the uptake. It also expressed concern about the nature and frequency of assessment for AS/A2. It recommended that a post be established within the Department for Education and Science (DFES) for special responsibility for mathematics. It further recommended that the assessment of 6 units introduced by Curriculum 2000 be reduced to 4 larger units to reduce the assessment burden, and the costs and timetabling difficulties. In order to address the falling numbers of mathematics students at A-level and university, the report recommended financial incentives for students and teachers. [Smith, 2004].

In 2005 the government published a white paper implementing the recommendations of the Smith report and appointed of Professor Hoyles as 'Maths Tsar'. The syllabus has been revised again and in 2008 the new A-level syllabus with 4 assessment units will begin to be taught in schools and colleges. It will include a new grade, A*, and material in the A2 examination to 'stretch and challenge' the best students. [AQA, 2007].

The continual changes to the A-level mathematics syllabus did little to address the claim by Higher Education Institutions that students were entering Higher Education with a poorer understanding of the subject and its rigours. Students still held the perception that mathematics was one of the hardest A-levels and the shortfall in mathematics and science teacher recruitment resulted in a severe shortage of mathematics specialists in schools. The impact of this shortage was leading to a change in teacher profile and the quality of learning that was taking place in the classrooms. This flux in the course style and content as problems were identified and addressed was the background to the research sample students' study of A-level mathematics. The quality of students' understanding of the subject was an important issue in the public debate. This research provided and opportunity to hear the students' voice and this lead to the research objective to investigate students own perceptions of their learning experience. [§4.2]

2.15 Change of Teacher Profile.

The shortage of graduate mathematics teachers has led to an increase in the number of students in mathematics A-level classrooms. Research documented by Hirst (1991, 1996) showed that the class size for A-level mathematics, pre-1986, had a mean of around 10 and a maximum of about 15. By 1996, class sizes had significantly increased with a mean between 15 and 18 students. In 2003 the average class size was 17 [Ofsted, 2003]. With the increased use of technology the teacher was no longer the only medium of delivery of content or understanding. Computer Assisted Learning (CAL) became universally used in A-level classrooms. Though there remains debate about the extent to which this is productive in the long term, research by Kadijev and Haapasalo in 2001 provided evidence that it can provide a link between the procedural knowledge and conceptual knowledge [See Chapter 3] that defines mathematical education.

Gordon (2005) points out that another change in the classroom is associated with the teacher profile. In 1996 only 15% of teachers were under the age of 30 and 63% over 40 [Porkiss (2000)]. Despite financial incentives to attract mathematics and science graduates, including paying off student loans and a bonus of £4000 initially (currently £6000) on starting teaching, recruitment is still well short of its targets. Where this has attracted new mathematics teachers, many are qualified in other disciplines and have undertaken a retraining course in mathematics to gain Qualified Teacher Status (QTS) [Smith, 2004].

2.16 University Entrance Tests.

The issue of the role of A-levels in assessing candidate's suitability for Higher Education courses is still unresolved. An independent review, headed by Professor Steven Schwartz, was commissioned by the Secretary of State for Education and Skills to investigate the options that English Higher Education Institutions should consider when assessing the merit of applicants for their courses. In September 2004 the Schwartz report, *Fair Admissions to Higher Education: recommendations for good practice,* was published. It recommended that where possible, a test should be devised that universities and colleges should use to predict undergraduate potential that is not reflected in level 3 (A-level) results. The report called for an evaluation of the tests currently set by Institutions of Higher Education and trials for a National Entrance test.

In September 2005 the National Foundation for Educational Research began a 5-year trial of university entrance tests called the Scholastic Assessment Tests (SATS) to investigate if they were a better predictor of able students than A-level results. Students take the test as they study AS courses and the results will be compared to A-level results and to eventual degree outcomes. The introduction of nationwide SATs indicates a deepening disconnection between A-level study and entrance to Higher Education Institutions.

2.17 Trigonometry at AS/A2 Level.

In my experience tutoring A-level mathematics, trigonometry has frequently been the topic that has been most problematic to students. Tutors see students from different schools that have been exposed to different teaching styles although the material they study is set by the syllabus. Indications of the confusions that students have are evidenced by comments such as:

"Are we talking about triangle trigonometry or circle trigonometry here?"

(Yr 13 student)

and:

"I used to understand it when it was just triangles but now I don't know where to start"

(Yr 13 student)

Sometimes students appear to be confused by the multiplicity of representations and how they are interconnected as evidenced by these comments:

"I don't understand radians – I can only do trigonometry in degrees"

(Yr 13 student)

"What is sine exactly? I thought I knew but now it is so confusing."

(Yr 12 student)

"I hate trigonometry. There is just so much to remember: all the diagrams and formulas. I never know which one to use."

(Yr 13 student)

This personal experience of students' specific problems with the study of trigonometry at A-level over and above other topics on the syllabus was the reason for this piece of research.

Trigonometry has been part of the A-level since its inception. As a set of functions of angle it is important when studying triangles and modeling periodic phenomena. The topic is developed through ratios of two sides of a right triangle containing the angle, to, more generally, ratios of coordinates of points on the unit circle, to, more generally still, infinite series, or equally generally, solutions of certain differential equations. The next section lists briefly the trigonometry component in each of the AS/A2 modules.

2.17.1 Trigonometric Content in the A-level Syllabus from 2005.

The full specification of the trigonometric content of A-level is provided in the Appendix. However over the four core units of A-level it is worth noticing that there is no trigonometric content within C1. Within C2 to C4 the content ranges from an exploration of the defined aspects of sine cosine and tangent (later in C3 secant, cosecant and cotangent and of arcsin, arccos and arctan) in surd form and radian equivalents, this is presented through a unit that include angle and radians. Though initially developing as the notion of a function of an angle by C4 notion of differentiation of a function is considered.

C1: There is no trigonometry in this unit

C2: Explores the idea of a function of angle. It moves from triangle to angle, to angle measured in radians, the graphic representations of the sine, cosine and tangent function with the properties, symmetries and periodicity of each, the unit circle and its usefulness, the representation of sine, cosine and tangent in surd form for 30°, 45°

and 60° or radian equivalent and the triangles that can be used to determine each of them via Pythagoras theorem. It then considers the nature of various transformations of the graphs and the identity for tan x and $\sin^2 x+\cos^2 x=1$ and combines all this knowledge to derive the solutions to simple trigonometrical equations, including those with a quadratic format, within a given interval.

C3: The course moves on to 'knowledge of the relationships between sine cosine and tangent, understanding of their graphs and appropriate restricted domains. Knowledge and use of the $\sec^2$ and cosec^2 identities. Knowledge and use of the double angle formulae for sin (A ± B), Cos (A ± B) and tan (A ± B) and for a cos x +b sin x in the equivalent forms of r cos (x+a) or r sin (x+a). Candidates should be able to prove simple identities such as cos x cos 2x + sin x sin 2x = cos x'.

In the development of calculus section we have the differentiation and integration of sin x, cos x and tan x, differentiation of cosec x, cot x and sec x. Skill is expected in the differentiation of functions generated from standard forms using products, quotients and composition such as $2x^4 \sin x$, $\cos x^2$ and $\tan^2 2x$.

C4: The integration of standard trigonometric functions is covered: 'such as sin 3x, $\sec^2 2x$, tan x and candidates are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$ $\cos^2 3x$. There is also differentiation of simple functions given parametrically or implicitly.

The full specification for the course post 2004 studied by the students in this research study is under Appendix 1.

2.18 Summary.

The development of A-level mathematics since its inception has seen a change in purpose, content, format and assessment procedure.

Initially its purpose was to provide a route to the study of mathematics-based courses at university. The syllabus was determined by Higher Education Institutions and the examinations were set and assessed under their control. The grades awarded to students reflected their achievement in comparison to that of the other students, in that year's cohort.

During the 1970's the traditional format of Pure and mechanics components was expanded to include statistics. The number of syllabuses available increased and the lack of uniformity in their content led to the re-introduction of a core curriculum in 1983. In 1984 the first Qualification-awarding body was established to oversee course content and assessment. The A-level was re-focused to be the final, and most advanced part, of the study of mathematics in school as well as an entrance examination for the universities. In 1989 modular assessment was trialled and AS mathematics was examined for the first time and in 1993 the core curriculum was rewritten to include the study of AS maths.

From the mid 1990's published research indicated that:

- Students considered maths and science subjects to be more difficult than other subjects despite mathematics grades being inflated to reflect levels of achievement comparable with other subjects (Fitzgibbon & Vincent,1994: Tymms Coe & Merrill,2005).

- The proportion of the A-level cohort opting for maths- based subjects was in decline and the proportion of these going on to study maths- based subjects in Higher Education was decreasing. [London Mathematical Society (LMS) & the Institute of Mathematics (IM) (1995), UCAS (1996), Gordon (2005)]

- Students lacked essential mathematical faculty (LMS&IM, 1995; Sutherland and Pozzi,1995: Standing Conference on Schools Science and Technology (SCSST),1996]
- Higher Education Institutions were having to change their courses and had introduced entrance exams to respond to this phenomenon (Kahn & Hoyles 1997; Sutherland & Dewhurst,1999)

- Maths departments in Higher Education Institutions were under pressure to reduce, amalgamate or close altogether due to the lack of student take up of maths-based subjects (LMS, 1995: Middleton, 2001; Office of Science and Technology (OST), 2007).

These concerns led to a review of 16-19 Education by Ofsted & SCA _ the results of which were published in 1996. It recommended a reformulated AS course, a more detailed specification of the new core curriculum, a move to modular assessment and recognised maths as more difficult than other subjects. It suggested other subjects be levelled up. The result of the review led to the introduction of Curriculum 2000 which introduced the separation of A-level maths into AS and A2 courses. Each course would entail the study of three modules, two pure and one applied, which would be separately assessed. The initial results of Curriculum 2000 were disappointing showing a high drop out rate from the A-level course and applications to study mathematics- based subjects in Higher Education dropping by 10%. The response by QCA was to revise the Pure content of the AS/A2 level renaming it Core maths. In 2004 the Shwartz report recommended that the proliferation of Higher Education entrance examinations be standardised into a single test.

The aim of modular AS and A-Level mathematics is to provide greater flexibility and to ease the burden of pressure that was a criticism of the traditional model that had a single assessment via two written papers at the conclusion of the two year course. By examining twice yearly it allows for one or two units to be studied and then assessed prior to moving on to the next unit.

In the first year two core components and one applied component must be studied and these are at AS level leading to an award of AS mathematics. In the second year two further units of core mathematics, which are dependent on knowledge of the AS core modules, and another applied unit are studied. These are at a higher level of mathematical knowledge and reasoning known as A2 level. Modules may be retaken without restriction and the best mark contributes towards the final grade. The final grade at A- Level is the sum of the best marks for the six modules. Synoptic assessment is expressly included to address the degree of drawing together that candidates have in knowledge, understanding and skills learned in different parts of the course. Trigonometry is studied at units C2, C3, C4 and in M1. Functions are defined at C3 though the word is used from the outset of C1.

In the face of this considerable upheaval to the A-level system in general and the mathematics A-level in particular, one reason for undertaking this study was to discover how students assess their learning experience and what they have, in fact, learned. To what extent does GCSE mathematics prepare the students for A-level mathematics currently and moreover to what extent have the stated core skills of developing understanding, coherence and mathematical progression been achieved. These were issues that were fundamental to the planning of this research of a group of students as they study of trigonometry.

This study was undertaken in the years 2004-2006. The pilot study group studied the Pure modules and the main study group studied the new Core modules. The trigonometry component in Pure 1 and Pure 2 studied by the pilot study group was covered in Core 2 and Core 3 in the revised syllabus. Ultimately the trigonometry component of the Pure course was covered in modules C2, C3 and C4.

Having considered the way in which the content of the A-level syllabus has been modified since its inception and highlighted some of the reasons for the change in and structure we now turn to consider broader issues on the nature of mathematical understanding and on trigonometry in particular.

Chapter 3

Research Literature

3.1 Introduction

This study seeks to discover how the understanding of trigonometry is transformed into a concept of trigonometric functions by a group of students studying A-level mathematics. The perspective taken is that knowledge is an individual's construction and thus the focus of the study is the student's cognitive organisation of new information into a coherent structure of links, axioms and procedures that together construct a concept of trigonometric function.

This chapter seeks to outline the constructivist viewpoint and provide definitions for the vocabulary that is commonly used in connection with mathematical development [§3.1.1]. It will then set out the main theoretical paradigms that are under consideration for the purposes of the study.

The study of trigonometry involves the use of spatial representations and diagrams to a greater degree than most other topics and within section 3.2 a theoretical position on the qualitative value of types of imagery commonly used in pedagogy is considered.

There is a section on some of the research that has been undertaken with students as they study trigonometry[§3.3] and finally there is a short section on the theoretical perspective of the cognitive changes that take place as students move into the study of mathematics at an advanced level [3.4].

3.1.1 Constructivism

Piaget's study of the cognitive development of children drew him to the conclusion that knowledge is actively constructed by each individual. He argues that what is crucial to intellectual development is a shift in focus from the properties inherent in real world objects as actions are applied to them, to a consideration of the actions themselves and the effect they have on objects. Through this shift in focus, knowledge is derived from the actions which the individual performs leading to a constructed abstraction of the action process. In his book on Genetic Epistemology, he wrote:

> The abstraction is drawn not from the object that is acted upon, but from the action itself. It seems to me that this is the basis of logical and mathematical abstraction.
>
> (Piaget, 1970, p16)

These abstracted actions may then become items of thought themselves supporting the establishment of a hierarchical cognitive structure that ultimately forms a mathematical concept/object. This initial concept then informs perceptions of further mathematical activities so that:

> The whole of mathematics may therefore be thought in terms of the construction of structures...mathematical entities move from one level to another...until we reach structures that are alternately structuring or being structured by stronger structures.
>
> (Piaget, 1972, p.70)

Constructivism as defined by Piaget is not the only theoretical framework of cognitive development. Alternative paradigms of knowledge development have been proposed. One such alternative is behaviourism which conceives learning as the transmission of knowledge from teacher to students (Even & Tirosh, 2001) but as the body of epistemological research has grown, the evidence appears to support the constructivist framework and since the mid 1980's this has been the prevailing, though by no means unanimous, developmental framework of the research community.

The implications of the constructivist paradigm may be summarized as:

> The view of learning as active construction implies that students build on and modify their current ways of mathematical knowing
>
> (Cobb, Yakel & Wood, 1992, p6).

The modifications that students make are personal and as such vary from one individual to another. In addition the modifications are made when the student is ready to make them. It is difficult to know what stimulates this cognitive development or indeed, what the timeframe for the reconstruction might be. Piaget said that it depends on the individual learner. Jaworski says that:

> A constructionist view of knowledge is that it fits experience. If the experience changes, the knowledge may need to be modified
>
> (Jaworski, 1993, p14).

From the perspective of mathematical thinking, however, this leads to the question: what is knowledge? For example can a memorised procedure be termed knowledge? Clearly there is a qualitative dimension to knowledge and knowledge with understanding is superior to knowledge of a procedure that has been learned by rote. The issue remains though: can a procedure that has been memorised but not understood be regarded as mathematical knowledge? Some researchers (Schoenfield, 1992; Hoffman,1989; Romburg & Carpenter,1986; Resnick,1989), believe that all mathematical knowledge is relevant, even the perfunctory, since it is the applications of mathematics that gives it power:

> One's mathematical knowledge is the set of mathematical facts and procedures one can reliably and correctly use.
>
> (Schoenfield, 1992. p3).

Mastering facts, formulae and procedures is indeed an important component of mathematics but mathematics teachers are aware that it is not necessarily the students who are best able to remember algorithms and formulae that have the greatest understanding of the underlying ideas. Teachers know that some students appear to understand the fundamental idea to the extent that they can short cut or circumvent lengthy procedures with alternative methods that are faster or more cognitively economic. These students have a cognitive flexibility that may be denied to those who have concentrated on learning mathematics by memory. It is evident that in the classroom there are two types of knowledge being learned by the students: one that is dependent on memory and one that is dependent on understanding.

3.1.2 Understanding Understanding

Skemp (1976) drew a distinction between the two different ways of understanding mathematics and termed them Instrumental understanding and Relational understanding. Instrumental understanding is considered to be the knowledge an individual has that can be described as 'rules without reasons'. Such knowledge is considered to be purely operational and relies heavily on memory. Relational understanding on the other hand is the knowledge an individual has when he or she understands why the rules apply, the limits of the context in which they apply and the ability to link knowledge or ideas to each other enabling the individual to compare and contrast them. People who have relational understanding appear to be able to condense knowledge so that its recall appears effortless and their means to a solution is usually far more cognitively economic than those who have only instrumental understanding. All mathematical activities may be conceived initially at the instrumental level but a reliance on remembering leads to an overburdening of the memory as more and more rules and procedures are encountered (Gray & Tall, 1994). By condensing instrumental procedural knowledge and connecting this knowledge to other procedural knowledge or known facts, relational thinkers seem to be able to construct for themselves some cognitive space that allows for further knowledge to be accumulated. In addition instrumental understanding requires a recognisable point at which to start a memorised procedure and finishes at a predetermined ending point. The individual has little or no control over the process. Relational understanding allows the individual to start in the middle of the procedure if desired and work backwards or forwards to an end that is determined by the individual. They can also start one procedure then switch to another if that is more cognitively economic.

> Learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point.
>
> (Skemp, 1976. p25)

By organising knowledge in an overall schema, it becomes less burdensome for the memory but, it also facilitates greater cognitive flexibility, for example, recognising that it may be easier to use an alternative procedure. Skemp pointed out that many mathematics teachers are instrumental thinkers and their teaching tends to focus on algorithms and formulae. This has implications for student understanding which is discussed below.

The understanding of mathematics is therefore qualitative. Relational thinkers are able to condense knowledge within a schema that includes facts, formulae and procedures and use these flexibly, whereas instrumental thinkers concentrate on applying one or more learned procedures until the solution is reached. There is a more limited ability to switch to an easier alternative or image which can provide a solution more directly.

3.1.3 Schemas

A schema may be generally defined as a system of concept images whose essences are structured into a unified, stable entity. (Skemp, 1971, 1987; Tall & Vinner,1981; Dubinsky 1991; Tall 1994,1995; Thompson, 1994). Skemp defined a schema as:

> A conceptual entity with its own name that has beyond the separate properties of its individual concepts three functions: it integrates existing knowledge, it acts as a tool for future learning and it makes possible understanding.
>
> (Skemp, 1987, p4)

This idea of schemas proposes that they are all-encompassing conceptual entities.

Other researchers in the field (e.g. Nickerson, 1985; Hiebert & Carpenter, 1992) argue that a schema is any form of knowledge or understanding of a mathematical idea including instrumental knowledge of a procedure, an isolated fact or a meaningless formula whose purpose has been forgotten. They argue that each student has a schema of some kind even if it is very basic. It might be argued therefore that the difference between high and low achievers is not so much whether they have or have not a schema but in the quality of the schema and, by implication, the quantity of the content and interconnections it contains (Steffe, 1996; Davis, 1984; Greeno, 1983; Sfard, 1991). Since schemas are cognitive constructions derived from personal experience they might be expected to vary from one individual to another and, through the passage of time, to become more varied. This is evidently the case with individuals who are high achievers. Mathematics teachers encounter, on a daily basis, students who are creative in their solution processes and comments such as “I would never have thought of doing it that way” or “That way is much quicker” abound in classes of high ability students.

Whilst empirical evidence is commonplace for the creativity and variation of high achievers with rich, flexible schemas, what is the nature of the schemas of low achievers? These students might be expected to think purely in an operational capacity. If their schema is nothing more than a rote-learned procedure then it might be expected that the schemas of all the students in a low ability group would be similar both in content and quality; after all they have all learned the same procedure. However, a study by McGowan (1998) found that even within a group of students who had historically been poor at mathematics there was a qualitative and quantitative spectrum in their schemas. Her results supported the theory proposed by Davis (1984) that success in the subject is a result of incorporating new information into an existing schema that is stable. Those students who progressed least showed clear signs of not having a stable schema and were inclined to repeatedly overwrite all past learning in the face of new information thereby denying themselves the opportunity to accumulate knowledge.

Though the quality of a schema is defined by Hiebert & Carpenter (1992) as the number of connections it contains, Tall (1995) argues that it is not the number of connections but the strength of these connections that is important. Though the schemas of the students in McGowan's study were distinguishable by their stability they varied from those that were sufficiently stable to incorporate new information to those that were so threadbare that they could hardly be considered to be schema at all — more simply a collection of unrelated procedures and facts.

This leads to the question of how mathematical schemas are constructed by students. In response to this question three different theoretical paradigms have arisen: Reification Theory (Sfard, 1991), a theory of encapsulation and its associated construction (Dubinsky, 1991) and Network Theory (Nickerson, 1985).

3.1.4 Reification

A mathematical procedure is initially learned as a step by step formulation that is sequential. Davis (1984) observed that as a procedure is practised, the procedure itself becomes an entity or object of scrutiny.

Sfard (1991) introduced the notions of 'operational' and 'structural' conceptions of mathematics and identified a three step process from operational knowledge of a procedure to the formation of a structural mathematical concept. This three step process she termed Interiorisation, Condensation and Reification.

She argued that knowledge of a procedure becomes interiorised by an individual, possibly through the promotion of some sort of mental image of the process that simultaneously attributes it with a beginning, an end and a purpose. Once this happens the process is then mentally condensed and linked to other condensed processes which ultimately are reified into a unified, complex, structural entity called an 'abstract object' (p27). Further learning of the activity may be incorporated into the structure creating new links to various different aspects and representations that are complementary. The object structure allows for the inclusion of additional, more sophisticated procedural manipulation which also has the possibility of being reified. As more and more facts, procedures and images are subsumed by this object schema it becomes increasingly more stable.

Throughout an individual's study of mathematics different concepts are reified and these are, in turn, connect to each other resulting in further interiorisation and condensation so that higher order concepts are developed with a range of operational procedures subsumed within them. The operational and conceptual understandings are complementary. Sfard stresses that a good understanding of operational, or instrumental, knowledge is a prerequisite for reification. However not all processes are reified and students who have reified some mathematical ideas into object schemas may not have reified others.

3.1.5 A Theory of Encapsulation

Another theoretical framework that describes the transition from procedural understanding to conceptual understanding is that of encapsulation which is associated with a four stage development. First introduced by Dubinsky (1991), it was later revised within Breidenbach et al (1992) and Cotteril, Dubinsky, Nichols, Schwingendorf, Thomas & Vidakovic (1996). This framework is more usually associated with the acronym APOS (An *Action* that is repeated turns into a *Process* which is then encapsulated as a mental *Object* to later become part of a mental *Schema*. Schema here is used to describe the complete edifice of cognitive knowledge or understanding in a similar way to Skemp's use of the word).

An Action is any physical or mental transformation of objects (such as numbers) to obtain other objects (or numbers). It includes recalling a fact from memory or may be a multi-step process but it has the characteristic that at each step the next step is triggered by what has come before rather than by the individual's conscious control of the transformation. The response to the stimulus may be automatic and mechanical with no reference to the intended objective. When and if a student comes to reflect on the action, he or she may be able to establish conscious control over it, interiorizing it, and at this juncture, the Action becomes a Process.

A Process is a construction of an Action that the individual student controls. It is rich in links that allow for alternative processes to be used, and remembered facts that can partially circumvent the process. The individual can reflect on the steps of the action without necessarily performing them. It may be reversed or it may be coordinated with other processes, facts and formulae. In some cases this leads to a new process (as in composition of functions) and ultimately, as these links are reflected upon, it can lead to the 'encapsulation' of the process into an Object.

An Object conception exists is when the student is aware of the totality of the process, and entails an ability to use the process in further processes (for example: $\int \sin\theta .\, d\theta$. The process of integration is applied to the object of $\sin\theta$). The encapsulated object is a concept construction that, in Dubinsky's (1991) view, may be de-encapsulated at any juncture to obtain the process from which it came. This would allow the student to constantly move back and forth between an object and a process conception of a mathematical idea. When connections between different objects are made a Schema is constructed.

Schemas, by this definition, maybe associated with a particular topic (geometry, trigonometry, functions, etc) or more generally, with the whole of the mathematical world. It is a mental map in which sub schemas are identified with a name and the interconnections between them vary in strength and number. Newly encapsulated objects may be linked into the schema and more connections may be constructed to enrich it further. It is unfortunate that the word schema is used in two ways in the mathematical literature; both to describe the whole edifice of mathematical knowledge that an individual has i.e. a network of reified objects, also to describe *any* knowledge that an individual has about a particular topic which may be a disjointed amalgam of processes, diagrams or formulae. Cotteril et al (1996) are precise in their use of schema in the former sense but other researchers such as Sfard and Nickerson frequently use it in the latter sense.

Thus some students appear able to construct these qualitative object structures of a concept whilst others keep their mathematical thinking strongly focused on the action/instrumental/procedural end of the spectrum. Why does this difference arise? Why do not all students construct an object conception of a topic? A study by Gray and Tall (1994) provides an insight into one reason for this difference in understanding.

3.1.6 Procepts

Mathematical ideas are mediated through a combination of symbolic language and abstract images. Cobb, Yackel & Wood (1992) proposed the basic principle that mathematical meanings given to such representations are the product of student's interpretive activity. Gray and Tall (1994) subsequently suggested that an underlying reason for the different relational and Instrumental thinking styles is a different perception of the symbols routinely used in mathematics. They identified two different ways that students perceive the symbols they encounter. The first way is as an instruction to undertake a specific action, and this perception they termed procedural. The second way is an anticipation of the result when the specified action is undertaken that is based on a conceptual knowledge of the action. This perception they called Proceptual (an amalgam of the words procedural and conceptual). It describes a perception of the symbol as an instruction to undergo an action *and* the result of that action. For example the symbol 3/4 suggests the action of dividing 3 by 4 and is also the result of that action, an idea that can be carried forward to other situations. Different interpretations of mathematical symbols would seem to be critical in the development of mathematical thinking. Gray, Pinto, Pitta & Tall (1999) claim that the human mind uses symbols to label and access a host of complex ideas — a symbol is a convenient short hand to communicate a multitude of thoughts that are interconnected. Mathematical symbols communicate mathematical ideas on two levels of understanding. For instance the symbol + communicates the requirement to take the *action of adding* the numbers either side of it yet it also communicates *the concept of sum* enabling an individual to comprehend expressions such as (x+y) where an addition can not be made. Any mathematical symbol which has the property of being interpreted proceptually Gray and Tall (1994) termed a ‘procept’ which relates to a thinkable concept and a process carried out by its corresponding procedures (Gray & Tall, 2001). They suggest that those who read procepts proceptually have a considerable advantage over those who read them procedurally and it is in the interpretation of procepts that is found the key to success or failure in mathematics. This distinction leads to the notion of a proceptual divide, the divergence between those who think proceptually and those who think procedurally.

New procepts are constantly introduced throughout the study of mathematics Post 16, for example $\int$, ∞, Σ, lim $x \to a$, $\partial x/\partial y$, dy/dx. In addition to this, elementary procepts (Gray and Tall, 1994) may be further combined to give higher order procepts such as lim $h \to 0$ (sin x) /x or sin $(2\theta + \pi/6) = \sqrt{3}/2$. Each procept that a student encounters must be understood proceptually or there is a danger that they will end up on the wrong side of the proceptual divide. The ability to anticipate a result is very important and will be returned to later. [§3.3.3] There is perhaps a chicken and egg situation here. Proceptual interpretation of symbols facilitates a duality in the understanding of mathematical ideas yet this duality of understanding is necessary for a proceptual understanding of mathematical symbols. The paradox is that the very symbols intended to make mathematics easier to understand and communicate also makes it very difficult for some.

Both Sfard (1991) and Dubinsky (1991) suggest that knowing how to do (procedural knowledge) can transformed into an object of thought through practice and the appropriate reflection on the procedure. However, an alternative argument to that of knowledge being reconfigured into a different format is the theory of networks.

3.1.7 Network Theories

Nickerson (1985) stated that the more connections there are between the various nodes of instrumental knowledge the better ones understanding. Davis (1992) described the development of understanding as fitting a new idea onto a larger framework of previously assembled ideas. Hiebert and Carpenter (1992) state that it is useful to think in terms of two metaphors. Networks may be structured like webs at one level, and the degree of understanding depends on the strength and number of connections. Moreover these networks may be hierarchical in that a general, part or whole, schema provides a set of higher order connections that can subsume other representations.

The connections are relationships of similarity and difference or may be based on inclusion. By thinking and talking about similarities and differences between procedures, students can construct relationships between them. Understanding is limited if only some of the representations of potentially related ideas are connected or if the connections are weak. The construction of new relationships may force a reconfiguration of affected networks and reorganisations are manifested both as new insights, local or global, and as temporary confusions. Understanding increases as the reorganisations yield more richly connected, cohesive networks and ultimately these become indistinguishable from a concept.

Reification/ encapsulation theories and network theory are not necessarily exclusive and indeed complement each other at different stages of understanding. Knowledge of a procedure frequently networks with knowledge of an alternative procedure and through reflection on the conditions for which one any particular procedure is preferable the procedures may be encapsulated into an object conception. The role of networking may be seen clearly in the linking of visual representations to algebraic ones. In the next section issues of visualisation and symbolic representation will be reviewed.

3.2.1 Visual Representations

Bruner, Oliver & Greenfield (1966) theorised that there are three different types of representation of human knowledge ‘enactive’ which is associated with a physical process, ‘iconic’ which has a degree of naturalistic resemblance and ‘symbolic’ which is an abstract marking bearing no audible or visual resemblance to that which it represents. Whilst the process of counting on fingers may be seen to be enactive and geometric images mostly iconic, representations of number such as ‘5’ or ‘3897’ and algebraic expressions are examples of symbolic representations,. Some representations are part iconic and part symbolic such as a random sketch of a right-angle triangle with two arbitrary side lengths given by the appropriate sides. The drawing of a freehand graph is an example of a representation that is symbolic and enactive. Iconic and symbolic representations can be drawn, written or described in words.

These representations are the symbols by which mathematical ideas are expressed and communicated to others, and ideally there should be a fluid and flexible movement between the different representations of a concept. For example the relationship between a diagram of a right angle triangle with an angle marked θ and the formula *tan θ =opp/hyp* should be almost one and the same thing. However the use of spatial visual images is not practised uniformly. There is evidence (e.g. Krutetskii, 1976; Moses, 1977, 1980; Surwarsono, 1982; Presmeg, 1985) that some students have difficulty dealing with spatial visual images whilst others have a preference for them. There would appear to a continuum of preference for their use between these two extremes. Students who prefer using imagery have been identified as visualisers by Presmeg (1985), but she indicated that visualisers do not appear to have an advantage over non-visualisers in mathematical ability. Indeed, she records that those pupils whose achievements were singled out as being outstanding 'were not merely often, but almost always, non-visualisers' (p297). Pitta and Gray (1999) observed that the ability to use images was not necessarily an indication of potential success in mathematics but, it was the ability to abstract from the images that distinguishes the potential to be high achievers. However researchers of visualisation argue that connection and abstraction maybe promoted by the use of different representations of an idea, that are complementary.

3.2.2 Visualisation Theory

There are many different forms of visual imagery mentioned in psychological literature but the definition that Presmeg (1986b) found most useful in her studies on the use of imagery in the solution of problems in the high school mathematics syllabus was that of visual image as *a mental schema depicting visual or spatial information*. Denis (1991) made a distinction between 'symbolist' and 'conceptualist' theories in the role of mental imagery in thought. Symbolist theories of thinking assume that thought is linked to mental representations (symbols) so that these symbols, which include images and verbal representations, are the medium of thought. Without the symbol there can be no thought. 'Conceptualism' assumes thinking involves mental entities of a conceptual and abstract nature so that symbols are a product of thinking. From this perspective symbols are the means of expressing thought but are not necessary for it, so alternative symbols for the same idea may arise. Bills (2002) claims the conceptualist paradigm concurs with the theory of constructivism. It is therefore the one that will be considered here.

The conceptualist viewpoint is not, however, a unified one. Kosslyn, Thompson, Kim, & Rauch (1996) noted that some people believe images to be 'epiphenomenal' which means that they play no part in cognitive processing: Images merely accompany thinking. Bills (2002) however, argues that this may be true, but mental visual imagery appears to have a role in *assisting* thinking in those situations where information about spatial characteristics is required. What assistance imagery provides and the nature of the image that provides it is considered in the next section.

3.2.3 The Role of Visualisation.

Duval (1995) claims that:

> The characteristic feature of mathematical activity is the simultaneous mobilisation of at least two registers of representation, or the possibility of changing at any moment from one register to another.
>
> (Duval,1995, p3)

He describes two distinct types of cognitive transformation that are evident when analysing mathematical activity from the learning and teaching perspective.

The first he calls 'treatment' and describes this as 'the transformation of representations that happen within the same register' (p4) giving examples such as solving equations using purely algebraic processes or, considering problems set in the visual spatial register only in spatial terms.

The second type of treatment he termed 'conversion' and this he described as "transformations of representation which consist of changing a register without changing the objects being denoted" (p4). An example of this is when a solution process begins with a diagram and then changes to a logic/algebraic formulation or vice versa. This conversion treatment, he conjectures, is the activity which leads to understanding. The reason is that two different representations of the same mathematical object do not have precisely the same content; indeed, the content of the representation depends more on the type of representation, algebraic or spatial, than on the object represented. For example compare the following algebraic formulation (Figure 3.1) and the spatial visual representation for considering the sine of θ°:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

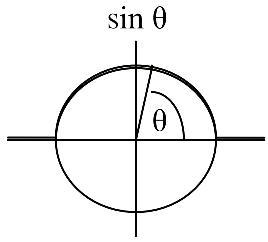


Figure 3.1. Complementary representations of Sine θ

The algebraic representation frequently gives problems for angles greater than or equal to 90° since determining the meaning of 'opposite' and 'hypotenuse' in this situation requires a re-evaluation of their meaning.

These diverse representations may help to promote the object schema of proportional reasoning involved in the calculation of sines, cosines and tangents by lifting it away from the purely operational action of processing. We might infer that visual imagery and spatial representations provide different insights to the concept than those provided by algebraic identities or manipulations. By transferring from one representation to another, other properties or aspects of the same object arise that give a greater depth of understanding and a more flexible attitude to problem solving. It also develops student awareness of the distinction between an object and its representation.

Duval's (1999) theoretical framework posits that connections both within and amongst different representations are absolutely fundamental to a deep understanding of mathematics. Presmeg (2005) however found that students are often reluctant to use visual representations if other symbolic representations are available. She claims that students who do use diagrams frequently have problems with generalization. Visual images, she argues, must be accompanied by rigorous analytical reasoning in order to be useful. This supports the claims by Healy and Hoyles (1996) that visual aids have a valuable role to play but it is not always self evident.

> Students of mathematics, unlike mathematicians rarely exploit the considerable potential of visual approaches to support meaningful learning...mathematicians know what to look for in a diagram, know what can be generalized from a particular figure and so are able to employ a particular case or geometrical image to stand for a more general observation.
>
> (Healy & Hoyles, 1996. p67)

Spatial images therefore may provide an alternative mode of thinking about a sub concept or, they can be the means of forming connections between different aspects of the concept that promote flexibility, create a deeper understanding and a greater insight. The issue remains though, that visual images do not necessarily provide these connections automatically. Further insight to this phenomenon was provided by Pitta & Gray (1999) who found that students focus on qualitatively different aspects of an image. The ability to filter out and see the strength of a visual device as a holistic representation of mathematical properties could be dependent on the ability for mathematical abstraction. Visual images therefore have a proceptual aspect. (p11) [§3.1.6] They suggest that it is an over simplification to say that the use of imagery is beneficial as it depends strongly on the types of imagery used and its purpose. This was borne out by difference in imagery used by the pilot study [§5.6] and the main study [§7.5]. The next section considers types of imagery used didactically.

3.2.4 Types of Imagery

Presmeg (1985, 1986a, 1986b, 1997b) identified five types of imagery used by the students in her studies. 'Concrete imagery', 'memory images', 'pattern imagery', 'kinaesthetic imagery' and 'dynamic imagery'.

She suggested that concrete imagery, where a specific image or diagram is drawn, was almost universally used by the students within her sample. These are spatial representations of concrete images that are used in a way that helps students to remember certain procedures. The image of a ladder against a wall is an example of a image that may be used to 'concretise' a particular trigonometrical problem that students may be required to deal with.

> Such imagery is characterised by its appearance in the absence of the objects to which it refers.
>
> (Mead, 1938, p224).

Presmeg (1986a) noted that this kind of image was not necessarily beneficial since it can be limited by the focus on the concrete aspects of the problem. For these images to be beneficial the one-case concreteness of the problem must be transcended to an awareness that this is merely one of a set of problems that have a common nature, but Presmeg believes that many students are not aware how to accomplish this task, especially those students who are naturally visualisers; a problem that non visualisers don't encounter. All the mathematical difficulties encountered by the visualisers in her study related in one way or another to problems with generalization. The universality of the situation promotes reification/encapsulation but a concrete image is not necessarily conducive to awareness of a universal. Many of the pitfalls associated with the use of concrete imagery were avoided however in the instances where visualisers were able to combine specific imagery and use of abstract non visual modes, for example algebraic formulations or thought processes based on logic. Dreyfus (1991), whose research findings supported the results of investigations by Presmeg (1991) and Dorfler (1991) made the point succinctly that:

> Only concrete images that were combined with rigorous analytical thought processes were an effective tool in mathematics.
>
> (Dreyfus, 1991, p6)

In a later study Presmeg (1993) again observed that efficient processing could result from the use of concrete imagery when it was 'alternated with a facile non-visual use of formula' (p297).

Symbolic imagery is an abstraction and this has been found to have the potential to aid further abstraction. Healy and Hoyles (1996) make the observation, also noted by others (e.g.Presmeg,1985, 1997b; Pitta & Gray, 1999), that particular images or diagrams not only provide connections between different modes of thinking but are actively involved in the service of mathematical generalization since patterns emerge that may have been obscured in another representation. These patterns may be the result of the students own reflections on the given spatial representations and are devised to aid the memory of particular operational facts or formulae. They help to condense the procedural aspects of the concept preparing the mind for the transition to object. Memory imagery, pattern imagery, kinaesthetic imagery and dynamic imagery are all types of symbolic imagery.

Memory images were used to remember formulae etc and over half of the students in Presmeg's study were observed using this form of visualization. An example would be "Cos Cos – Sin Sin" to support memorising Cos (A+B) = Cos A Cos B – Sin A Sin B.

Pattern imagery is where the essence is embodied without the structure. One observed example of this was a student's use of + + - - for the period of sine (compared to + - - + for cosine and + - + - for tangent). Presmeg notes that the student who had devised this said it is the regularity of the pattern that is the important feature of the image, a pattern the student abstracted from the more concrete although, still representative, images of the four quadrants.

> Pattern imagery, and use of metaphor via an image, are two significant ways by means of which a static image may become the bearer of generalized mathematical information for a visualizer.
>
> (Presmeg ,1986a, p209)

This kind of imagery incorporates elements of what Dorfler (1991) referred to as ‘relational image schemata’ and was a strong source of generalisation for the learners who used it. In Presmeg’s study one student in three developed some kind of pattern image.

Kinaesthetic imagery could be considered as a precursor to dynamic imagery [cf §3.2.1 enactive imagery]. It is where some form of movement is involved such as drawing a triangle or circle in the air.

The final classification is dynamic imagery which is identified in situations where the image moves in some way. Presmeg found this kind of imagery was rarely used by any of her students but since Presmeg’s studies, research has been undertaken on the use of computers for teaching trigonometry and the development of dynamic imagery that resulted through the pedagogic use of computers will be described later in this chapter [See Blackett §3.3.2].

There is now a large body of evidence (Gray, 1991; Dehaene & Cohen, 1994; Pitta & Gray, 1999; White & Mitchelmore, 2004; Pitta, Pantazi, Gray & Christou, 2004) that shows that there is a qualitative dimension to spatial imagery. Imagery that is concrete or, is seen to represent a single case scenario, may be counter productive to the aim of seeking to generalise situations and can promote compartmentalisation rather than challenge it. There is however, a further point to make here: the issue of the use of imagery might not end with the types of imagery used but is subject to the perception of the individual. Images, like symbols, can be perceived procedurally or proceptually. Pitta and Gray (1997) found a correlation between the focus of mathematical imagery used by students and their mathematical achievement. High achievers focused on the abstract qualities of the image and there was free movement between the abstract and descriptive aspects. Low achievers, often describing superficial characteristics, held fast to the procedural associations of an image.

Gray and Pitta summarized their findings by concluding that the importance of images is not what it is, but what can be done with it. However, the notion of image, identified as the concept image, has also been used in the context of total cognitive structures.

3.2.5 Concept Image

Concepts may be defined in two ways, by concept definition and by concept image. Tall & Vinner (1981) defined concept definition as 'the form of words used to specify that concept' (p 152). This could be a formal definition constructed by the mathematical community or a personal description constructed by the student. The concept image they describe as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." (p152). Images of trigonometry would include concrete images of triangles, right angled or otherwise, memory images of formulae, images of the unit circle and images of the graphs with their attendant properties. There should also be images of function. Students can construct their mental images as they experience the function concept in different representations. Vinner (1983) observed that students preferred to use images to think of a function concept rather than use of concept definition. He suggests that 'memory works better with pictures than with words' and therefore that teaching strategies should allow students to use experiential opportunities to construct process images before the concept definition is given. He warned however that the concept image should not be given priority over concept definition since teaching styles which prioritise concept images do not guarantee a better understanding. They can restrict students understanding of the concept to the particular images with which they are presented — just think of the influence that the image of a ladder against a wall has on a student's early understanding of trigonometry. Concept definition and concept images are not alternative ways to represent the function but should be used to complement each other.

3.2.6 Imagery in Trigonometry

Traditionally the imagery of trigonometry is introduced through association with concrete images: ladders against walls, flying kites, the angle of elevation of tall trees or buildings are typical examples of the concrete situations that a diagram of a right angle triangle aims to represent. Whilst the research described above has shown the shortcomings of this type of imagery without accompanying links to general situations, at GCSE there is a strong emphasis on an informal concept definition e.g. sine equals opposite over hypotenuse. Those students who are able to, in Presmeg's words "alternate the use of the concrete image with the facile non – visual use of formula" are probably the ones who find trigonometry at GCSE non problematic but it is argued in this study that this is only a temporary respite; the transition to a wholly symbolic series of representations that characterises the study of trigonometry at AS/A level can cause a cognitive conflict that was averted in the initial stages.

One of the first spatial representations that most students meet that is symbolic is the graph of the sine wave, followed closely by that of the cosine and tangent graphs at Higher level GCSE. Although some students may have explored graph transformations and the way these are indicated algebraically it could be argued that for most students the trigonometric graphs are instrumental. They are strongly linked to a procedure for evaluating sines, cosines or tangents for specified angles greater or equal to 90°.

Evidence obtained at the start of the two investigations reported in this study (See Chapters 5 and 6) indicates that despite all the students having gained at least an A at GCSE, none of them regarded trigonometry as anything more than a set of procedures accompanied by diagrams of right angle triangles or non right angle triangles that were concrete images. This might suggest that the transformation from process to concept of function could take place, if indeed it would take place, in the future, that is, during the A-level course.

3.3.1 Research in Trigonometry

There has been little research examining the learning of trigonometry at 'A' level but experimental studies investigating different teaching styles at GCSE level have produced interesting results. This next section describes the research of Blackett (1990), Pritchard (1998) Weber (2005) and Delice and Monaghan (2005), which provide some insight into issues of student's learning of trigonometry and possible pedagogic alternatives.

Kendal & Stacey, (1997) argued that traditional instruction of trigonometry that emphasises a ratio conception of trigonometric functions does not support students understanding of these operations as functions. Hirsch, Weinhold, and Nichols (1991) contend that instruction in trigonometry of this kind emphasizes procedural, paper-and-pencil skills at the expense of deep understanding.

Orton (1987) believed that trigonometry is a difficult concept that is dependent on an understanding of two other difficult concepts:

> In terms of knowledge, both similarity and ratio are difficult ideas, and may not be adequately formed. One must, however, acknowledge that they might become better formed through a study of elementary trigonometry, but if there is no relevant knowledge there at all, sines and cosines would once again have to be learned by rote.
>
> [Orton, 1987, p146]

Weber (2005) claims that when trigonometry is taught with a focus on the ratio aspect, students develop a concept of trigonometry that is little more than a formulaic association with some form of algebraic manipulation. The development of trigonometry tends to become the selection of the correct formula from a choice of three and then the correct substitution of values into this formula with a possibly transposition before finally seeking a value on the calculator by pressing the correct keys.

For example: To find the length of an adjacent side of an angle of 50° in a right angled triangle where the hypotenuse is 6cm. The solution process can take the form of:

- Adjacent and hypotenuse so CAH so cos
- Cos 50 = x/6
- Rearrange formula to x= 6 x cos 50
- Type expression into calculator and press =

This operational procedure of evaluating sines, cosines and tangents does not naturally lend itself to, for example, being able to estimate sin 15° or, to any instinctive understanding of whether sin x is an increasing or decreasing function in any particular quadrant (an important sub concept in the object of trigonometry that has implications at 'A' level when calculus is applied to trigonometric functions). He argues therefore that by introducing trigonometry via such a process deters development into an object concept because the outcome may not be anticipated. Anticipation is recognised to be of fundamental importance to the development of understanding. The National Curriculum in Teaching Mathematics [NCTM] (2000) states that understanding an operation involves being able to estimate the result of that operation (p. 32-33). An experimental study by Blackett investigated whether the use of dynamic imagery on computers could be used to initiate an understanding of trigonometric process that would be more beneficial in the long term.

3.3.2 Trigonometry and the Pedagogic Use of Computer Graphics

Blackett noted that:

> Students of high ability can memorise procedures which are only instrumentally understood and recall them with some success but this facility is not evident with any but the most able children.
>
> (Blackett, 1990, p338)

[see §5.8.2, §5.11 and §6.2.2 for students who were of high ability and had memorised instrumental procedures at GCSE.] Blackett predicted that even the high achievers who could memorise the procedures sufficiently to use them at GCSE, would experience problems later in properly understanding trigonometry. [See§5.7, §7.9]. Their instrumental knowledge may be a necessary expedient to gain the best grade possible at GCSE but would not provide a stable schema which could be expanded into a conceptual object of trigonometric function.

Blackett wanted to see if using computer graphics with an experimental group would encourage ideas of similarity and promote a conceptual understanding of the trigonometric ratios. His experimental groups were encouraged to consider the specific nature of the right-angle triangles they were looking at on the computer, such as the length of the opposite side in relation to the adjacent, and as a result transpose that experience to a sketch of a right angle triangle that had the lengths indicated by placing numbers beside them. As a consequence the diagram may be recognised as a symbolic representation of a given situation. The experimental groups were encouraged to interpret diagrams not only as an alternative means of conveying information which is then processed directly into a numerical or algebraic procedure, but to use the symbolic features of the diagram to visualise more accurately the situation being represented.

He found a significant difference in the conceptual understanding between his control groups (who followed the GCSE syllabus) and the experimental groups who had worked on computer graphics to link the visual aspects of trigonometry with the numerical. The experimental group were able to predict the sines, cosines and tangents of given angles to a far greater degree of accuracy than the control group, less able experimental groups performed far better then less able control groups; and the students in the experimental groups remembered the process better when assessed again after a time delay of 2 months.

Blackett concluded that the experimental group had an instinctive grasp of trigonometric processes that was richer and longer lasting than the understanding constructed by the control group. He went on to predict that the control group members of his study would have difficulty in forming a conceptual understanding of the trigonometric functions in pure mathematics as well as the extensive trigonometric work associated with applied mathematics.

Here is evidence of the improved understanding that resulted from using computers as a pedagogic tool. Apart from being able to provide a large number of different examples instantaneously computers also have an important role in that they are a suitable medium to dynamically transform images to reinforce ideas of similarity and difference. Such images were found to be of greater benefit than static pen and paper images for exploring the fundamental concept of trigonometry as a geometric function. They helped promote the construction of useful imagery and provided a foundation for visual reasoning. Dreyfus (1990) claims that computers can promote flexibility of visual reasoning, an area of mathematical development that is often neglected in the classroom.

A study by Tall (1986) with 'A' level students found the use of computer graphics as part of a planned teaching strategy was seen by pupils and teachers alike to have been very successful. By using the computer to dynamically link the unit circle and the trigonometric graphs, zoom in to the gradient of a trigonometric function and explore the nature of sin (A+B), he found that the study group had a greater understanding of the geometric aspects of the trigonometry without any detriment to their algebraic competence. He surmised:

> The graphical images which students can now visualize in their mind's eye will be appealed to even in quick blackboard and chalk explanations.
>
> (Tall, 1986, p18)

The research by Tall and Blackett provides evidence that the power of diagrams is greatly enhanced when they are recognised by the students as a symbolic representation. The diagrams themselves are not the issue but a means of representing the issue. Emphasising the distinction between the real situation in all its complexity and the symbolic nature of the diagram representing its abstract features, but only those pertinent in this instance, may facilitate an ability in the learner to transcend the one-case concreteness of the image. [§3.2.3]

However the issue of computer use is not entirely straightforward.

Sinclair (1993) noted that extensive studies of Cabri have shown that a geometry problem cannot be solved simply by perceiving on screen images, even if these are animated. The student must bring some explicit mathematical knowledge to the process "that is, an intuition about a generalization involves more than observed evidence." (Sinclair, 2003, p.192). Hadas & Arcavi (2001) noted that visual learning associated is not exempt from the difficulties resulting from prototypical mental images and inscriptions.

For computer aided dynamic imagery to be a beneficial pedagogic instruments it is suggested that they should be used imaginatively for tasks that explore different aspects of the concept; for instance using a range of triangles drawn to scale, not as a replacement for pen and paper sketches labelled with side lengths and/or angles.

Another study that sought to investigate how a more meaningful understanding of trigonometry might be promoted and associated with the notion of procept was undertaken by Weber (2005).

3.3.3 The Promotion of Trigonometric Functions as Procepts

Weber (2005) reporting on earlier work by Kendal and Stacey (1997) suggesting that:

> Students who learned trigonometric functions in the context of a right triangle model performed better on a post-test than those who learned about the subject using a unit circle model.
>
> (Kendal, M., & Stacey, K., 1997, p 4)

claimed that this observation was contrary to theoretical research. This conventional method of introducing trigonometry, he said, focused on procedure that is not conducive to an understanding of trigonometry as a geometric function. He undertook an experimental study that involved a group of students being given a lecture, in which they were given graph paper, drew the unit circle and marked off angles using a protractor and ruler. A control group followed the conventional course centred on algebraic procedure. After completing the procedure all the students were asked a series of questions such as:

"Which is bigger sin 37° or sin 23°? Explain why."

"Without measuring, estimate the sin of 170°"

"Is sin 145° positive and why?"

"Explain why sin θ can never equal 2"

Weber found that in the control group, zero marks were obtained by 26 of the 40 students and none of the students could give any justification for the reason why sin θ could never be 2. In contrast, within the experimental group at least 30 in the class of 40 gave correct answers and adequate justifications. When asked why sin θ is a function, the students in the experimental group described the process of drawing the angle and determining its sin and then typically explained that for each angle there was only one possible point of intersection with the circle. None of the control group was able to give any explanation despite being reminded that the definition of function is 'for each input there can be only one output'. Weber concluded that reasons why the experimental group outperformed the control group were associated with an emphasis on the performance of geometric processes and the opportunity to reflect on the actions of those processes. However, he is careful to note that 'one method of teaching is not necessarily superior to another' and that

> Using the unit circle does not necessarily guarantee that substantial learning will occur but what is critical is to give the students the opportunity to think of sines and cosines as a process regardless of the model.
>
> (Weber, 2005, p18)

This implies that to give the students the opportunity to interiorise and condense the process themselves is more successful than attempting to give them an already condensed process [See § 5.12]. The ramification of allowing students to interiorise and condense the process themselves is that students were able to visualise the process subsequently in a way that encouraged visual reasoning. The images they used were meaningful. Many students however are introduced to trigonometry through symbolic images that are supplied for them by a text book or teacher and have no intrinsic meaning to an action process. The dangers of using images that have not been embedded in a meaningful concept are shown by the research of Pritchard.

3.3.4 The Role of Visual Images

A study by Pritchard (1998) into the role of visualisation in the understanding and application of trigonometry by year 10 students following the GCSE syllabus found that:

> The hope that some students would think of using visualisation was soon dispelled by the fact that only one of them applied a visual method as an initial means of solution. Even in this instance, it could be suggested that she was not displaying visual ability. She did not choose the correct triangle because she was adamant that the numbers supported her choice and thus her decision was based on the symbolic process that she had also carried out. For the other students, looking at the triangles was very much a secondary consideration.
>
> (Pritchard, 1998, p83)

Pritchard concluded that students in her group were unable to visualise situations that were non-routine. She noted that the students regarded SOHCAHTOA as a formula and the thinking took place at the point of deciding which lines were opposite, adjacent or hypotenuse so that their lengths might be substituted into this ‘formula’. Otherwise the diagrams played no part. They were unable to identify a similar triangle to the one given in the question or construct a diagram that would represent an unfamiliar scenario. When listening to one of the students talk through her solution, Pritchard noticed that when the student spoke of inversing, she was referring to the need to rearrange the formula, rather than connecting to inversing the function in the sense of $\tan^{-1}$. To find a required angle the students relied on remembering the key sequence necessary on the calculator such as:

> I remember it’s shift then cos.
>
> (Pritchard, 1998, p68)

And,

> When you are working this out you have to write cos, that’s there, you don’t need to explain what it is. I don’t know what it is either. I know it’s a button on the calculator.
>
> (Pritchard, 1998, p78)

Pritchard concluded that the students had a limited network of concepts at their disposal and they adhered to the algebraic procedure rigidly.

> The option to visualise the function and thus think of it as a whole entity rather than as a ratio is not open to them because switching to visualisation is not an immediate or intuitive thought process and they do not have an image that can be drawn upon easily.
>
> (Pritchard, 1998, p 91)

Pritchard considered that the way in which the trigonometric functions and its notation were presented to the students in her study lead to misconceptions and confusion with the algebra. She concluded by indicating that visualisation is under-used in schools leaving students reliant on the mnemonic and denied a deep understanding of the ideas. If the procedure was incorrectly remembered the students were unable to make any progress in solving the problems.

The above descriptions of research into trigonometry all focus on students in the 14-16 age range. An interesting comparative study by Delice and Monaghan investigated the pedagogic methods and material used to teach trigonometry in England and Turkey to students of senior high school (16-18 year old students) and the effects these techniques had on the respective students' ability to answer questions.

3.3.5 What You Get Is What You Teach

Delice and Monaghan (2005) compared the techniques and content used to teach trigonometry in the Turkish Republic to those teaching the English A-level course. They found that the English curriculum continues the pre-16 focus on right-angle triangles whereas the Turkish curriculum focuses on the unit circle. Turkish lessons focused on answers in surd form and centred on simplification, solving equations and inequalities, and solving geometric problems. This was a feature of English trigonometry lessons as well, but they had an added emphasis on 'real world problems'. Turkish teachers encouraged their students to employ a number of ways to solve a problem whereas English teachers provided a fixed set of steps to solve a problem. This difference was particularly marked in the use of diagrams: English teachers directed students actions associated with drawing diagrams whereas Turkish teachers provided little direction. The results were that Turkish students were better at algebra and trigonometric simplification and English students were better at real world problems. Delice & Monaghan found that neither system had the advantage over the other and concluded that 'you get what you teach'. The student in a trigonometry lesson, they said, is required to do certain things such as understand something or complete an exercise and this is done with 'culturally sanctioned tools and associated techniques'. The culture is defined by the learning environment and they point out:

> The student is not a lone agent; the teacher directs the student's activity and actions, and the voice of others, e.g. curriculum designers, is present in the teacher's voice.
>
> (Delice and Monaghan, 2005. p8)

[See §5.6,§7.6]. Their study of the two systems led them to the conclusion that:

> Classroom activities differ; there are considerable differences in the tools and techniques used; mathematical actions related to tool use differ; and the rules of behaviour regarding activities and tool use differ. Trigonometry in the Turkish republic and trigonometry in the UK are related but distinct trigonometries.
>
> (Delice and Monaghan, 2005. p8)

The study implies that the concept of trigonometry constructed by an individual is directed by the emphasis placed on the content by the teacher [See §3.4.2]. This concurs with the conclusions of researchers (Resnick, 1989; Sfard, 2004; Schoenfield, 1992; Merttens, 1997;... etc) who argue that attention should be given to the importance of making connections not only for oneself but with those of one's peers.

3.4 The Importance of Cultural Perceptions

The importance of responding to the mathematical thinking of others emphasises the importance of interaction within a socially constructed learning environment (Elliot, 1993). In short individuals develop their understanding of any enterprise from their participation in the community of practise within which that enterprise is practised. The lessons students learn about mathematics in our current classrooms are broadly cultural, extending far beyond the scope of mathematical facts and procedures that they study. Thus of paramount importance is the need for the teacher to focus on the growth of mutual understanding and coordination between the learner and the rest of the community by the establishment of a culture that has the same values and perceptions communicated by a common language. It is these perceptions that are critical (Gray, Pitta and Tall, 1997). They are the stimulus for thought patterns. This in turn relates back to Skemp's claim that teachers who focus on the operational aspects of the concept direct a concept formation that is operational whereas teachers who focus on the relational aspects promote a conceptual perception. It is this that allows students to read symbols and imagery proceptually. (Gray & Tall, 1994)

Two further points are worth noting about the teaching of mathematics at advanced level. These are the role of the teacher in the understanding of functions and the change of focus of delivery.

3.4.1 The Role Of Teachers In The Understanding Of Function.

In Sfard's (1992) study 67% of the students described a function as a computational process and this she interprets as evidence that students possess a cognitive tendency towards an operational conception of function. She claims that students have difficulty when a concept is introduced through a definition (such as "the sine of an angle is opposite over adjacent") and suggests a computational approach would give students a greater chance to develop improved understanding. This is in keeping with reification/encapsulation theories which concur on the need to understand mathematical ideas instrumentally in the first instance before an object concept can be formed. However development from procedure to object conception should be the ultimate aim of mathematics teachers and an over-emphasis on procedure through out the topic can be also problematic.

Bayazit's (2005) study of the relationship between teaching and learning of functions suggested that:

> Action oriented teaching leads to action conception of functions…and the way to promote a conception of function is to deepen and strengthen their understanding of the sub concepts of function.
>
> (Bayazit, 2005, p4)

He concluded that:

> Student's qualitative differences in understanding the function concept are attributable to teachers' instructional approaches.
>
> (Bayazit, 2005, p267)

It is necessary, he concluded, to expose students to the range of representations of each sub function in order for the student to construct an object concept of function. This includes algebraic definitions which are linked to visual representations and a concept definition that is complemented by a concept image. Language also plays an important role. Action–orientated language which avoids reference to definitions does not encourage students to construct an object concept of function that is a process that transforms inputs to outputs. Instead it encourages the students to remain focused on the procedural aspects. Tall Davis, Thomas, Gray & Simpson (2000) similarly observed the importance of language in mathematical discourse noting that when a schema is process-based the description favoured is a narrative but when it is object based the description is likely to be descriptive. (p11).

Bin Ali (1996), in his study on symbolic manipulations and interpretations of graphs concluded that good symbolic manipulation, in other words algebraic manipulation, does not necessarily imply reasonable interpretation for the same object in different forms. Whilst examining how students undertook solving integration problems he found that some students who were good at algebraically solving the problem were unable to discern whether another solution obtained was the same or different. The students had little flexibility in their approaches to problem solving and by clinging to specific strategies were unable to short cut lengthy procedures. He concluded that more capable students with strong links between different representations were able to carry out conceptual preparations that gave them access to simpler alternatives for solving problems. The less able were more likely to plunge straight into a procedure and often this led to greater difficulties and a higher incidence of error. Finally he concluded that students who depended heavily on a single procedure to process a function often had problems reversing the process. The development of a process to inverse a function entails either strong conceptual links or, relying on a further procedure that must be committed to memory. The conceptual links were indicated by the teacher in the use of different representations for the same object. Bayazit (2005) and Bin Ali (1996) both noted the key role of the teacher in the development of process to object.

The research indicates that for an object concept of function to be developed functions need to be introduced as a process that transforms an input to a unique output. Once this is understood good teachers start to introduce complementary representations, alternative procedures, formulae, definitions and concept images derived from the known facts and alternate these accordingly.

The factors that affect the transition from process to object are both external, such as the instruction style of the teacher, and internal such as personal interpretation of the mediating symbols. The socialisation issue that explores the way in which an individual becomes a part of the mathematical community through language and the sharing of common experiences has been well researched by Vygotsky (1978), Resnick, (1989), and Sfard (2004) amongst others and is interesting but the focus of this study is the internal reorganisation that takes place in the minds of individual students and whether an object concept of the function of trigonometry is achieved.

3.4.2 The Change of Focus and Delivery at Advanced Mathematics

There would appear to be a substantive difference in the way that mathematical concepts are presented to learners within the elementary school and within the A level classroom. At the elementary level actions that lead to processes are introduced and many of these are conceived, by linking different representations, into an object concept. Just think of the notion of 5 or of the addition of 3+4 (See also Gray, Pinto, Pitta and, Tall 1999). At advanced level a concept definition is usually given, first through axioms, and then the exploration explore the concept processes with reference to the axioms. Ideally examples are given initially which satisfy the definition but each has additional qualities which each of the others may not share. In this way an overall umbrella concept is constructed from known properties instead of properties deduced from manipulating processes.

> This didactic reversal - constructing a mental object from "known properties, instead of constructing properties from "known" objects causes new kinds of cognitive difficulty.
>
> (Gray, Pitta, Pinto and Tall, 1999)

Trigonometry is a singular case in that it is one of the few subjects pre-GCSE that is introduced with a definition first. The definition "sine x equals opposite over hypotenuse" and corresponding definitions for cosine and tangent are given at the outset. Problems are then selectively introduced to allow students to explore the applications of the concept as defined. This can be problematic for students as anticipated by Gray, Pinto, Pitto and Tall (1999) and evidenced by the research by Blackett (1990), Weber (2005), Pritchard (1998).

Another feature of advanced level mathematics is that currently 50% of the syllabus is devoted to mathematics in the abstract, termed Pure or Core mathematics. The ability to abstract properties from processes into an object concept that can then be abstracted into the greater mathematical schema is a key aim of the A syllabus as evidenced by the assessment objective to:

> Develop an understanding of coherence and progression in mathematics and of how different areas of mathematics can be connected.
>
> (Edexcel A- level aim (d) (2004)

The shift in emphasis from the problem-solving, process-driven GCSE entails a reconstruction of schema for most students. Harel & Tall (1991) describe three distinct types of schema modification that lead to abstraction or generalisation: Expansive, Reconstructive and Disjunctive.

> Expansive generalisation occurs when the subject expands the applicability range of an existing schema without reconstructing it. Reconstructive generalisation occurs when the subject reconstructs an existing schema in order to widen its applicability range. Disjunctive generalisation occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available.
>
> (Harel and Tall, 1991. p38).

Harel & Tall are of the view that Disjunctive generalization is not a true abstraction and whilst it may be enough to see a student through, for the time being, it is limiting in the long term.

> It increases the number of procedures required to solve the more general class of problems (and) gives the weaker student an additional burden to carry under which he or she is prone to collapse.
>
> (Harel & Tall, 1991. p38).

Studies that have attempted to analyse such changes (Steffe & Cobb, 1988; Hiebert, Wearne & Taber, 1991; Schoenfield, 1992) have recorded difficulties due to the chaotic process of building understanding. Growth can be sometimes manifested as temporary regressions as well as progressions. Hiebert & Carpenter (1992) observed that changes appear to be intermittent and somewhat unpredictable. Understanding is built sporadically rather than through smooth monotonic increases.

3.5 Summary

This study is predicated on the belief that mathematical concepts are constructed by an individual. It draws on the distinction made by Skemp (1976) that there are two forms of understanding: instrumental understanding and relational understanding and that these in turn produce schemas that are operational and conceptual. The greater proportion of school based mathematical concepts begin at the operational level but some individuals are able to transcend operational understanding to a relational or conceptual level. The means by which this may happen is posited by two theoretical frameworks: Sfard's (1991) operational/structural framework which proposes that operational concepts are interiorised and condensed then, through a process of reification, become a structural object conception. These two aspects of the concept, the operational and the structural, then work in tandem, each part augmenting the other. The second framework is the APOS theory (Dubinsky, 1991), whereby an Action is developed into a Process which is in turn developed into an Object which links, ultimately, into a mathematical schema. Mathematical symbols play a fundamental role in mathematical thought processes through their ambivalent nature. They are at the same time both an instruction to carry out an instrumental process and the concept of the product of that process. The ability to interpret symbols 'proceptually', that is in both senses simultaneously, indicates an object concept.

Imagery is used in trigonometry to promote understanding to a greater extent than many other topics. It is frequently used to beneficial effect but imagery also has a qualitative dimension to it; it can be used as an instrumental aid or it may enhance understanding by providing an alternative representation of the object concept. The ultimate image of the object concept is the concept image which symbolises all the properties and instrumental processes associated with the object concept.

Research by Blackett (1990) found that dynamic computer graphics focusing on similar triangles and measuring the lengths of the opposite, adjacent and hypotenuse sides gave students a better understanding of trigonometry as a function of angle than the traditional approach. Research by Weber (2005) found that students who were introduced to trigonometry by the pen and paper method of drawing the unit circle and reading off the sines and cosines of specified angles had a more profound concept of trigonometry as a function of angle, linking it directly to the definition of a function. They were also able to reason when the function was increasing and decreasing. This was in contrast to the control group. Research by Pritchard (1998) found that visualisation played no beneficial role to students attempting trigonometric problems via an algebraic method. Their focus was on selecting the right part of the SOHCAHTOA 'formula' and remembering the process of computation. The students had no meaningful understanding of the concept of inverse sine etc. and confused it with the algebraic process of transposing formula.

Delice and Monaghan (2005) found that student construction of trigonometry is strongly connected to the manner in which it is taught. The perceptions and links are coordinated by the teacher and this leads to a mutual understanding of the concept being developed by the students in the class. Instrumental teaching leads to an instrumental concept of function (Bayazit , 2005) but students who were skilful at algebraic manipulation did not have an advantage when confronted with different representations of the same object (Bin Ali 1996). Those students who had developed strong links between different representations were more flexible in their approach to problems.

Finally the delivery of mathematical ideas changes at the level of advanced mathematics. Instead of constructing properties from known objects the process is reversed to constructing known objects from properties and this can cause cognitive conflict. Trigonometry is one of the few topics where this occurs pre GCSE. Initially this has an operational dimension but at some point the operational schema has to be reconstructed into an object schema. The nature of the reconstruction shapes the concept construction into something which can be expanded or something which will prove to be limiting in the long term.

It is an aim of this study to investigate whether or not student conceptions of trigonometry change as they transfer from the essentially triangular approach to Trigonometry at the GCSE level to consider alternative ways of representing similar ideas and are introduced to the notion of trigonometrical function at A-level. The next Chapter (Chapter 4) considers this issue in detail and the way in which it was investigated.

Chapter 4

Research Design and Methods

4.1 Introduction

The core research question for this phenomenographic study was concerned with the organisation of knowledge that leads to the development of a function concept of trigonometry. The two theoretical frameworks guiding the description of development were the process/object encapsulation theory proposed by Dubinsky, (1991) and Cotteril et al (1994) [§3.1.5] and the theory of reification associated with operational and structural conceptions of mathematical thinking proposed by Sfard (1991) [§3.1.4]. Research by Delice and Monaghan (2005) [§3.3.5] that compared the teaching of trigonometry in English and Turkish classes found that the nature of student schemas is a direct result of teacher emphasis. This, and the results of the pilot study showed that other important issues that need to be taken into consideration are (1) teacher emphasis and (2) learning objectives [see Chapter 5]. The literature review was selected to highlight 3 themes that could guide the response to the main research question. These themes are:

- The development of a flexible object conception of mathematics.
- The use of spatial visual images.
- The role of teacher emphasis in the construction of trigonometrical function.

These give rise to several issues that are at the heart of the study:

- How do students individually perceive trigonometry at the end of the 2 year A- level course.
- The role of the teacher in the delivery of key ideas and how they are linked.
- The impact that regular assessment has on the long term accumulation and students own reflection of their knowledge

4.2 Research Questions

The purpose of this study was to use a phenomenographic approach to determine the knowledge structures a group of students develops during their study of trigonometry in years 12 and 13.

The pilot study, essentially an exploratory study (See Chapter 5) found that after one year of study, the evidence, from both classroom observations and the concept maps, indicated that the quality of the schemas that the students in the group had was operational or at the action stage. An aim of the study was to identify the quality of the schema at the end of 2 years study and the results of the pilot study suggested that it would be suitable to frame the main study around the following four questions:

1. Is there any evidence that students are linking together different sub concepts?
2. Is there any evidence of students curtailing procedures and flexibly using different forms of representation?
3. Is there any evidence that students can de-encapsulate concept images?
4. What is the extent to which students think their own perceptions of trigonometry have changed over the course of their A-level study?

The pilot study also found that the focus on success in the examination had a prevailing influence on the type of learning that was taking place. In connection with this, students were presented with sub-concepts that were condensed by the teacher and it is possible that this led to problems in the long term with students failing to interiorise them. By short-cutting the operational process that leads to a specific generalised maxim, the schemas developed by the students in the pilot study relied heavily on remembering facts and formulae. To investigate this phenomenon two additional objectives were added:

5. An appraisal of the students own perceptions of their learning experience.
6. The identification of opportunities presented to the students so that they may interiorise and personally condense trigonometrical ideas.

The investigation of the quality and the degree of connections (question 1) is through an analysis of concept maps since within the pilot study these were shown to be effective in providing an external representation of the mental organisation of ideas. Further insight into the students' ability to link together sub concepts was obtained through the analysis of integrated questions typical of the examination process. The ability to curtail procedures and flexibly switch from one representation to another (question 2) was investigated through observation of student's responses to four integrated questions. During this observation the interviewed students were asked to talk through their thinking. The pilot study group had considerable difficulty with these questions and three of the students subsequently revised their method of learning to try to emulate the learning style of the student who had achieved the highest mark [see §5.11.4]. All the students in the main study group were asked the straightforward content-uptake questions and four selected students were asked to attempt to answer the integrated questions [§7.9] The investigation of the nature of any concept images the students have and whether a representation of an expression of multiple functions such as 2 sin (2x + 5/6π) can be de-encapsulated into a series of functions applied to sin x algebraically and geometrically (question 3) as set out in Section 3.1.5 was investigated through interviews with selected students. A direct question asking the students to give their personal perception of trigonometry [§5.8] was substituted by the question:

> **A year 11 student asks you the difference between trigonometry at GCSE and at A-level. What would you say to them? Include its uses then and now.**

It was hoped that this would focus the student's responses on a holistic comparison of the courses at GCSE and A-level and allow them to indicate how they thought their study of trigonometry had developed since the beginning of the course.

Students own perceptions of their learning (objective 5) were investigated through interviews with selected students and are discussed in Chapter 8. The opportunities that students were given to interiorise and personally condense trigonometric ideas (objective 6) were investigated by observing the teacher's style of delivery.

In order to assess students' conceptual development it was necessary to know what their preliminary understanding of trigonometrical terms and operational abilities were at the start of the course. The students' schemas prior to starting the course would be investigated by asking them to draw an initial concept map. An issue that needs to be considered with regard to the concept maps are the changes that take place over time. Why are some items subsequently omitted? The theoretical literature suggests that this could be a schema modification brought about by a reappraisal of content and its connection to other content (Harel & Tall, 1991) [§3.4.2]. It was therefore necessary to clarify the reasons behind any omissions or additions to the maps as they arose. This was done by interviews with the students concerned.

4.3 The Phenomenographic Approach

Phenomenographic research was initiated in Sweden in the mid-70s, with the work of Ference Marton, Lennart Svennson, Roger Saljo, and Lars-Owe Dahlgren, the term 'phenomenography' only appeared in the mainstream literature in 1981, when Marton proposed that the study of variation in conceptions of phenomena can be a research specialisation in its own right (Marton, 1981). Traditionally, phenomenographic research has been defined in terms of the object of study (Marton, 1981), commonly described as variation in human meaning, understanding, conceptions or, more recently, awareness or ways of experiencing a particular phenomenon (Marton and Booth, (1997). Most phenomenographic research has focused on mapping variation in experience, in terms of the range of qualitatively different ways of experiencing particular phenomena and the inclusive relationships between the different ways of experiencing. However, with the development of a stronger theoretical base underlying the research approach, there has been a growing emphasis on identifying the structure of awareness underlying the varying experience of phenomena, in terms of key dimensions of variation in experience and aspects of the phenomenon. (Marton, 1994; Marton and Ming Fai, 1997; Marton and Booth, 1997).

Phenomenographic interviews are typically tape recorded and transcribed verbatim, making the transcripts the focus of the analysis and phenomenographic analysis is often described as a process of 'discovery' (Hasselgren and Beach, 1997), in the sense that the set of categories or meanings that result from the analysis cannot be known in advance but must emerge from the data, in relationship with the researcher.

4.3.1 Referential and Structural Components of a Phenomenon

Within this study there is an attempt to identify student perceptions of trigonometry prior to, during and after a period of instruction associated with an A-level mathematics course. It is from this perspective that the notion of phenomenography is perceived to be the most useful paradigm from which to work. Experiences of a phenomenon may be seen to have two components: a referential component and a structural one. Both of these are considered through the concept maps. Whilst the referential component takes account of which items are referenced and the degree of definition given, the structural component considers the structure of the maps, any links, groupings or other overriding phenomenon.

Though the concept maps are core sources of information it was recognised that more direct questioning would also add information to the way in which students perceived trigonometry. Thus the developing method applied to study of the research questions was guided by a model presented by Järvinen (2004), and included the recording and transcription of interviews associated with defined tasks and through the analysis of the outcomes and reference to the concept maps the phenomena identified were placed within categories of description.

4.4 The Sample

The sample for the main study was taken from the same school as the pilot study sample in order to maintain consistency as far as possible. The school was a state grammar school of approximately 1000 students in a Buckinghamshire town, population approximately 70,000. The students are selected by ability at age 11 years and then again by ability for study aged 16-18 years. According to the school website, in 2005, the first year of the two year investigation that was to inform the main study, 44% of students studying A-level mathematics achieved A or B grades and 88% achieved A to E grades studying the 2004 syllabus [§2.12.] The main study group were the first year group to study the 2005 syllabus [§2.17]. In 2006, the second and final year of the main study the statistics were 60% for A or B grades and 92% for A to E grades the national average for 2006 was 65% and 97% [Mathematics Grades 2006 www.bbc.co.uk]. Therefore the mathematics A-level results for this school were broadly in line with the national average for the years in which this study was taken. Specific details of the sample and their teachers are given in §5.3.and §7.2

4.5 Data Gathering

The data obtained for the study was obtained in four distinct ways: concepts maps [§4.5.2] student interview [§4.5.3] and classroom observation [§4.5.4] with students homework and test results used as supporting evidence for analysis [§4.5.5]. An informal teacher interview was held before the trigonometry course began [§4.5.6]. The class chosen for observation for the main study was too large, at 17 students, for a meaningful in-depth study of the cognitive development of all the students so four students were selected from the teacher's assessment of those who were more able or less able within the group. The term 'able' is used throughout this study to describe students who indicate a greater depth of understanding by making cognitive links between different processes. There are issues associated with teacher expectation of performance and this next section briefly describes some of the main issues.

4.5.1 Teacher Expectation of Student Performance and Lesson Structure

There is a large body of research regarding teacher expectation and the impact this can have on student performance (Brophy, 1983; Ruthven, 1987; Sells, 1978; Cooper, 1979; et al).The greater part of the research finds that ability stereotyping can lead to different behaviour in the teacher when dealing with students of perceived high or low ability. Typically the differences are the type of questions asked of each type of student and the restriction of opportunity for students perceived to be of lower ability. In addition higher ability students were more likely to experience a conceptually demanding curriculum (Boaler, 1997).

This last point becomes relevant in the light of the pilot study teacher's comments on student expectation [§5.9].

A further related issue is the style of content delivery and the opportunity given to students to construct their own understanding [§3.1.4]. A study by Boaler, Wiliam & Zevenbergen (2000) of 8 schools in the UK and USA found that high ability students were more likely to be asked to discuss the questions, and consider the meaning of possible solutions with each other.

> This act of negotiation and interpretation meant that mathematics did not appear to the students to be an abstract, closed and procedural domain, but rather was seen as a field of inquiry that they could discuss and explore.
>
> (Boaler Wiliam & Zevenbergen , 2000, p6)

The students' reported beliefs about the nature of mathematics and learning varied according to the extent and nature of mathematical discussion in their classes [§3.3.5 & §7.5], those students participating in the pilot study who were simply exposed to teacher directed input and then encouraged to work individually practising procedures associated with this input, unanimously describing mathematics as a procedural, rule-bound subject.

4.5.2 The Concept Maps

A key source of information for this study was the student construction of a concept map through which evidence of the way in which the referential and structural components of trigonometry might be considered and through which the phenomenographic nature of trigonometry for the identified sample of students may be considered.

It is hypothesised that a concept map represents a person's structural knowledge about a certain concept or subject [§4.3]. Crucial terms are related by means of explanatory links between those concepts. According to constructivist learning theory, the learner's growing understanding of content knowledge is considered a process of enlargement and enrichment of the interrelations established between different kinds of information and eventually their integration into his/her existing knowledge framework.

The analysis of the concept map could take one of two forms: it could either be quantitative or it could be qualitative. Of course it could also be a combination of the two and this will be the way in which the maps are analysed within this study.

A quantative analysis of a concept map is associated with counting the number of links. Shaka and Bitner (1996) and Ruiz-Primo et al. (2001) suggest that a quantative assessment of a concept map depends on the assignment being well structured and of a 'closed format' i.e. the map structure and the concepts are provided by the evaluator and students merely fill the map. Though a quantitative assessment will be used in this study it will not have this form of structure but instead will merely be an assessment of the frequency of occurrence of particular items on the maps. Though quantitative scoring may be generally seen to be more objective than a qualitative analysis, it is generally accepted that they do not fully represent the potential of a concept map as a demonstration of a student's knowledge structure. Thus a more detailed picture of the student's understanding can be extracted from a qualitative analysis.

A qualitative concept map may fit more appropriately with the phenomenographic outlook of this study. It would aim to include both the perceptions of students of a particular phenomenon and give some indication of different kinds of shortcomings such as straightforward mistakes, faulty or vague descriptions, misconceptions and either completely or partially deficient relationships. It is knowledge of these shortcomings which may become the basis of important feedback for the researcher, enabling the pinpointing of students' difficulties and the tracking of recurrent patterns or areas of vague or clouded student understanding. In comparison to a quantitative approach to the analysis of concept maps, a qualitative approach may rely on the expertise of the evaluator for its interpretation.

Concept maps can be a powerful tool for discovering the extent to which the concept is being developed and augmented by identifying these changes over time. Changes in the student's way of interpreting certain aspects of the concept are often a crucial component of cognitive growth. As new knowledge is acquired it may be dealt with in several ways:

(1) It may be used temporarily while the focus of attention is directed towards that particular aspect, and then discarded.

(2) It may be added on to the existing knowledge structure and used when prompted by an external stimulus such as a routine problem in a specific format.

(3) It may be fully incorporated into the knowledge structure by means of numerous, rich links to other aspects so that it becomes a core entity.

It is for this reason that the nature of student concept maps will be considered prior to the start of their A-level course, at the end of their first year and at the completion of the course.

From the researcher's perspective it is inevitable that through the analysis of a particular concept map the researcher will identify uncertainties and the emergence of missing links (forgotten or unknown or indeed simply unrecorded). The nature of such uncertainty etc may only be clarified through one-to-one interaction with the student.

4.5.3 Interviews

An assumption that is extremely important to phenomenographic research is that a person's conceptions are accessible in different forms of actions, but particularly through language (Svensson, 1997). The method of discovery is usually an open, deep interview (Marton & Booth, 1997). One of the main interview tools of phenomenography is the suspension of judgement by the interviewer of the interviewee's report. This is called 'bracketing' and its aim is to give the interviewee as much cognitive space as possible to reflect on the subject as he or she understands it. Instead frequent encouraging remarks are made such as "Tell me what you are thinking" and "What makes you say that? or "How do you know? These sort of questions and requests should be made throughout the interview, at points that are obvious to both the interviewer and the interviewee as well as those points were there is a lack of clarity in thought, in order avoid alarming the interviewee and, to gain as much insight as possible of the cognitive schema they are utilising and possibly constructing.

To gain such insight initially (within the pilot study) the interviews consisted of an attempt to gain supporting evidence for the referential and structural components of the concept maps initially drawn [§4.3.1] and to gain insight into the students' perception of the content currently being studied and how that is being incorporated into the students schema [§3.1.3]. However, the analysis of the concept maps within the pilot study indicated that changes to the students' original schemas were not easily discernible by looking at the second maps even after a second follow up interview. Consequently the interview process within the main study was modified to include a) questions that established the initial skills level of the selected students within the group, b) a predetermined questionnaire [see appendix] that sought to investigate the selected students interpretation of commonly used expressions such as sin x or $\sin^{-1}x$ and c) a supervised observation of the students as they attempted questions that required a more integrated knowledge. At the end of the year studying the AS components the pilot study group attempted a series of past P1 questions selected by their teacher that were designed to test the ability of students who had completed the course. In the event the pilot study students found these questions very difficult. Although a direct comparison between the teaching styles of the pilot study teacher and the main study teacher was not an initial objective of the study, the difference in didactic styles of the two teachers became a significant issue [§5.6 and §7.7]. It was decided therefore to set the same questions to the main study group in order to investigate the impact of teaching style. The responses to the questionnaire and the students' responses to the past P1 questions were used to support patterns indicated by the concept maps.

4.5.4 Classroom Observation

The pilot study sample and the main study sample were observed during their classes on trigonometry. For the pilot study this entailed observing the group as they studied the trigonometry component in the P1 module. The main study group was observed as they studied the trigonometry component in modules C2 (during year 12), and C3 and C4 (during year 13). The classes were timetabled as double periods i.e. 100 minutes long, three times a week. The applied modules were studied concurrently during two further double periods during the week. The lessons were recorded on a small digital device and then transcribed either later that day or the following day. This was necessary as some times background noise impaired the quality of the record of student enquiries during classroom expositions and by transcribing as soon after the lesson as possible the researcher was able to supplement from memory and classroom notes any partially inaudible comments or enquiries or comments and which student was making them. The teacher mainly used the board to illustrate his expositions. All the teacher's board work was carefully copied down contemporaneously and kept for later analysis along with the recordings of his expositions. After the initial exposition, the students were usually set problems to work on from the text book and during this time the researcher either withdrew one of the selected students for interview to the corridor outside the classroom or observed and recorded them (or another student who was raising interesting issues) as they attempted the problems.

4.5.5 Analysis of student reinforcement/consolidation material

The students were set homework at the end of each lesson. This was submitted to the teacher and then returned the following lesson. The homework of the four selected students was photocopied by the researcher who recorded observable patterns in the students solution processes with particular reference to the use of imagery, including graphs, and any indication that procedures were being flexibly curtailed and linked to alternative knowledge and sub-concepts. The students' assessment results were recorded for additional evidence of the students' ability with trigonometric problems.

4.5.6 Informal teacher interview

The teachers of both the pilot study group and the main study were interviewed before the course started to gain an indication of their assessment of the students' ability and their philosophy of teaching trigonometry. [§5.3.4 and §7.2].

4.6 Data Analysis

Marton believes that the different ways of experiencing different phenomena or concepts are representative of different capabilities for dealing with those concepts. Some ways are more productive than others. So the perceptions and their corresponding descriptive categories can not only be related but hierarchically arranged. The ordered and related set of categories of description is called the 'outcome space' of the concept being studied.

The categories possible for the concept can be discovered by immersion in the data from the maps and supporting evidence i.e. transcriptions of the interviews. By looking for similarities and differences among them it is the researcher's aim to seek to identify patterns of perception and focus that could indicate the nature of a student's trigonometry schema [§3.1.3].

The need to handle the data set in manageable components, without reducing its integrity, is obvious and has been approached in different ways by different researchers. The emphasis on an iterative process involving looking at the data from different perspectives or foci at different times is the most common method.

This study has a longitudinal element to it in that it seeks to discover the changes that take place in students' schema during their study of trigonometry at an advanced level. The assessment and evaluation of the data is based on

- The processes, facts, formulae and images that students initially refer to.
- The extent to which any of the above continues to be included as the students experience new ideas within the course.
- The extent to which these new ideas are included.
- The connections between new material recently learned with content previously covered.
- The structure of links.
- Whether there is an emphasis on procedures, facts and formulae.
- Whether there is any evidence of knowledge being condensed.

The issue of identifying where knowledge has been condensed needs careful consideration in order to distinguish between genuine condensing and an omission of detail that is the result of an idea being only partially understood, remembered or, an indication that this item is of lesser value in the student's perception and may soon be dismissed altogether. A concept map that includes only core ideas needs corroborating evidence to determine if the missing details have been subsumed in the student's structural concept but are available to him or her through decomposition [§3.1.5]. This evidence is provided by the recorded interviews and by the students talking through their thinking as they solve problems designed to probe their understanding.

The interviews were transcribed verbatim as soon as possible after they took place. These transcriptions were then repeatedly scrutinised for evidence that supported or contradicted indications shown in the concept maps. Where there were contradictions, a follow up interview was arranged to investigate the reason for the discrepancy. There were few contradictions between the interviews and concept maps but where they did occur they were mostly concerned with items omitted from the concept maps that the students subsequently referred to e.g the sine and cosine rules. The students' explanations for these omissions took the form of 'I know it but didn't think it was relevant anymore' and 'I only put down the new stuff not the things I already know' [§7.7.7]. This seemed to indicate that these rules were not forgotten but were being subsumed.

Researchers have questioned the reliability and repeatability of phenomenographic studies; however once the patterns have been found, they must be described clearly so that all researchers can understand and use them. The aim of the researcher is to describe the range of patterns and foci and this is necessarily subjective. A different researcher may interpret the data differently but this does not invalidate the first description. Considerations of better or worse descriptions refer to matters of judgement of relevance not right or wrong thus, an individual researcher can, at the least, make a substantial contribution to our understanding of a phenomenon, even if further research might take that understanding further. Consequently, it is acknowledged that the final description produced need not be the *only* possible outcome from the data.

The issue of the impact of the researcher is a well documented problem. By seeking to articulate thoughts and thinking processes the students are inevitably reflecting on them more than they might otherwise. Since reflection upon actions and procedures is theoretically one of the key stages of development this requires careful consideration. To offset the effects as far as possible the researcher must strive to remain as passive as possible in the interviews, and explore all thought processes emerging from the student without judgement or intervention. Bracketing in this manner is often counter instinctive to a teacher especially when a student has misunderstood a concept or is ploughing through a long and arduous procedure when a simple alternative is at hand. The researcher needs to quell the desire to teach, and promote the desire to discover. Similarly restraint is needed on the part of the researcher when observing the construction of concept maps. Having determined that the maps would be the students' own constructions and thus the practice of posting up key words, images or concepts for inclusion was deliberately avoided, it would have been inconsistent to have then sought more detail in the maps. Omissions and explanations were probed during the interviews and linkages were explored as the students undertook the integrated problems.

4.8 Validity.

In a study of this kind the issue of validity is less easy to resolve than that of reliability. The only way that the interpretations that are made may be seen as valid is whether or not they inform a coherent whole. If some of the features that are identified cannot be understood as part of the structural whole the final conclusions and a subsequent model can be unintelligible. However, there is another sort of validation, the validation of response given by the subjects. How do we know that the response given is a manifestation of the conception held by the subject and not simply a 'random' answer or an explanation or an answer relevant to that moment? To clarify this question we could refer to Piaget's (1978) way of identifying genuine conviction by consideration of consistency of one respondent and the consistency among several respondents and then the occurrence of some sort of evolution in responses. Both of these issues can be considered in the responses made overtime in this essentially longitudinal study. Identifying the cause of a particular response in this study is somewhat peripheral since teacher/class observation is used to add background detail and is not of itself a key component of the study. However, the objective within the study is to consider the subjects responses and to collectively draw these into a phenomena which may provide some account of the way the subjects conceptualise the notion of trigonometry during their A-level course.

4.9 Ethics.

All the students in both the pilot study and the main study partook in the research voluntarily. Since they were above 16 years it was not necessary to seek parental consent. The teachers of both groups were happy for their lessons to be observed and recorded and for the students to be interviewed for this research. They both said that in their opinion the nature of the research would have no detrimental effects and may be beneficial to the students by causing them to reflect more deeply on their learning. An issue for the researcher as described [§4.11]. The students' names have been changed to maintain anonymity.

4.10 The Research Schedule

The pilot study was undertaken during the first three weeks of the spring term, January 2004. The main study was undertaken over three separate sessions. The first was during the first two weeks of the spring term, January 2005, when the trigonometry chapter in C2 was studied. The second period of observation, the trigonometry component of C3, was during the autumn term of year 13, October 2005 and the third and final observation of the trigonometry component of C4 was during the Spring term of year 13, February 2006 [Chapter 7]. The students were asked to draw their first concept map prior to the start of the first lesson on trigonometry in the C2 module. During the first two lessons on the C2 module the selected students were withdrawn to undertake the questions intended to investigate their initial competence skills [Chapter 6]. During the subsequent two lessons the same students were withdrawn to undertake the questionnaire. At the end of the C2 module all the students were asked to attempt questions designed to assess their application of the new course content and the responses are detailed [§7.7] and draw a second concept map [§7.7.6]. At the end of the first year of study (year 12) the students had completed four modules: C1, C2, M1 and S1 and sat the AS assessment. During the study of trigonometry in C3 the three remaining students (one, a low achiever with poor AS result had withdrawn from mathematics) attempted the integrated questions from past A-level papers under supervision [§4.5.3] and were asked to 'think aloud' as they attempted to solve the questions [§7.9]. At the end of C3 a third concept map was drawn [§7.10]. The trigonometry component of C4 was mainly concerned with techniques for integrating different types of trigonometric functions. The pressure of forthcoming module assessments left little time for the students to do anything more than their scheduled work. There was no time for the students to draw a 4^{th} map and give an appraisal of their perceptions of the difference between trigonometry at GCSE and A-level so a decision was made to omit the final map. Therefore at the end of the 17^{th}, and last lesson, studying trigonometry, all the students were asked to write down a response to the question in §4.2 which sought to investigate student perceptions of the way in which the study of trigonometry had developed over the course.

After completing modules C3 and C4 the students sat their A2 assessment and retakes [see Chapter 2].

4.11 Limitations of Study

The main issues that need to be considered here are first and foremost, whether or not the thoughts and the words of the subjects actually respond to their way of thinking. How can we be sure that responses reflect this? The different research instruments, concept maps, interviews, and questions have been employed to attempt to limit any deficiency but ultimately we still cannot be sure. An issue associated with this is the impact of the research on the students. In particular when attempting the integrated questions the students frequently sought reassurance from the researcher that their chosen solution process would be successful and was error free. It is natural for a teacher to point out errors, incomplete answers or redirect students to significant results when they get lost. Great discipline was required on the part of the researcher to refrain from offering beneficial advice on how to proceed with a solution process or indicate calculation errors. If there is no constraint on time it might be preferable to seek to go over the questions again after the research observation and show the student his or her errors and allow them to reconsider their solutions.

Secondly, defining the sample was a little problematic since it was reliant on the good will of the teachers and their subsequent judgements of the subjects [§4.5.1]. A ramification of the large size of the main study group was that not all the students could be closely observed. The final concept maps indicated students who were not amongst those chosen for closer study [§7.10] who had they been, may have provided greater insight into reification processes. With hindsight it may be easier to identify such students from concept maps that are increasingly pared of instrumental detail.

Thirdly, the pressure of time on teachers prevented real in-depth interviews. Whilst teachers were aware of and fully supportive of the study they did not have time to communicate at a more informal level about their teaching style and belief. This was a weakness of the study. It would have been informative to know how the individual teachers personally linked trigonometry to function and whether the definition of function was considered implicitly, if not explicitly, in their delivery.

4.12 Conclusion.

The research tools chosen for this phenomenographic study were concept maps to investigate referential and structural aspects of the nature of students' schemas validated by interviews with students based on a predetermined questionnaire. The maps were evaluated for evidence of similar themes and these themes were used to indicate qualitative differences in the students constructed schemas. The qualitative nature of students' schemas was established by their ability to think flexibly when attempting integrated questions. The students' perceptions of their own learning of trigonometry during the two year course was investigated and recorded. The emphasis of the teacher was noted and the impact this had on the students subsequent schemas was evaluated.

In the pilot study the researcher did not ascertain the students' initial skill level, and on further consideration, it was thought that this was an important omission. Therefore a series of questions designed to determine the skill level of the main study students was designed [See Chapter 6]. Also the very limited structure of the interviews during the pilot study, designed with openness in mind, resulted in interview data that could not be easily compared. With this in mind it was decided that the questions that would be put to the main study group would forsake total openness for the benefit of identifying how different students perceived terms and ideas that would be predetermined in order for a comparative analysis to be made. In line with this a questionnaire was designed for this purpose.

It is the goal of teachers to help students develop conceptions that are consistent with those held by experts. However students often have multiple conceptions that are not necessarily consistent, in themselves, or with the experts. As Marton (1988) says "A careful account of the different ways people think about the subject may help uncover conditions that facilitate the transition from one way of thinking to a qualitative better perception". The use of phenomenography as a research method can give at least four different benefits. First, a researcher is able to see different views of reality. Second, by using phenomenography it is possible to question the prevailing and catch unexpected views. Third, views are comparable with other views. Fourth, a researcher might see future views and predict future developments. It is the aim of this research to provide empirical evidence of the nature of the knowledge construction of trigonometric functions by striving to reliably record the data emerging from the students and constructing a comprehensive outcome space for the articulated thinking processes that are discovered within the group.

Chapter 5

Pilot Study

5.1 Introduction

A small pilot study was undertaken on a group of six year 12 students at a local grammar school where the main study was to be researched. Its purpose was to explore student's concept of trigonometry before, during and after a course on trigonometrical functions. It was also to serve as a measure of the suitability of the research methods that would be applied to gather information to answer the research questions of the main study.

Although the full A2 course is two years long it was thought the pilot study would provide insight in a methodological approach even if it only covered the first year. Study of the students during the second year would only prolong the development of the main study. Consequently the pilot study followed a group of Year twelve students (median age 16.5 years) through the trigonometrical component of Pure 1, P1, (See appendix) of the AS course which also covered modules M1 (Mechanics) and S1 (Statistics). P2, P3 and a second applied module would be taken in the A2 year (§2.17). Assessment of the P1, M1 and S1 components were taken in June 2004.

The pilot study took place in the second term of the AS course, that is between January and April 2004. During the first term the students had studied algebraic techniques and processes. This chapter describes this pilot study and considers its research instruments [§5.2], and the Pilot study sample details [§5.3]. It then shows the students initial concept maps [§5.4] with observations and interviews. The lesson content for the 11 lessons is shown in Table 5.1 [§5.5] .This is followed by a section on the teacher's emphasis in delivery [§5.6]. The student's development of trigonometry during the course is then described from three perspectives: the results of the task questions [§5.7.1], the second concept maps [§5.8], an analysis of the past P1 answers and the AS assessment results [§5.9]. Then there is a conclusion which sets out what was learned from the pilot study and what refinements were needed for the main study [§5.10].

5.2 Research Instruments

The aim of this study is to investigate how students within the upper years of an English Secondary School change their perceptions of trigonometry as its properties of function are explored. To investigate this issue and bearing in mind the discussion of Chapter 4, the means for investigating it were:

5.2.1 Concept Maps

These were used to provide some insight into the mental schemas that students have from GCSE studies prior to starting the course (§6.2.1) and again on completion of the trigonometry chapter studied during the P1 module on the AS course. The aim of using concept maps is twofold:

1) To examine the Content i.e. to establish the nature of the items the students choose to identify: for example, imagery, axioms, formulae etc.

2) To examine the Structure i.e. to determine the structure of the schema students' possess and how the new images and axioms of trigonometry are being linked to their existing schema.

This second feature was intended to discover whether or not there is evidence to support the theoretical proposition for the construction of a connected conceptual object that can facilitate cognitive flexibility [§3.1.2]. Features identified from the concept maps were supported by, and clarified through, the use of interviews with each student.

It was the purpose of the study to investigate the initial schemas of the students and how they were adapted with regard to the new content and the development of trigonometry as a function. Thus it was necessary to find a way to specifically identify the initial schemas and any differences in the new schemas at the end of the unit on trigonometry. An option was to present the students with a list of previously covered rules, identities, diagrams and graphs and ask them to include them in their maps with relevant links – in other words to establish a quantitative concept map (§4.3.1). However it was decided at the outset of the investigation that in order for this to be a truly representative description of the student's schemas, the students must be given the freedom to draw the maps according to their own perceptions. A quantitative map could impose a focus, structure, or some content, that they might not have included themselves. If the maps were to be revelatory it was critical that the structure, inclusions and omissions that characterised them should be entirely the students own and therefore, bearing in mind the phenomenographic nature of the study, it was felt most appropriate that the maps should be qualitative in nature. Interviews [§4.5.6] were used to follow up the construction of the first and second map and consider points that they raised.

5.2.2 Lesson Observations

Lesson observations were used to monitor the content of the lessons and the emphasis placed on this content by the teacher. The teacher's exposition and student interaction were recorded on tape for analysis and contemporaneous notes were made to identify which students were making observations and comments. After the exposition the students were given questions from the text. As it was such a small group the students sat around a group of three double tables. Whilst Pupil 2 (P2) and Pupil 4 (P4) worked quietly by themselves, the other four (later three) students conferred between each other. The researcher sat at the table and recorded the discussions on disc for later transcription. The interviews were also recorded and later transcribed.

5.2.3 Task Questions

On completion of the sequence of lessons associated with trigonometry, the students were presented with a series of questions that were designed to investigate how much of the content of the module the students had assimilated. They were deliberately similar to the style of questions practised in class. Subsequently the teacher gave the students a test that consisted of past P1 questions. These questions required a more integrated understanding of the core concepts of trigonometry. In the event the students found these questions very difficult and this was to have a dramatic effect on the way the students chose to study the subject thereafter [§5.7.3].

5.3 Pilot Study: Sample Details

5.3.1 Administrative Issues.

The purpose and methods of the research were explained to the students and their consent for involvement within the study was obtained just before they started the trigonometry component of their AS course. As a preliminary each student was asked to draw a concept map for trigonometry and how the different aspects of it linked up in their minds. Only one of the 6 students was familiar with concept or mind maps so the researcher gave a demonstration of a possible map for circles that included formulae, spatial images and possible applications to cylinders etc including links as an indication of how a map may be constructed.

During the following four weeks, three lessons a week were observed as the class covered the chapter in the module. After each of the first three lessons two of the students were asked to wait behind and participate in interviews through which some insight into what each student had learned and understood about the aspect of the module that had just been taught was gained. For example the students were asked 'Tell me about what you learned today?' then depending on the response further follow up questions were asked such as ‘How does that fit in with what you know about trigonometry?’ or ‘ How does that relate to what you know about sines, cosines and tangents?’ etc. The intention was to gain an indication of whether or not what had been learned contributed towards a modification of their original schema. Towards the end of the trigonometry component of the course (lesson 9) the students were given the set of task questions [§5.7.1] and invited to draw a second concept map. The following lesson (lesson 10) the teacher gave the class an ‘End of Module’ test which he had constructed from past P1 questions. During lesson 11 the test of the previous day was considered and after this a further interview, to investigate issues that arose from the second map, was then held with each student.

5.3.2 The School.

The school is a mixed, State grammar school within Southern England. It has approximately 1 000 pupils, of which 250 are in the 6th form. Pupils are awarded a place in the school on the basis of the outcome of an examination in their final year of primary school. This exam, nationally known as the 11+, is intended to consider students achievement and potential on the basis of mathematics, verbal and non verbal reasoning and written English.

The examination is no longer supported nationally but the local authority has opted to maintain it against most of the prevailing trends over the past thirty years. Approximately 30% of the children within the area administered by the Local Authority currently attend the grammar schools but this includes a large proportion within the 6th forms (median age 16.5 -18.5). The Grammar schools have large 6th forms in comparison to with other secondary schools. College numbers do not contribute to the education authorities figures. Thus the number of children selected for the grammar schools from the state primary schools is far less than 30% and represents children at approximately the 75% percentile or above. All may be identified therefore as the more able.

At the end of key stage 4 students (age 16 years) take 10 or 11 subjects at GCSE level. Within the school reported within this study the pass rate for students achieving A* to C grades is currently 88% compared to the Local Authority average of 59.1% and a national average of 45.8%. Pupils who have performed well, that is gained at least 30 points at GSCE (A*-5 points, A-4, B-3, C– 2 and D-1) across the range of their subjects with a B or above required in the subjects that would be studied at AS and A2 level, may apply for entrance into the 6th form. In line with most schools a pre-requisite to studying mathematics at A-level is that the Higher paper [§2.2.2.1] was taken at GCSE (available grades: U, C, B, A and A*) and a grade of at least a B obtained.

5.3.3 The Students.

The students who participated within the Pilot Study were awarded grades of A or A* at GCSE or their equivalent. The consequence is that the sample within this study represents a small subsection of the population of English students who were selected by ability at 11 and then again at 16. In this particular instance they were all confident about their own ability in mathematics particularly at the start of the course but this was to change. The pilot study sample was selected by the Head of Mathematics at the school on the grounds that he thought they would be more interesting since they were not "full time mathematicians" but were taking mathematics AS/A2 level as part of a mix of subjects that included Psychology, Art, PE, English, French and History. The sample consisted of four students who had been at the school from the outset of their secondary education and two Chinese students who had entered the school at the start of the year. The Chinese students were able to understand English but had more difficulty explaining clearly their thinking and the reasons for taking a particular action. They paid little attention to expositions but were alert to any resultant formulae or axiom. The English students, in contrast, engaged in and contributed to the expositions and took notes of the background rationale as well as copying down notes from the board.

5.3.4 The Teacher.

The teacher was just starting his second term teaching after retraining via a Graduate Training Scheme, his degree was in Business Studies. He had previously had a career in market research which culminated in him running a small company. He was very hard working and had an informal, relaxed attitude in the classroom that was appreciated by the group, but his focus was to teach the students what they needed to know to get the best grades possible in the examinations. He frequently made comments such as "I'll start you off but I won't be able to sit in the exam with you"; "make sure you learn this question as there is a very good chance it will come up in the exam"; "Unfortunately you will have to learn this for the exam" [§5.6.4]. To achieve his objective he taught specific procedures that were detailed and served to solve a particular type of problem [§5.6]. Each lesson considered a new type of problem and an associated procedure to achieve an answer. He was not, he said, a visualiser and so preferred to use algebraic means to solve problems whenever he could. He explained that he didn't like the topic associated with graph transformations and had never found much point to them. Where it was clear that images were a preferable alternative in part of a solution process, such as the use of the unit circle to find obtuse or reflex angle solutions, the image was presented complete [§5.6.4]. Privately the teacher thought the Chinese students were the strongest in the group and would perform best in the examinations. Interestingly he also was diffident about his own ability in the subject evidenced by a comment made about the Head of mathematics “He’s a real mathematician!”

5.3.5 Lesson Format

In the classrooms where the group was observed, there was a board and a projection screen. The lessons were timetabled as doubles and were 100 minutes long. The pilot study teacher followed a lesson format that started with an exposition, either at the board or using power point projected on to the screen, followed by questions chosen by the teacher from the set text. The questions in the main were technique exercises. The class either worked on the questions by themselves or conferred quietly whilst the teacher moved around the room and answered queries put by individual students. Sometimes the teacher felt the need to call the attention of the class to consider an issue that had arisen that he felt was relevant to the whole class but mostly the groups were left to explore the problems, and attempt the solutions themselves. Homework was set at the end of each lesson and usually took the form of finishing the set questions which were to be handed in to the teacher at the start of the next lesson.

5.4 Outcomes of Pilot Study

5.4.1 Initial Concept Maps

The initial concept maps for each of the 6 students are shown below (see Figures 5.1-5.8). These were drawn by the students in the second term of the AS Pure 1 course during the first lesson of the first A-level component on trigonometry. The trigonometry component of the GCSE syllabus covered: finding lengths and angles in right angle triangles using the ratios, finding lengths and angles in non right-angle triangles using the sin and cosine rule, finding lengths and angles in 3-D, the sine, cosine and tangent graphs and how to use them to find multiple solutions to simple equations such as cos x = -0.65. Interviews supplemented students perceptions of the core ideas of trigonometry prompted by the initial concept maps and investigated links being made to the current work in the lessons.

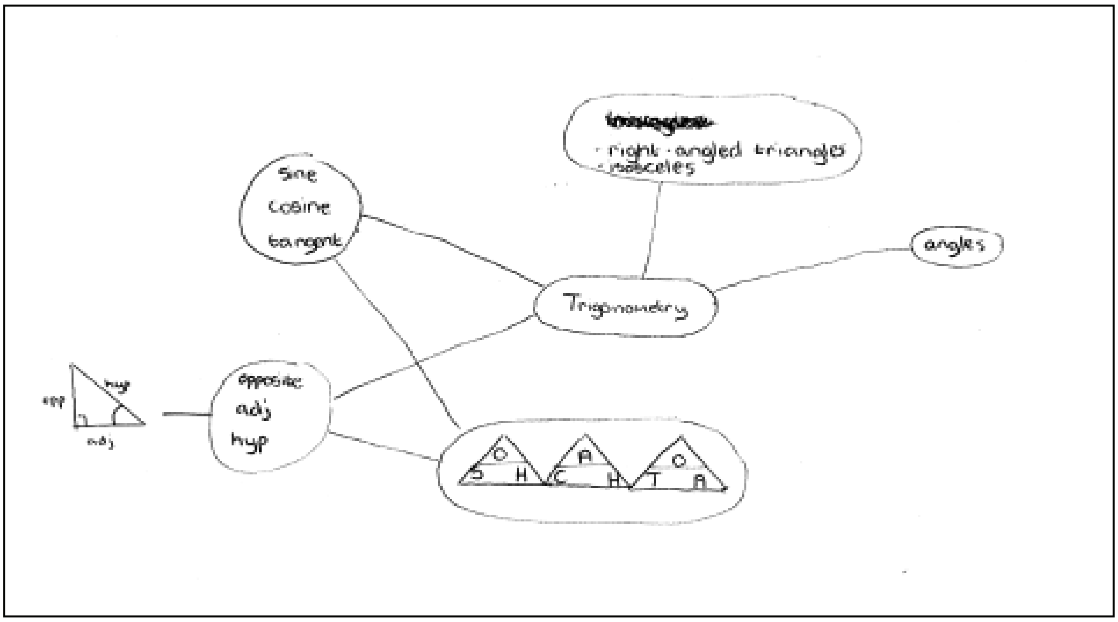


Figure 5.1: P1's Concept Map 1

P1's map (figure 5.1) is very sparse. It appears to consist of mostly notes (right-angled triangles, isosceles triangles, sin, cos, tan, angles). The two core ideas that are specifically defined are a right-angled triangle with an angle identified and the sides opposite, adjacent and hypotenuse marked relevant to the angle and the SOHCAHTOA triangles used as an operational device for choosing the correct formulation for working out unknown lengths and angles. SOHCAHTOA is a commonly used mnemonic for 'Sin = opposite over hypotenuse, Cosine = adjacent over hypotenuse, Tangent = opposite over adjacent.' During the interview after the first lesson P1 was asked what she understood by the sine of an angle:

I: What does sine θ mean to you?
P1: Sine is an angle or you can find it by opposite over hypotenuse
I: So if I draw a triangle such as this... (Figure 5.2)

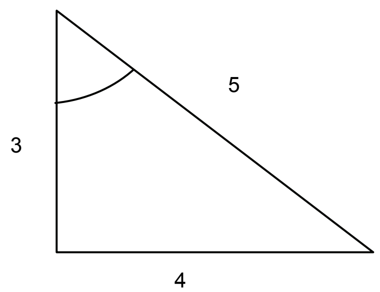


Figure 5.2. Diagram of Triangle to Identify Sin of an Angle

P1: The sine is four over five!

I: How big is the angle?
P1: You work out four over five and that gives the angle
I: Well four over five is 0.8 so is that the angle?
P1: [Diffidently] yes … I think.
I: Talk me through again how you find the size of the angle in a situation like this
P1: It's SOHCAHTOA you want sine so you do opposite over hypotenuse
I: Right. So that is 0.8
P1: Then you must do something else but I can't remember it now

P1 appeared to be recalling her knowledge entirely from memory. She was focused on remembering procedures but had no deep understanding of sine θ as a function of the angle θ, indeed she seems to think at this point that opposite over hypotenuse gives the value of the angle. In Sfard's terms the procedure had not yet been interiorised whilst in APOS terms this student had yet to fully reach the developmental stage of Process and is still at the Action stage.

In contrast to P1, and indeed in contrast to the concept maps prepared by the other students, P2's map [figure 5.3] contains the most detail.

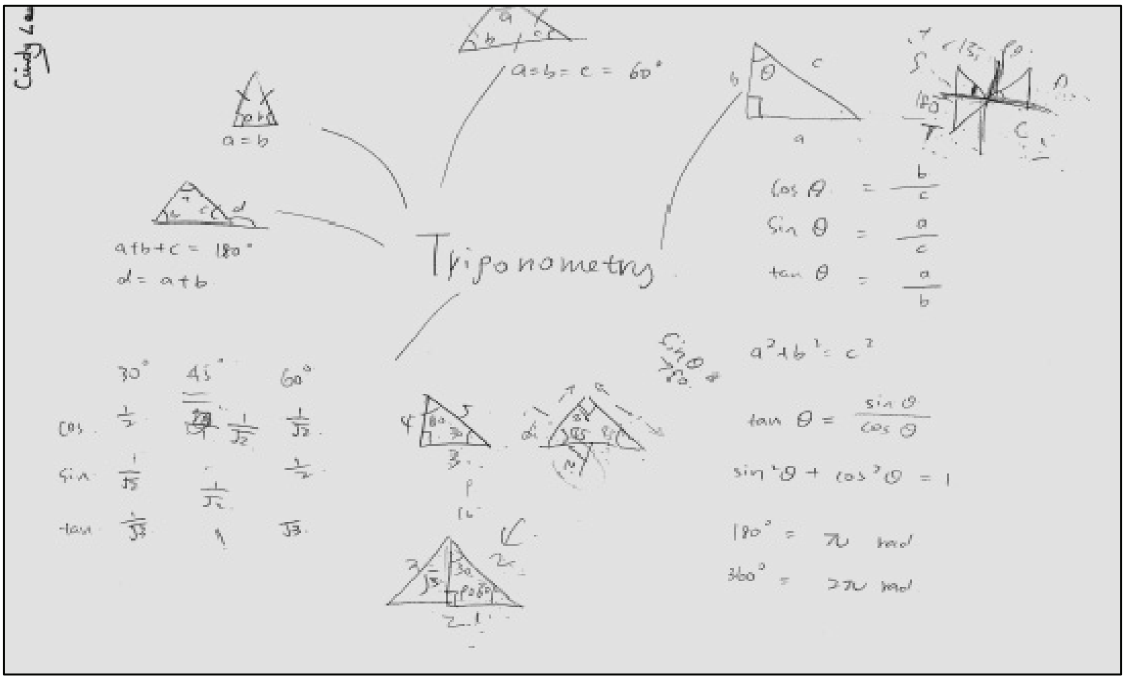


Figure 5.3: P2's Concept Map 1

However, within this map (and the one drawn by P4, Figure 5.6) we see a collection of random facts and axioms, many of which though related to properties of triangles are unconnected to trigonometry: for instance she has given a definition of an isosceles triangle and noted that the sum of the angles in a triangle total 180°. It is noticeable here that P2 has included seven diagrams of triangles, three of which describe properties of triangles but not trigonometric properties. P2 has evidently already covered some of the work contained in the course the class is about to start for instance: the identities for tan and $\sin^2\theta + \cos^2\theta = 1$. She has also encountered radians before, although she defines 180°=70 radians and 360° =270 radians indicating, that she does not properly understand the definition of radians. She included the surd values for the special angles of 30°, 45° and 60° though most of these are incorrect. She has included a diagram of the special angles triangle which is correct but has clearly not linked this to the values in her table. She has included a spatial representation of the angles in the four quadrants but not the graphs. There is a noticeable absence of any links between the items identified.

During the course of the interview it became clear that P2 focussed on remembering processes at the cost of a deeper understanding.

I: You seem to have done some of this before; did you know the things you did this morning?"
P2: Yes we learn all the graph and sin (-x) cos (-x) tan (-x) to remember.
I: So you remember each one?
P2: Yes
I: I see also in your concept map you put down a table of values for sin, cos and tan in square root form. How did you learn those?
P2: From the calculator
I: But you can't get the results in square root form from the calculator, you can only get decimals. Did you learn them from triangles?
P2: Some learnt from the triangle but I just learn them because in exam we usually just use the result so I only learn that. I just learn one over square root two and others to remember.

It is evident from the concept map and the interview that P2's objective is to do well in the assessment rather than understand the mathematical ideas behind trigonometry.

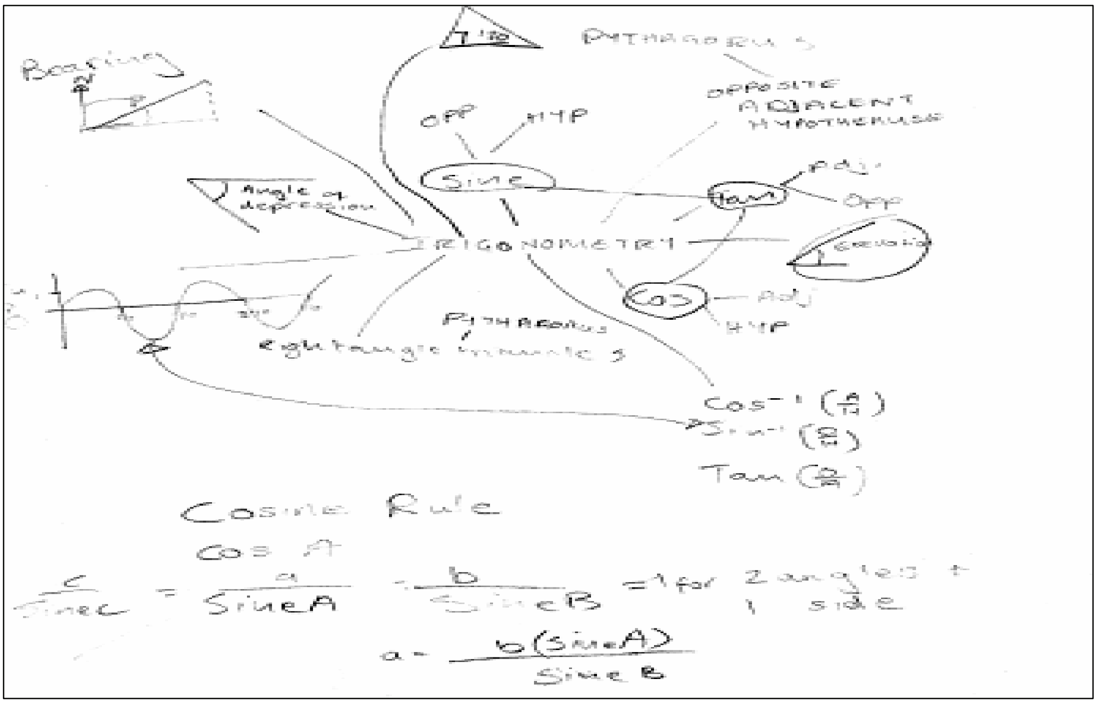


Figure 5.4: P3's Concept map 1

In P3's concept map (figure 5.4) there is again an emphasis on the words opposite, adjacent and hypotenuse and memory prompts such as the inverse sin, cos and tan formulae but this is linked to a graph, albeit only drawn for the sine function.

There are more links between the different facts and formulae and inclusions of the situations where certain formulae are applicable such as angles of depression, bearings, and the cosine rule. Trigonometry is connected to a right angle triangle which is then connected to Pythagoras implying that Pythagoras is not perceived as a sub-function of trigonometry by this student, but both are applications for right angle triangles. It would appear that applications are the main focus for this student which may be seen also by her inclusion of "angles of depression".

During the interview P3 indicated some cognitive conflict with the content of the day's lesson which had been on the development of the unit circle, and her previous conception of sine:

I: Okay so can you tell me what you understand by the sine of an angle?
P3: I don't know, I thought it was something to do with the height but …I am not sure
I: The height of what?
P3: Well in the triangle but I got a bit lost when we started on the circle and now I'm not sure
I: What do you mean the height of the triangle? Any particular kind of triangle?"
P3: Well a right angle triangle with an angle given like this… (Figure 5.5)

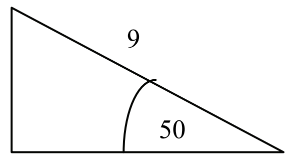


Figure 5.5 P3's Triangle to Demonstrate Sin θ

...And then you have to know the hypotenuse say that's 9 and then you can find the height because x/9 = sin... lets say this angle is 50 or something... so x/9 = sin 50 and then we can change it round so the height, that's x equals 9 times sin 50.

I: Okay. What about triangles that don't have a right angle; just any old scalene triangle ...
P3: Oh well then you would have to do the sine rule or cosine rule!
I: To find the height?
P3: No they find a length or an angle. It depends what you are looking for ...um but then... Oh sine finds a length or an angle doesn't it! So in the circle ... when you know the angle... and the radius becomes the hypotenuse... and that's one... so then you find the length of the line by doing the sin times one. So sin finds a length, or an angle!

Here is a demonstration of P3's understanding. She is thinking entirely in an operational sense; in terms of what 'the sin' does. She is familiar with the core concept that sin x = opposite over hypotenuse and how that is used to find lengths or 'heights' of triangles but was unable to link that understanding initially to the lesson's focus on the circle. During this demonstration she appears to have found a connection between her previous knowledge and that day's lesson on the unit circle. However there is no evidence here that P3 identifies sin as a function of angle. Her focus seems to be on its operational use to find lengths.

P4's concept map (Figure 5.6) is extremely sparse. It has only three items mentioned, two of which are facts associated with triangles, Pythagoras' theorem and a random fact about the lengths of the sides of triangles, but not core concepts of trigonometry.

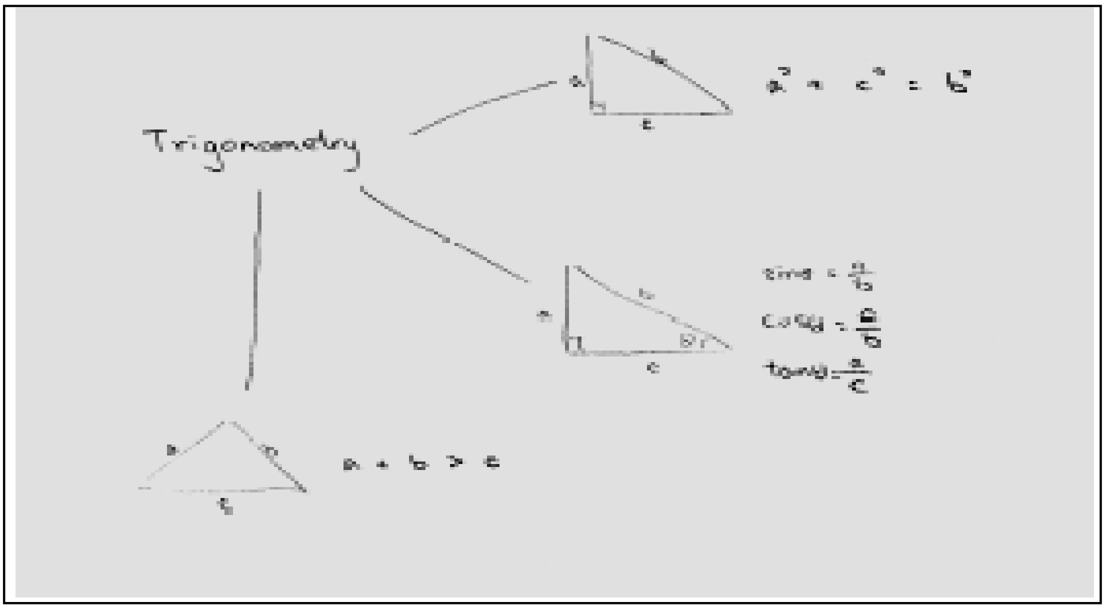


Figure 5.6: P4's Concept Map 1

During the interview it became clear that P4, like P2, prioritised memorising theorems and facts over understanding the mathematical ideas behind them. This was highlighted in the following extract.

I: What about sin over cos does that mean anything?
P4: Hmm! I don't know
I: It is an identity. Do you know what that means?
P4: I think tan. We did it in lesson
I: What do you think an identity is?
P4: Sin 60 = $\frac{1}{2}$, sin 30 = $\sqrt{3}/2$
I: How do you know those?
P4: I learned them.

Firstly it is interesting to note that the values given for sin 60 and sin 30 are wrong. They have been 'learned' without any understanding of the increasing nature of sin θ for angles $0 \le \theta \le 90$ and that $\sqrt{3}/2$ gives a higher value than ½. Here also is an insight into this student's perceptions of the words 'identity' and 'learn'. To P4, identity means 'what it is equal to' without distinction between one-case examples like the value of sin 60, and identities in the true sense which are true for all values of θ. 'Learn' here is synonymous with remember. It is a superficial means of engaging with the mathematics but one that concurred with the teacher's didactic methods.

P5's concept map (Figure 5.7) is interesting in that it gives the impression of being ordered into groups.

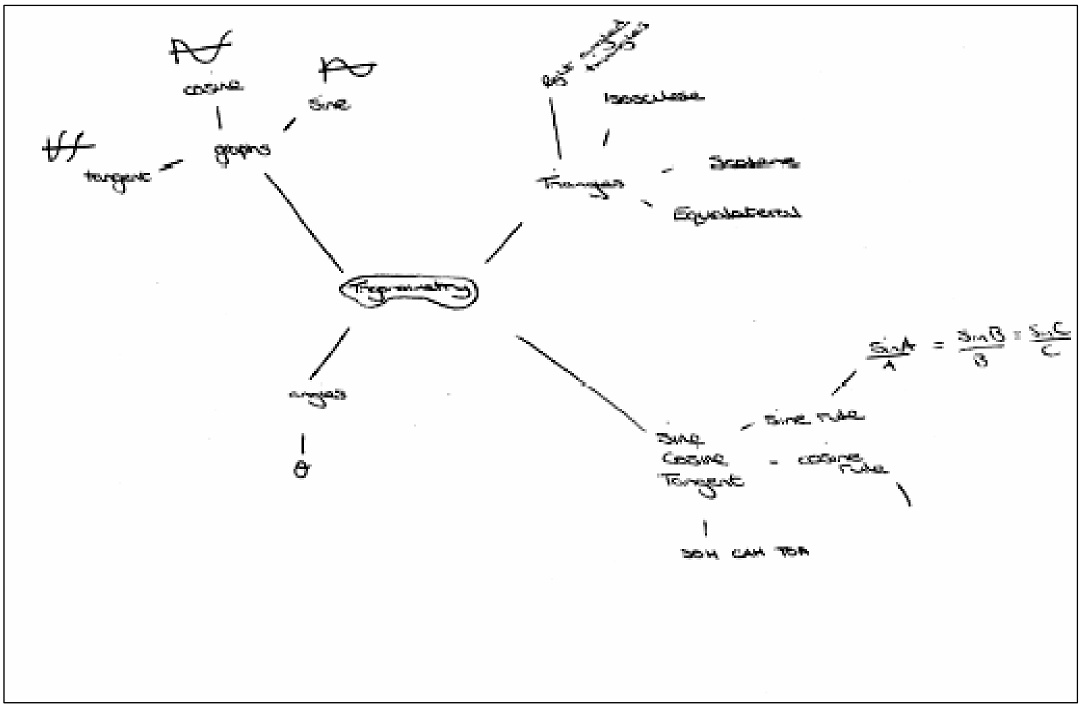


Figure 5.7: P5's Concept Map 1

There are four sections: graphs, triangles, operational uses of trigonometry and angles. This could point to a schema such as that described by Sfard in that it has both an operational dimension and a conceptual properties dimension. However there is very little in the content in terms of definitions, only the sine rule and SOHCAHTOA a mnemonic for Sine equals Opposite over Hypotenuse, Cosine equals Adjacent over Hypotenuse Tangent equals Opposite over Adjacent. Otherwise it appears to be more a series of notes. These notes appear to be focused on different types within the subsections, for example the different types of triangles are listed, and the different graphs are shown. This ordering of information into groups, especially distinct groups for core properties and operational applications is an indication of thinking mathematically. However P5 found the AS trigonometry course problematic. This may be seen in the following extracts.

P5: Well what we learn in class is different to what I learnt at GCSE

I: In what way?

P5: For example CAST diagrams! At GCSE it was all related to triangles so when you started introducing circles it confused me

There appeared to be some reluctance on P5's part to use the new methods and images.

I: To find the other angle we can use the circle or the sine wave. Which do you prefer?

P5: The sine curve [note: this is interesting as they have been using the circle and All Silver Tea Cups exclusively in class]

P6 did not complete the course as he gained a sport scholarship to study in the USA. So there is only one map drawn by him (Figure 5.8).

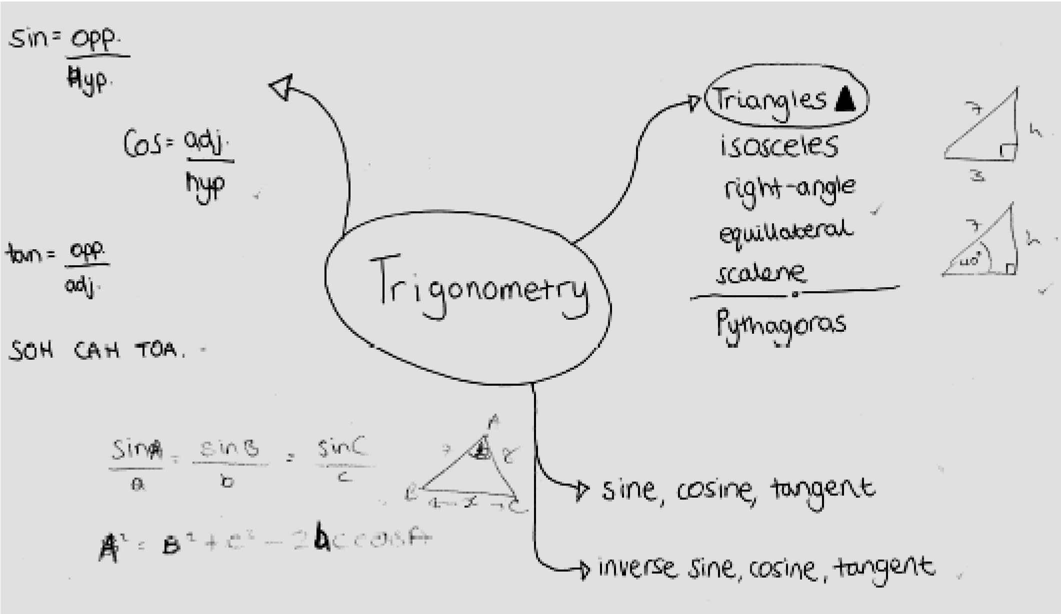


Figure 5.8: P6's First Concept Map.

P6, like P5, appears to be organising his map into groups. There are three groups: triangles, definitions of sine, cosine and tangent and operational formulae. In this group he has connected sine, cosine and tangent to inverse sine, cosine and tangent. In the operational formulae group he has included the sine and cosine rule with a spatial image showing the how angles A, B and C relate to each other and to side lengths a, b and c. This is another example where there are indications of a schema that has both an operational dimension and a core properties dimension as described by Sfard. However the overall impression his map gives is that the focus is on triangles.

In the interview it transpired that there were many omissions that he "didn't bother to include" such as familiarity with the graphs. When asked about this in the interview he explained:

I: Why didn't you include the graphs?

P6: Yeah but I know those and I put down the things that I need to remember

I: Tell me what you understand by sine.

P6: I'm not sure now. I thought it was opposite over hypotenuse but now I don't know… I'm a bit muddled."

I: Do you know anything about it, for example what is its maximum value?

P6: One

I: And does it have a minimum value?

P6: Minus one

I: How do you know that?

P6: From the graph

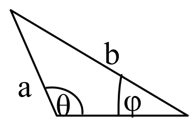
I: At which angle is the sine one?

P6: That must be 90° or $\pi/2$ radians [smiles shyly; they had done radians the previous lesson]

Clearly P6 is aware of the graphs and their properties though he did not indicate them on his map. This brought into sharp focus the issue of the extent to which the concept maps were a true representation of the students' schemas. The maps drawn by the students clearly did not show the whole picture and the interviews would have to be more probing in order to investigate the true nature of the student's conception of trigonometry. This was a refinement that was required for the main study. However the concept maps did throw light on aspects of the student's schemas that were informative. They appeared to show that four of the students: P1, P3, P5 and P6 had grouped items together whilst P2 and P4 had not. In P6 and P5's map there was an indication that operational procedures were distinguished from the core properties although these were not specifically identified by P5. The tendency to group, and distinguish between core properties and operational formulae could indicate the beginnings of a schema that has both an operational aspect and a structural object as proposed by Sfard [§3.1.4]. The perceived importance of remembering values suggest a schema that does not recognise the hierarchical ordering of the components of trigonometry. These initial concept maps also clearly indicated that, for these students, trigonometry was associated more strongly with triangles than with the angles in the triangles.

5.5 Lesson Content and Delivery

As indicated earlier [§.5.3.1], during this component of the study 11 lessons were observed over a 4 week period. These lessons focused upon radians as a measure of angle, the sine, cosine and tangent graphs and the connections between graph transformations and the algebraic representations of composite trigonometric functions, the sine, cosine and tangent of the special angles, the identities for $\tan \theta = \sin \theta / \cos \theta$ and $\sin^2\theta + \cos^2\theta = 1$ and the use of these identities in solving simple trigonometric equations. A breakdown of these observed lessons is given in Table 5.1 (below). The table is constructed to illustrate the topic being covered and phases that identified lesson delivery: whether or not there was consolidation or revision of previous work, the development of the core content, practice and reinforcement associated with this content and the emphasis when the lesson came to and end. The text book used was Mannall, G. & Kenwood, M. (2000) Heinemann Modular Mathematics for London AS and A-level: Pure Mathematics 1

Topic	Consolidation /Revision	Core Content	Practice	Closure
Lesson 1 1. Construction of initial concept maps. 2. Introduction of radians		Conversion of degrees to radians and vice versa. Construction of $l=r\theta$ where l is length of an arc and θ is angle in radians.	Construction of sine, cos and tan graphs with angles marked in radians,	Homework: text questions requiring the evaluation of sin, cos and tan of various angles between 0 and 2π.
Lesson 2 1.Use of radians to calculate the area of a circle **2.** Defining sine and cosine in any quadrant of the circle	Definition of sin θ in the circle. Reinforces SOHCAHTOA in context of triangle formed within unit circle	1.Derivation of $A={}^{1}/_{2}r^{2}\theta$ for area of circle where θ is radians. 2.Ilustration of sine rule to establish that within:  $sin\theta=\dfrac{b\ sin\varphi}{a}$ 3.Defining $Sin\ \theta$ and $cos\ \theta$ in any quadrant in terms of $sin\ \varphi$ and $cos\ \varphi$. Construction of relevant table of values	Use of formula	Homework set to learn the results in the table thoroughly.
Lesson 3 Introduction of the identity $tan\theta=\dfrac{sin\theta}{cos\theta}$		1. Algebraic substitution: sin θ =b/c cosθ=a/c sinθ/cosθ =(b/c)/(a/c) =b/a =tanθ 2. Illustration of the nature of tan in each of the four quadrants of the circle and a summary of relevant formulae. 3. Introduction of mnemonic **A**ll **S**ilver **T**ea **C**ups.	Practice finding the values of sin, cos and tan θ for all values of θ given in degrees and radians.	Homework set which continues the practice by responding to questions within the text book.
Lesson 4 Graphical representation of trigonometrical functions.		Use of computer. Distinction between continuity and non-continuity. Emphasis on limit. Periods associated with sin 3θ, cos θ, cos 2θ and cos 3θ. How does graph of y=(-sinθ) differ from y=sinθ.	Class discussion identifying graph of y=sin(θ +90	Emphasis on recording summary of outcomes as notes. so that it can be learned.

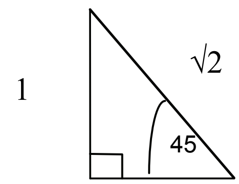
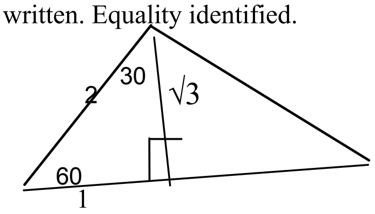
Topic	Consolidation /Revision	Core Content	Practice	Closure
Lesson 5 Emphasis on working "the sort of questions within the examination		Textbook exercises. [1]1. Alternative angles within the range 0<x<90 that would provide same values as, for example, sin 260, cos 140. 2. Reemphasised mnemonic **A**ll **S**ilver **T**ea **C**ups. (Lesson 3). 3. Finding values of sin, cos and tan θ given in degrees and radians.		Homework set that required completion of text exercises.
Lesson 6 "Trigonometrical identities and their use"		 Surd values for sin, cos and tan 45° written. Equality identified.  Sin, cos and tan values of 30& 60° Written underneath diagram. Pupils then introduced to $\sin^2\theta + \cos^2\theta = 1$ and examples of use discussed: Given θ is obtuse and cos θ =-4/5 find sin θ and tan θ. $\tan^2\theta = 2 - \tan\theta$	"There is a table of these values you can learn on page 75[2]." Class asked to learn these.	Homework was set on selected questions from the text on solving trigonometric equations by using identities.
Lesson 7	Consider previous homework questions	Core of lesson associated with responded to homework questions presented in previous lesson. All students had expressed difficulty with the questions.	Students copied solutions.	
Lesson 8	Finish trigonometry today — start afresh next week. "I want to go through some more questions with you"	Solving trigonometric equations on the board.		Homework set. Pupils advised that they would be set a test for the next lesson based upon P1 questions

Table 5.1: Individual lessons of the Y12 Trigonometry Course

[1] Mannall,G. & Kenwood,M. Heinemann Modular mathematics for London AS and A-level: Pure Mathematics 1

[2] Ibid

Three further lessons formed part of the observation programme. Lesson 9 was a test on the work covered, during lesson 10 the teacher went through the test and during lesson 11, as a part of the current study, the class were invited to answer the questions designed to investigate how much of the course content they had assimilated and construct new concept maps for trigonometry (§5.7.2)

It is worth noting that, with the exclusion of lesson eleven, of the other ten only five introduced new content, three were directly associated with responding to examples and two were associated with doing a test and responding to its results. Other issues worth noting include:

- The sense that they are fragmentary —discrete presentations within the general topic under consideration but few if any links are made between each component unless such links were associated with revising or consolidating the way to deal with particular examples. Reinforcement of the notion of SOHCAHTOH, a mnemonic for the ratios 'sin x = Opposite over Hypotenuse, Cos x = Adjacent over use Hypotenuse, Tan θ = Opposite over Adjacent' (first met in year 11) proved an exception to this general form of delivery. However, during lesson 1 the teacher referred to Sine being a function; this was not questioned by the class and the teacher did not expand on the statement.

- There was little clear exposition of the objectives of each lesson. Only within lessons 5 and 6 were explicit objectives given and only one of these, lesson 6, made a clear reference to an aspect of trigonometry that would be studied during the lesson. Within other lessons this was made in a more oblique way. Typically the teacher began the lesson with an exposition that was new content without explaining its purpose, uses or links to other lessons.

- Core content was always presented through the medium of teacher exposition. Although there was evidence of pupil/teacher interaction [§ 5.6.2] the essential form of delivery derived from teacher instruction and partial reaction from the pupils.

- Whilst opportunities to practice skills directly associated with new concepts were apparent within lessons 2, 3, and 8, within most lessons a distinction between the presentation of core ideas and the opportunity to practice were limited. Essentially lessons were devoted to the core idea although there was an opportunity to consolidate these ideas (lessons 1and 4). As may be seen from within Table 5.1 lesson 7 was devoted to practice but this was generally in the form of procedural application.

- Practice was supplemented by individual work at home but as seen, lesson 7 was totally devoted to remediating, on a class basis, difficulties experienced by the pupils, lesson 8 gave opportunities to consider a range of questions, again on a class basis whilst lesson 9 was devoted to working through past AS assessment questions.

- Some considerable emphasis was placed on memorising particular features of the topic. We see this within lesson 3 when the emphasis was placed on the mnemonic **A**ll **S**ilver **T**ea **C**ups, re-emphasised within Lesson 5, and on the learning of tables of identities (lesson 6)

- Pupils interpreted remediation associated with homework difficulties (Lesson 7) as an opportunity to copy 'model' answers.

The last point identified within this summary of the structure of the lessons taught raises an issue associated with the underlying objectives associated with teaching and learning the topic — the AS examination. This will be considered within the next section, whilst §5.6 will consider pupil attitude and understanding from first a qualitative perspective as evidenced from the exchanges and comments obtained from the classroom observations and then from a quantitative perspective in terms of their achievement in the two tests, the one presented by the classroom teacher and the one presented as part of this research study.

5.6 Teacher Emphasis in Delivery

The research by Delice and Monaghan (2005) found that students' trigonometrical schemas are dependent on the teacher's emphasis in delivery of the content. Bayazit (2005) concluded that students' conception of function is dependent on the teacher's emphasis, use of language and the way in which the teacher deals with students misconceptions. This next section describes the teacher's methods in these regards and his teaching emphasis.

5.6.1 Teaching Objective

It was clear from many of the teacher and pupil comments that the AS examination dominated teacher actions/words and student reaction. Indeed many of the responses of the former to questions associated with understanding were linked to examination requirements and needs.

After instructing the class to look at a question from the text book, the teacher gave a demonstration of the solution process on the board. He pre-empted his demonstration of the solution to this question by saying:

This is the sort of question you will get in your exam so I will do it with you.

However, when the students did experience some difficulties the teacher was sympathetic as instanced by his opening comment during lesson 7 after identifying that the students had had difficulties with the previous lessons homework.

Teacher: I know trigonometry is hard at first but hopefully the penny will drop before June.

The mention of June triggered the notion of examination within one of the students:

P3: Is that when the exam is?

Teacher: Yes. It's the first week in June

The thought seemed to dominate in the mind of this particular student as the following exchange, prompted by attempts to solve a quadratic trigonometric equation could not be factorised, illustrates:

Teacher: Does it factorise... I don't think so, so we have to use the quadratic formula [writes the formula on the board]

P3: At this point we walk out of the exam as its getting ridiculous!

Teacher: ... find the answer then inverse cos then find all the solutions using the circle.

P3: Dear God! How long do we have to do this exam!

It may be noted here that the teacher is listing a set of procedures [§5.6.3]. It was possibly in recognition of his student's uncertainty that the class-teacher indicated at the start of lesson 8 that:

Teacher: I want to go through some more of these questions with you. I'll start you off but I won't be able to sit the exam with you,

Only to receive the response:

P1 & P3: Why not! We need you!

The dominance of the needs of the examination and its relationship with the topic area appeared to be summed up in the following exchange at the end of lesson 8:

Teacher: What do you think of trig?

P6: Can I tell you after the exam?

It is clear that an explicit objective of the teaching and learning that is taking place in this class is the desire for the students to do as well as possible in the forthcoming assessment. Though this may be a general teaching objective of all teachers in this instance; the pressure of examination performance appears to be foremost in this teacher's mind.

5.6.2 Clarifications Leads to Confusion

During the sequence of lessons the pupils illustrated their effort to understand in their attempts to engage in responses to issues raised and in their attempts to seek clarification for some of the nuances associated with trigonometry. However, the teacher's response frequently caused frustration.

An example is shown here of one students attempt to clarify the need for radians at the end of the first lesson and the teacher's response to the enquiry.

P6: Why do we need radians Sir, when we have degrees?

Teacher: Mathematicians prefer them because it makes life so much easier. You see, instead of having to deal with awkward numbers like 90 and 180 we can use π and $\pi/2$ instead.

P6: And that makes life easier does it Sir.

The teacher was attempting to be light hearted but it was misplaced as this student's enquiry was genuine and ultimately unresolved. Exchanges of this sort appeared to permeate these lessons. Another example is given here associated with the use of Greek letters.

P1: Why do we have to use Greek letters? It makes understanding even more difficult.

Teacher: I'm sorry but you will have to get used to it, the angles in trigonometry are often given in Greek letters and on the exam they will be almost definitely.

P6: I suppose the 26 in the English alphabet will have all been used up by the time we get to the trigonometry question.

Clear confusion demonstrated by the whole class was displayed during lesson 4. During this lesson the class were asked to look at the graph of $y = (-\sin \theta)$ and asked how it differed from the graph of $y = \sin \theta$. P6 suggested that one was the inversion of the other so the teacher proceeded by asking the class to compare them with the graphs of $y = \cos \theta$ and $y = (-\cos \theta)$. However, P1 who was confused asked:

P1 : Why?

Teacher: Because the φ angles in the unit circle become positive then negative then negative then positive as the radius goes round clockwise which is the same as when you go round in the positive direction.

The class looked at each other bemused.

The teacher's attempts at clarification may be the result of his inexperience however the effect it had on the students was to increase their anxiety with trigonometry and promoted the desire to memorise results which he strongly advocated [§5.6.4]

5.6.3 Action Orientated Language

One of the issues raised in the research literature associated with the teaching of mathematics is the use of language. Tall et al (2000) and Bayazit (2005) make the point that Action–orientated language which avoids reference to definitions does not encourage students to construct an object concept of function as a process that transforms inputs to outputs. Instead it encourages the students to remain focused on the procedural aspects. [§3.4.1]

The teacher's language was action orientated. An example is shown above [§5.6.1] Another example is the following problem set for the previous lesson's homework which the students had difficulty with. The teacher went through it during lesson 7:

$$\mathbf{3\sin^2 x - 2\sin x - 1 = 0, \quad 0 \leq x \leq 2\pi.}$$

Teacher: What do we do first?

P6: Factorise

Teacher: So that gives (3sinx +1) (sin x-1)=0, sin x =-1/3 or 1. Now we inverse sin these answers on the calculator and we get x =-0.34 which is the same as 5.94 but there is always another solution.

Later during this solution process consideration was given to sin x = 1.

Teacher: So we inverse sin 1 which equals 1.57 rads

P5: Isn't that π/2?

Teacher: Yes, so those are the solutions.

P3: Why aren't there other solutions?

Teacher: Because there is only one solution for that.

This explanation is entirely action-orientated. The teacher began the exposition asking what do we need to do and then continued with predetermined procedure. There was no specific reference to the equation as a quadratic in sin x. The students could have solved this by factorising, using the quadratic formula or completing the square. Factorising is probably the most efficient method but no mention was made as to why this method is preferable to any other. The first solution had been found, with the teacher here emphasising the process of inverse sin (-1/3), but it was out of range. This was not emphasised. Finally no explanation was given for the number of solutions to each component part of the solution. The teacher's emphasis is on a step by step procedure that is undertaken without pausing to reflect on how any part might link to other representations or processes.

5.6.4 Teacher's Emphasis on Remembering

There were numerous examples during the lessons of the teacher's advice that the students should memorise the results of his expositions. A few of these are given here with the students' responses.

After giving the class the identity tan θ = sin θ/cos θ, (lesson 3) the teacher continued by listing definitions of tan θ as follows:

Tan θ = sin φ/ cos φ = tan φ (1st quadrant)
Tan θ = sin φ/ -cos φ (2nd quadrant)
Tan θ=- sin φ/ - cos φ (3rd quadrant)
Tan θ = - sin φ/ cos φ (4th quadrant)

Figure 5.9: Table of Tan φ (the Acute Angle) in the 4 Quadrants.

And then provided the following a summary:

	1	2	3	4
Sin	+	+	-	-
Cos	+	-	-	+
Tan	+	-	+	-

Figure 5.10 Summary of the Nature of Sin, Cos and Tan in the 4 Quadrants.

This exchange then took place:

Teacher: Unfortunately you have to learn that.

P5: Is that as well as everything else you teach us Sir?

Teacher: Well there is a mnemonic to help you.

The teacher drew a unit circle without any angle specified but with the four quadrants labelled as follows:

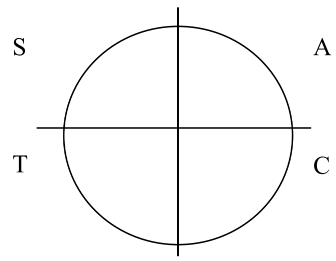


Figure 5.11 Diagram for mnemonic ASTC

The teacher explained that this summarised where the angles where positive with A indicating All, S indicating Sin, T indicating Tan and C indicating Cos the teacher continued:

Teacher: Now the mnemonic to help you remember is All Silver Tea Cups.

But this didn't appear to help P1 who gave some evidence of attempting to understand the mnemonic but was confused with its orientation

P1: Sir, All Silver Tea Cups makes me want to put All where Silver should go.

Teacher: Always start at the x axis and go round anti-clockwise.

No explanation was given as to why the letters ASTC are placed in the specified quadrants. The recommendation to remember seemed to dominate teacher delivery and this had an effect on the students' perceived method of learning. After an exposition on the solution process for solving the equation $\text{Sin}^2\theta - \cos\theta = 2,\ 0 \leq \theta \leq 4\pi$, (lesson 8) the following exchange took place:

P3: In that one Sir, that's like everything we have ever learnt in one question. If I just learn this would that be alright?

Teacher: Yes learn that one.

Though the indications here are that there appears to be a reliance upon a model solution that could fit all questions of a particular type there was also evidence to suggest that individual students attempted to see things from a relational perspective — they did not only desire outcomes but also how these outcomes were arrived at. Where appropriate they could use modifications of established mental representations to bypass the need for memorizing fundamental aspects of the course. This was perhaps best exemplified in an exchange between pupils and the teacher during lesson 6 which focused upon trigonometrical identities and their use. After using a triangle as a representation to illustrate the surd forms of the sin, cos and tan of 45° the student were particularly interested to note that the sin and cos were the same:

P5: That's interesting they are the same!

P6: It's where they meet on the graphs.

Teacher: There is a table of these values you can learn on page 75.

P5: No I prefer the triangles; it's easier to hold it all in my head.

5.6.5 Summary of Teacher Emphasis

The overall impression received from the observation of teacher and pupil engagement during this series of lessons was that:

- Aspects of this syllabus were covered in a hierarchical way but there was seldom explicit cohesion in terms of their relationship. Links between geometric and algebraic representations of trigonometry were not identified specifically. The exception was the lesson on graph transformations but this was not referred to after that lesson.
- Though the chapter referred to trigonometrical functions the explicit nature of a function was never addressed. There was never any attempt to specifically identify the similarities and differences associated with each of the functions.
- Concepts were introduced as a representation that had already been condensed, either as a formula or spatial visual diagram that students were expected to memorise.
- The teacher's emphasis was on the procedural and his language was action-orientated.
- The ultimate objective for this teacher was for his students to pass the assessment as well as possible. The means to this were to practice solution procedures with past paper questions or typical questions from the text exercises.

What this all led to could be summarized in an exchange that took place in the final lesson after the teachers admission that the test was "much too hard"

P3: I see it now but I wish I could keep it all in my brain. I can't seem to remember it all.

Teacher: I think you all understand trigonometry alright but you just need to do some more problem solving.

P3: I thought that might be coming.

5.7 The Students Development of Trigonometry

The development of the students' trigonometry schema during this course was considered from four different aspects. The responses they made to the questions on content designed by the researcher [§5.7.1], the concept maps that they drew at the end of this course on trigonometry, the interviews which were conducted after the course [§5.7.2] and the students results and comments after attempting the past P1 questions set by the teacher [§5.7.3].

5.7.1 Task Questions

This set of questions, unlike those set by the teachers, was not designed to give experience in answering the type of question that would appear in the examination. Instead it was designed to investigate how much of the syllabus content [§5.5] had been added to the knowledge that each student had previously specified in their concept maps [§5.4.1]. P6 had left England by this time so only the remaining five students attempted these questions.

The first question was set to investigate the student's ability to use the radian formulae: $l=r\theta$ and $A=1/2r^2\theta$ (lesson 2)

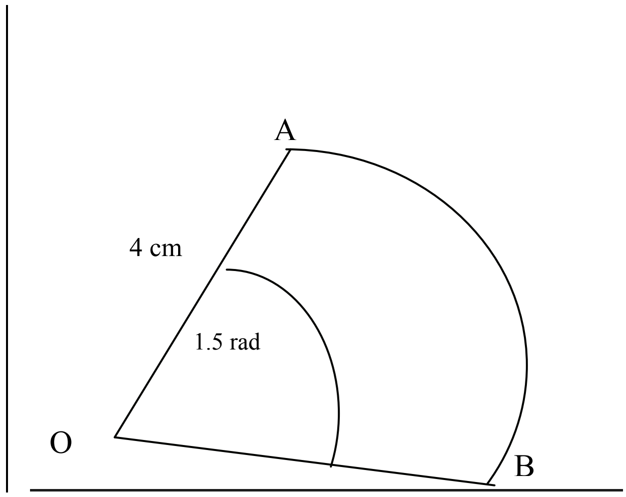


Figure 5.12 Question 1

Q1 a) Find the length of the arc AB

b) Find the area of the circle sector

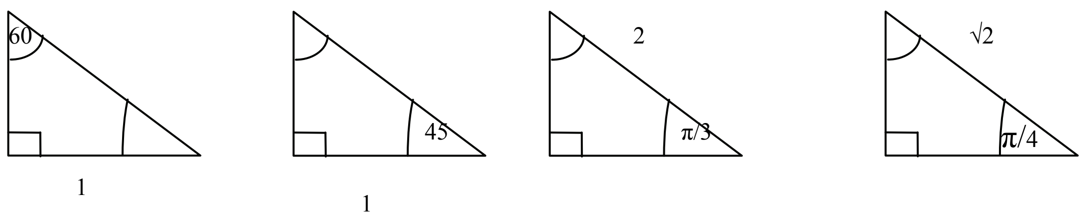
P2 and P4 use the radian formulae correctly. P1 and P3 found the right answer to a) by using ${}^{1.5}/_{2\pi}$ x2πx4 (P3) and ${}^{\theta}/_{2\pi} = {}^{AB}/_{2\pi r}$ (P1). This is relational knowledge about the ratio of the angle to 2π is in proportion to the length of the arc to the circumference of the circle.

P5 divided the circumference of the circle by the radian value which gives the wrong answer. The students approached part b in a similar fashion to part a.

It may be concluded that P2 and P4 could use the two formulae correctly. P1 and P3 had not remembered the formulae but had remembered that there are 2π radians in a circle and could deduce the correct answer by considering how the ratio of the angle given to number of radians in a circle corresponds to the arc length and the circumference of the circle.

This next question was set to see how well the students were able to recall or recompose the special angles, and then whether or not they were able to use this knowledge to find the sin, cos and tan of other related angles in degrees and radians (lessons 6 and 3).

Q2. a) Complete the triangles by marking in all lengths and angles. (Diagrams not drawn to scale)



b) Hence or otherwise determine the exact value of:

(i) Sin 120°

(ii) Tan 315°

(iii) Tan 5π/6^c

(iv) Cos (-π/4)^c

Figure 5.13 Question 2

P1 and P2 completed the triangles in part a) correctly. P2 also found three of the four angles in part b). For tan 315 she gave the answer 1 instead of -1. P4 and P5 used their calculators to find decimal values for the lengths in the triangles. P4 gave some of the angles but not all and P5 found none of the angles. In part b) P4 correctly identified the corresponding acute angles using formulae such as sin 120 = sin (180-120) etc and whether they were positive or negative but did not use his answer to part a) to give the values. P5 gave the answers as decimals to 2dp.

P1 gave decimal values for part b) as she explained subsequently:

> I was couldn't remember how to relate these angles with the triangles.
>
> (P1)

P3 correctly completed three of the triangles but did not get the correct lengths on the first triangle. She used the unit circle and her answers from the triangles to find the angles given in degrees and cos (-π/4) in part b) but was unable to find the answer to (iii) though she did recognise that 5π/6 has a connection to 30°.

We can conclude that P2, P1 and P3 were, for the most part, able to reproduce the special angle diagrams. P3 gives evidence of relating her answers to part a) with the questions in part b) and used a spatial visual image to help her determine whether they were positive or negative. P4, and P5 were dependent on their calculators to answer part a) and P1 and P5 were dependent on their calculators to answer part b).

The following question was set to investigate how well the students were able to connect the algebraic representations of trigonometric functions with geometric graph transformations (lesson4).

Sketch the following curves for $0 \le x \le 360°$ showing any maximum and minimum values and where the graphs intersect the x-axis.

a) Y = Sin 2x
b) Y = -cos x
c) Y = 3 sin x
d) Y = Tan (x-90)

P2 and P1 gave four correct answers. P3 and P4 drew three correct. P3 drew no graph for part b) and P4 drew y=sin3x for part c). P5 drew part c) correctly only. Instead of y=sin 2x he drew y=sin 1/2x. For y=-cos x he drew a graph that could be described as an inverted cos graph between 0 and 90° but the its maximums and minimums were randomly placed at 0°,135°, 225°, and 315.

The conclusion is that, apart from P5, the group had some knowledge of the connection between algebraic representations and graphical representations of trigonometric functions. The questions, however, were very simple involving only one transformation in each case and the point should be made that all the students should have been able to do these without error.

The following questions was set to investigate the students ability to use the 2 identities learned in solving simple equations (lessons 6, 7 and 8)

Q4. Solve for $0 \leq x \leq 360°$

a) $\text{Sin } x = 2 \text{ Cos } x$

b) $\text{Cos}^2 x - \sin x + 5 = 0$

P2 and P4 answered both questions correctly. P1 and P5 did not attempt them. P3 wrote tan x =sin x /cos x but finished with tan x =1/2 instead of 2 and from this found 4 solutions instead of 2. She found the correct answer to the second equation but failed to give all the solutions.

In conclusion we can say that despite 3 of the 8 lessons being devoted to solving equations of this type, only P2 and P4 were able to find all the correct solutions.

The results for the task questions were as follows:

P2: 83%

P3: 66%

P4: 61%

P1: 49%

P5: 20%

The mark achieved by P2 was significantly better than those of the others. The group concluded that she was brightest in the class and her style of learning was thought to be the most effective and it subsequently became the model for the others. P1 and P4 resolved to make more effort to "learn" all the formulae better and P3 abandoned her attempts to 'understand' what she was learning in a meaningful way. She explained:

P3: It takes too long to learn it properly…I mean it's better to learn it properly but it just takes too long and we have the first module coming up soon so I think P2 has a better method…I mean look at her mark 83%!

The question now becomes what is it that P2 was doing, how did it differ in quality to what the others are doing and did it justify being a model of learning of mathematics.

P2's learning focus was to memorise key results from the lessons and the text. To prepare for the assessment she said she:

> Learnt it then do lots of questions to get faster. (P2)

In the class she paid little attention to the expositions given by the teacher and only became focused when the result was given. The teaching style suited her learning style as the teacher went through the expositions in a way that had been predetermined and led directly to the result. He then recommended that the result be memorised. P2 never asked questions during the class expositions nor did she confer with the other students. She was able to recall and apply the correct procedure in almost every case. She was however, unable to think creatively when presented with a problem that she had not met before nor was she able to answer questions that required an interconnected knowledge of trigonometry as a whole as may be seen from the results of the past P1 questions. [§5.7.3]

5.8 Post Course Concept Maps

We now turn to consider the development in understanding as represented on the concept maps completed at the end of the course. [§4.6.] These were completed during the final lesson of the trigonometry component. First we consider P1's map (Figure 5.14).

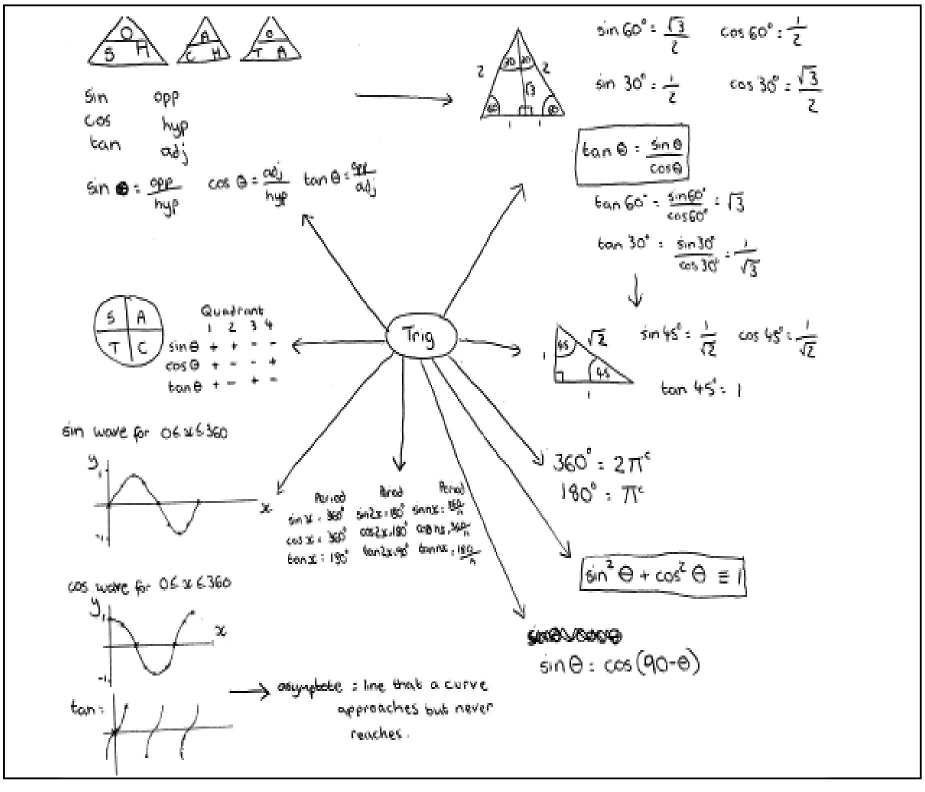


Figure 5.14:P1's Concept Map 2

P1's second concept map contained much more content than her first pre-course map. Now she included illustrations of trigonometrical functions and their graphs, tables of values and an illustration of the mnemonic taught within the class. It includes new facts and formulae, surd values, identities, the unit circle, a concept graph and the teacher's table of the positive and negative values of the three trigonometric functions in each quadrant.

It is interesting to see that although the concept image graphs of sine, cosine and tangent have been drawn on the left it is evident that they have been plotted.

The final point to make about the second concept map is the complete lack of connection between the different items identified. In the first map, despite its sparsity, the different items are shown as linked; yet in the second map there are no links of any kind apart from those connecting the items to the word trigonometry. The overall impression is of many legged spider with the item at the end of each leg distinct and isolated rather than the hoped for network of connected sub-concepts, images and related facts and formulae. When questioned about the lack of connection P1 explained:

P1: Well they are sort of connected because it's all trigonometry but at A- level trigonometry is different. It's a lot harder and there is more to remember
I: What do you think is the key to being good at trigonometry?
P1: Having a good memory.

When questioned about plotting the graphs, P1 explained:

P1: I can never remember which one is sine and which is cosine so I always work out the values to make sure.

The inclusion of so many values is a sign that was to re-emerge in the main study [§7.10.]

Figure 5.15 illustrates P2's second concept map.

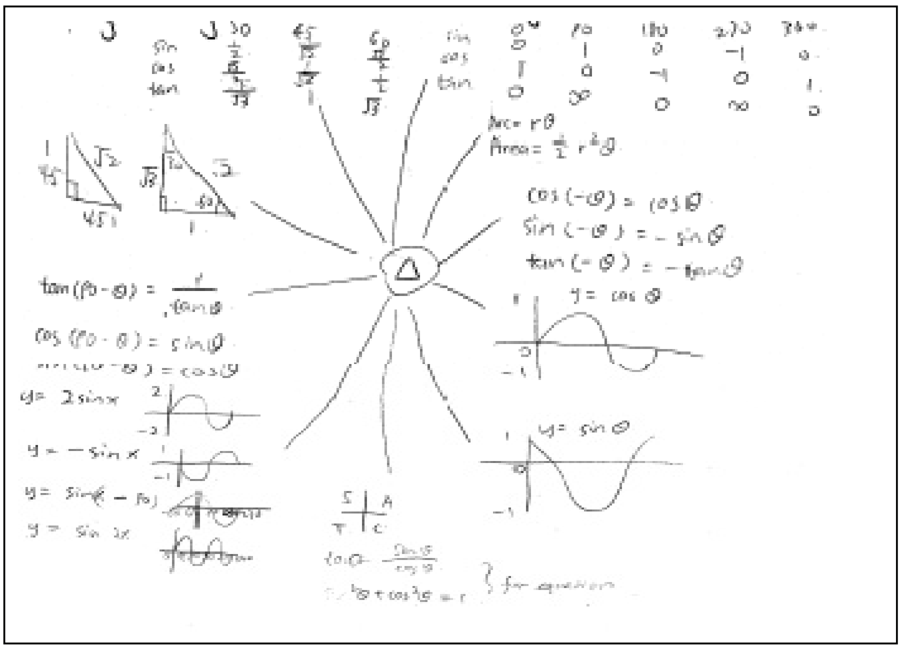


Figure 5.15: P2's Second Concept Map

P2's second map includes many of the core properties of trigonometry. She has omitted the general triangle properties but it is interesting to note that in place of the word 'trigonometry' she has drawn a triangle so the connection is still being made. She is the only student to include the radian sector arc and area formulae or the correlation between cos, sin and tan (-θ) and cos, sin and tan θ. She has defined tan, cos and sin (90-θ) in terms of sin, cos and tan θ but interestingly she has drawn the graphs of y= sin θ and y=cos θ incorrectly. The graph of y= sin (x-90) is also incorrect. The conclusion may be made that she has not connected the characteristics of the graph of y=sin θ to the graph of y=2sin θ or y=sin 2x etc. Of the six graphs she has shown three are incorrect. She has included two tables of trigonometric values, one for the special angles in surd form (the correct values given this time) and one for the values of 0, 90, 180, 270 and 360°. These values are correct therefore she has not connected this table of values to the graphs. This points to a lack of connection between the geometric aspects of trigonometry. P2 has included the two identities covered in this chapter and a spatial image of the CAST diagram with a note 'for equations'. This points to an operational concept of the identities. Finally there is no indication of linkage either explicitly or implicitly in this map. This was supported by her comments in interview where she repeated the comment made above [§5.7.1] and went on to say:

> You only have to learn the things that the teacher tells you at the end of the lesson. In the assessment you don't have to explain how it is but you just use it. I don't bother to learn it all- just the conclusions and remember those.
>
> (P2)

P4's second map (Figure 5.16) shows the two identities covered in this chapter, the graphs of sin, cos and tan x, a spatial image of a triangle with an angle marked θ with the ratios for sin, cos and tan defined beside it. There is a spatial image of the CAST diagram and a table of correct values for the special angles. There is also a formula for Pythagoras theorem. There is no explicit indication of connection between these different representations but the results mentioned are consistent unlike P2's second map. In the follow- up interview, P4 said that (like P2) he just learnt the results from the lessons.

> They are what is important. To pass the test you have to learn those things. It doesn't matter if you don't understand everything just learn the important things that are in the test.
>
> (P4)

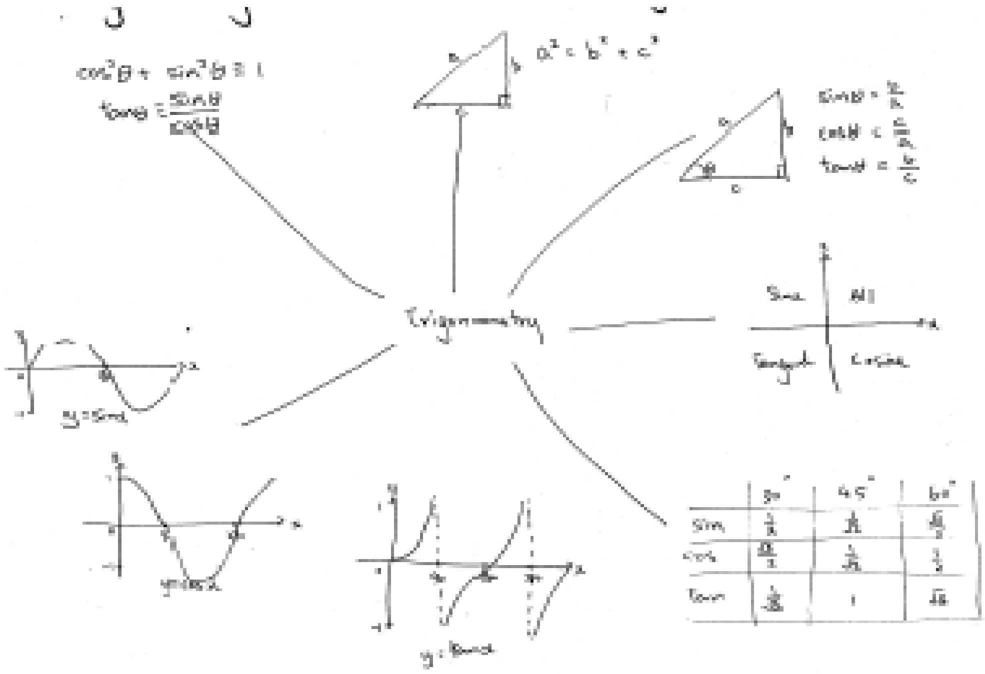


Figure 5.16 P4's Second Concept Map

Unfortunately the reproduction of P3's second map (Figure 5.17) is poor.

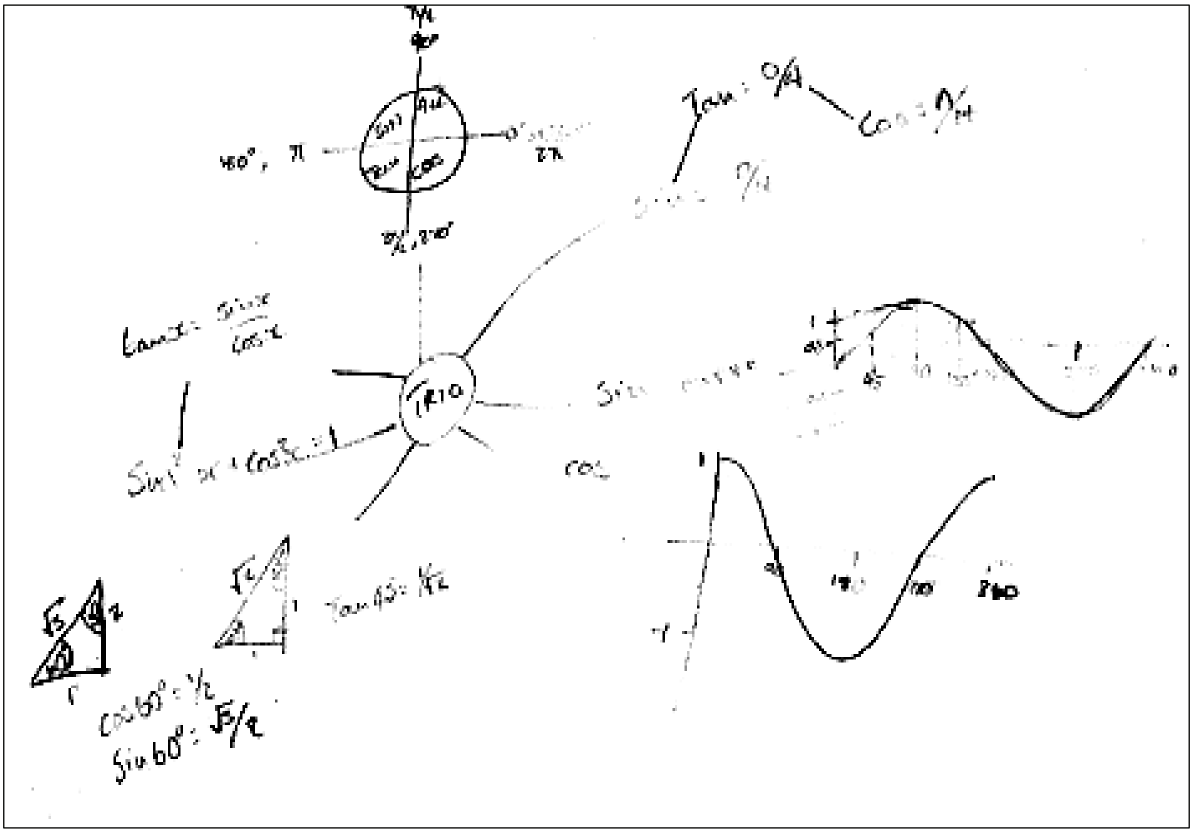


Figure 5.17: P3's Second Map

However, we can see that she has included the two identities, a representation of The CAST diagram in degrees and radians and the ratios for sin, cos and tan. She has also included diagrams of the special angles triangles, though the one for 30° & 60° is incorrect, and underneath are two values for the sin 60° and cos 60° clearly derived from the incorrect triangle. The graphs of sin x and cos x have been included with details of the maximum and minimum values and the angles where these occur. The main feature is the lack of connections between the items mentioned in comparison to her first map (Figure 5.4) however there is some attempt at grouping. Many of the items that were mentioned on her initial map have been omitted.

The second interview with P3 revealed that the reason she had left out items mentioned in her first map was that she knew them. She said that her understanding of trigonometry had changed because now it included the circle as well as triangles but there was still no clear indication of perceiving trigonometry as a function of angle. This extract exemplifies this:

I: Could you explain in a sentence the connection between say sine and an angle?
P3: Well sin will find the angle or if you know it already it will find the length or what ever you want to find.
I: What are functions?
P3: That's like… when you have f(x) equals something, like f(x) = 2x say
I: Do functions always have to have f(x) equals something?
P3: Well the ones we've done have always been f(x). No sometimes we have g(x)… Yeah either f(x) or g(x) that's what we learned. It's another way instead of writing y equals something, just say f(x) equals it…it's the same thing.
I: Why do you think this chapter is called trigonometric functions?
P3: I don't know… it should say f(x) = sin x then it would be a function.

It appears that P3's recognition of functions is determined by the use of function notation. In the lessons 'trigonometric function' is a phrase frequently used by the teacher for example, "today we are going to graph trig functions" "what happens to the graph of the sine function when we make it negative?" "the thing about these trig functions...". The students, including P3, also used the word, for example, "it's a trig function" "the sin and cos function oscillate between 1 and -1 but the tan function can go to infinity", but the concept of trigonometry and the concept of functions appear quite distinct to P3. It is worth reiterating here that the relationship was not made explicit by the teacher and the students were not given any indication of the properties of function.

Functions had been part of the work covered the previous term, along with Algebraic processing skills and Equations and inequalities (see Mannall & Kenwood, (2000) Heinemann Modular Mathematics for London AS and A-level, Pure Mathematics 1). The definition of a function is given as:

> A mapping between two variables, usually called x, the independent variable, and y, the dependent variable. The function, or mapping, is given by f and written as f: x→y
>
> (Ibid, p51)

This could explain P3's reference to the notation. It may be noted however that P3 still does not appear to have an informal understanding of sin x as a function of x. She still thinks of it as operational tool or finding lengths and angles.

Figure 5.18 shown below shows P5's second concept map.

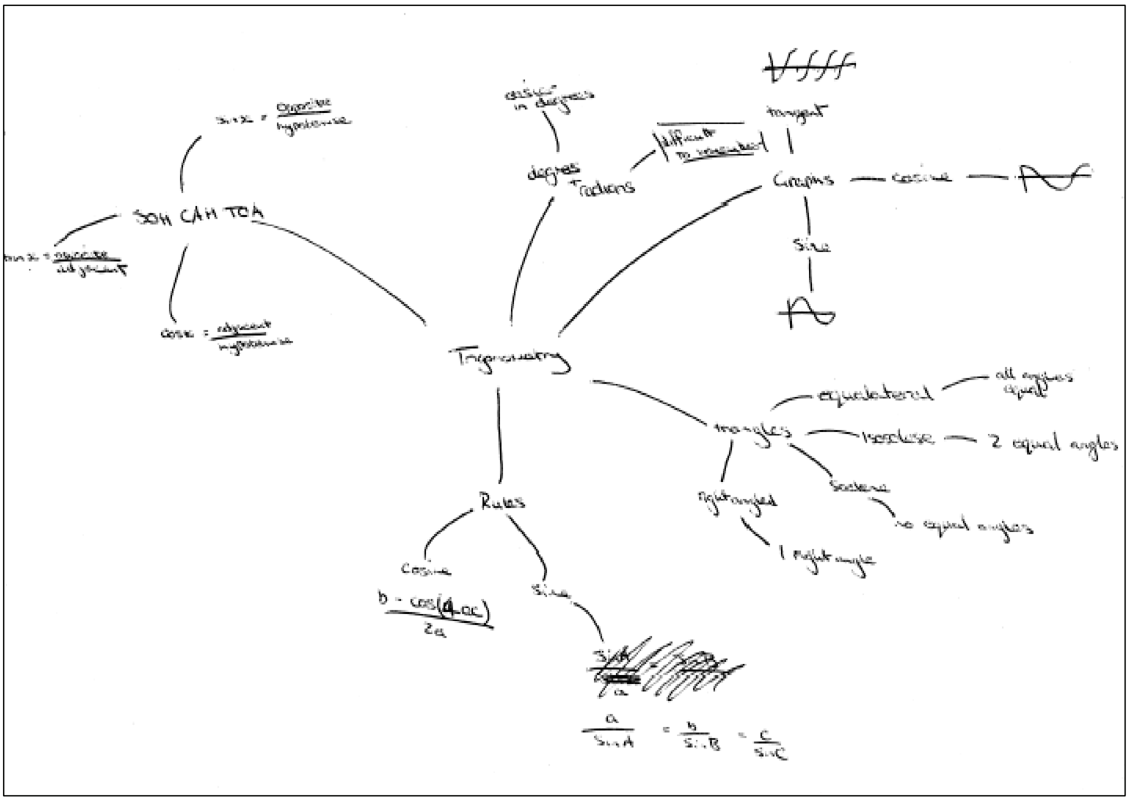


Figure 5.18: P5's Second Map

P5's second map (Figure 5.18) is consistent with his first (Figure 5.7) in that it appears to be a set of notes. It has maintained a grouped structure although the ratios and SOHCAHTOA have been put in a separate group this time. The triangles group is indicated and expanded indicating a strong association with triangles. The angles group now shows radians as well as degrees with notes stating that radians are 'difficult to remember' and its 'easier in degrees'. Under Rules the sine rule and a version of the cosine have been defined. The graphs group shows an incorrect graph for 'tangent'. There is no mention of the identities, the circle or the special angles; either represented spatially or as values.

In interview P5 was asked about how his understanding of trigonometry had changed after studying this component. He said:

> It hasn't really. What we learned in class was different to what we learned at GCSE, for example, CAST diagrams; I mean what is that all about. Trigonometry was always related to triangles before so when you started introducing circles I was lost. I remember the graphs from GCSE and we did some work on surds which were fine on their own but surds related to trigonometry were something else.
>
> (P5)

The evidence of the two concept maps P5 has drawn appears to indicate that P5's schema after studying this course on trigonometry is unchanged. This is supported by the evidence from the task questions and the follow up interview.

5.9 Assessment at the End of the Component

The test set by teacher consisted of the trigonometry questions from past P1 papers [See §7.9]. These questions required analysis to determine what was required, an ability to disseminate a trigonometrical function into its component parts and familiarity in dealing with radians and surds. This was an accumulation of the different aspects of the topic presented and the students were required to consider the different aspects within the whole. As they answered the questions the students were observed by the teacher and the researcher however, since the teacher wanted the students to attempt the questions under examination conditions i.e. no conferring and within a time constraint, it was not suitable for the researcher to ask the students about their thinking processes as they went. This was a shortcoming of the pilot study that was reconsidered for the main study [§5.8]

However the student's papers viewed after the lesson showed that P2 had no understanding of the period of the sine wave, wrote out procedural answers in full taking 10 lines or more showing an inability to flexibly short cut procedures. She did not use spatial visual representations flexibly and was muddled by a question where the graph y=A +B Cos 2x was shown with coordinates given for a maximum and minimum value for which the values of A and B had to be determined. Surprisingly she factorised $3\sin^2 x + \sin x - 2 = 0$ wrongly and did not give any values for sin x=-2/3.

P3 started most of the questions with a correct strategy but had problems with algebraic misconceptions, for example, she wrote 2sin (2x+k) = 2sin 2x + 2sin k. When she could not proceed with a method she tried an alternative or used her calculator. She used the unit circle and the graphs to refocus her thinking.

P4 did not attempt most of the questions. When he did he frequently made errors in algebra manipulation. His solution strategies were random, for example, in the question: Given that $f(x) \equiv 3+2\mathrm{Sin}\,(2x+k)$ and $y=f(x)$ passes through $(15, 3+\sqrt{3})$ show that $k=30$, he substituted 15 for x but failed to substitute for y. This suggests a stimulated Action approach that has not been properly thought through as to the purpose of the Action.
P1 started parts of the questions except the one where the graph was shown (described above) which she did not attempt at all, but frequently abandoned her solution process even when it was correct. She used the unit circle to find solutions. P5 was absent this lesson.

The students did badly in this test. Their marks were:

P3: 20%

P1: 12%

P4: 10%

P2: 8%

The results indicated that none of the students would pass the pending AS level examinations.

Despite the fact that in this test, rather than being the highest ranking student, P2 achieved the lowest mark and P3 the highest, P3 remained convinced that P2 had a better style of learning since it involved the memorization of formula. P3 therefore still rationalised that she should follow P2's style:

> 20% is not going to get me very far is it! If I am going to pass this exam I need to learn the formulae like P2 does. I mean it's no good liking maths if you fail the exam. From now on my priority must be to get serious about this exam!
>
> (P3)

P2's perceived style of learning appeared to match the general philosophy that pervaded the teachers admonishment to "learn" and that of the class, explicitly at least, to subscribe to it. This became the perceived route to success for P2 and P3.
Despite the fact that all of the students within this sample had been awarded an A* or A in mathematics within the GCSE or its equivalent, the class teacher gave the following explanation for the poor performance. He said that he could:

"… put this down partly to the fact that the class are not 'mathematicians' because they are doing a mix of A-level subjects and partly down to the hardness of the subject".

The conclusion of these results is that the past Pure1 questions were beyond the ability of this group. They were unable to consider trigonometry as a whole, unable to connect the component parts of an algebraic representation of a compound function with the series of geometric transformations it represented. They were unable to switch between representations and poor at dealing with radians. The also had poor algebra manipulation skills.

However, the AS mathematics results for this group generally present a different picture, one that confirms P2's approach to mathematics and confirms P3's approach to elements of the P1 course.

	Pure 1	Statistics 1	Decision 1	Total mark	Final Grade
P1	64	62	82	69	C
P2	100	76	90	89	A
P3	68	47	48	54	D
P4	79	77	62	73	B
P5	20	30	44	31	F
P6	46	55	80	60	C

Table 5.2 Pilots Group's AS Mathematics Results

The exceptionally high marks achieved by P2 and to a lesser extent P4 appear to show the benefit of practising past paper questions extensively as a preparation for the assessment. However the marks here give indications of ability that are in marked contrast to those indicated by the concept maps. It is worth noting here that P1 decided to continue the course through A2 but not into Higher Education. She said it would look good on her CV to have A-level mathematics but she did not like mathematics enough to study it thereafter.

5.10 Conclusions and Refinements

The emphasis of this pilot study was to see the way in which trigonometrical concepts develop. To do this a knowledge base was established in terms of the initial concept maps and then another in terms of the subsequent maps. In between the content and the way it was delivered was considered. With regard to issues of the development of understanding, P1, P2 and P4 provide evidence in their concept maps and interviews that they are making little effort to understand trigonometry relationally [§3.1.2] and for them an instrumental understanding is all that is required. Their knowledge of core properties has not been derived by consideration of the process but learned by rote. It is a point of interest whether their teacher's style of delivering operational knowledge that is already condensed [§3.1.4] and his emphasis on operational problem solving procedures suits their learning method or dictates it. There is little evidence of a Process conception as defined by Dubinsky [§3.1.5] being developed and little evidence that the spatial-visual imagery introduced during this component is being employed to give a deeper awareness of the core concepts [§3.2.3].

The evidence in P3's first concept map pointed to an interlinked schema and during the lessons she clearly attempted to find a meaning to what she was learning in the light of what she already knew. This points to an attempt at relational understanding. She constructed the special angles triangles each time using Pythagoras theorem to find the lengths rather than memorising the details but consistently made the error of giving the perpendicular a value of 2 instead of the hypotenuse. This suggests an attempt to understand the spatial–visual representations as procepts that simultaneously represented both operational and conceptual elements as described by Gray & Tall [§3.1.6] but the teacher emphasis and P2's high score in the task questions led her to abandon these attempts at relational understanding in favour of memorising facts and formulae [see Delice & Monaghan §3.3.5]. Despite the fact that P2's test scores were very erratic, P3 insisted this would give her a better chance of success in the assessment. The conclusions drawn from the evidence in P5's maps show that no schema modification has been made to include the new content. [See Harel & Tall §3.4.2].

With regard to issues of the assessment objectives, the conclusion that may be drawn from the second concept maps and the second interviews is that the reality of what is being learned in this class room may be falling short of the assessment objectives stated in the AS/A level modular syllabus (2004) to 'Develop an understanding of coherence and progression in mathematics and of how different areas of mathematics can be connected' and again under synoptic assessment which states:

> 'Synoptic assessment in mathematics addresses candidates' understanding of the connections between different elements of the subject. ...Making and understanding connections in this way is intrinsic to learning mathematics'
>
> [UA014391 –Specification –Edexcel AS/A GCE in mathematics (2004). p8]

It may be noted here that on careful consideration of the QCA's 2002 draft proposal for the new AS/A Level Mathematics, the post 16 mathematics advisory group stated that:

> We wondered whether it would be appropriate for the word *"function"* to be properly defined at AS since many students will have used it loosely at GCSE.
>
> [Porkess, R. Commentary on QCA's draft proposal for AS/A Level Mathematics (2002) p 12]

The evidence shown here is that for these students the lack of a definition for function has led them to believe that it relates to the format in which it is presented i.e. f(x) =sin x is recognised as a function but y=sin x is not. There is no connection between functions and properties in any way.

With respect to the methodology, the concept maps had given some good insights into the student's schemas [§3.1.3] and these were confirmed by the interviews. However the interviews required a greater structure in order to compare and contrast answers so that evidence of any patterns in the students developing schemas might be identified. It was therefore decided that a questionnaire would be devised that would provide the framework for investigation of students understanding of core issues. In addition a failing of the pilot study was that no analysis had been made of student's initial competence in trigonometry. An assumption had been made that the student's ability in the subject were roughly comparable because they had all achieved high grades at GCSE (or equivalent). In the main study this needed to be verified and not assumed. Finally although looking over the students attempts at the past P1 questions was interesting it would have been more useful to have asked the students to talk through their thinking as they attempted the questions. This could have provided a deeper insight into any tentative links, algebraic or spatial- visual, that the students were aware of but did not pursue and, if this was indeed the case, why they did not pursue them. It was therefore decided that the same questions would be put to the students in the main study but under supervision where they would be encouraged to talk through their thinking.

Chapter 6

Students Initial Knowledge of Trigonometry

6.1 Introductory Remarks

A consequence of the pilot study was that it would have been helpful if the students' initial skills at trigonometry had been ascertained. It had been assumed that students starting the A-level course were competent at finding required lengths or angles within a given right angled triangle but this was not always the case [§5.4.1] Therefore at the start of the main study the students' initial knowledge of trigonometry was investigated. As with the students in the pilot study [§5.3.3] all of the students had followed a course in trigonometry prior to the start of their A-level component. The purpose of this chapter is consider both the knowledge they had acquired from this course and the way they perceived concepts acquired in the context of their trigonometrical schema.

In this preliminary investigation four aspects of students' knowledge were focused upon:

(1) The way in which student schemas of trigonometry were connected

(2) Their ability to handle pre-requisite skills necessary for their development of trigonometry as a function.

(3) Their interpretation of the representations of sin, cos and tan.

(4) Their informal ideas of function.

To respond to these issues the students within the main study sample were asked to draw concept maps of how their understanding was linked.

This chapter reports on the outcome of this part of the study. It provides an indication of the approach used to investigate student knowledge and how students who were perceived to be representative of extremes and groups were selected (§6.2).

An analysis of the students construction of these initial concept maps is considered with §6.2.1. The evidence suggests that none of the students indicated a possible relationship between the individual components presented on their maps but instead appeared to rely more extensively on a construction associated with discrete components of trigonometry.

Section 6.3 provides an indication of the students' ability to deal with pre-requisites of their A-level course. This was assessed through the students' responses to three questions that required the students to interpret a trigonometrical graph, find a solution to a trigonometrical question within a given range and to identify a length and angle within an isosceles triangle. Section 6.4 shows the students responses to a short questionnaire that seeks to discover how the students interpreted some of the key terms associated with the topic of trigonometry. Section 6.5 then details the students' responses to questions that sought to investigate sub concepts of function within a trigonometric framework. The National Curriculum to GCSE does not include formal definitions of function however sub-concepts such as inverse function and function limits can be recognised on an informal level. The chapter finishes with a summary of the findings in Section 6.6.

6.2. Sampling Students Initial Knowledge of Trigonometry

Since three students were absent from the class that morning only fourteen undertook this element of the study. It was felt that a random selection of students at this point of the study would not have provided a reasonable insight to the subject knowledge of the group as a whole and so to overcome this problem the concept maps were used. McGowan (1998) had shown that even within a group that is of broadly similar ability there is a range of conceptual understanding (§3.1.3). It was this range that was of interest but also of interest was whether or not there appeared to be qualitative differences in the understanding of students who were perceived to have potentially different levels of achievement. It was therefore decided to categorise the students into two broad levels. Not withstanding reported the difficulties associated with stereotyping students (Ruthven, 1987) the teacher, who was familiar with all the students and had some knowledge of the mathematical ability of each of them, was asked to use his judgement to broadly categorise the students into those who were relatively more able, and less able, within the group. Though there are documented issues associated with teachers and their ability to identify over a range of ability [Thompson (1984), Carpenter, Fennema, Peterson & Carey (1988) Guskey & Passaro (1994) etc] however within the constraints of time this method was thought to be the most suitable. This enabled the researcher to select from the outset four students from within the two broad categories of ability for closer observation. The teacher was also asked to identify those students who in his consideration were the highest and lowest achieving within the class. These were identified as S1 and S4. Two others were randomly selected from the two categories, S2: above average and S3: below average relative to the group. S1 and S2 were studying Further Maths.

The 14 students present were asked to draw a concept map of their understanding of trigonometry before the start of their AS course. They were then asked to complete the skills questions. During the following two double lessons the four selected students were withdrawn from the group, in turn, and asked to respond to the questionnaire consisting of

15 questions [§4.5.3]. This chapter considers the results of these initial concept maps and questions and responds to the issues identified above,

6.2.1 The Initial Concept Maps.

In the pre-course concept maps up to 10 items trigonometrical ideas were identified by the 14 students in the group. Table 6.1 shows the frequency with which these items were correctly defined, incorrectly defined or not indicated by students.

	Correctly defined	Incorrect or not defined	Not indicated
1. Sine graph	12	2	0
2. Cosine graph	10	4	0
3. Tangent graph	6	8	0
4. Sine rule	12	2	0
5. Ratios	7	7	0
6. Cosine rule	8	4	2
7. Pythagoras theorem	7	3	4
8. Tan identity	8	2	4
9. 1 or more Surd values	5	1	8
10. 1 or more Decimal values	3	0	11

Table 6.1 Items of content in first concept maps

In completing the frequencies for Table 6.1 it was judged that a ‘correctly defined’ required an element of clarification. For example a sine or cosine graph required the identification of at least one maximum or minimum together with a correct shape. The tan graph required either a good shape or the angle intercepts. The correct formulae for the sine rule and cosine had to be itemised. With regard to ‘ratios’, seven of the students wrote the definition as sin = opp/hyp etc but the remaining seven just wrote SOHCAHTOA which was not considered to be a definition.

The first six items on the list are covered in the Higher GCSE syllabus. Item 8 on the list which is not part of the GCSE syllabus but a key part of the trigonometry component in C2 was mentioned by S1, S2, and S3 [See concept maps below]. The concept maps drawn by the students illustrated little evidence of connections between the ideas identified as may be seen from the concept maps of the four ‘selected’ students (see Figures 6.1 to 6.4).

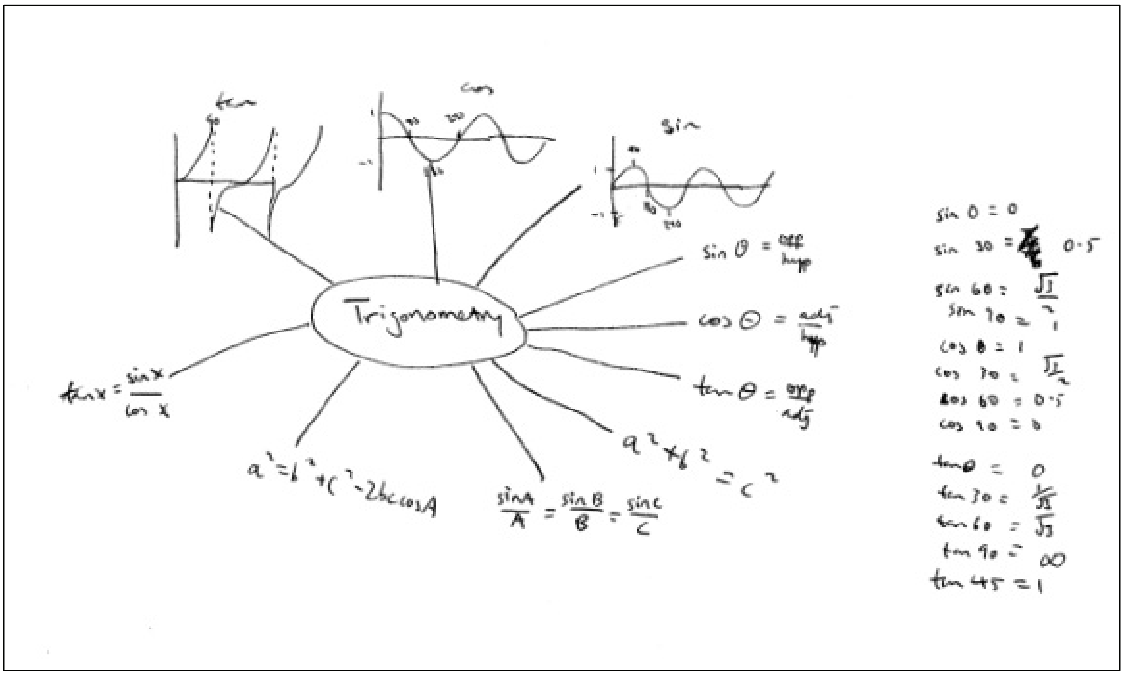


Figure 6.1: Concept Map 1

The concept map drawn by S1 (Fig. 6.1), regarded by his teacher as the best student, illustrates essential material derived from his GCSE course of the previous year. In particular we see formulae derived from a triangular perspective of trigonometry associated with Pythagoras; sin, cos and tan. Additionally we see the formulae for the sin rule and the cosine rule, an identity for tan x and a list of values for the sines, cosines and tangents of the special angles in surd form. The concept map has a ‘spider construction’ (apart from the surd values) and the essential feature to be determined from it is that whilst it may be an indicator of the students knowledge, superficially at least, there is little to indicate that one piece of knowledge is actually related to another. However there are signs that the data is being grouped: graphs, ratios, rules and then the identity which suggests that the identification of one item within a theme triggered the identification of other items.

The teacher identified S2 as of above average achievement. In her concept map there appears to be no evidence of any grouping (Fig. 6.2).

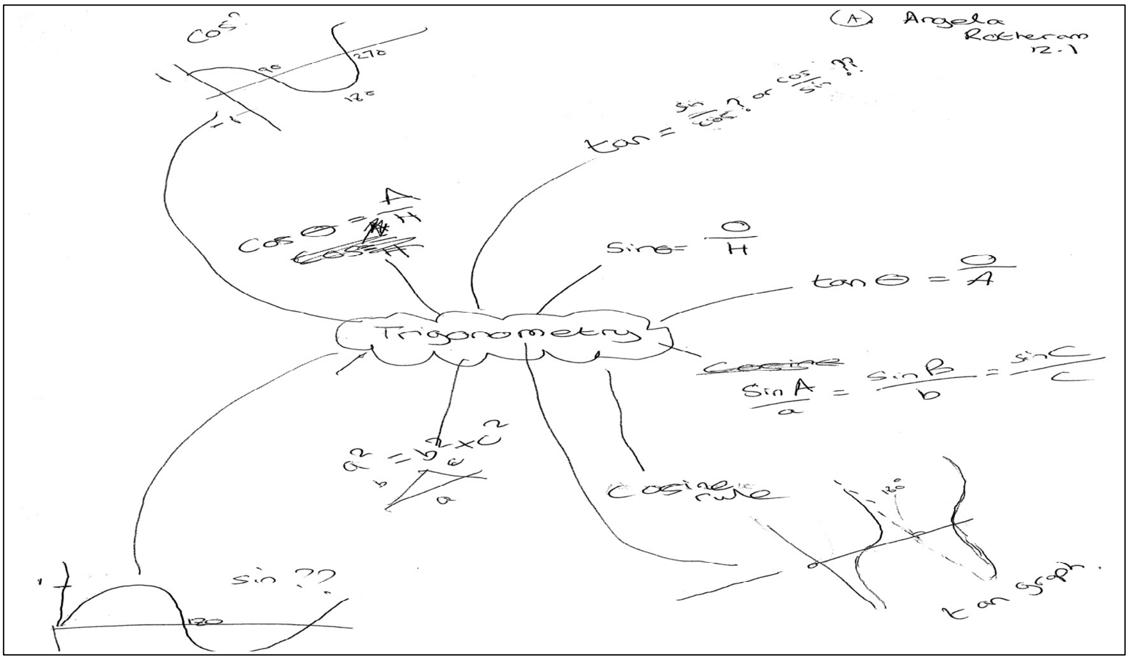


Figure 6.2: S2 Concept Map 1

S2 places items on the map in an almost random way, and the map contains less information than that given by S1. However the graphs of sin x and cos x show maximum and minimum values and the angles where these occur, these are properties of the functions. There are indications, from the inclusion of the question marks that the graphs are not remembered with 100% confidence. No surd values are included and the items that are mentioned are core properties of the function: the ratios, the graphs and the tan identity. Once again, though the essential feature of the concept map suggests the discrete rather than related aspects of trigonometry and given the somewhat random specification of ideas the interpretation suggests a less relational aspect of thinking than that evidenced by S1.

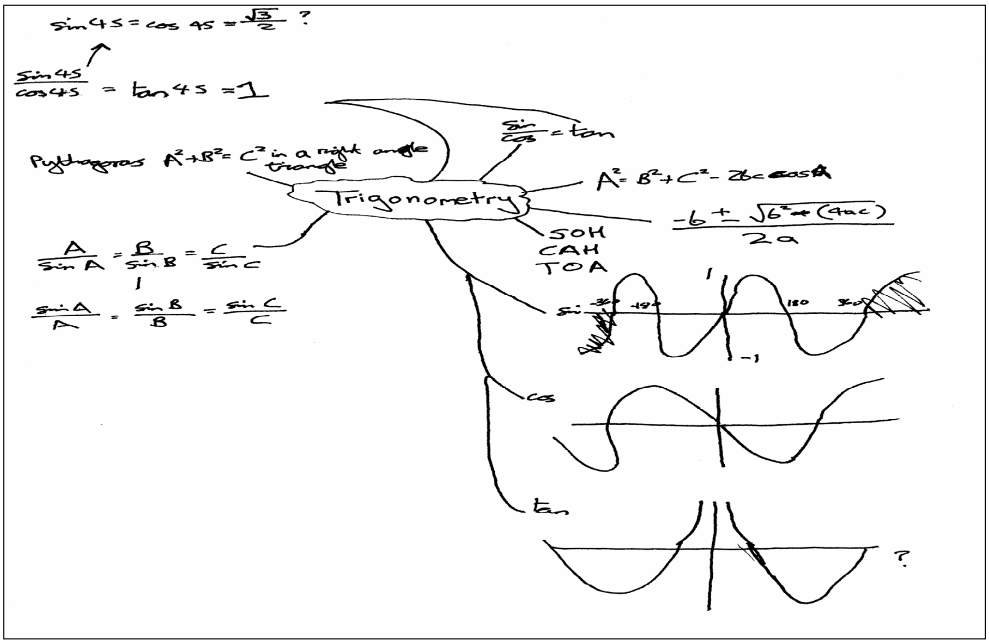


Figure 6.3: S3 Concept map 1

Interestingly, the concept map drawn by S3 (Fig. 6.3), identified as a less than average pupil by the teacher could lead to a similar interpretation as that given to S1.

There is some indication of grouping: graphs, operational formula and connections between the tan identity and values. However, essential differences lie in the fact that the graphs for sin x cos x and tan x are drawn but only the sine wave is correct. He shows Pythagoras theorem and a note of when it may be applied. Both versions of the sine rule, one for finding unknown lengths and one for finding unknown angles are shown, as is the cosine rule. An identity for tan x is shown and the fact that cos 45 = sin 45 though the surd value given is incorrect; also given is the fact that tan 45 =1. There are question marks beside the surd value for sin/cos 45 and the tan graph. The quadratic formula is also shown hinting at an operational approach to trigonometry. The implication from this map is that this student is more comfortable with algebraic representations than spatial visual ones.

The concept map drawn by S4 (Fig. 6.4) contains far fewer items of specific content than those created by the other three students.

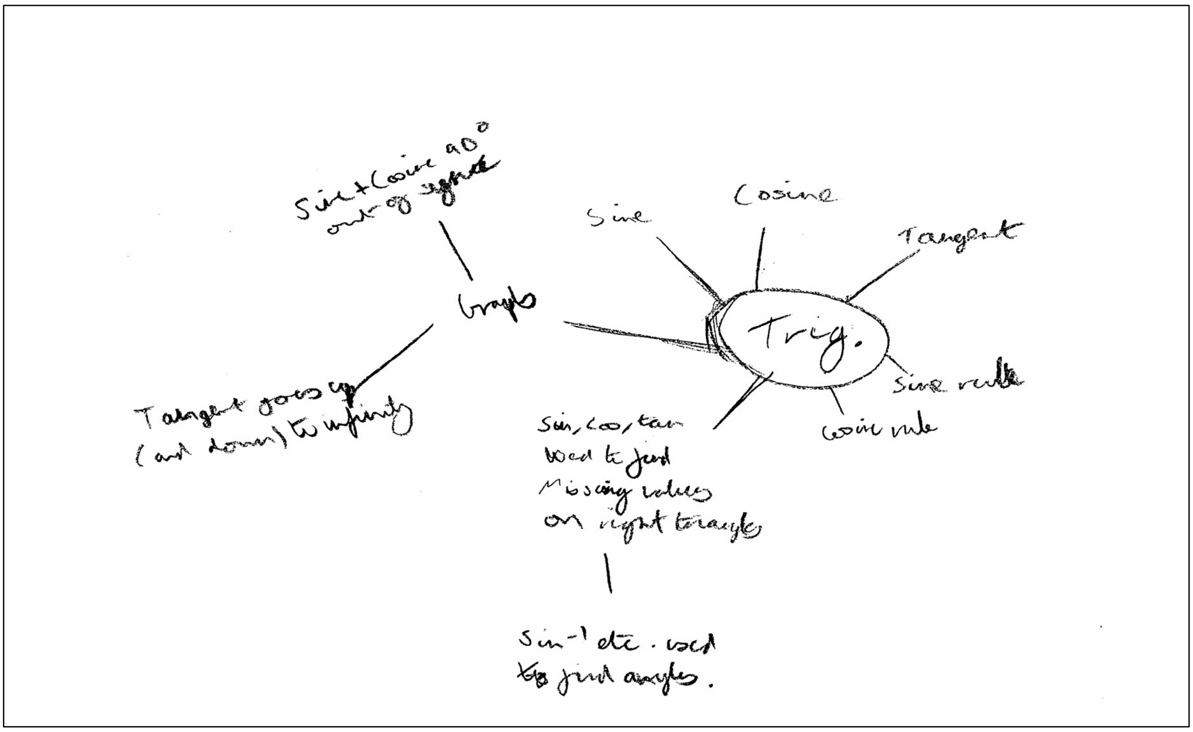


Fig. 6.4: S4 Concept Map 1

There is evidence of grouping but the items mentioned appear to be a series of notes. Graphs are mentioned but no attempt has been made to draw them though there are two comments about the properties of the graphs. The words sine, cosine and tangent are noted but there is no ratio formula given for these; the sine rule and cosine rule have been mentioned but there is no attempt at the formulae. There is a note about the operational uses of sin, cos and tan and another note about the operational use of the inverse functions sin $^{-1}$. The essential features of this map are names without any reference to definitions.

6.2.2 Summary

Initial inspections of the maps show that the maps drawn by S1, S3 and S4 show signs of grouping. The maps by S1 and S2 include Pythagoras theorem indicating a strong connection to triangles. The map drawn by S3 implies that less importance has been given to the graphs by this student at this stage and a greater emphasis has been placed on remembering the formulae for operational use. The inclusion of the quadratic formula may also suggest that the student posses an operational schema rather than one related to the concept of angle. However S3 has indicated links between some aspects such as the tan identity and values. The map drawn by S4 is sparse in detail but it does appear to have a structure that has two centres of activity: the word ‘trigonometry’ is linked to operational notes whilst the word ‘graphs’ which has notes on properties linked to it. A point that should be considered here is whether the concept maps provide evidence to support the teacher’s judgement of each student’s ability. By considering operational and structural aspects of student perception described in the theoretical frameworks of Sfard and APOS theory it may be seen from the maps that S1 and S2 include both algebraic representations of sin, cos and tan and the spatial representations of the graphs whilst S3 and S4 do not. S3 focuses primarily on formulae and S4 has written notes on operational uses of sin, cos and tan. This hints at a difference in perception between students S1 and S2 and students S3 andS4. S1 and S2 are showing signs of a focus on conceptual structure whilst S3 and S4 are showing no signs of this but a strong focus on the operational aspects of trigonometry. Therefore the teachers judgment of a qualitative difference between the students appears justified at this stage. Moreover despite the fact that these students all achieved similar GCSE assessment grades this early difference in focus would have increasing significance as the course progressed.

6.3 Students ability to handle pre requisite skills

At the end of the GSCE Higher course students are expected to be proficient in using the symmetries of the graphs to find alternative angle solutions to the one given by their calculators and find required lengths or angles in scalene triangles using the sine and cosine rules. It is assumed on commencing the A-level course that all students have this capability. Therefore prior to starting the course the following three questions were distributed to the group separately to assess the students’ skill in:

- Interpreting a graph
- Finding multiple solutions

- Finding lengths and angles in a non right angled triangle using the sine and cosine rule.

Since students had no formal input of the function concept, the technical properties of the concept of function such as one to one and onto were not considered as aspects of the students' knowledge. The four selected students answered the questions, in turn, during the second and third lessons. These lessons, as shown in table 7.2 [§7.5.1] were on graphs which may have had an influenced on some of the responses. It should be noted that S1 spoke English as his second language and frequently did not expand on his comments. Statements of encouragements by the researcher for the students to expand on the answers given (such as 'Could you explain further?' and 'How do you know?') have been omitted but are indicated by the pauses. The student's responses are set out underneath each term.

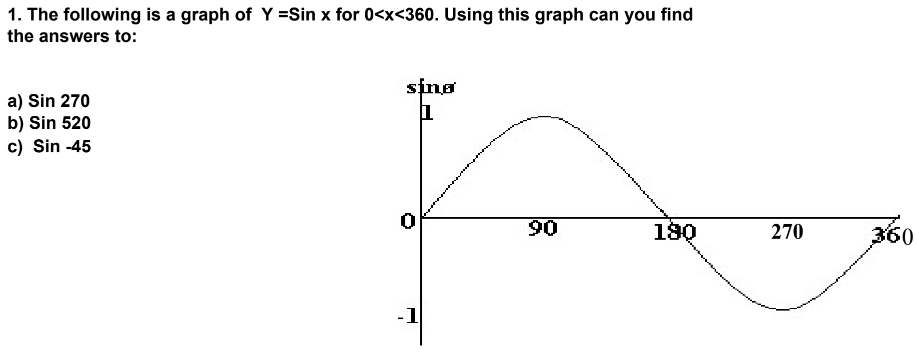


Figure 6.5 Question 1

When responding to the first question (Figure 6.5) all of the students found the correct solution to part a) whilst 11 of the students, including S1, S2 and S4, found the correct answer to part b). The 11 students each extended the graph to 720° and used the fact that 520° is 540°-20°. They then used the calculator to find the sin 20°. S3 and two others extended the graph but worked from 360° and made an error with the arithmetic. All the students correctly answered part c): 12 of these extended the graph for angles <0 and deduced that sin (-45) was equivalent to –sin 45. The remaining two students, including S3, found the answer on their calculators.

2. Find all the solutions to $\tan^{-1}$ 1.75 in the ranges -180<x<180

All the students used their calculators to find the positive solution, rounding the answer to 1 or 2 decimal places i.e. 60.3° or 60.26°. 11 of the students correctly found the second solution sketching the graph of tan x for 0≤x≤360 and extending it for values of x in the range -360≤x≤0. They then used their calculators to subtract 180 from their first solution.

The remaining students found a second solution in the range 180≤x≤360. When asked about this they said they all said they had not noticed the solution range and just assumed it would be 0 to 360 as usual.

3. Find length AB and hence determine angle ABC

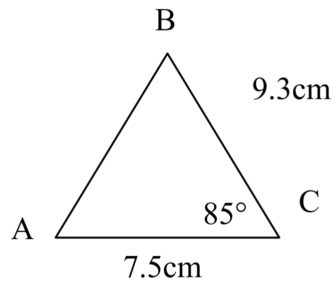


Figure 6.6 Question 2

12 of the students used the cosine rule ($c^2 = a^2+b^2-2ab\text{Cos } C$) to find the length AB. They then used the sine rule (a/Sin A = b/Sin B = c/Sin c) to find angle ABC. The rules were remembered from the GCSE course. The remaining 2 students had misremembered the cosine rule in some way: for example, writing 2+a+b instead of 2ab or a^2-b^2.

The overall evidence established from the students' response to the questions indicated that:

- All the students were able to interpret the graph and 79% were able to extend the graph to find correct solutions.
- All the students were able to find multiple solutions though 21%, including S3 and S4, those identified within the 'low achievement' group, had automatically found solutions in the 'usual' 0 to 360 range.
- All the students were able to recognise the need to use the sine and cosine rule and 86% had remembered the rules correctly.

These results lead us to conclude that all the students started the course with the skills necessary to study trigonometry at A-level.

6.4. Students' Interpretations of the Representations

To investigate further the student's informal knowledge of trigonometry the selected students were final asked to partake individually in an informal interview with the researcher over the following two lessons (lessons 2 and 3). The questions used to identify the students' knowledge of trigonometry considered the students':

- Interpretation of sin, cos and tan
- Interpretations of the inverse functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$
- Interpretations of the graphs
- Interpretation of function

It should be recorded here that lessons 2 and 3 had begun with a revision of the core ideas introduced at GCSE level trigonometry. The lessons focused particularly on the graphs. As seen below this influences some of the answers given by the students however the exercise was still valid as this was not new work but a consolidation of work already covered and significantly the students differed in the meaning they extracted from the graphs. Note that for S3 in particular the exercise of constructing the graphs has little, if any, relevance to his trigonometric schema.

6.4.1 Interpretation of Sin, Cos and Tan.

The first question invited the students to describe as fully as possible what they understood by the following terms.

4. Sin 30°

S1: Like ½
S2: Doesn't make me think of anything. Usually a triangle. Anything to do with trig makes me think of triangles though I might now think of the graph...try and picture it in my mind ... Sin 30 is ½
S3: Is it √3/2... That's just how I learned it. Decimals are harder to remember.
S4: Curve on a graph...

The students were asked how they would find sin 30. Each of the four said they would use the calculator. They were then asked how they would find it without using the calculator. S1 and S3 said they just remembered what it was. S2 said that

> It's opposite over hypotenuse isn't it. So I would draw a triangle with an angle of 30° and divide the opposite side by the hypotenuse.
>
> (S2)

S4 said he would use the graph but with no calculator he said he couldn't draw the graph.

It should be noted that all of responses are qualitatively different. S4 refers to a curve rather than a point, whilst S2 switches between an image of a 'generic' triangle and an image of the graph, sharpening focus to arrive at a conclusion. S1 and S3 respond with 'remembered facts' although in the case of S3 the answer is incorrect.

5. Sin 120°

S1: It's √3/2

S2: I think of the graph...I know sine 90° is 1 so it's the negative side of the curve... like the bit that comes down again after 90...because it is symmetrical about 90° so it would be the same as 180 – 120 which is 60.

S3: Doesn't ring a bell. I would find it on the graph... well I would draw the graph and then find it.

S4: Curve on a graph... You just draw the graph and its part of the curve.

Asked how they would find this value without the use of the calculator S1 said he just remembered it. S2, S3 and S4 all said they would use graphs to find it although only S2 gave any indication of being able to visualise the graph and its characteristic shape and symmetries. S3 and S4 said they would draw the graph by plotting the points using a calculator. Neither of them could think of a way to do it without a calculator. Again these answers are supported by the details shown on the student's concept maps where S1 has listed values, S2 is thinking in terms of the graph symmetries as she had with question 1, S3 is thinking of undertaking the procedure of drawing the graph and S4 gives a vague description.

6. Tan 90°

S1: Impossible/infinite. In a triangle you can't get 2 angles of 90°

S2: That's were the asymptote is... so there is no value for it.

S3: Sin 90 over cos 90 or cos90 over sin 90... It's infinity because it's one over zero

S4: Tan is the one that goes up to infinity isn't it. Basically a different curve. It's infinity.

S1 has answered this question by referring to the image of a triangle and using opposite over hypotenuse. This appears to indicate that is thinking predominantly in terms of a ratio conception. S2 and S4 used their knowledge of the tan graph. S3 used an identity. This appears to indicate that he is predominantly relying on an algebraic representation.

This again links with the concept maps where S1 indicated perceptual links with triangles and S3 indicated a preference for algebraic representations over spatial visual ones.

8 of the 15 students had written the tan identity of sin x/cos x including S1, S2 and S3. This next question sought to investigate this further.

7. What does identity mean?

S1: Equivalent to. It's always the same like $(a+b)^2 \equiv a^2 + 2ab + b^2$.
S2: It has no meaning in maths.
S3: Don't know.
S4: I don't know of any meaning it has in a maths context.

S1 shows here that he has a good understanding of the meaning but S2 and S3 are unable to connect the word with the tan identity they mentioned on their maps. Other students in the group who wrote the tan identity on their maps were asked this question and the answers given below are typical:

> It's an identity which means it's a formula that can be used instead.
>
> (S2)

> 'It's another way of thinking of tan x. It's opposite over adjacent but that is the same as sin over cos because the sin is the opposite over hypotenuse and cos is adjacent over hypotenuse so if you divide sin by cos and times by the hypotenuse you get sin over cos. It's the same thing'
>
> (S14)

This implies that different students had different interpretations of the word from their GCSE classes. Some students were able to closely define it; others thought it had an operational use whilst S2, S3 and S4 were typical of a sub section of students who were unfamiliar with the word.

In conclusion it may be seen that even at this early stage there is a difference between the thinking of these students. S2 simultaneously recalls different representations, spatial and algebraic, and switches between them. S1 and S3 rely heavily on 'remembered facts'. These take the form of learned numerical values and formulae. When S3 is unable to recall the answer to sin 120:

S3: Doesn't ring a bell. I would find it on the graph... well I would draw the graph and then find it

He resorts to two processes that he thinks will provide the answer: draw the graph; use the graph to find the value. However in order to draw the graph he needs to plot the values of sin x which he presumably would get from his calculator. If he has the value he does not need to draw the graph. The processes can be described in APOS terms as actions over which he has not yet reflected and has no means to shortcut. S4 is showing every indication of not engaging with the mathematics but passively going through the motions.

6.4.2 Interpretation of Spatial Images.

The purpose of the next set of questions was to identify students' interpretation of the spatial visual representations used in trigonometry some of which had been shown on their concept maps. Graph transformations had been covered in the Higher GCSE syllabus. As previously mentioned the lessons during which these interviews were undertaken, lessons two and three, were covering transformations of the trigonometrical functions and this naturally influenced the students replies.

8. What comes to your mind at the mention of Sin x?

S1: The sine wave....
S2: I would think of the graph.
S3: The graph comes in my head.
S4: Unknown value on the curve.

The lesson that morning had been on graphs so it is unsurprising that the expression Sin x should bring this to the students minds immediately.

9. What is an Asymptote?

S1: It's the bit where the graph will never reach.
S2: The dotted line... The graph tends towards it. It never actually reaches it.
S3: I can't remember the meaning of it. Is it related to circle theorems?
S4: That's another type of curve... I can't remember (what type) but I just remember hearing of an asymptote curve.

S1 and S2 appear to be aware of the meaning of asymptote though not S3 or S4 despite having drawn the graph of tan θ during the lesson again indicating a lack of engagement in the exercise.

10. What is a Graph transformation?

S1: How we stretch graphs sideways, upwards or shifting left or right, up and down.
S2: How different numbers affect the information displayed on the graph... its shape or maximum values or intercepts.
S3: It's where you alter the graph by changing the formula... You add or take numbers in the bracket like f(x) + 1. We did some of it last year.
S4: Co-ordinate geometry and the curves.

Again this had been the subject of the lesson that morning. S1 has indicated the geometric transformations possible but no mention is made of how this is linked to algebraic representations whilst S2 has directly linked changes in the graph to changes in the algebraic representations. S3 has also speaks of this connection but his response indicates a memory of work learned previously and he makes no reference to the work currently being studied. When asked how f(x) might link to trigonometric functions he said he hadn't leaned that yet. S4 indicates no real understanding of the term.

The next question was to see if any of the students were aware of the unit circle diagram at this point in their studies. Later this would become a key trigonometric image and the manner in which it was introduced could have important long term repercussions on the students understanding of it. [§5.6.4]

11**. What is the Unit circle (Sometimes called the CAST diagram**).

S1: Never heard of it.
S2: Don't know.
S3 Never heard of it.
S4: Never heard of it

6.5 Investigation of ideas of function

The purpose of the next set of questions was to investigate the students' idea of function in general and some of the properties of trigonometric functions specifically. The previous questions had been designed to find supporting evidence for the constructions of the concept maps albeit implicitly. These questions now considered deeper issues of function conception with specific regard to trigonometry (it should be noted here that the course syllabus refers to the topic specifically as 'Trigonometric Functions'). As stated above the students had not covered any formal definition of function and as such these questions were designed to investigate if they had any informal function concept.

6.5.1 Interpretation of Inverse Functions

This question was adapted from McGowen, DeMarois, & Tall (2000) It was used to explore whether or not the students had an informal conception of inverse function. The students were asked to explain their interpretation of:

12. Cos^{-1} 0.5

S1: That would be 60°... Like what number of cos would make 0.5.
S2: It's like a triangle when you have to work out the angle instead of the length... You do inverse cos to find the angle... It's on the calculator. You shift cos 0.5 to get the angle.
S3: Sounds like it would be negative... Inverse usually means negative.
S4: You would use it for finding angles wouldn't you?

The answer given by S1 suggests he recognises that he has an output and is now seeking to find the input. This shows an informal grasp of the idea of inverse function. S2 appears to be thinking in terms of an operational understanding of finding angles. S3 is unable to give any clear idea of either the purpose or meaning of the term inverse $\cos^{-1}0.5$. Unlike S1 and S2, he did not show the ratios in his concept map, merely SOHCAHTOA which is the mnemonic for 'Sine equals Opposite over Hypotenuse, Cosine equals Adjacent over Hypotenuse, Tangent equals Opposite over Adjacent'. He may have forgotten how to use the mnemonic to find angles and although he has just been drawing the graphs he has not made any informal function connection between an angle and its cosine. S4 is aware of its purpose but can give no further indication of how it relates to the cos function.

13. Sin^{-1} 2.5

S1: Impossible...It must be less than 1
S2: You can't do it because the sin only goes up to 1 so you would get error on the calculator. There is no value for Sin^{-1} 2.5
S3: It says error on the calculator...I am not sure why it just says it.
S4: Well the graph only goes to 1 so you can't inverse sin any number greater than 1.

All the students except S3 appear to be aware that the maximum value sin can take is 1 and therefore Sin^{-1} 2.5 is impossible. This implies a link between the operational process of using the calculator and the graphs which had been the subject of the morning's lesson. S4 has explicitly linked the question to the graph of sin θ suggesting some idea of reversing a process. S3 shows evidence of not connecting the limits of the graphs with the result given on the calculator. This supports the evidence of the previous question that he has not recognised the relevance of the graphs.

6.5.2 Functions Links

The next set of questions was designed to investigate student's interpretation of the word function and how that linked to trigonometric function. It was anticipated that these students would think of function in spatial-visual terms since the notation is primarily used with graph transformations at GCSE. It was therefore decided to see how the students linked this with ideas of differential calculus which had been covered previously on the AS course.

14. What comes to mind at the word 'Function'?

S1: How the graph can be transformed...like y=f(x) which I did at GCSE.
S2: It's a graph; the function of x or whatever is in the brackets. ... How it maps onto the graph...I don't know how to explain it.
S3: It's $f(x) = x^2$ or $f(2) = 2^2$. In differentiation instead of using Y we used functions sometimes... It's just another way of writing Y= something.
S4: It's an equation that does something to the value of something else... like f(x) in graphs. The function of x is basically a graph or the result of a graph... It's co-ordinate geometry.

All the students have linked the term to the algebraic representation f(x) though S1, S2 and S4 have made a connection between the algebraic representation and the spatial images of graphs. S4 has indicated further that functions do 'something to something else'. S3 has identified the notation as a synonym. It may be noted that since there has been no formal coverage of properties of a function or its uniqueness of an output for any given input, these students experience of function at GCSE has been in terms of the transformation that, for example, f(x) + 1 has on f(x). This explains the responses 'do something', 'transform a graph', 'map onto a graph'. S3's response supports the previous comments about his preference for algebraic interpretations above spatial-visual ones. He makes no reference to graph transformations but describes function as an alternative algebraic representation to 'y= something'. The connection made by students between graph transformations and 'functions' is partly supported by the response to the next question through which the students attempt to give specific examples rather than talk in generic terms.

15. What does Trigonometric function mean?

S1: Just like sine, cosine and tangent.
S2: Don't know really...The graphs we have been drawing?
S3: Sin x? I'm not really sure.
S4: Probably stuff about graphs- what we were doing this morning. There is some association between functions and graphs... we did it this morning and it came up on the last unit as well.

S1 has linked the phrase expressly with the terms sin, cos and tan. S2, S3 and S4 have made the connection to graphs but are less confident with it.

The following was an extra question to investigate the students understanding of the role of radians which had been covered in the previous lesson. It is indirectly related to this sections investigation of the concepts of function in that it is a measure of angle and trigonometry is a function of angle. Therefore an understanding of radians can develop trigonometry from an operational schema of triangles towards a conceptual understanding as a function of angle.

16. What do Radians mean?

S1: It's a different system to degrees. In a circle there are 2π radians
S2: It's a different way of separating angles in circles... I radian is equivalent to 180°
S3: We did it on Monday. I can't remember it.
S4: It's an alternative to degrees. A radian is bigger than a degree but I forget now how many degrees there are in a radian something to do with π.

S1, S2 an S4 appear to be aware that radians are an alternative measure of angle to degrees further, S1 and S2 have directly related radians to circles although S1 is the only student of the four who has correctly identified the conversion details. S3 later developed problems in dealing with radians [§7.9].

The following questions were asked to investigate students understanding of differentiation and how they might extend this knowledge to anticipate its meaning in the context of trigonometric graphs. The students had covered differentiation but not yet with trigonometric functions. These questions were designed to investigate how the students connected different concepts together which, to date, had been unconnected.

17. What does dy/dx mean?

S1: It's the gradient function or differentiation.
S2: Differentiation or the gradient of a curve.
S3: That's differentiation. It's nx^{n-1} the gradient of a line.
S4: Differentiation. It's something I can't quite do... I don't know what it's for that's why I can't do it.

The responses by S1 and S2 give both the correct term for the symbolism and a definition. S3 again defines it by an algebraic formula, though this is only valid for polynomial functions. The response by S4 indicates serious problems for his continued study of mathematics.

18. What would d/dx [sin x] mean?

S1: It means nothing. I have never tried that before.
S2: You would need a value (traces sine wave in the air). The gradient changes so you would need a point.
S3: That would be the gradient of the sin x curve at a rough guess.
S4: Nothing.

S2 and S3 show evidence of being able to extend their knowledge of the use of differentiation to find gradients of curves to the curve of sin x. The response of S1 is interesting and may provide some explanation for his poor result in the final C4 module [§7.11]. S2 and S3 have provided qualitatively more sophisticated responses than S1 here despite S1 at this point being considered by the teacher and the class to be the most able student in the group.

This series of questions designed to investigate student understanding of function on an informal level found that the word function is strongly associated with graphs for three of the students. S1 thinks in terms of input and output and can therefore define inverse functions. S2 and S4 think about inverse functions operationally but not for values outside the range of the graphs.

6.6 Students Knowledge and Skills as a Foundation for the AS/A2 Course in Trigonometry

This chapter has dealt with three aspects of students' knowledge, the representations used by students to show aspects of trigonometry and the extent of connection between the different representations, their pre-requisite skills and, their informal knowledge of trigonometry and function prior to the formal course on trigonometry. The evidence suggests that these students had developed an understanding of trigonometry that was operational and defined predominantly as a ratio conception before they started to take the formal course, though S1 and, to some extent, S2 also indicated meaningful knowledge of the functions being described by graphs. The concept maps indicate a schema of discrete entities at this stage however S1, S3 and S4 indicated signs of grouping. In the interviews there were indications that some connections were being made to other representations, for example tan 90 prompted S1 to think in terms of the a triangle, S2 and S4 to think in spatial visual terms and S3 to think in terms of substitution into a formula.

S1, S2 and S4 linked the word function strongly to graphs whilst S3 thought in terms of an algebraic representation. With regard to inverse functions S1 thought consistently in terms of input-output. S2 and S4 were less consistent. They indicated a multi representational perception understanding of $\sin^{-1} 2.5$ linking it to the maximum values of the graphs and recognising that it was out of range but gave a procedural response to $\cos^{-1} 0.5$. S3 thought almost entirely in procedural algebraic terms. The investigation of how students would link differentiation to trigonometric functions showed that only S2 and S3 were able to connect the two ideas together. Therefore we can conclude that although, at this point, S1 has the most developed concept of function, he is unprepared to think creatively about the mathematical ideas he has.

Without a direct comparison we can not say if this group had any significant superiority over the pilot study group, though their GCSE results indicated they were comparable. Overall the pre-test results suggest there was no substantive difference in the pilot study and main study groups initial knowledge of trigonometry. This conclusion is particularly important because the fundamental purpose in conducting pre-test and follow-up interviews with selected students was to establish their initial levels of knowledge and skill as a foundation for the consideration of their growth and the way in which it developed. However within this group there is evidence that the students S1 and S2 and S3 and S4 show, already, a qualitatively difference of focus in their thinking.

Chapter 7
Teacher Delivery of Content and Instructional Outcomes

7.1 Introduction

This chapter reports on the main study. It describes the teacher's method of delivering the content of the trigonometry components in modules C2, C3 and C4 and the result of studying these components on the trigonometric schemas of selected students.

Section 7.2 describes the teacher and section 7.3 describes the trigonometry component of the AS/A2 course. The lesson format is considered within section 7.4 whilst the teacher's style, specifically which aspects of the course he emphasises, is considered within §7.5. Section 7.6 provides a summary of the key characteristics of the teacher's pedagogic style. In section 7.7 details are given of the students' skills and abilities during and after the course as indicated by their responses to the questionnaire and the course content questions. The first concept maps, drawn prior to starting the course, were shown in Chapter 6 [§6.2.1] the second maps, drawn after the trigonometry component of the C2 course, are shown in this chapter [§7.7.6] and an analysis of changes and student feedback is shown in section 7.7. The results from the AS assessment are shown in section 7.8 and then section 7.9 shows the selected students responses to the integrated questions where they talked through their thinking as they went. The third concept maps are considered within section 7.10 with observations and the A2 results indicated in section 7.11. Finally a summary of the chapter is given in section 7.12.

7.2 The Teacher

The teacher had been Head of Mathematics for 10 years and was regarded by the pilot study teacher as a "real mathematician". He was respected by the students and staff for his mathematical ability and deep understanding of the subject. He was very relaxed in his classroom manner and respected the students' ability to think for themselves allowing them to develop their own means to a solution. He valued the utilisation of any valid thinking process that simplified solution methods and frequently transferred from one thinking register to another and back again: for example from spatial visualisations to algebraic representations or known facts (§7.5.6). In short he was able to simultaneously mobilise at least 2 registers of representation and transfer between them (Duval, 1995, §3.2.3). Prior to becoming a teacher he had been in engineering. It was the belief of this teacher that trigonometry is primarily a geometric topic and he thought it was the only way he could think of to teach it.

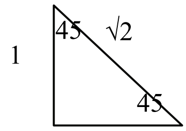
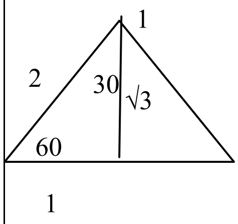
7.3. The Course

The trigonometry component of the AS/A2 course was covered over 4 modules. [See appendix] The C4 module did not dedicate a specific chapter to trigonometry; rather integration of trigonometric functions was included in the integration chapter [See appendix and table 7.1]. The timetable for covering the syllabus was based on the school text book [§5.9] though the teacher decided how long to spend on each topic and how to deliver the syllabus content. This teacher's delivery of the syllabus content has been put into table 7.1 below.

7.4 Lesson Format

The lessons were structured in the traditional format of exposition, class activity and plenary. Homework was set in the majority of the sessions though this was less rigorously adhered to during the latter part of the course when students were given 5 minutes 'time out' approximately half way through the session when they could listen to their iPods or cross the room to talk to each other. The remainder of the time the group were expected to be focussed and the observations showed that this proved to be the case. When the teacher thought that the students needed time to consolidate knowledge the session was given over entirely to problem solving sessions when the structure of the session was to begin with routine problems and then a number of non-routine problems were set that the students discussed with each other and the teacher.

As with the pilot study the students were asked to draw a concept map of their understanding of trigonometry before the course started [§5.4.1]. The lessons were recorded and expositions from the board were noted [§4.5.4].
The lesson content has been put in a table [table 7.1] below along with the syllabus content and the teacher's consolidation of related material.

Syllabus	Topic	Consolidation /revision	Core content	Practice	Closure
Sine & Cosine rules ½abSinC Radians	**Lesson 1** *(term 2 Spring 2005)* Construction of initial concept maps. Introduction of radians Introduction of special triangles.	Sine graph. Max & min values of sin θ and values of θ when they occur. Tan graph →∞	Definition of θ=l/r 360°≡ 2π rad Special case angles 	Construction of $l=r\theta$ and $a={}^{1}/_{2}r^{2}\theta$ where l is the length of an arc and θ is the angle in radians. Changing radians to degrees. Constructing special case angle triangles with angles in radians Construction of sin, cos, tan graphs at intervals of ${}^{\pi}/_{6}$	Homework: Text questions requiring the evaluation of sin, cos and tan of various angles between -2π and 2π Questions on finding the length &area of a circle segment
Sine, cosine & tangent functions, their graphs, symmetries & periodicity	**Lesson 2** Graphs of sin ax, asin x, sin (x±a) between -2π and 2π	Graph transformations Manipulation of fractions Sin (π-θ)=sin θ cos(π-θ)=-cosθ tan(π-θ)=-tanθ	Trigonometric graph transformations. Unit circle Solving simple equations	Construction of $y=\sin^2x$ Number of solutions each function has within given range. Using graph sketches to find all solutions to simple equations.	Homework: exercise from text on graph sketching and solving simple equations
Identities tanθ=sinθ/cosθ $\sin^2\theta+\cos^2\theta=1$	**Lesson 3** Trig identities $\sin^2x+\cos^2x=1$ and Tan x = ${}^{\sin x}/_{\cos x}$ Unit circle Solving equations	Pythagoras theorem Trig equations have multiple solutions Special angle triangles	Trig identities Solving eqns such as 3sinx=5cosx	Solving trig equations within different ranges for angles in degrees or radians	Homework Text questions on solving equations
Solution to simple equations in a given interval	**Lesson 4** Solving equations	Consolidation of solving equations using identities, radians, graphs or unit circle and special angle triangles.		Solving a wide variety of equations	Homework text questions on solving equations that are more varied
	Lesson 5 Solving more complex equations	Further consolidation and experience of core concepts		Solving a wide variety of equations	Homework text questions on solving equations that are more varied

End of section on trigonometry in C2 module

Syllabus	Topic	Consolidation /revision	Core content	Practice	Closure
Sine, cos& tan(A±B)	**Lesson 6** (Term 3, Summer 2005) Sin (A±B), Cos (A±B), tan (A±B) Sin 2θ = 2sinθcosθ Cos 2θ $=\cos^2\theta-\sin^2\theta$ $=2\cos^2\theta-1$ $= 1-2\sin^2\theta$ Tan 2θ=$\frac{2\tan\theta}{1-\tan^2\theta}$	Special angles from triangles Graphs of sin 2θ, cos2θ and tan2θ	Double angle formulae And derived identities	Sin (θ+60)=cos (θ-60) using graphic calculators to investigate graphs of 2sinθcosθ etc Solving equations using these identities Double angle formulae derived	Homework text questions on solving equations using these identities
Relationship between asinθ±bcosθ to Rcos(θ+α)	**Lesson 7** Sinθ= $2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ Cosθ=$2\cos^2\frac{1}{2}\theta-1$ $=1-2\sin^2\frac{1}{2}\theta$ Tanθ=$\frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$ Asinθ±Bcosθ =Rsin(θ+α) =Rcos (θ-α)	Graphs of $2\cos^2\frac{1}{2}\theta-1$ and $1-2\sin^2\frac{1}{2}\theta$ Encouraging students to have spatial visual representation in tandem with algebraic representation	Half angle formulae and derived identities for sin θ etc Identities for Rcos (θ+α) etc	Derived ½ angle formulae Used graphic calculators to verify identical nature of functions Solving equations such as 3sinθ+4cosθ=2	Homework text questions on solving equations using these identities
	Lesson 8 Consolidation and practice	Identities, graphs, special angle triangles		Solving a wider range of problems utilising processes and graphs	Homework: text questions
Secθ, cosecθ, cotθ Graphs of arcsin, arcos, arctan Graphs of above and restricted domains	**Lesson 9** Secθ=1/cosθ Cosecθ=1/sinθ Cotθ=1/tanθ Graphs of sec,cot & cosec Graphs of arcsin, arcos, arctan	Radians Connection to sin, cos tan graphs	Sec, cosec and cot functions	Solving equations using sec, cot & cosec identities. Using the graphs to find all solution within stated range	Homework: text questions
Identities: $\sec^2\theta=1+\tan^2\theta$ $\text{Cosec}^2\theta=1+\cot^x$	**Lesson 10** Derivation of identites	Tan θ=sin θ/cos θ $\sin^2\theta+\cos^2\theta=1$	identities	Proving identities Solving equations	Homework: text question
	Lesson 11 Consolidation & practice	Graphs, unit circle, identities, radians, double & half angle formulae		Proving identities Solving equations	
This was in the differentiation chapter but the teacher thought it better to cover it here	**Lesson 12** Differentiation of y=sin θ, y=cos θ and y=tan θ	Radians Differentiation of composite functions using chain rule, product rule and quotient rule	Differentiation of trig functions	Derivation of f(θ) = tanθ=sinθ/cosθ Practise of differentiating various composite functions of sinθ, cosθ and tanθ	Homework: text questions

End of section on trigonometry in C3 module

	Lesson 13 *(Term 5, Spring 2006)* $\int\sin\theta\, d\theta$, $\int\cos\theta\, d\theta$ $\int\sin^2\theta\, d\theta$ $\int\sin^{2n}\theta\, d\theta$ and $\int\cos^{2n}\theta\, d\theta$	Integration by substitution Derivatives of $\sin\theta$, $\cos\theta$ and fundamental rule of calculus. $\text{Cos}2\theta=2\cos^2\theta-1 = 1-2\sin^2\theta$	Integration of trig functions	Derivation of $\int\sin^2\theta\, d\theta$ using identity for double angle formula. Practise of integrating functions of $\sin(a\theta)$, $2\cos\theta$, $\cos^2 3\theta$ etc Consideration of integrating sin or cos to an even power	Homework: text questions
	Lesson 14 $\int\sin^{2n+1}\theta\, d\theta$ and $\int\cos^{2n+1}\theta\, d\theta$	Fundamental theorem of calculus	Fundamental theorem of calculus	Consideration of integrating sin or cos to an odd power	Homework: text questions
	Lesson 15 $\int\sin\theta\cos\theta d\theta$	Integration by parts	Integration by parts	More stretching problems that needed 2 or 3 applications of integration by parts to reach solution	Homework: text questions
Differentiation of simple functions defined implicitly or parametrically	**Lesson 16** Find dy/dx given: $y=3\sin\theta$, $x=\cos\theta$ $\sin(x+y) =\cos y$	Differentiation, identities	Functions defined parametrically	Problems involving parameters t & θ	Homework: text questions
Volumes of revolution for functions given parametrically	**Lesson 17**				Homework: text questions
		End of section on trigonometry in C4			

Table 7.1: Lesson Timetable

It should be mentioned that lessons 12, 16 and 17 were not part of the trigonometry chapter. The teacher thought it would be better to cover the work of lesson 12 at this point. Lessons 16 and 17 were covered with strong reference to the work covered in the trigonometry chapters and the researcher went in for these lessons specifically.

7.5 Teaching Style: Introduction

The effective teaching of function requires the teacher to have strong pedagogical content knowledge and an awareness of the sorts of difficulties and misconceptions that students may encounter. Importantly the teacher needs also to be aware of how to help students overcome such obstacles and this may require the teacher to have an idea about students' way of thinking (Shulman,1986). Knowledge and expertise is needed in using suitable representations and providing appropriate explanations to align the logic of the function concept to students' comprehension (Ibid). This section examines the teacher's delivery of pedagogical content with respect to: connections between the representations [§7.5.1], utilising the power of visual representations [§7.5.2], encouraging students' visual thinking [§7.5.2.1], the emphasis of infinite nature of the trigonometric functions, local and global solutions [§7.5.3], the prioritisation of conceptual properties over operational processes [§7.5.4], the precise use of language [§7.5.5] and the use of problems to encourage flexible thinking [§7.5.6].

7.5.1 Connections Between the Representations.

A crucial feature of the teacher's teaching style was his use of different representations to promote understanding on two levels. In his expositions he continually changed from an algebraic representation to a spatial visual representation and back again. An example of this was the teaching of the identity tan x = sin x /cos x [lesson 3]. Initially the teacher started with the idea of tan x = opp/adj, an algebraic representation which all the students were familiar with. He then drew a triangle with the length of the hypotenuse equal to 1 and an angle marked θ as shown below. Students were initially asked to evaluate the length of the opposite and adjacent sides. Responses took the form of "sin θ over 1 which is just sin θ" and "cos θ over 1 or cos θ"

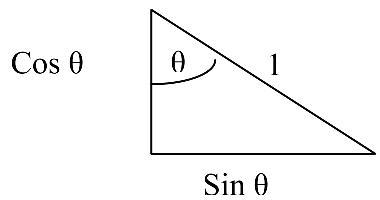


Figure 7.1: Spatial Representation of Tan θ=sin θ/cos θ

Subsequently students were asked to find an expression for tan θ using sin θ and cos θ. The outcome was the establishment of a link between the tan θ = opp/adj and tan = sin θ/cos θ for all values of θ, an idea returned to later in the lesson when the unit circle was drawn and the nature of sin θ and cos θ in each of the four quadrants was identified: for example in the first quadrant sin θ is positive and cos θ is positive, in the second quadrant sin θ is positive and cos θ is negative etc. Using the knowledge that sin θ /cos θ =tan θ the students were asked to determine the nature of tan θ in each of the four quadrants.

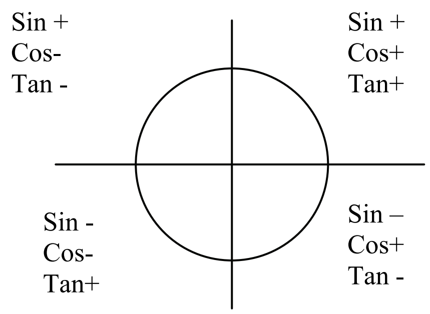


Figure 7.2: Diagram indicating when Sin, Cos and Tan are Positive or Negative.

It may be noted here that the process of determining the nature of each of the functions in the four quadrants was explicitly undertaken by the students in this group in contrast to the students in the pilot study group who were given the result without undertaking the process [§5.6.4]. Thus these students were given the opportunity to interiorise the process, an opportunity denied to the pilot study group. The results of this exercise were then linked to the graphs of the three functions by then considering the positive and negative nature of the graphs in each of the four quadrants in this representation. Each of the representations of trigonometric function had been considered in the analysis of the identity creating links between all the representations [see Nickerson (1985) §3.1.7] and complementing the algebraic representation with the spatial visual representations [see Duval (1995) § 3.2.3].

7.5.2 Utilising the Power of Visual Representations.

It was the belief of this teacher that trigonometry is a geometric topic and needs to be understood geometrically in the first instance. He commented that

> The algebra only describes the fundamental ideas, they are really geometric ideas and have to be understood in those terms; I don't think you can teach trigonometry from an algebraic perspective, I don't think it would make much sense if you did.
>
> (Teacher)

He believed that the representational systems should be used hierarchically and his stated aim was to encourage the students to think primarily in geometric terms.

> The algebra is usually an operational tool but the real power of the subject is in the spatial representations which summarise the whole picture. You need both aspects of course but the algebra isn't where the thinking takes place.
>
> Teacher

The next section describes the teacher's attempts to encourage students' visual thinking.

7.5.2.1 Encouraging Student's Visual Thinking.

Here is a short extract from lesson 6 from the C3 module on the double angle formula.

Teacher: I now want to go onto the double angle which just means things like Sin 2A, Cos 2A and tan 2A. Now when we did transformations you remember what happened when we had sin 2A…Peter would you like to remind us? …What happens when you have Sin *ax*?

S8: Amplitude!
Teacher: No not amplitude
S5: Frequency!
Teacher: Yes. When you have something like that it is squashed by *a* or expanded depending on whether it's bigger or less than 1. When you have 2Sin θ…
S13: It goes up!
Teacher: Yes the amplitude is changed. So that's the difference between 2 Sin θ and Sin 2θ. Okay. So this is this sort of function: Sin 2θ.

The language used in this discourse describes spatial visual characteristics: "It is squashed" "It goes up" and the notions of amplitude and frequency that many of the students would be familiar with from their physics lessons. The emphasis is on the spatial characteristics of the graph and then linking these to the algebraic representations.

In lesson 7 on the identity $a\cos\theta+b\sin\theta = R\cos(\theta-\alpha)$, before introducing the identity, the teacher distributed graphical calculators and asked the class to use them to draw $y = \cos\theta + \sqrt{3}\sin\theta$ and compare the image with $y = 2\cos(\theta-60)$. The students noted they produced the same graph. The teacher then asked the students to suggest how this could be used to solve an equation such as $\cos\theta+\sqrt{3}\sin\theta=1$. The students suggested drawing the graph and then drawing a line at y=1 and finding the points of intersection. [Vinner, 1983; §3.2.5]. Thus a spatial–visual solution process was found before introducing the algebraic method. Another example from lesson 9, where the reciprocal functions of sec θ, cosec θ and cot θ were introduced, is where the teacher asked the students to attempt to draw the graphs of these functions based on their knowledge of the shapes of the graphs of sin θ, cos θ and tan θ.

The evidence suggests that the teacher thought initially in terms of spatial visual representations and then transferred to algebraic representations when required. This was in line with his perception that trigonometry is primarily a geometric topic [§7.5.2.1].

7.5.3 The Infinite Nature of the Sin, Cos and Tan Functions are Emphasised

The teacher repeatedly emphasised the infinite nature of the functions graphically. When sketching a graph he would specify the domain either verbally or using notation for example in lesson 9:

Teacher: So when we consider the graph of cosec x for, say, $0 \le \theta \le 4\pi$...

This drew attention to the global and local nature of the graph of the functions. He deliberately used different domains to emphasise that the function was not only defined on one positive revolution through 360° for example: $\pm\pi$, -270° to 90° etc. When solutions to equations were required he made a point of checking the domain and if none was specified, for example, in a problem that a student was working on where the student had not specified the domain, he would pronounce the number of solutions to be infinite. In connection to this he took time to explain the way calculators have been programmed to give answers in the range $-180 \le x \le 180$ for the inverse functions for example $\mathrm{Sin}^{-1}(-1/2) = -30$. The teacher explained:

> This means the result given by the calculator must be viewed in the context of the graph.
>
> (Teacher)

7.5.4 Operational Processes are Subsumed Within Conceptual Connections

Once an operational process had been practised, the teacher tended to refer to it as if it was a contained process that could be undertaken to achieve a specific result. He rarely undertook the process himself but asked the students what they could do to achieve a result and then left them to undertake it themselves. This had two effects: the first was that the process was perceived as a means to a specific end and the second was that the teacher implied a hierarchical aspect to operational and conceptual ideas. For example, to solve the equation $2 \sin \theta + 3 \cos \theta = 1$ the teacher suggested to a student:

Teacher: Where have you encountered the format something $\sin \theta \pm$ something $\cos \theta$ before?
Student: In the sin A+B formula?
Teacher: That's right. Expand it and use the fact that component parts are equivalent.

The role he played was to suggest links that could be profitable rather than undertake the solution process himself. Thus a conceptual element was emphasised that was predicated on knowledge of previously encountered identities and operational features [see Sfard (1991), §3.1.4; Dubinsky (1991) §3.1.5; Gray & Tall (1994) §3.1.6].

7.5.5 The Precise use of Language

The teacher made a point of defining terms precisely. At times he would pause in his exposition to emphasise the point. An example is the following discourse during lesson 3:

Teacher: One of which we already know is $\sin^2 \theta + \cos^2 \theta = 1$. This is an identity. And the point about identities is if I have $3x = 2$ how many solutions to that are there?
Student: One
Teacher: One. If I have $x^2 = 4$. That is an equation but this time there are how many solutions?
Student: Two
Teacher: Two. So none of these could be considered identities because they are not true for all values of x. But identities are very useful because they are true for all values of x

The emphasis on precise definitions by this teacher contrasts sharply with the use of language by the pilot study teacher [§5.6.3] and is fundamental to the construction of concept [see Vinner (1983) §3.2.5].

7.5.6 Using Tasks to Encourage Students' Flexibility of Thinking.

Many of the lessons were started by the teacher writing a problem on the board and then inviting suggestions from the class on how to solve it. All suggestions were considered though students were encouraged to think creatively about possible solution processes. The discourse below from C4, lesson 14, is an example of the teacher attempting to create links that would facilitate an alternative solution process. The following integral was written on the board:

$$\int \cos^2 x \sin x.\ dx$$

Teacher: How might we solve this?
S11: By parts?
Teacher: Well we could but there is an easier way.
S14: By substitution?
Teacher: No, no! Think, think, think!. What do you know about differentiating cos to some power?
S6: Oh we get the differential as well.
Teacher: Yes when we have some function f(x) that is cos x to a given power then it differentiates to...
S6: The power times cos x times sin x so the integral of this is just $^1/_3 \cos^3 x$.

Here the teacher has averted the start of a stimulated procedure of integration by substitution by emphasising the connection to other known concepts. This has indicated how lengthy processes may be short cut by thinking flexibly.

Another way of using tasks to encourage flexible thinking was the use of problems. For example at the end of a C3 lesson he set homework saying:

> We did some exercises using compound angles (θ+φ) or (A+B). I'd like you to have a go at questions 5 and 6 page 66 but I'd particularly like you to have a go at question 6 because there's an element of proof in that.
>
> (Teacher)

The emphasis here is on question 6 which requires the element of proof. This is an important mathematical idea that has recently been under discussion in the mathematical community due to perceived concerns about students' lack of ability to demonstrate rigorous proof [See Chapter 2]

7.6 Summary of Section

This section has considered in detail the teacher's pedagogic content knowledge and style of delivery where we see him implicitly drawing upon the positive teaching qualities identified within other studies.

- The teacher continually connected different representations of the same idea so that the idea linked all the representations (Duval, 1995,§3.2.3; Tall& Vinner, 1981 §3.2.5; Arcavi, 1999, §3.2.3)
- The teacher encouraged visual reasoning by the students whenever possible. (Presmeg, 1986, §3.2.4)
- The infinite nature of the functions was continually emphasised.
- Operational processes were subsumed within conceptual properties and links (Sfard, 1991,§3.1.4; Dubinsky, 1991, §3.1.5; Weber, 1995, §3.3.3)
- The class were encouraged to fully interact with the exposition; questions and queries were encouraged and were answered as they arose. This would seem to indicate that the teacher was comfortable with a degree of flexibility when delivering the exposition. His exposition stayed focused on the concept as defined and linked it to other mathematical knowledge, and he encouraged the students to perform the instrumental processes themselves (Blackett, 1990; §3.3.2 and Weber, 2005; §3.3.3 Shulman ,1986 §7.5)
- The teacher educated the class in the language that is used by the mathematical community and defines it precisely (Delice & Monaghan, 2005, §3.3.5, Vinner, 1983,§3.2.5)
- The method of continually asking questions requires the students to give the answers themselves rather than passively watching as knowledge is presented to them. The students appeared to be active in their own learning (Piaget, 1972, §3.1.1).

- The teacher used problems to create links to other concepts that the students are aware of in order to encourage flexible thinking (Hiebert & Carpenter, 1992; Tall 1995, §3.1.3; Nickerson, 1985, §3.1.7).

Although this study is not designed to be a comparative one of teaching styles but a focus on the schemas the students are creating, it is interesting to note this teacher's emphasis on visual spatial images in contrast to the pilot study teacher's emphasis on algebraic representations (Duval, 1995, §3.2.3; Presmeg, 1986b, §3.2.2; Blackett, 1990, §3.3.2)

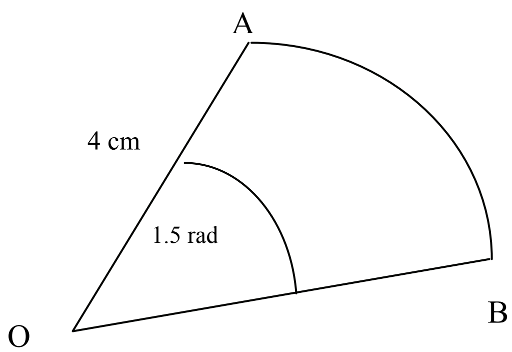
7.7 Instructional Outcomes After C2

On completion of the C2 module all the students in the sample were asked to complete a pre-prepared set of questions that sought to examine their content knowledge. This was intended to be an investigation of how much of what had been taught during the course had been learned by the students.

7.7.1 Students Skill with Radians

The first question is similar in style to those given in the text book [Exercise 7A, pp125-6]

Q1 a) Find the length of the arc AB



b) Find the area of the circle sector

Figure 7.3: Diagram of circle sector

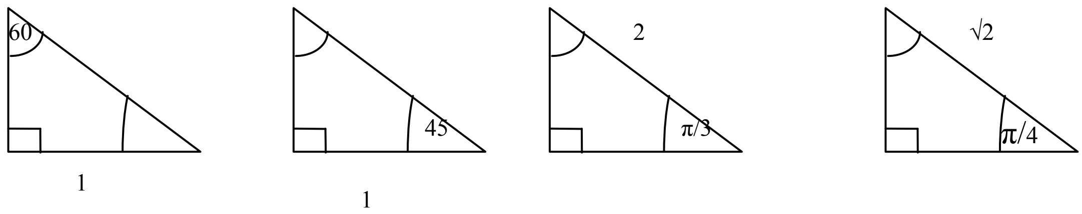
4 marks were allocated for this question, I mark for method and I for a correct answer to at least 2 decimal places for each section a) and b). 5 of the 15 students scored the full 4 marks, including **S1** and **S2** [§6.2]. The remaining 10 students scored 0 marks and 4 of these students, including **S4**, did not attempt the question. Those who did attempt the question but received no marks attempted to solve the question by converting radians into degrees. By writing their answer as a fraction of 360° they

found this proportion of the circumference and area of a circle using the formulae $C=2\pi r$ and $A=\pi r^2$. This method would find the correct answer but the students did not convert the radians to degrees correctly. **S3** divided 360 by 1.5. The remaining 5 students thought that I rad =180° and so found 1.5 rads to be 270°.

7.7.2 Students Knowledge of the Special Angle Triangles

This question was devised to examine their knowledge of the special angle triangles in degrees and radians. The aim was to investigate how much of the reasoning behind the construction of these triangles the students were able to reproduce.

Q2. a) Complete the triangles by marking in all lengths and angles. (Diagrams not drawn to scale)



b) Hence or otherwise determine the exact value of:

(i) Sin 120°

(ii) Tan 315°

(iii) Tan $5\pi/6^c$

(iv) Cos $(-\pi/4)^c$

Figure 7.4: Diagrams of the Special Angle Triangles.

Only one student (S1) scored full marks for part b) although another student gave decimal answers to 3 decimal places.

There was evidence of pattern within the answers given. For section (a) these are listed below with the number of students who made this error shown in brackets afterwards:

- All correct (2 including S1)
- Wrote radian angles as a decimal (4)
- Wrote angles in degrees but not radians (3 including S3 & S4)
- Determined lengths but not angles (3 including S2)
- Lengths given as a decimal (2)

- Did not give any answer (2)

It may be noted that the total number of students is more than the 15 students who completed these questions as some students made more than one type of error (for example wrote radians and lengths as a decimal).

The patterns of answers for section (b) were as follows:

- Unable to find angles in radians (7 including S2 and S3)
- Answers given as a decimal (4)
- Only able to determine tan 315 (2 including S4)
- Gave wrong +/- sign (1)
- All correct (1, S1)
- Wrote nothing (1)

Answers given as a decimal included: $\sqrt{0.75}$ instead of $\sqrt{3}/2$ and $\sqrt{0.5}$ instead of $1/\sqrt{2}$ which implies the students found the answers on the calculator and then found values that were equivalent. This observation was verified later by the relevant student.

One student gave the answer to (iv) as $-1/\sqrt{2}$ when it should have been positive. Further questioning of this student found she is aware of the symmetrical nature of the Cos wave each side of the vertical axis yet she gave a negative value. This implies that this student is not yet linking together different sub concepts.

The general poor quality of answers for this question is noteworthy. Clearly at this juncture few of the students had given much attention to the special angle triangles and many had not mastered working in radians. Most of the students were dependent on the calculator to find values.

7.7.3 Students Ability to Connect Graphical Images to Different Algebraic Representations

The next question was set to investigate the extent to which students could connect algebraic representations to graphs transformations. They were examples of questions from Exercise 7E in their text books.

Q3. Sketch the following curves for $0 \leq x \leq 360°$ showing any maximum and minimum values and where the graphs intersect the x-axis.

a) Y = Sin 2x

b) Y = -cos x

c) Y = 3 sin x

d) Y = Tan (x-90)

4 students scored full marks including S1 and S4. The answers given by S4 here, and in question 5 below, indicate that he had a good grasp of the nature of the graphs and could relate the graphical transformations to the algebraic representations.

S2 and another student wrote incorrectly that the intercepts on y =-cos x were 90°,180° and 270° instead of 90° and 270°. S3 incorrectly identified the intercepts for y=sin 2x. He put the values as 90°, 180° and 360° instead of 90°, 180°,270° and 360°. Both these errors are fundamentally the same in the respect that period of the graph has not been consistently drawn.

3 students made errors in recognising the type of transformation indicated by the algebraic representation: 4 drew Y=sin 3x instead of Y=3 sin x, another drew Y= cos x instead of Y = Sin 2x. Other errors included 2 students drawing y=cos 2x instead of y=sin 2x. S3 and another student drew Y= Sin 3/2x instead of Y= Sin 2x and the other drew an unending wavy line. 2 students failed to draw Y=Tan (x-90) correctly. There was an attempt in both cases at an asymptotic graph but the intercepts were at 90° in both cases and the sections of curves drawn were more akin to wiggly lines than a recognisable tan graph.

One student failed to indicate the maximum and minimum values of any of the graphs, and did not draw the last graph. One student, S5 [see Chapter 8] who was one of the most able in the class and achieved 100% in his AS assessment [§7.8]; scored O marks for this question. He drew Y= Sin 5/2 x instead of Y=sin 2x. He drew a random wavy line for Y=-Cos x and he drew nothing at all for Y=Tan(x-90). The graph of Y= 3 sin x was approximately the correct shape but there were no intercepts shown and no indication of its maximum and minimum values.

Again the confusion over the graphs and their algebraic representations is noteworthy as the teacher had allowed the class some considerable time to explore these graphs.

7.7.4 Students Ability to Solve Simple Trigonometric Equations

The following question was set to investigate the students' ability to recognise

a) sin x/cos x as an identity for tan x and b) a quadratic function in tan x. It is similar to the style of questions in exercise 7D in the text.

Q4. Solve for $0 \leq x \leq 360°$

a) Sin x =2 Cos x

b) Tan^2x-tanx-6 =0

5 of the 15 students, including S2, scored full marks on this question.

S1 and another student gave only three solutions within the given range. A third student gave additional solutions for Cos x =0 and the fourth student gave five solutions instead of 4. S3 and another student attempted part a) by substituting (1-Sin x) for Cos x. Clearly they had not recognised that a simple rearrangement of the equation would invoke the use of the identity for tan x and further, mistakenly thought that sin x + cos x = 1 instead of $\sin^2 x + \cos^2 x = 1$. In part b) S3 solved the equation by substituting into the quadratic formula. He found the two correct values for x when tan x =3, but then solved Tan x = 2 instead of -2 although he had originally found tan x to be equal to -2. This was a careless slip. He evidently had the ability to solve this equation. One student solved the equation in part a) correctly and correctly factorised the expression in part b) however she failed to find 4 correct angles as her final solution. Another student correctly factorised and solved for tan x but was unable to find the correct values for x. One student correctly deduced that $\tan^{-1} x = 2$ but only gave one solution and he barely attempted part b). S4 and S5 [see Chapter 8], mentioned earlier, did not attempt the question.

The most common error here was the inability to deduce the correct number of solutions within the given range.

7.7.5 Students ability to recognise identical functions.

This question is related to question 3 in that it is requires knowledge of the graphs. The previous questions had been given to the pilot study group however in the light of the memorised responses given by Cyndi it was thought by the researcher that it could be illuminating to ask this group an extra question that they had not attempted before.

Q5. Which of the following are equivalent? Put a circle around them and join them with a line.

Sin x　Cos x　Tan x　Sin (x-90)　2Sin x　Cos(x-90)　Sin 2x　Sin (x+1)

Sin (x+90)　Sin x +1　1+ Sin x　Sin x/Cos x

All the students, identified tan x ≡ Sin x/ Cos x. 10 students, including S2 and S4, identified Sin x ≡ Cos (x- 90). 9 students, including S2 and S4, identified Cos x ≡ (Sin x +90) and 8 students, including S1 and S2, identified Sin x + 1≡ 1+ Sin x. Thus S1 identified two, S2 identified all four, S3 identified only tan x ≡ Sin x/ Cos x and S4 identified three of the four correctly.

This indicated that at least ten of the students were able to make a meaningful connection between the different functions.

Overall 4 students, including S1 and S2, scored at least 75% of the marks available. 6 students including S5, scored in the range 50 -75% and the remaining 5 students, including S3 and S4 scored in the range 25-50% marks. The highest mark was 93% and this was scored by S1. The lowest mark was 35% and this was scored by S4.

The students themselves commented on how difficult they found these questions, especially the first two questions which involved working in radians. They were conscious that the AS assessment was only a few weeks away and the following comments were made on receiving their results:

> S5: I didn't learn any of this properly. I didn't know there was going to be a test.
>
> S3: It's trigonometry. It doesn't make sense any more.
>
> S2: Are these the sort of questions we will get on the exam?

7.7.6 The Second Concept Maps

We have seen that the student had constructed their first concept maps before the course had started (See Chapter 6.) Now, after completion of this component students were asked to complete their second concept maps. Those prepared by the four students (S1 –S4) who shared in the deeper interview process are shown within figures 7.5 to 7.8.

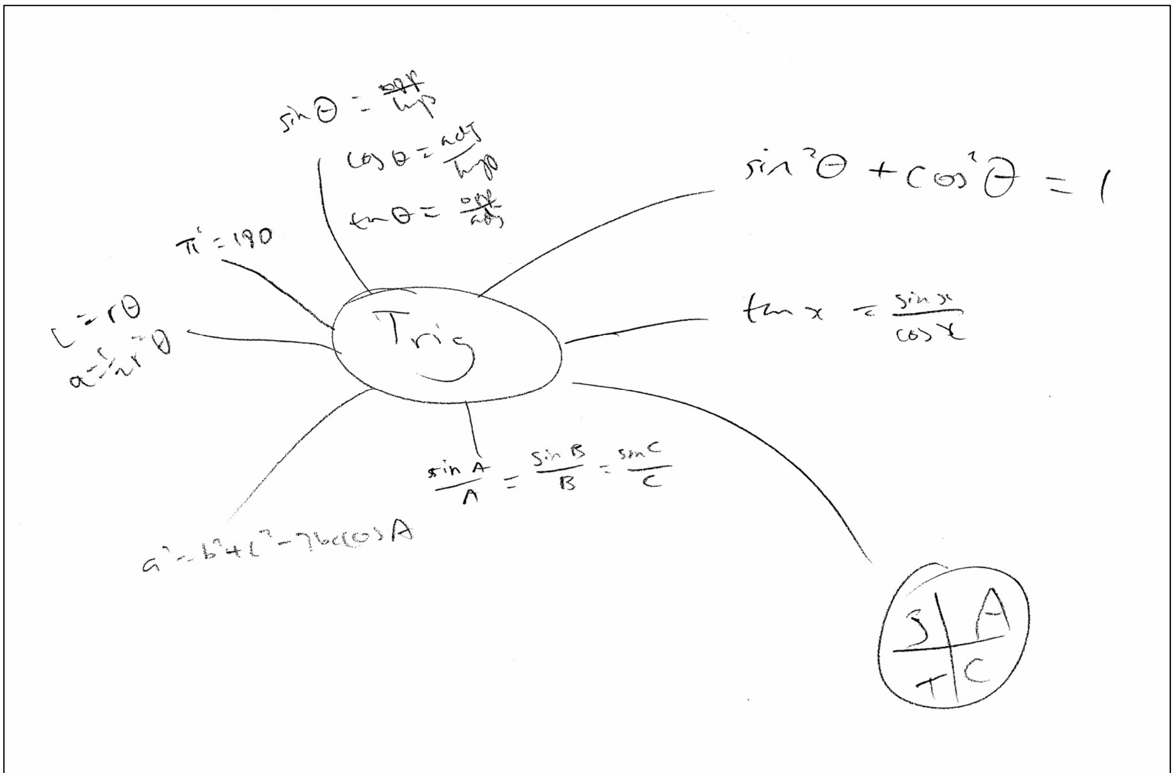


Figure 7.5: S1 Concept Map 2

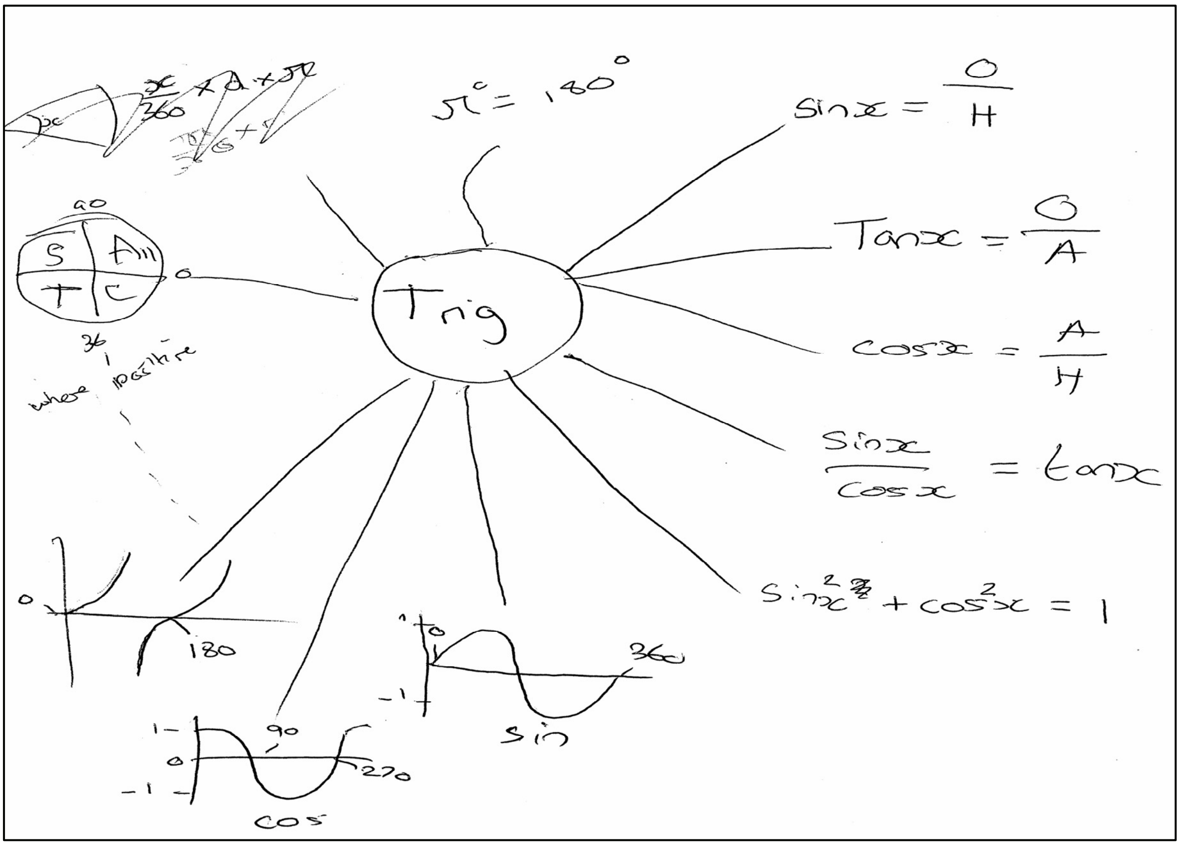


Figure 7.6: S2 Concept Map 2

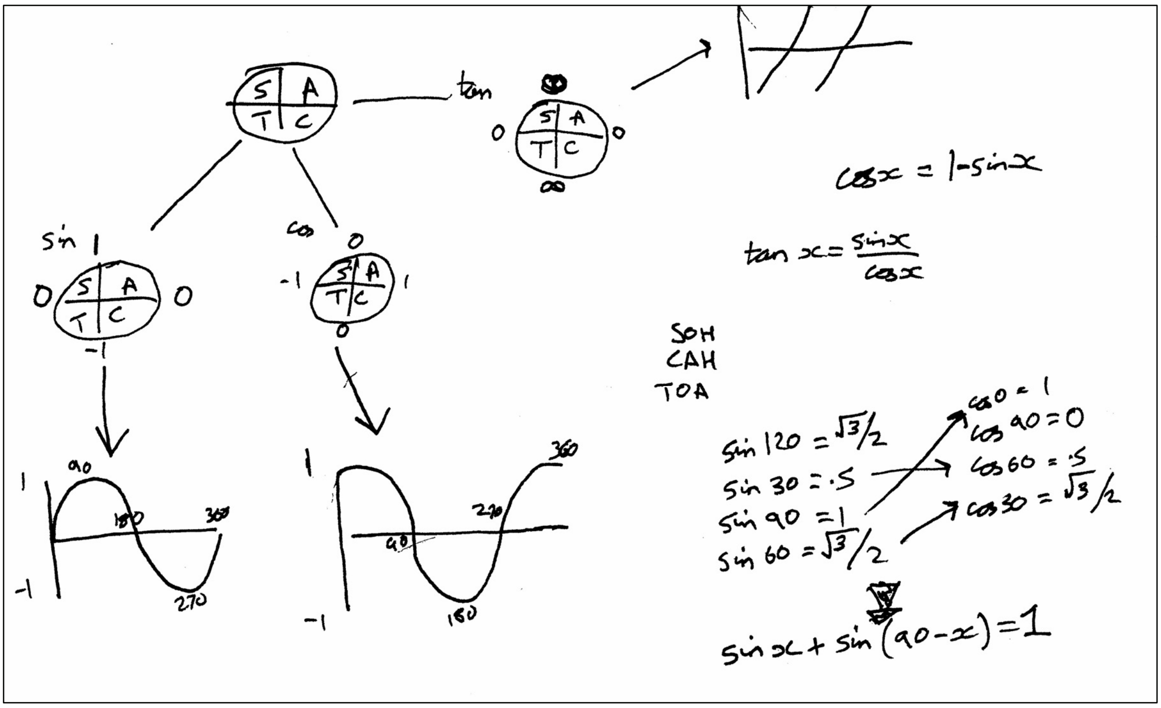


Figure 7.7: S3 Concept Map 2

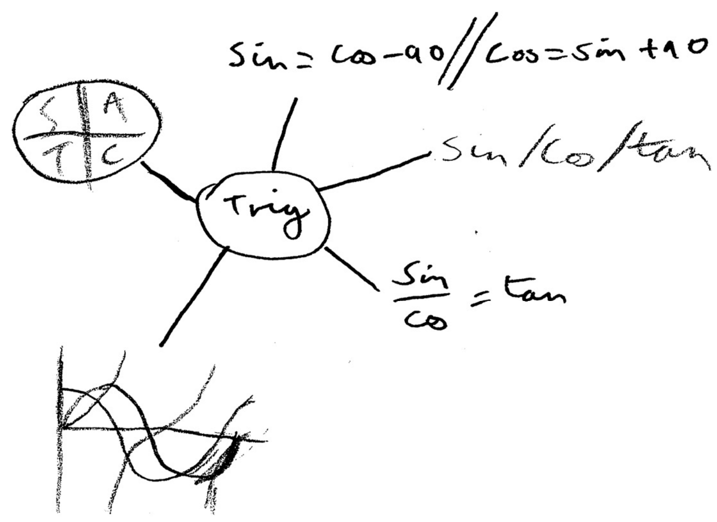


Figure 7.8: S4 Concept Map 2

The content of the concept maps may be grouped under 4 sub headings: Core function properties, formulae, values and spatial images. Formulae that are operational in nature such as the sine rule are distinguished from identities such as tan θ =sin θ / cos θ which is identified as a core function property. Thus the Pythagorean formula $a^2+b^2=c^2$ will be grouped under formulae whilst it's trigonometrical equivalent $Sin^2\theta + Cos^2\theta=1$ will fall into the core function properties. Using these classifications a table may be constructed of the difference in content between each of the four students' first and second map. Where material was identified in the first concept map and is also within the second is shown in bold.

S1	Map 1	Map 2
Core properties	**Sin = opp/hyp etc** **Tan x =Sin x/ Cos x**	**Sin = opp/hyp etc** **Tan x =Sin x/ Cos x** $\sin^2\theta+\cos^2\theta=1$
Formulae	**Sin rule** **Cos rule** Pythagoras	**Sin rule** **Cos rule** $L=r\theta$ $A=1/2r^2\theta$
Values	Surd or decimal values for Sin, Cos and Tan 0, 30, 60, 90° and tan 45°	$\pi=180°$
Spatial images	Sin, Cos and Tan graphs	CAST circle
Other		

Table 7.2: S1's Concept Map Content.

S2	Map 1	Map 2
Core properties	**Sin =O/H etc** **Tan =Sin /Cos**	**Sin =O/H etc** **Sin x /Cos x =Tan x** $\sin^2\theta+\cos^2\theta=1$ $\pi=180°$
Formulae	Sin rule Cos rule Pythagoras	
Values		
Spatial images	**Sin, Cos and Tan graphs**	**Sin, Cos and Tan graphs** CAST circle
Other		

Table 7.3: S2's Concept Map Content.

S3	Map 1	Map 2
Core properties	**SOHCAHTOA** **Sin /Cos =Tan**	**SOHCAHTOA** **Tan x =Sin x/Cos x**
Formulae	Sin rule Cos rule Pythagoras	
Values	Sin 45/Cos45=Tan 45 =1	Sin and Cos 30,60,90° Sin 120° Cos 0°
Spatial images	**Sin graph** Incorrect Cos and Tan graphs	**Sin graph** Cos graph Approximate Tan graph 4 CAST circles
Other	Quadratic formula Sin 45=Cos45=√3/2 ?	Cos x =1-sin x Sin x + sin (90-x)=1

Table 7.4: S3's Concept Map Content

S4	Map 1	Map 2
Core properties		Sin /Cos =Tan
Formulae	Sin rule Cos rule	
Values		
Spatial images		Sin, Cos and tan graphs CAST circle
Other	Sin^{-1} used to find angles 'Sine &cosine 90° out of sync' 'Tangent goes up (and down to infinity'	Sin = Cos -90 Cos sin + 90

Table 7.5: S4's concept map content.

S1 and S2 have increased the number of core properties in their second maps. S3 has exactly the same core properties and S4 mentioned none in his first map and 1 in his second map.

S1 has increased the number of operational formulae in his second map whilst S2, S3 and S4 have omitted all these from their second maps.

S1 has omitted values from his second map whist S3 has increased the number of values in his second map.

All four students included the CAST diagram in their second maps. S1 did not include the graphs in his second map but S2, S3 and S4 gave the CAST circle in addition to the graphs.

7.7.7 Feedback on Concept Map Issues

Several of the students omitted items on the second map which had been mentioned on the first. Students S1, S2 and S3 were asked to explain this phenomenon (By this time S4 was increasingly missing classes). Their replies are given here:

> It was second nature and you know it already so I left it off and I put down the things I learned this year rather than what I learned last year like the graphs and that. We didn't use it that much so I didn't put it down.
>
> S1

> I didn't include the sin rule etc because I was thinking deeper into it and not thinking of the stuff on the surface. They are a part of trigonometry but the bit that comes straight to me so I just put down the new stuff.
>
> S2

> Well I thought the sine rule and cosine rule are a part of trigonometry but not a part of the syllabus.
>
> S3

Two further questions were asked about their use of the spatial images and the reason for not using the 2 radian formula for the sector arc length and area.

Do you prefer to use the CAST diagram or the graphs to find angles greater than 90° and why?

> CAST um...both. CAST is good for finding angles but for something like theta minus something or theta plus something the graph is better. Also the graph is better for maximums and minimums.
>
> S1

> I prefer to use the graph because I missed the explanation on the CAST diagram.
>
> S2

The CAST diagram makes more sense because I don't remember the graphs too well. Once you know what each bit means it is easier.

S3

In the task questions why didn't you use the formulae $l=r\theta$ and $A=1/2\ r^2\theta$ to find the answers to questions specifically written to test this knowledge. Some people didn't attempt these questions and some used other, equally valid, means to find the answer which involved more work. Why do you think this was?

I forgot these formulas at the time! (S2)
I didn't know anything about those two formulas. (S3)

7.8 AS Results & S4 Drops Out

The students sat the C1 assessment at the end of term 1. At the end of term 3 all the students sat C2, S1 and M1 and some re-sat C1. Both scores for the C1 assessment are shown for these students. Some of the students sitting retakes fared worse on the resit than in their first attempt however the better of the scores has been credited. The two Core module results and the better of the two applied module results are added together to determine the grade. The grade boundaries are A-240, B-210, C-180, D-150, E-120 and U for less than 120 marks. The marks awarded to the students in the whole group, with those interviewed identified, are set out in the table below.

Student	C1	C2	Best Applied module grade	Final grade
S1	95	97	91	A
	89	89	97	A
	95	92	80	A
	97	84	83	A
S2	84	92	88	A
S5	100	89	70	A
	(84) 87	83	81	A
	89	79	80	A
	92	80	70	A
	(61) 78	83	89	A
	(66) 79	73	63	B
	(54) 79	69	64	B
S3	85	71	64	B
	93	58	54	C
	(65) 66	73	46	C
	(69) 64	44	34	E
S4	21 (12)	27	45	U

Table 7.6: AS results.

Student S5 was well known for doing well in the assessments however as the AS /A2 course progressed his marks went down. See Chapter 8 for further discussions with this student about learning and understanding.

S4 dropped maths at the end of the first year. He was interviewed about his experience studying mathematics at AS level.

S4: Right from the start I never got into it especially the algebra and trig. It was all different to what I expected.
R: How was it different?
S4: I just didn't feel it was worth learning.
R: How did you decide what was worth learning?
S4: I don't know...I haven't really thought about that.
R: What did you think trigonometry was about at GCSE?
S4: Well er that was always one of my weaker parts of maths at GCSE.
R: What things were you good at?
S4: Mechanical stuff mostly...I don't know...I just... the mechanics side of things was something I did find easier. That was the module I did best at last year as well. Yeah mechanical things I am better with.
R: You did very well at GCSE. What happened last year?
S4: I pretty much got to the point where I couldn't see anyway of learning everything so I didn't bother, like the formulas and that... I thought I could leave it till the end and then revise hard.
R: Is that what you did at GCSE?
S4: Yeah. But I never thought I would go onto the A2. I thought I would just do it for a year.

This dialogue brings to mind the comments by A-level mathematics teachers that:

> Students find trigonometry one of the hardest subjects on the A-level. They have difficulty with it as it becomes more abstract.
>
> Trigonometry at A-level is the topic that sorts the sheep from the goats. It is a good indicator of true mathematical ability and if they can master it then it indicates that they have the potential to become a real mathematician. [Chapter 1]

S4's reflection on 'learning' implies that his perception of 'learning' mathematics is based on remembering formulas; a method which was to prove increasingly inadequate.

7.9 Responses to Integrated Questions.

C2 was studied in the third term of year 12 and C3 was studied during the second term of year 13. The AS results were received during the summer break between year 12 and year 13. During the study of trigonometry in the C3 module the students were asked to answer the past paper questions previously set to the pilot study group. These questions were taken from previous P1 [See Chapter 2] papers and covered work now studied in C2 and C3. The students attempted these questions individually, in turn, in the corridor of the maths department away from the other students. They were asked to explain their reasoning as they progressed.

7.9.1 Students understanding of composite trigonometric functions and use of visual imagery

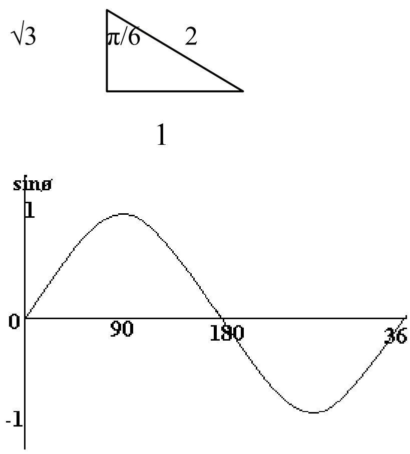
The following question was set to investigate the understanding these students had of composite trigonometric functions.

Q1. (a) Find the coordinates of the point where the graph of y=2sin (2x+5/6π) crosses the y-axis.
(b) Find the values of x, where 0<x<2π, for which y=√2 "

Each of the students used a different approach to solving the first part of this question.

S1 wrote:

Y=2sin (2x0+5/6π)



[Using calculator] x=0, Y=1

Figure 7.9: Spatial Imagery Used by S1

When asked about the connections between these different spatial images S1 said

It is just how I know how to get the answer… I dunno

S2 also used visual spatial images but her thinking was less focused.

S2: Well x=0, so y=2sin (5/6π) Umm the sine graph goes like that and that'll be 2π so 5/6π will be about here where x is zero. Mmm Cos it's 2 here does it mean you're stretching the y values by 2 and because it's zero it will be zero anyway....Umm so when it has something like this in the brackets, if it's plus it usually means you shift it to the left. Oh! Okay so it's 5/6 π...[uses calculator] sine of that is 1!

[*This result was clearly a surprise to S2. She then went back to verify it]*

So that means the sine of 5/6π is a half so that when you double it you get 1. So there is a triangle for this. [draws triangle] so it's opposite over hypotenuse...so it's there...so it's 30. Of course! 1/6 of 180 is 30°!

R: So in the end what is your answer?

S2: You have to give the coordinates of where it crosses the y axis so x is zero and y is one.

S2 premised her solution method by thinking initially in terms of the geometric transformations of the sine graph. When she had a sufficiently satisfactory spatial image of the function in front of her she was able to apply herself to the specific question asked and using the calculator gave the correct answer. When she had obtained this answer she then recognised its connection to the spatial images of the triangles and went on to verify to herself that the answer was correct. This appears to show evidence of links between the visual aspects of trigonometry. However it seems at this stage that the images are themselves Actions that are automatically triggered by a stimulus. The drawing of the graph seems to have been stimulated by the expression y=2sin (2x+5/6π) and the drawing of the special triangles seems to be stimulated by finding that sin (5/6π) =1/2, a value which she clearly recognised. These conjectures were verified later by interview when S2 explained that she "did things without really understanding the connection at first but afterwards she did". S2 is therefore able to reverse the operational procedure from the solution back to the triangles. It could be concluded therefore that this student is still apparently at the Action stage of development but able to reverse the procedure.

Below is the answer from S3.

S3: When it crosses the y axis x is zero. So, that's not going to be too difficult

Writes; y= 2sin (2x+5/6π)

Well I am just going to expand now

Writes: 2(sin2xcos5/6π+ sin 5/6πcos2x) = 0

Logically since x is zero then that just equals zero. Logically! [Sighs]

Writes x=0 y=0

R: So the answer to part (a) is that it crosses at the origin: at x=0, y=0?

S3: Well it makes sense because if x is zero multiplied by cos5/6π [points to expansion] then you'll just get zero zero

R: That's right and here? [points to sin 5/6πcos2x]

S3: It's the same again.

R: Cos of 0?

S3: Oh! Of course yeah. Tut. [reaches for calculator] Oh it's radians. Oh I always hated radians, I could never see the point of them.
writes new solution x=0, y=π, values found on calculator. This answer is not correct but student moves on to next part of question.

Here appears to be evidence that the thinking of S3 at the Action stage. He spent time remembering the formulae shown in his concept map for the expansion of sin (A+B) and was prompted to use it when he saw the stimulus of y= 2sin (2x+5/6π). This could have resulted in the correct solution although it would take rather longer to reach, however S3 then made the expanded version equal to zero. There was no pause for thought as this was written out and this line was written automatically without any conscious control of the procedure. Once written out the error was not detected and student 3 then concluded that as well as x equalling zero, his starting point, y also equals zero. He did not arrive at this conclusion by a logical deduction such as:

$$Y = 2\sin(2x+5/6\pi)$$
$$Y = 2(\sin 2x\cos 5/6\pi + \sin 5/6\pi\cos 2x) = 0$$
$$\text{Therefore } y=0$$

but by substituting zero for x in the expansion. This again could have resulted in the correct answer if the mistakenly placed zero on the right hand side had been detected but the substitution also contained an error. Once this error was hinted at, S3 realised his mistake and recalculated the value of the expansion on his calculator but still obtained the wrong answer. In the subsequent interview S3 said that he "mainly used methods that he had been taught in class" and "didn't really know until he got to the end how he was going to get the answer"

The conclusion drawn from this evidence is that S3 is undertaking a series of learned procedures with little or no recourse to other trigonometric concepts. He works entirely within an algebra representation without reference to any conceptual spatial images and his thinking has a single direction linear dynamic. His trigonometric schema is not interconnected and he does not have the cognitive flexibility associated with the Process stage of development.

7.9.2 Qualitative Differences in Students' Solution Processes

These 3 students exemplify the diversity of approaches shown by the students in answering a trigonometric problem. S1 appeared to recognise a simple route to solution immediately. S2 took some time setting the scene for the problem in a spatial visual context. She thought operationally but was able to reverse procedures, an important developmental stage according to the theoretical paradigms. S3 immediately applied procedures showed little evidence of recognising the purpose of the procedures he was implementing or how any one of them would progress the problem to solution.

7.9.3 Students' Ability to Think Flexibly

The second of the task questions showed that two of the three students were able to switch from one procedure to another in a manner that indicated they were aware of alternatives and could switch between them. This was the second question attempted by the students.

Q2. $f(x) \equiv 3 + 2 \sin (2x + k)°$, $0 \leq x \leq 360$, where k is a constant and $0 \leq k \leq 360$. The curve with equation $y = f(x)$ passes through the point with coordinates $(15, 3 +\sqrt{3})$.

(a) Show that k=30 is a possible value for k and find the other possible value.
Given that k = 30,
(b)Solve the equation f(x) =1,
(c) Find the range of f,
(d) Sketch the graph of y=f(x), stating the coordinates of the turning points and the coordinates of the point where the curve meets the y-axis.

Source: P1January 1999 Q10.

There were indications of different types of switching taking place. One form was to switch from an algebraic procedure to using known facts.

In the question above, parts (a) and (b) were solved by each of the students by algebraic means. (For part (a) the students substituted 30° into the expression for k, and 15° for x then evaluated the expression to get y equal to 3+√3. For the other possible value the students substituted 15° for x and 3+√3 for y and determined k using the sin graph. For part (b) the students substituted 30° for k and 1for y and solved for x.)

Therefore all the students used algebraic representations to solve the first two parts of the question. However part (c) was attempted differently by different students:

S1: [writes] $1 \le x \le 5$
R: How do you know that the range is less than 5?
S1: Because the sin of anything is less than 1 and times 2 is 2 and then add 3 makes 5.

There was a similar response from S2 to attempt the questions.

S2: Well the biggest sine can be is +1 and the smallest is -1 and it's 2 so 3 plus… going to be 5 or 1. Is that right?

However the response from S3 was less direct.

S3: I'm not exactly sure ... Um range is the minimum value of x. So I could differentiate it and then make it equal to zero and then see what x is....or not!
Okay Plan B. Do you make the sin of that -1 or something, like it's minimum value?
[looks at researcher but gets no response so goes on to differentiate it]
[writes: 2[sin2xcos30] +2[sin 30 cos 2x]
[f'(x)=] 2[(cos30cos2x+sin2xx-sin30) +(cos2xcos30 + sin 30x-sin2x)]
2[2cos2xcos30)+2(sin2x)(-sin30)]
4[(cos2xcos30]+4[sin2x(-sin30)]=0

Cos2xcos30=-sin2x sin30 *(sic)*
Cos30=-sin2x/cos2x + sin 30/cos2x
√3/2=-tan2x+1/2cos2x [stops]
R: What rule are you using here?
S3: Umm the product rule. [Continues with working though to a solution then sighs]
There is probably a much simpler way of doing it than this... [But I will] just sort of keep scribbling, ask for another page, keep scribbling, until eventually I have it worked out.

S1 and S2 switched from the algebraic manipulations they were using in the earlier parts of the question to the recognition of the properties of the sine function. It is noteworthy here that when S2 read this part of the question she said "Oh this is functions isn't it". When asked to explain she said "well before, we were just doing trigonometry but now it's gone on to functions." This would seem to imply that this student has not personally interiorised the fact that trigonometry is a type of function despite the term 'trigonometric function' being in common usage in the class by both the teacher and the students. When questioned about her idea of functions S2 explained that they "are like f(x) equals something". This indicates that functions are being defined by this student by their format rather than their essential properties.

S3 however continued working within an algebraic representation and applied procedures successively. Although he did briefly contemplate using the maximum and minimum values of the sine function he appears to be stimulated by the word *minimum* to the take the Action of differentiation and putting the derivative equal to zero. This is a routine procedure for finding maxima and minima that is learned when studying differentiation and its uses.

Having chosen this route he could have differentiated the function f(x) directly but appears to have been stimulated again by the sight of *sin (2x+ 30)* to the action of expansion using the sin (A+B) formulation that was mentioned on his concept map. After expanding he has attempted to differentiate using the product rule a third Action possibly stimulated by sight of the products in the expansion. It seems that this student is thinking in an Operational manner that is Action based. This was confirmed when he explained that when he sees certain things he is prompted to take a particular Action. This implies his response is not within his conscious control. Ultimately he failed to reach an answer because he ran out of stimulated Actions to perform, or in his words

> "I am not really sure what to do now. I have never answered this sort of question before"
>
> S3

Another form of switching between different representations was to switch from an algebraic procedure to a spatial image. Evidence of this was found in the students' response to part (d) where part of the instruction is to find the coordinates of all the maximum and minimum values.

> S2: And now I need to differentiate to get these. [Points to turning points on graph] But! I could just use the graph. It is easier that way.

Here S2 has switched from implementing a procedure to find maximums and minimums through differentiation, an Action possibly stimulated by the words maximum and minimum, to using the graph that she has already drawn. She appears to have recognised an alternative means to solution and consciously switched to it as a preferable means in this situation. This was confirmed in the interview when she explained that normally to find a maximum or minimum she would just differentiate but then she realised that she had the graph so she didn't need to.

In summary it may be seen that students S1 and S2 have shown evidence of switching between algebraic formulations and awareness of trigonometric properties; and switching between algebraic representations and spatial representations. This flexibility has its reward in cognitive efficiency. S3 on the other hand did not switch but implemented a series of procedures that involved lengthy working and ultimately failed to reach solution. It may also be seen that functions are recognised by S2 at this point by the notation 'f(x) = something'.

7.10 Third Concept maps

After completion of the C3 module the students were once again asked to complete concept maps. This is now at the end of the various units that comprised the A level course on trigonometrical functions. We consider only the concept maps completed by S1, S2 and S3 (S4 having now withdrawn from the course). These maps are indicated within figures 7.7 to 7.9. Towards the end of the section maps some additional students are also presented since they illustrate features of particular interest.

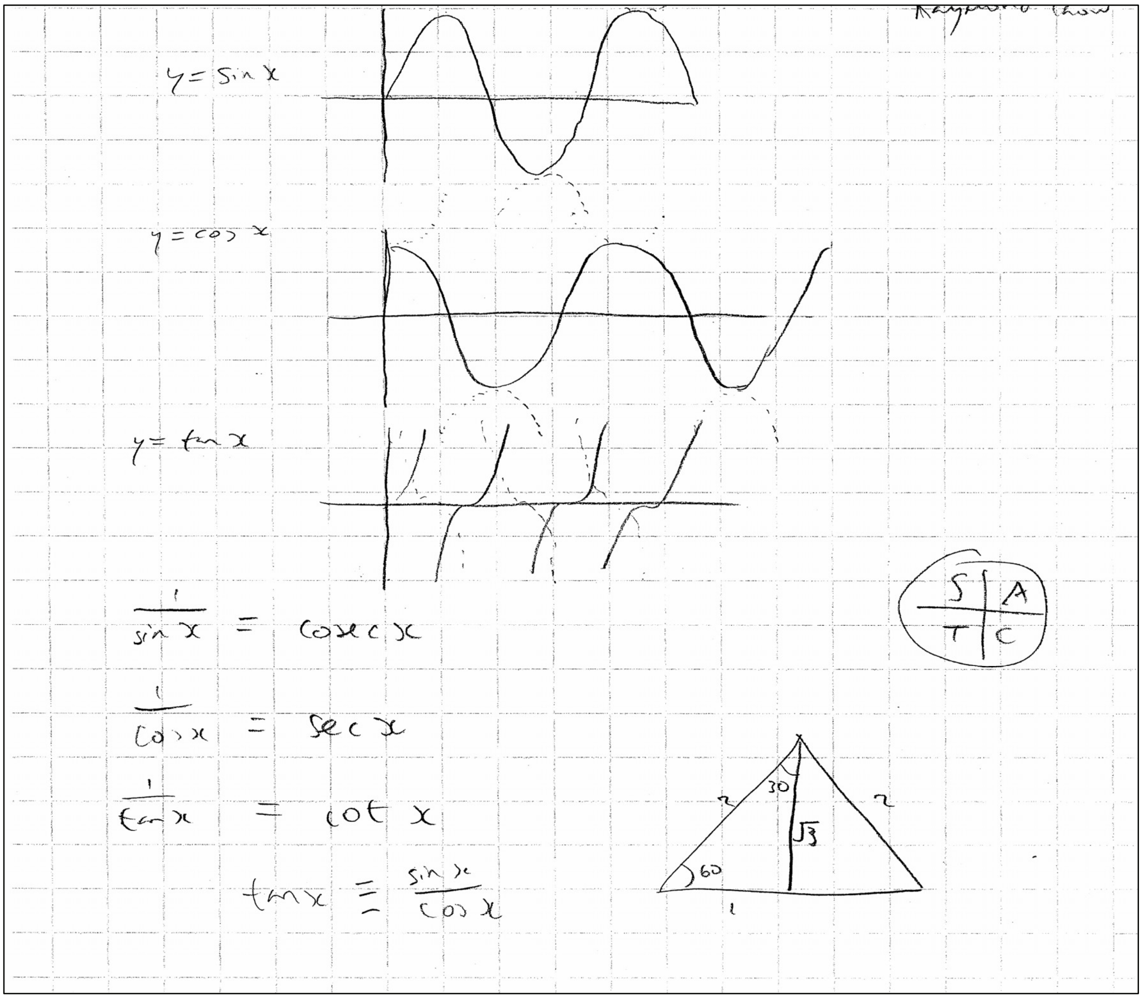


Figure 7.10: S1's Concept Map 3

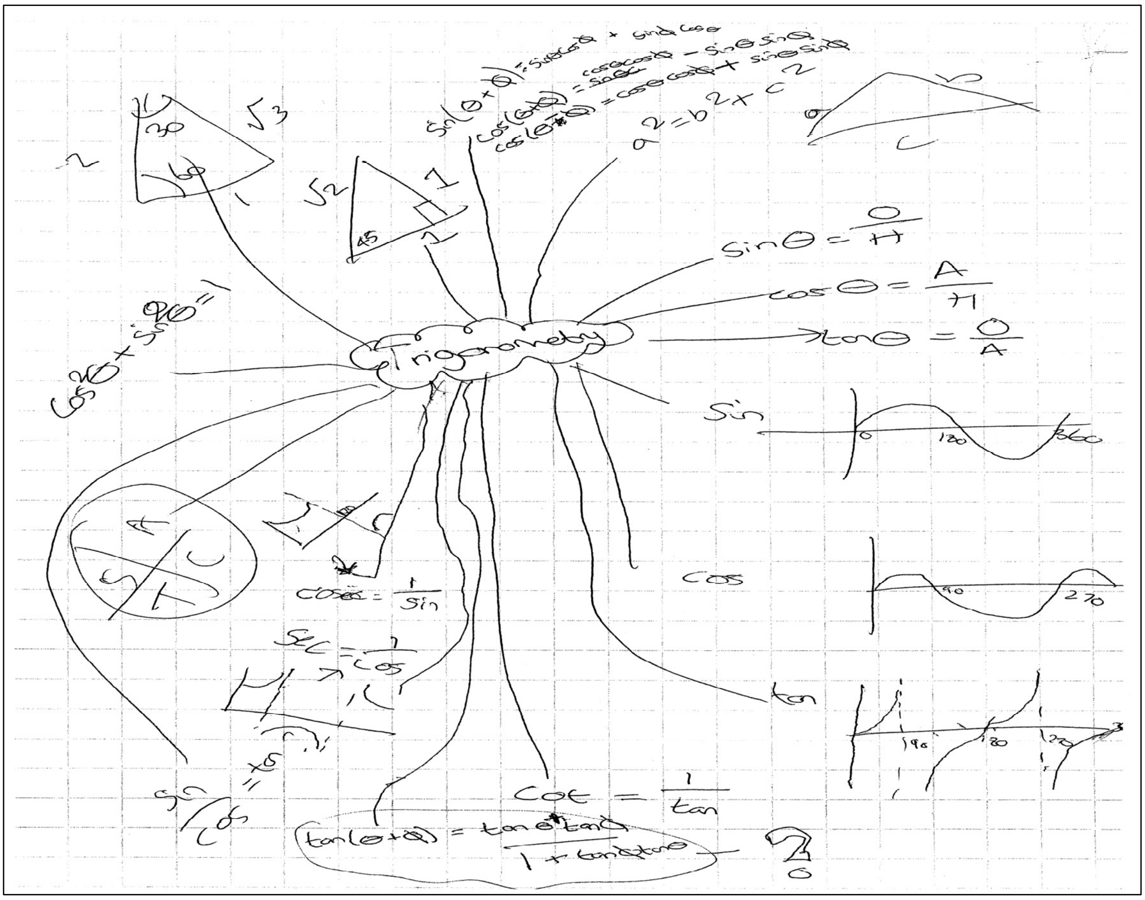


Figure 7.11: S2's Concept Map 3

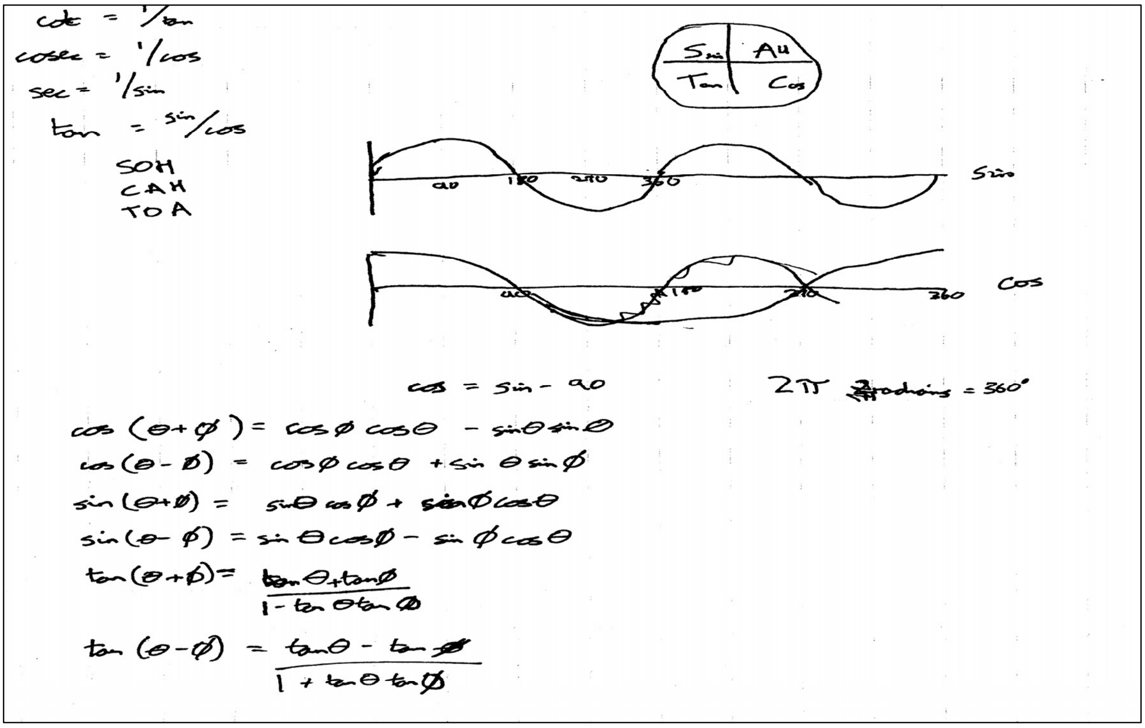


Figure 7.12: S3's Concept Map 3

7.10.1 Observations of Third Concept Maps

The content from the three maps drawn by these students is shown in the tables below. The bold type indicates where content has been repeated.

S1	Map 1	Map 2	Map 3
Core properties	**Sin = opp/hyp etc** **Tan x =Sin x/ Cos x**	**Sin = opp/hyp etc** **Tan x =Sin x/ Cos x** Sin$^2\theta$+Cos$^2\theta$=1	1/sin x=cosec x, 1/cos x=sec x, 1/tan x =cot x **Tan x=sin x/cos x**
Formulae	**Sin rule** **Cos rule** Pythagoras	**Sin rule** **Cos rule** $L=r\theta$ $A=1/2r^2\theta$	
Values	Surd or decimal values for Sin, Cos and Tan 0, **30, 60,** 90° and tan 45°	π=180°	**Surd values for 60° and 30°**
Spatial images	**Sin, Cos and Tan graphs**	**CAST circle**	**Sin, Cos and Tan graphs** Cosec, sec and cot graphs **CAST circle** Special angles triangle

Table 7.7: S1's Concept Map Content.

S2	Map 1	Map 2	Map 3
Core properties	**Sin =O/H etc** **Tan =Sin /Cos**	**Sin =O/H etc** **Sin x /Cos x =Tan x** **Sin$^2\theta$+Cos$^2\theta$=1**	**Sin =O/H etc** **Tan θ =Sin θ /Cos θ** **Sin$^2\theta$+Cos$^2\theta$=1** Cosec θ=1/sin θ etc
Formulae	Sin rule Cos rule **Pythagoras**		**Pythagoras** Sin ($\theta+\varphi$) Cos($\theta+\varphi$), cos ($\theta-\varphi$) Tan ($\theta+\varphi$)
Values		π=180°	Surd values for 30°, 60°, 45°
Spatial images	**Sin, Cos and Tan graphs**	**Sin, Cos and Tan graphs** **CAST circle**	**Sin, Cos and Tan graphs** **CAST circle** Special angle triangles Cosec & Sec graph

Table 7.8: S2'sConcept Map Content.

S3	Map 1	Map 2	Map3
Core properties	**SOHCAHTOA** **Sin /Cos =Tan**	**SOHCAHTOA** **Tan x =Sin x/Cos x**	**SOHCAHTOA** **Tan x =Sin x/Cos x** Cot=1/tan,Sec=1/sin (sic) Cosec =1/cos (sic)
Formulae	Sin rule Cos rule Pythagoras		Cos (θ+φ), Cos(θ-φ) Sin(θ+φ), Sin (θ-φ) Tan(θ+φ), Tan (θ-φ)
Values	Sin 45/Cos45=Tan 45 =1	Sin& Cos 30, 60, 90° Sin 120° Cos 0°	2π rad =360°
Spatial images	**Sin graph** Incorrect Cos and Tan graphs	**Sin, Cos graphs** Approximate Tan graph 4 **CAST circles**	**Sin, Cos graphs** **CAST circle**
Other	Quadratic formula Sin 45=Cos45=√3/2 ?	Cos x =1-sin x (sic) Sin x + sin (90-x)=1 (sic)	

Table 7.9: S3's Concept Map Content.

Each of the three students now added the secant, cosecant and cotangent identities and S2 and S3 added the formulae for sin, cos and tan (θ±φ). Therefore there is an increase in the core properties mentioned by each of these students.

S1 and S2 added the sec, cosec and cot graphs to their spatial images and the special angles triangles. So there is an increase in the spatial images that S1 and S2 have included since their previous maps. This is not the case with S3.

S2 and S3 have included the double angle formulae. None of the students have made any mention of RCos (θ+α).

S1 and S3 have not indicated any kind of structure but merely recorded a list of items. When S1 was asked about his map he said:

> It is just what comes into my head... to help me remember.
>
> (S1)

S2 has shown some indication of a structure but there is no evidence here of the development of an interconnected structural concept as described by Sfard (1991) [§3.1.4] or an object construction as described in the APOS theory [Dubinsky (1991) §3.1.5].

Other students in the group however did draw maps that pointed to the development of an interconnected schema. For example here are the first and third maps of one of the other students in the group (S6) who also studies Further maths.

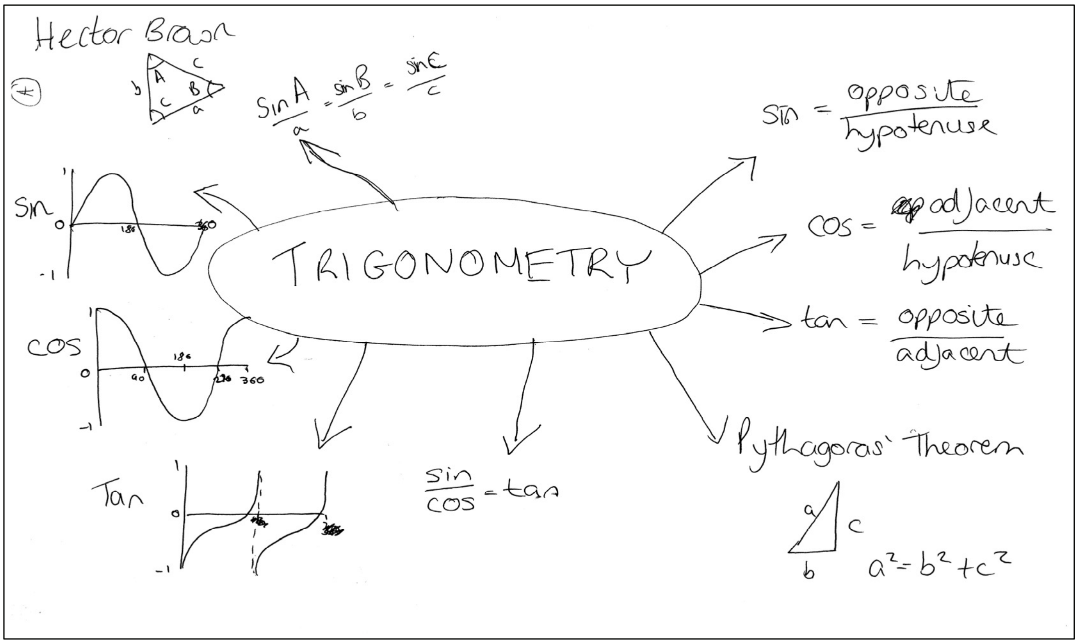


Figure 7.13: S6 Concept Map 1

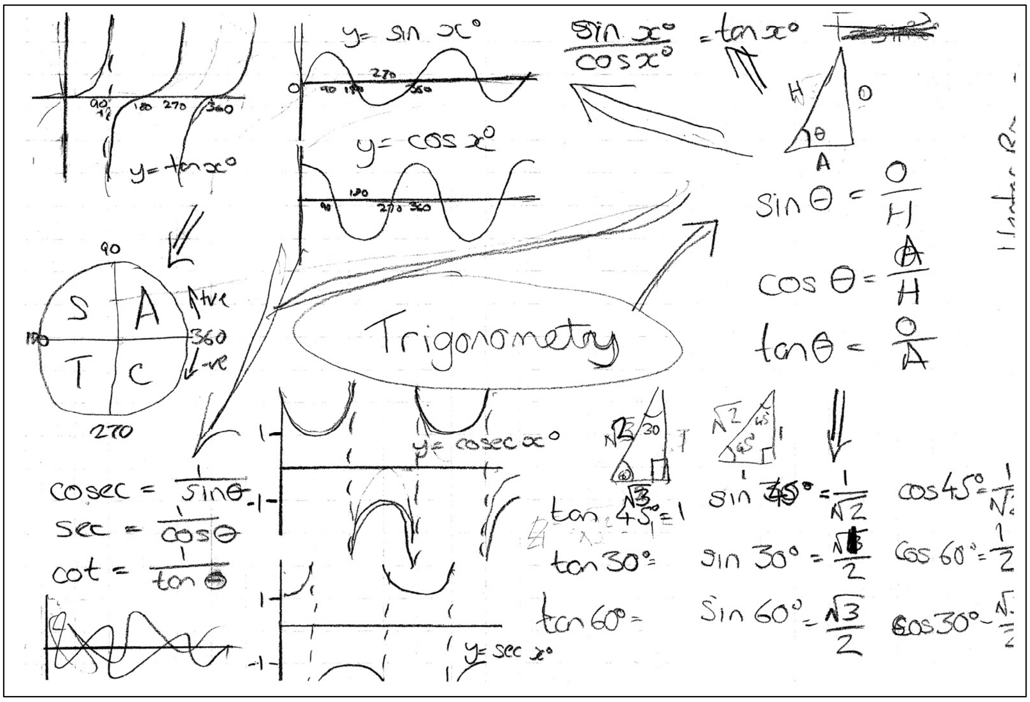


Figure 7.14: S6 Concept Map 3

The 1st map is comparable with the maps drawn by the all the students initially with regard to, both the content indicated, and the way that the items mentioned appear to be discreet and unconnected. This is the 'spider' style that was predominant at the start of the course.

The 3rd map shows increased content and a variety of concept images in the form of visual spatial representations. The student has clearly attempted to organise the content as evidenced by the way in which items such as the graphs or identities are grouped together within the overall schema. However the arrows indicate that the different components are interlinked. Again some of the items originally included in the 1st map have been omitted in the 3rd map. These items are the formulae for the sine rule and for Pythagoras theorem. When interviewed about these omissions the student said that he didn't include the sine rule because

> I know it and can use it if I need to. (S6)

This points to an interiorisation of process [§3.1.4]. He said he didn't include Pythagoras theorem because:

> Although it is used in trigonometry it's not really about trigonometry because it's about lengths and not angles. In the beginning trigonometry was just about triangles, like finding lengths and angles but now it's about angles; angles in triangles and in circles... anywhere really.
>
> (S6)

This points to a schema reconstruction.

Another example that may indicate the development of a schema that is organised is shown here. This student started to study Further Maths but dropped it.

.

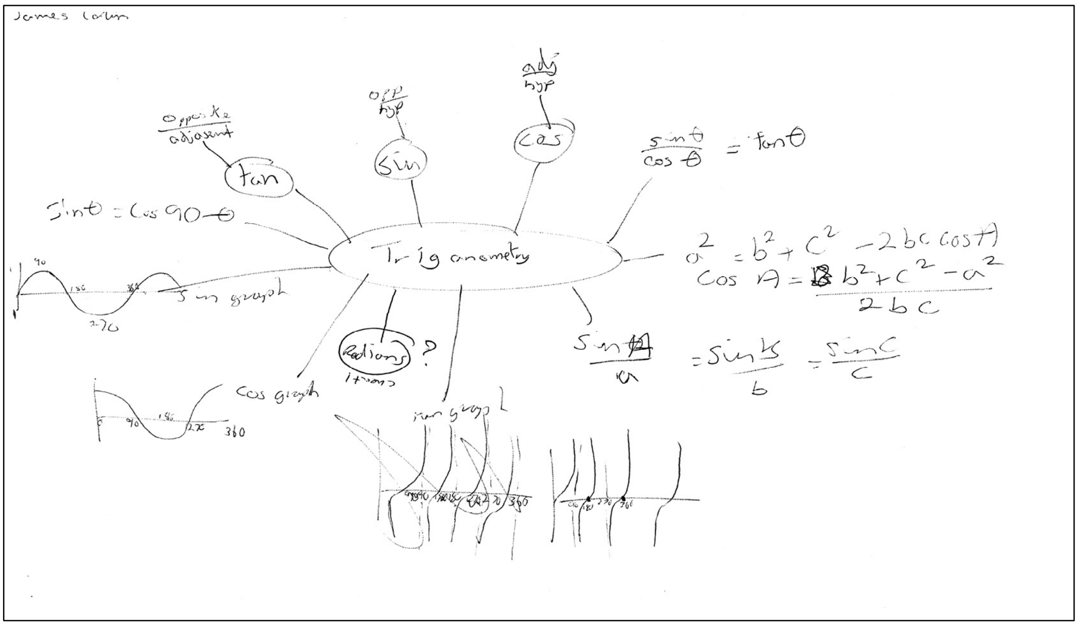


Figure 7.15: S7's Concept Map 1

Again the first concept map is drawn with the items radiating from the centre and it shows little indication of a concept schema that is richly interconnected.

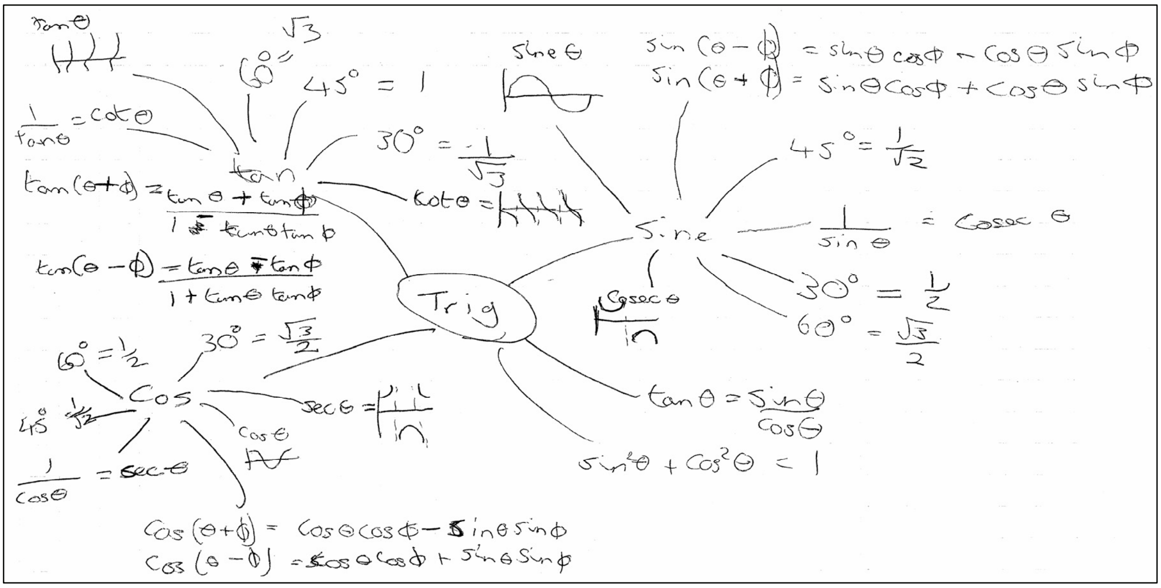


Figure 7.16: S7's Concept Map 3

The third map shows clearly an organisational structure with the words sine, cos and tan placed as centres or nodes to which their relevant graphs, identities and values are linked. However it may be observed that the content appears to be focused upon items that this student feels the need to remember. This observation was borne out by this student's response when asked why he had dropped further maths. He said:

It was just too much to remember it all. I felt that life is just too short!

7.10.2 Evidence of Concept images

Tall and Vinner (1981) defined concept image as “the total cognitive structure that is associated with the concept” [§3.2.5]. Three of the students in the group drew maps that predominantly featured spatial visual images. Here are the first and third maps of one of these students (S8) who, like S6, also studied Further Maths.

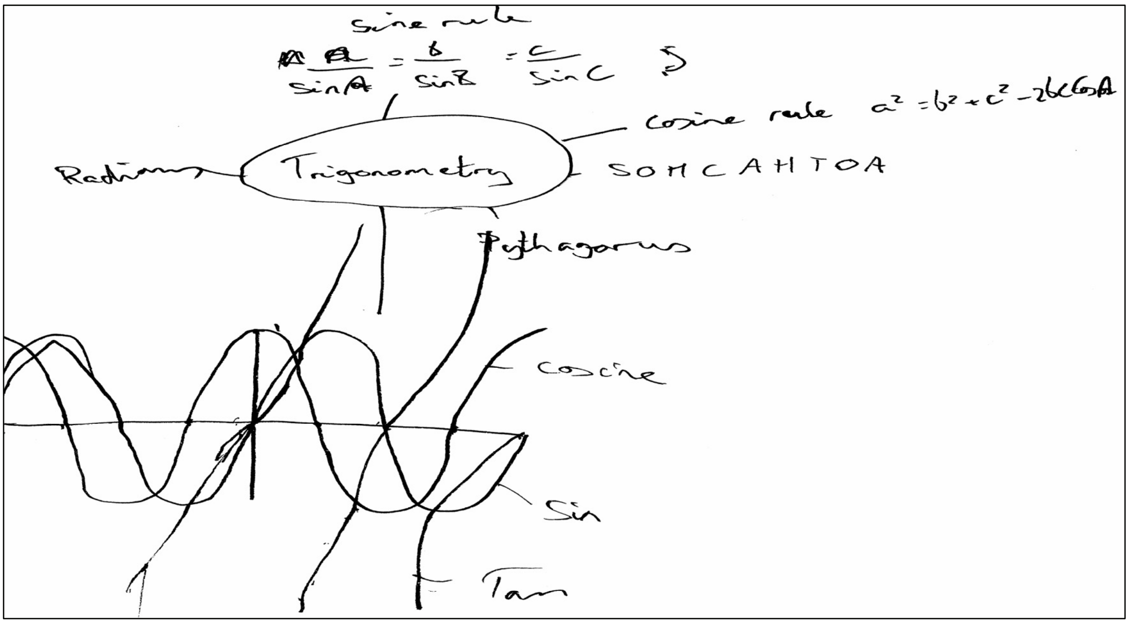


Figure 7.17 S8’s Concept map 1

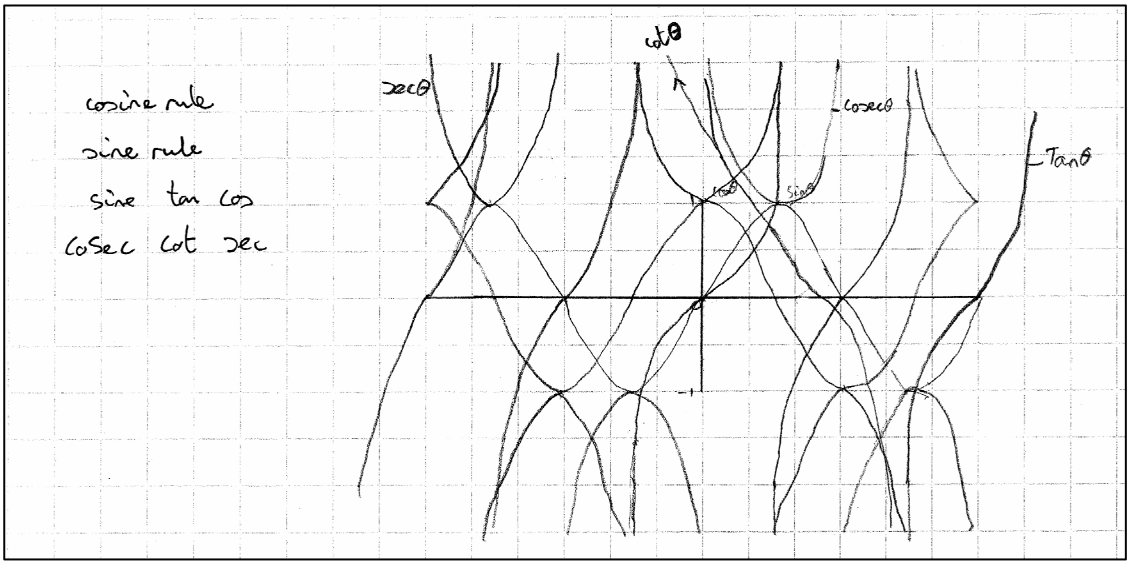


Figure 7.18: S8’s Concept Map 3

These maps seem to be trying to portray an overall image of the concept [§3.2.5], especially the third concept map where all the trigonometric graphs and the reciprocal trigonometric graphs are shown layered over each other in the same plane. Indeed the graphs appear to replace the word 'trigonometry' as if for this student the two are synonymous. Vinner (1983) observed that frequently students appear to prefer a concept image to a concept definition such as 'sine x = opposite over hypotenuse'. This seemed to be an example of this. When questioned about this map, S8 explained:

> Well it was the patterns and symmetry of the different functions that I was trying to portray, like how they sort of connected to each other. The things here are all that you need to say really".
>
> (S8)

It may be observed that there is very little content mentioned on this concept map. [*cf* Figure 7.15 S7's third concept map]. The key aspects that S8 has focused upon are the spatial symmetries and connections between the functions. In the third map he has noted what appears to be a list of processes. This appears to support Sfard's (1991) theoretical framework which describes a reified object as having a duality of concept and operational aspects [§3.1.4].

7.11 A-level Mathematics Results

The total number of marks available on C3 and C4 was 75 for each paper. In C3, 33 of the 75 marks (44%) were allocated to the three trigonometry questions and on C4, 18 marks (24%) were allocated to the trigonometry questions. The final A2 results for the students in this group are shown in Table 7.10. Where students have re-sat C3 the original marks are given in brackets beside the final mark for this assessment. In the far right column their hopes for future study are given along with conditional grades required. The grade boundaries out of 600 for the 6 papers (AS and A2) are: 480– A, 420–B, 360–C, 240-D, 180-E.

	C3	C4	Final grade	The future
S6	97 (74)	78	567 A	Physics @ Southampton ABB
S8	89 (83)	90	544 A	Maths @ Bath/Warwick AAA+A in Further maths, STEP or AEA grade 2
S2	91	98	540 A	Chemistry @ Oxford AAA
S13	86 (72)	100	534 A	Electrical & Electronic engineering @ Manchester ABB But if 3 A's she will get a grant of £1000pa
S11	90 (73)	93	528 A	Economics & management @ Oxford AAA in maths, economics & chemistry
S1	84	51	506 A	Medicine. No offers (possibly due to poor interviews)
S5	87 (69)	80	506 A	Chemical engineering @ UCL. BBB or IC AAA
S9	78	31	484 A	Computer Science @ Birmingham BBC
S14	89	68	472 B	Physics @Southampton ABB
S12	76	54	447 B	Music @ Brighton
S15	75	73	415 C	Financial management & Risk analysis @ City university AAB
S16	67 (47)	65	411 C	Physics @ Warwick, A in physics & maths
S10	71 (64)	69	411 C	Art Foundation @ Northampton Unconditional.
S3	67 (58)	51	393 C	Business studies @ Aston ABB, B in maths
S7	67	52	362 C	Accountancy & Finance @ Loughborough ABB
S17	53 52)	31	270 E	Mechanical Engineering @ Aston BBB

Table 7.10: A2 Mathematics Results

The results show that S6, who drew the complex interconnected third concept map shown above [§7.10] gained the best mark overall. The second best mark was gained by S8 who drew the pared down map dominated by a concept image. S2 gained the third highest mark. In the next chapter there is an account from the student who gained the fourth highest mark about her change in perception of trigonometry for GCSE to A2. In C4, the hardest paper, this student scored 100%. S1 and S5 identified by the teacher and other students as the best students in the class at GCSE, came sixth and seventh overall in the group at A2, though still gaining A grades. This suggests some correlation between the complexity of concept map in structure and the final attainment however, there are two strong counter examples to a clear link: S2 drew a third concept map that was predominantly spider shaped and yet she gained the third highest mark in the group, averaging 90% overall; and S7 gave every indication of ordering sub concepts within a general structure but scored the second lowest mark in the group. These issues are discussed in Chapter 9.

7.12 Summary

The theoretical paradigms propose that operational knowledge is developed into a richly inter-linked cognitive structure that provides flexibility. However this does not always happen with all students [Chapter 3]. In this study it was found that concept maps can provide some insight into the types of schemas [§3.1.3] that students are constructing. At the start of the course all the students drew 'spider' style maps whose content was a mix of concept properties, operational formulae, and images though not all of these had been correctly remembered. Some students also included lists of values. By the end of the course many, though not all, of the students had modified the style of their maps [Harel & Tall (1991) §3.4.2]. The 'spider' style was less in evidence and was replaced, in the case of the students who obtained A grades, by a structure that in some cases that was interlinked [S6] or had a predominant concept image [S8]. S8's explanations for these maps implied that these images had some intrinsic meaning to him possibly serving as a codified reminder of the essential features of trigonometry [Gray & Tall (1994)].

The content of the final maps tended to show an increase in concept properties and spatial visual representations. Operational formulae were briefly mentioned in a manner that suggested that the process had been condensed [Sfard (1991) §3.1.3]. Other students indicated no particular structure in their final concept maps but rather listed items of content. These students performed less well in the final assessments than their classmates. The content of their maps also changed by including more core properties and spatial visual representations and, in the case of S3, a list of formulae.

The deep open interviews indicated that the students interviewed had no connection between trigonometry and functions in either an informal or formal way.

The task questions answered by the students after C2 seemed to indicate that little of the content that was delivered in the lessons had been interiorised by the students at the stage of interview.

There was a range of responses from the students who were selected to undertake the integrated questions. S1 was able to switch from one representation to another flexibly. S2 was able to undertake a process in two directions whilst S3 responded in an Action sense, stimulated to respond by a signal in the question but unable to control his response.

S2 was seen part way through the course to be using spatial visual images in an Action sense by drawing out a graph of the function specified before taking stock of the situation and implementing a suitable solution procedure. She then showed signs of connecting known facts to her procedure and could reverse from the procedure to the fact which could indicate that she was, at that point in time, close to a process conception of these images as described by Dubinsky (1991) [§3.1.5].

Chapter 8
Student Perceptions of Their Own Learning

8.1 Introduction

This study has sought to describe changes in the trigonometric schemas of a group of 17 to 18 year old students from a phenomenographic perspective. Chapter 7 considered types of schema construction and the qualitative difference between them. The benefit of a schema that utilised spatial representations simultaneously with algebraic processes was evidenced by their responses to the integrated questions and the A2 assessment results. All the students studied the same course with the same teacher so we suggest the differences in their resultant schemas were based upon a difference in the students' focus of attention when studying the course. Prior to starting this course all the students had developed a style of learning mathematics that had been successful as evidenced by their GCSE results. However the study of trigonometry at this level appeared to qualitatively divide the learning styles of the students.

This chapter provides some insight into the students' perceptions of their own learning experience. These insights are based on the replies students gave to the researcher at the end of the 2 year AS/A2 course. Firstly students reflect on the difference between trigonometry at GCSE and at A-level and how their understanding of it has changed (§8.2). Following that is a section on students comments on learning for assessments (§8.3). The chapter finishes with a summary (§8.4).

8.2 Student Perception of the Difference Between GCSE Trigonometry and A-level Trigonometry

At the end of the trigonometry component in module C4, that is, at the end of the study of trigonometry at A-level (revision aside), the students were asked to respond to the question:

> *A year 11 student asks you about the difference between trigonometry at GCSE and A level. What would you say? Include its uses then and now.*

Though all of the students itemised the increased content that was covered at A-level, there were two other perceptions that many of the students referred to: the shift in focus and the greater level of difficulty.

8.2.1 The Shift of Focus From Triangles

The students made comments about the transition from triangles to 'something more general' such as:

> At GCSE, trig is mostly used in triangles with degrees whereas at A-level, radians and degrees are used. Trig in general is a lot more freely used, for example, in quadratics or negative values. Sin cos and tan are manipulated in much more complicated ways at A level. (S8)

> At GCSE it's just triangles, some rules like the sine rule and cosine rule and a bit on graphs. At A-level there are identities, radians, the CAST diagram and you have to solve trigonometric equations and the graphs using them all at the same time. (S5)

> At GCSE you do SOHCAHTOA, the sine rule and cos rule but at A-level there is cot, sec, cosec, radians where $360° = 2\pi$ and more generalised applied problems. (S1)

> GCSE mainly uses triangles but at A-level it relates more to graphs and the CAST diagram. You understand more about how it works and there is more algebra involved. (S6)
>
> GCSE is mostly SOHCAHTOA, sine and cos rule and its all about triangles with a little bit on graphs whereas at A-level you learn how to solve new types of trigonometric equations with identities and radians and these are used to get more than one answer to the same question. (S2)

It should be noted that none of the students used the word function to describe the trigonometry they were studying at A- level.

8.2.2 The Perception That it is More Difficult

Some of the students commented on the difference in difficulty, such as:

> It's a lot harder at A level! Radians are included at A level and things like cosec, sec and cot and you have to solve equations sometimes with an unknown angle like $R\cos(\theta \pm \alpha)$. (S10)

> There are a lot more extra things at A level e.g. cosec etc radians, differentiation of trigonometric functions... so it is harder. (S7)

> For a start it is a lot harder! I don't really know its use aside from triangles but now we have a lot of formulas to remember like the double angle formula. (S3)

One student wrote that trigonometry was the most difficult topic in the mathematics A-level. When asked to explain he said:

> There are a lot of identities to remember and it is hard to remember which one to use and manipulate in the right way. I find it difficult to get my head around it. It is harder than other topics because there is a bigger variety of questions and it is not as clear cut as 'use this formula to differentiate' or 'use logs to solve this'. There is a lot more to think about. I didn't find it difficult at GCSE but at A-level it takes more understanding.
>
> (S11)

Another student commenting on the different uses of trigonometry at GCSE and A-level said:

> We used to use it to find the height of like... they said here's a ladder up against the wall like that [indicates a slope with her hand] how high will it go up. So there were those problems whereas now ... at GCSE they worded it to real life problems whereas here they're not. Here it's just solving the numbers.
>
> (S12)

This student said that she now found trigonometry easier than she had at GCSE. In the following comment she appears to describe how she had internalised and condensed the procedure learned at GCSE for evaluating the sin of an angle. At GCSE, she had had to undertake the procedure step by step whereas, now she could evaluate it almost instantaneously with little or no mental effort, which delighted her.

Researcher: Which do you like better: the trigonometry you did then or now?

Student: Umm prob-a-bly better now. At GCSE I don't think I learnt it. I dunno I don't think I learnt the er ... you know where we have the angle when you work out if it's cos or sin. At GCSE I had to think 'Uh Oh Sin equals opposite over hypotenuse' yeah? and work it all out whereas it's sort of like 'Oh that's fine, that's cool'. So now I like it better and it's like Ooh! [noise of self congratulation]

(S12)

It is worth noting this student's idea of 'learning'. She was clearly able to use trigonometry competently at GCSE but this, by her own account, was at an operational level. At the end of the A level course she now reflects on this as not really having 'learnt it'. This would seem to imply that she now considers her learning to have a qualitative dimension to it.

8.3 Student Comments on Learning for Assessment

One of the issues that arose during the course of this study was the student's continual awareness of forth coming assessments. The students were asked about the influence of the syllabus on their learning, and the impact of regular assessments. They identified different types of teaching, for learning and for the assessment and different types of learning.

8.3.1 The Influence of the Syllabus on Learning

One of the issues with the concept maps was the students' selection of which items to include or not include. Some of the reasons for this have already been discussed. (§7.7.7) There was however a third reason given for items that were originally mentioned but subsequently omitted and this implied an awareness of the syllabus and the needs of the assessment. Here is a student comment from interview:

Researcher: Can you explain why you left some things off your recent concept map that you put on your original one? For example: you'd mentioned the sine rule and cosine rule on the original one but when you did the later one you didn't include those things.

Student: Oh! No because it wasn't …well it wasn't really on the syllabus. We didn't need to use it so I didn't put it down.

Researcher: So it's not that you don't think they are not part of trigonometry anymore.

Student: Oh they are part of trigonometry.

(S5)

This would seem to infer that this student was compartmentalising aspects of trigonometry according to the criteria of the syllabus. There are two implications that arise from this: Firstly this student's focus of trigonometry is circumscribed by the requirements of the syllabus and secondly that, if this is the case, the use of concept maps as a research tool has limitations. This student, and others, deliberately did not include or show connections to all the aspects of trigonometry of which they were aware.

The purpose of the syllabus could be described as directing learning and there is clear evidence that all the students were conscious of it as may be seen by the following comments taken from interviews and during the lessons:

It's on the syllabus so we have to learn it; I am not sure why. (S3)

I don't remember doing that. Is it on the syllabus? (S13)

We learn about it because it's on the syllabus but it's not really connected to anything. (S10)

There were occasions however when the syllabus appeared to be driving the learning. Some insight is shown by the following comments made by students:

> But if the syllabus required us to work it out I'm sure I'd find a way. (S5)

> Well I haven't thought about that because it's not on the syllabus but if it was I would probably think about it more. I only really concentrate on what's on the syllabus. (S3)

> The syllabus really makes us think about things that we haven't thought about before. It sort of gives us new problems that we have to think of solutions to. (S2)

It could be concluded, therefore, that for the students in this sample, the syllabus had the effect of re-focusing their trigonometry schemas. Those aspects of trigonometry that were not included on the current syllabus under study were, however, still recognised by the students as part of the subject. It may be further concluded that the syllabus was perceived by some of the students to take the form of an intellectual challenge.

8.3.2 The Impact of Regular Assessments

The students were asked if they thought it was to their benefit to have module assessments twice a year. These students had sat the C1 assessment at the start of the second term of year 12 (January 2005). They would sit the assessments for C2, the first module that included the study of trigonometry, and the first two applied modules S1 and M1 in June 2005 that is during the third term of year 12. The responses they gave showed an awareness of the advantages and disadvantages.

> Well it's better because you don't have to study for everything all at the same time...In the old days it was one exam and that's it! Whereas now we get the option to retake modules if the grades aren't good enough. (S3)

> The good thing is that we get more chances to get the good grades but the bad thing is that we don't get so much time to learn things properly because we have to start revising. (S5)

These questions led to the identification by the students of different styles of teaching: for the exam and for learning.

8.3.3 Different Teaching Styles Identified

Most teachers have to balance the dual agenda of teaching for knowledge and the need to ensure students gain as high a mark as possible in the examinations. High achievement in assessments was also an important priority for the students in this study. The students drew a distinction between the teachers who prioritised 'teaching to expand your knowledge' and those who prioritised 'teaching to pass the exam'. This was illustrated by the following comment made by a student in interview.

> There's a difference. Yeah definitely. Because you get teachers who are more like a lecturer so you do lots of extensions right but don't concentrate on like exam practice. We all know the stuff but we end up not doing as well as we should. Our understanding is deeper because those teachers are more concerned with our knowledge but some teachers, their teaching is really based on dictation of past papers and you learn more about what is going to be on the exam.
>
> (S5)

Here the student has identified the two types of teaching style. This distinction was identified by another student in interview who thought that both styles were necessary. He explained:

> It's not good if you have this kind of teacher or the other kind because we have to know two kinds of knowledge: One for depth and one for passing the exams so the best thing is when you have a teacher who can teach both.
>
> (S8)

This comment indicates that this student recognises the need for instrumental knowledge and relational knowledge. He also appears to consider the two types of knowledge as qualitatively distinguishable when he says 'one is for depth'.

Many of the students commented during informal conversation in the classroom on the two types of learning that they are required to master. The next section shows this in more detail.

8.3.4 Different Learning Styles Identified

Most of the students identified that there were two types of learning taking place in the classroom. These they referred to as 'learning for the exam' and 'learning for depth of knowledge'. They thought that the key to passing exams was to do lots of practice papers. This is illustrated by the following comments:

> I do better in the exam if I have done more practice papers…because everything's a constant surprise to you if you haven't done practice papers.
>
> (S5)

> But when it comes to exams I just look at past questions we've done and try to remember them.
>
> (S3)
>
> For the exams you just have to remember all the formulas and do the past papers. It's mostly the same sort of questions you get each time.
>
> (S1)
>
> To get the top grades they (practice papers) are key. They are key. If you don't do any practice papers then you probably know everything and you'll probably get about 70 or something like that, but if you keep doing practice papers then you refine it because you know the things that can come up and then you can get the top grades.
>
> (S12)

It is worth emphasising that the students' self-identified learning tools for the examinations are memorising formulae and being familiar with the answers to routine questions.

The other type of learning that they identified as taking place in the classroom was frequently described as learning with 'understanding'. Examples are:

> You learn it so that it all makes sense. You just understand it better.
>
> (S2)
>
> When you study it carefully for a while you get to understand it more.
>
> (S6)
>
> In the beginning its sort of you do this or this but then you start to understand why you do that and you don't have to strain to remember it anymore.
>
> (S12)

When students were asked to describe the methods they used to learn with understanding, their responses emphasised understanding what they learned and also, identified the importance of the teacher.

I try to focus in lessons and try to enjoy it as well as learning. When I get home I just do the homework set and mostly I find like it's quite logical. But with the hard things I look at specific things that the teacher emphasises.

(S5)

When you have a good teacher they like make it all seem simpler somehow and you just seem to remember it.

(S2)

At A level there is an expansion of knowledge with harder questions than at GCSE but the teacher shows you how it all links up.

(S13)

A few of the students also noted that learning for depth took more time.

When you learn doing practice papers it doesn't actually reflect your knowledge, but it takes time to understand all the theories.

(S8)

Sometimes I don't have the time to learn it properly so I just learn enough for the exam.

(S3)

The exam doesn't always show who understands it better because it takes time to do that but on the exam they might not test that understanding.

(S14)

These comments seemed to imply that the students recognised a qualitative dimension to their learning however they were also pragmatic about it.

Researcher: Which of these two types of learning do you think is better?

Student: The learning for knowledge is better for university but how are you going to get to university if you aren't going to get the grades?

(S5)

8.4 Summary

Many of the students noted a shift in focus from triangles to something more general. This something was described variously as just solving the numbers, solving equations, solving different kinds of problems or finding multiple solutions but no mention of function was made. This could imply that the students had not interiorised trigonometry as a function.

The general impression of the students was that trigonometry was harder at A level than GCSE. One student stated that it was the hardest subject on the A level syllabus due to the bigger variety of questions and the fact that it was not as clear cut as being asked to differentiate this or solve that using logs. There was a need, he said, to understand it better than other topics.

There was evidence that at least one of the students appeared to identify a shift in her own understanding from the operational to an interiorised, condensed form during the course.

The students were fully conscious at all times of their learning being dictated by the syllabus. Some of the students said they had no idea of the connection of some aspects of the subject but learnt it because it was on the syllabus. Other students appeared to regard the syllabus as an intellectual challenge.

The students identified that the teaching style of their teachers were of two distinct types. One type was teachers who taught for knowledge and the other was teachers who taught for the exam. The students thought both type of teaching was necessary and the best teachers were those who could teach in both ways. This supports Skemp's description of relational and instrumental teaching. He identifies instrumental teaching by its focus on algorithms and formulae [§3.1.2].

The students identified that two styles of learning was required of them: learning for knowledge and learning for the exam. Learning for the exam was centred on practising past papers to gain experience of the general nature of the questions and remembering formulae. Learning for knowledge required understanding. Also of importance according to the students, was the teacher both in his emphasis and the way he 'linked up different things'. It may be noted that none of the students mentioned that the teacher showed them how to do the work. The skill of the teacher in giving the students the opportunity to interiorise operational procedures is discussed further in the next chapter.

Chapter 9

Conclusion

9.1 Introduction

This chapter considers the main issues of this study. The themes of the study were:

- The students' development of a flexible object conception of trigonometry.
- The extent to which there is evidence of an integrated use of spatial imagery
- The role of teacher emphasis in the construction of function.

These issues lead to the following research questions:

1. What opportunities do the students have to interiorise and personally condense trigonometric processes? [§9.2]
2. Can the quality of students' schemas be identified? [§9.3]
3. Is there any evidence that A-level students are linking together different sub-concepts in their study of trigonometrical functions as specified within the A-level syllabus of study? [§9.4]
4. Is there any evidence of the students being able to curtail procedures and change from one representation to another? [§9.5]
5. Is there any evidence that the students can de-encapsulate concept images? [§9.6]
6. To what extent do students think their own perceptions of trigonometry have changed over their A-level study and since their study of the topic for GCSE? [§9.7]
7. What are the students' perceptions of their learning experience? [§9.8]

These questions are here set out according to the order in which they will be analysed which is different to the order they were originally listed in Chapter 4 [§4.2]. It may be also be noted that at the outset of this research there were 6 research questions whilst above there are now 7 listed. The additional question is question 2 above. During the course of this research it became clear that the concept maps provided an indication of the nature of the schemas the students had constructed that was over and above the researcher's expectation. The strength of these indications led to the inclusion of this additional question.

The chapter starts by considering the opportunities presented for students to interiorise and personally condense sub-concepts [§9.2. See also §3.1.4] and then considers the extent to which concept maps provided an indication of the quality of students' schemas [§9.3].

This is followed by a detailed analysis of the empirical evidence in response to the remaining research questions [§9.4 to 9.8]. This study was predicated on the theoretical frameworks proposed by Sfard (1991) [§3.1.4], Dubinsky (1991) [§3.1.5] and the theory of mathematical networks as proposed by Nickerson (1985) [§3.1.7] and the extent to which these frameworks fit the empirical evidence of this research and their usefulness to teachers of trigonometry is discussed [§9.9]. Limitations of the study are then noted [§9.10] and followed up by suggestions for further investigations [§9.11].

9.2 What Opportunities were Presented for Students to Interiorise and Personally Condense Sub-Concepts?

The teaching of trigonometry, as with other mathematical concepts, is enriched through the use of words, ideas, diagrams, examples, explanations and applications. Interiorisation and condensation are personal processes. The teacher's methods of facilitating these processes involved the following six interconnected approaches:

9.2.1 Emphasis of sub-concepts.

Graphs, the CAST diagram and the special angles triangles can be identified as spatial visual sub concepts of trigonometry. The teacher within the main study made frequent use of the special angles triangle always drawing a sketch of it when it was required and he never indicated that he had personally memorised the values even if he had. He emphasised their specific use for evaluating the sin, cos and tan of the special angles either in degrees or radians (and vice versa) and also through reflection on each function's symmetries, their value for determining other angles of a more general nature such as sin 390 or tan $-5\pi/4$. This emphasis was conducive to the students' development of a proceptual perception of the trigonometric ratios in the special angle triangles (Gray & Tall, 1994, §3.1.6).

He continually referred to the CAST diagram or graphs both as an operational device to find angles greater then or equal to $^{\pi}/_{2}$ radians or 90° (or less than 0) and to emphasise an idea such as the infinite nature of the functions or stationary values. When the class were being initiated into the idea of radians this continual referral emphasised the idea that regardless of the system of angle measurement the trigonometric functions of the angle had the same properties. i.e. the geometric properties were consistent regardless of the numerical representation of the angle. The benefits of this approach are documented by

Bayazit (2005) who concludes that teachers who established connections between representations and between the ideas obtained better learning results from their classes than those who presented the ideas as discrete elements and did not establish connections between the representations.

9.2.2 Links

The teacher continually made links to other mathematical concepts such as: proportional thinking when considering angles in radians, differentiation when considering integration, and spatial-visual representations but remained focused on the sub-concept that was being explored. This approach encourages in the students what Nickerson (1985) terms 'Deep structure' understanding which empowers students to (1) comprehend the structure of relationships in a problem and to clarify what is needed to solve the problem and (2) better judge the plausibility of solutions obtained [§9.2.5]. This was particularly evident at the start of lessons when he wrote a problem on the board for consideration such as $\int \sin^3\theta \cos\theta.d\theta$ and asked the students to recall what they had found out about differentiating functions such as $\sin^n\theta$. He did not, for example, ask them to remember the fundamental rule of calculus. The emphasis was on linking to understanding rather than remembering formulae. The value of this approach is documented by Bin Ali (1996) who concludes that strong links between different symbolic representations enables students to move easily from one way of thinking to another and greatly increases their chance of solving a given mathematical problem.

9.2.3 Operational vs. Conceptual Thinking

The teacher encouraged the students to evaluate solutions that were required by any operational means of their choice [see §6.6]

Throughout the observed class activities the teacher stressed the difference between operational procedures and conceptual knowledge by emphasising the process aspect of operational procedures. A specific example was when he used graphic calculators to show that, $4\cos\theta+3\sin\theta$ is equivalent to $5\cos(\theta-\alpha)$, where α is $\tan^{-1}3/4$. This led to the understanding that the use of $R\cos(\theta-\alpha)$ is an alternative process that can be added to the solution processes the students were familiar with. The benefit of emphasising this distinction is noted by Sfard (1991) who argues that by combining processes with other processes, making comparisons and generalising become easier and results in a growing readiness to switch between different representations of the concept.

9.2.4 Direct Equivalence

The teacher emphasised notions of a direct equivalence. Examples were between the two systems of angle measurement or between identities, for example $\cos^2\theta$ and $½(1-\cos2\theta)$. Remarks such as "So how big is this angle here? Yes 60° or π/3" reinforced notions of direct equivalence. In another example the use of graphic calculators to draw the graphs of $\cos^2\theta$ and $½(1-\cos2\theta)$ enabled students to experience for themselves their congruence. This encouraged the development of an understanding of the underlying unity between different representations and helped clarify cognitive/semiotic confusion. There are two links here with the recorded literature: Firstly Gray &Tall (2001) note that encapsulation or reification is not an automatic development but is greatly enhanced by 'configuration of the base objects as a precursor of the sophisticated mental abstraction.' (p7). The teacher's emphasis on equivalence therefore helped the development of an abstracted object that is more than process alone. Secondly, Duval (1995) conjectures that it is the conversion between representations without changing the objects being denoted that leads to understanding. Another issue associated with this was the teacher's precise use of language; for example, he made careful distinction between the angle that 2π *corresponds to* and *the value* of 2π. Bayazit (2005) notes that the language used by a teacher should be consistent with the epistemology of the concept being taught so that a teacher's verbal explanations can contribute to student's learning [See §5.6.3].

9.2.5 Accuracy of Answers

Associated with ideas of equivalence was the choice of forms of answer. The teacher drew attention to the different natures of the different forms of answer. For example: a surd value or a fraction of π could be evaluated to the degree of accuracy required but an answer given in decimal form from the calculator was, at best, only accurate to 10 decimal places. He drew attention to the different applications of trigonometry in the real world and highlighted the need for different accuracies according to the circumstances of the situation; building a dome or a bridge across a river, compared to the angle of the Hubble telescope which is controlled by very powerful computers. The effect of this was twofold: firstly, it put an emphasis on the importance of trigonometry in engineering projects and secondly, it challenged the notion that the degree of accuracy of an answer was arbitrary and deterred the students from routinely writing all answers to two decimal places. Once the importance of accuracy was recognised by the students they were more motivated to try solution processes that were less dependent on the calculator. As with the reinforcement of links [§9.2.2] this style of teaching supports the development of 'deep structure' understanding as identified by Nickerson (1985).

9.2.6 Encouragement of Creative Thinking

After writing the problems on the board the teacher encouraged the class to explore different possible processes to solution until they had exhausted all possibilities and drew attention to the reasons why these processes ultimately failed. By doing this the students were able to justify for themselves the inappropriateness of the suggested processes and the need for a new solution process. Sometimes there was no need for a new process if the question was an unusual one but the teacher wanted to encourage the students to think about how they could solve it with what they knew (perhaps by reframing the question). The teacher's method here could be described as encouraging the students to reflect on the nature of the problem and consider the conditions that describe it. He was encouraging the students to undertake an abstraction and then reflect on the suitability of familiar solution processes. His emphasis was predominantly on the concept properties and operational procedure was frequently referred to only in passing and then as an entity.

An example was the problem:

Given that θ is obtuse and $\sin \theta = 5/7$, find the exact value of $\sin 2\theta$.

Student: I'm not sure how to solve this Sir
Teacher: Draw a triangle for θ, use Pythagoras and think about what you know.

The teacher's answer advocates two procedures which are described as finite entities: 'draw a triangle for θ' and 'use Pythagoras' then he tells the student to 'think about what you know' in other words use conceptual understanding. When a new operational procedure was taught, for instance, the expansion of cos (A±B the teacher gave the class the chance to initially practise it with simple questions but soon moved them on to questions where the operational knowledge was subsumed within the question. The focus of the lesson was not only to master the expansion correctly but to use it when it was suitable i.e. to increase the arsenal of tools available to the students.

He continually asked questions leading the students to actively engage in the activity of finding the answers themselves rather than requiring them to passively watch as the exposition were played out in front of them.

This set of interrelated approaches contrasted markedly with the approaches used by the pilot study teacher who concentrated on specific questions rather than the properties of sub-concepts, the assessment rather than the wider applications of trigonometry and step by step procedures rather than links to other sub-concepts.

To summarise the pedagogic style of this teacher it may be seen that he naturally incorporated aspects of best practice as documented by the literature. Specifically he:

- Encouraged a proceptual perception of sub-concepts such as the special triangles.
- Established connections between representations that facilitated a transference from one representation to another
- Encouraged 'deep structure' understanding which again promotes transference between representations
- Encouraged flexibility of learned procedures and comparisons between them with regards to economy
- Encouraged abstraction of the problem type by reflecting on its conditions and properties which is a precursor for reification/encapsulation
- Encouraged reflection about representations and appropriateness of answers
- Defined mathematical terms precisely to promote understanding of concepts
- Referred to procedures as finite entities that could be applied (or not) and promoted considerations of choice for subsequent procedures.

9.3 To What Extent did the Concept Maps Provide an Indication of the Quality of Student's Schemas?

The concept maps were selected as the main research tool for this study as they were thought to have two main advantages over questions of content knowledge, as in assessments or questions of an open ended nature. These were that: firstly, they would identify the content associated with trigonometry that is the focus of the students' attention (§9.3.1) and secondly, they would provide some insight into schema structure by the ways the students link together, if at all, different items of knowledge content (§9.3.2). To this end the results were informative. The maps provided indications of the nature of the schema the students had constructed.

9.3.1 Indications from Representations of Content Knowledge

Indications of the usefulness of the concept maps in providing insight into representations of content knowledge may be exemplified by three students, P1 from the pilot study group and S2 and S3 from the main study.

P1's initial map shows the trigonometric ratios in the form of triangles that have an operational purpose [§5.2.1]. S2 gave the ratios as sine = O/H etc [§6.2.1] which are procepts that describe simultaneously both an operational and a conceptual interpretation (Gray & Tall, 1994). This indicates the possibility of a qualitative difference of concept. P1's representation is solely operational whilst S2's representation has the possibility to being dynamically operational and a structural object as described by Sfard (1991). Obviously a specific example is not in itself sufficient evidence of a difference in thinking however the difference in presentation of these ratios suggested grounds for further investigation [See also §9.4]. After her initial map was drawn, P1 was interviewed and questioned on her understanding of the term sin θ. She said:

> 'An angle or you can find it by doing opposite over hypotenuse'
>
> (P1, §5.4.1)

This comment suggested she was unable to distinguish between the angle θ and sin θ as a function of the angle. Though she was able to describe an operational procedure to find sin θ, there was no indication at this point that P1 perceived sin θ as a mathematical object that is distinct from the angle θ. The focus of her explanation is on a dynamic description of a process rather than an object constructed from complementary representations that have personal meaning.

This contrasts with the response given by S2 when asked to evaluate sin 30°

> 'Anything to do with trig makes me think of triangles though I might now think of the graph...try and picture it in my mind ... Sin 30 is $\frac{1}{2}$'
>
> (S2, §6.4.1)

Here the expression 'sin 30' appears to stimulate complementary images rather than a process. It is significant that S2 made no reference to a procedure but instead employed at least two representations, a triangle and a graph when reflecting on sin 30. This initial reflection appears to have stimulated recourse to further knowledge that then led S2 to give the answer without undertaking a step-by-step procedure such as drawing out the special angle triangles and undertaking the process of dividing the opposite side by the hypotenuse. The indications are that S2 is showing signs of constructing a structural object for sin θ which she employs to determine the specific value of sin 30. This implies a deeper level of understanding than the process-driven understanding described by P1. However this structural conception is only partially formed and there are indications that some aspects of S2's schema are still operational. An example is S2's description of her understanding of Cos^{-1} 0.5. She says:

> It's like a triangle when you have to work out the angle instead of the length... You do inverse cos to find the angle... It's on the calculator. You shift cos 0.5 to get the angle.
>
> (S2)

Here S2 describes a step by step process rather than referring to complementary images. This implies that S2's sub-concept of inverse function is process-driven rather than proceptual. These differences in schema construction were flagged by the representations given by these students in their concept maps.

Another example of where the concept maps signalled areas for investigation is the concept map drawn initially by S3 [§6.2.1] who presented the ratios in the form of SOHCAHTOA, which is a mnemonic for the ratios Sin equals opposite over hypotenuse etc. In the subsequent interview S3 was asked the question: What does sine 30° mean to you? He said:

> Is it √3/2...that's just how I learnt it. Decimals are harder to remember.
>
> (S3)

S3 did not recall any meaningful representations and also did not describe a procedure but specifically referred to 'remembering'. Many of the students wrote SOHCAHTOA in their initial maps and its inclusion in the maps was not in itself conclusive evidence of a particular type of schema but SOHCAHTOA is a device to aid memory and its appearance in the maps drawn by the students gave a rough indication of the students who were possibly using memorisation as their main tool of learning. It was the presentation of the ratios in the concept maps that pointed to a possible contrast in the student's schemas.

In addition to providing an indication of possible differences in schemas between students the second and third concept maps also indicated a personal development of individual schemas. An example is in the students' presentation of graphs. [§5.4.1 and §6.2.1]. Graphs were included in the first concept map by all the students in the main study (though only two students included them in the pilot study) but many of them were incorrect implying that for some students the graphs had not yet been interiorised as a meaningful description of the function. It is possible that these graphs were memorised from GCSE where they are often used in an operational sense to find obtuse or reflex angles for a given value of sin, cos or tan x. The graphs drawn in the main study's second and third concept maps were more accurately drawn. This would seem to imply that an initial reliance on memory had been superseded by an interiorised spatial visual representation of function that suggests a deeper understanding.

9.3.2 Structure

The development of students' personal schemas was also apparent in the change of structure of an individual's maps. Initially all the students drew maps with the word trigonometry at the centre of the page with lines of connection linking this word to the items of content. No connection lines were shown between the content items resulting in the overall impression of a 'spider' image with the word trigonometry as the body and legs of connection to different aspects of the topic. The items of content at the end of the legs were mostly operational formulae such as the sine and cosine rule, and the graphs [§6.2.1 §9.2.2]. This suggests that the students' initial trigonometrical schemas consisted predominantly of an amalgam of instrumental processes that were not interlinked but were assigned to the general heading of trigonometry. However, by the end of the C3 module there was evidence that the students' focus had changed from an operational conception to one which was illustrating relationships between core concepts. Now, most students drew maps that focused upon the graphs and the identities as evidenced by the tables of map content [§7.10.1]. However, the analysis of the outcomes from the whole group illustrated a spectrum of development. At the centre of the spectrum were the maps that presented a balanced relationship between visual spatial images and algebraic identities as evidenced by the maps of S2, S6 and S7 [§7.10]. At the extremes maps illustrated either a prevailing focus on algebraic representations, especially formulae as evidenced by the map of S3 [§7.10], or a prevailing focus on visual images of the concept as evidenced by S8 and to some extent S1 [§7.10]. It is interesting to note that the maps in the centre of the spectrum included operational images such as the special angles triangle but those at either end did not, although S3 did include the CAST circle. This study was unable to specifically determine a qualitative hierarchy to this spectrum of focus however, it was evident from the responses to the integrated questions and the final assessment results that students (such as S1 and S2) who included spatial imagery, had a cognitive advantage over S3's preference for predominantly algebraic representations [§7.9]. This would appear to justify the teacher's philosophy that trigonometry should be taught as a geometric subject [§7.2].

In conclusion we can see that the use of concept maps was of considerable value in providing an indication to whether a sub-concept was being considered proceptually by a student or process-driven. The maps also gave a rough guide to those students who might be attempting to memorise key features. Finally the students subsequent maps indicated a change of structure in individual schemas over the course and led to the identification of a spectrum of schemas that ranged from those that predominantly relied on spatial visual representations for meaning to those that contained a mixture of spatial visual and algebraic representations to those that mostly consisted of algebraic representations of sub concepts.

We now turn to the research questions of the study.

9.4 Is there any evidence that students are linking together different sub concepts?

We have already seen an example of a student linking together different images of a sub-concept in S2's reflections on the meaning of sin 30 (§9.2.1). The lesson observations provided several further examples of students linking together different sub-concepts.

One example was the use of alternative spatial-visual images to find all the angle solutions to a problem within the given range. By the time the students were studying C3 most of them in this group were comfortable using either the graphs or the unit circle/CAST Diagram. It should be noted that the teacher used either representation without favour. However, there were two interesting exceptions to the general observation that students were comfortable. The first is S3 who always used the CAST diagram. He said he found it "easier" because he "had learnt it better" than the graphs. It is worth recalling S3's second concept map at this point.

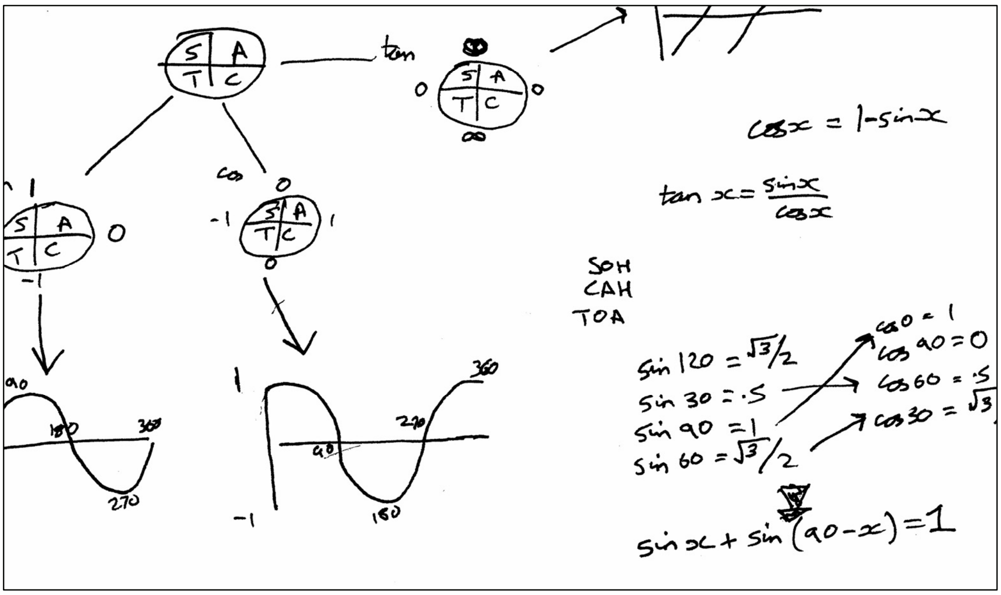


Figure 9.1 S3 Second Concept Map Drawn after C2.

There are two points of significance to note about this map. Firstly there is not one CAST diagram but four. There is a general one that then leads to three more that have been specifically labelled for each of the three functions. The values at 0°, 90°, 180°, and 270° have been stated. This implies a perception of trigonometry as being formed of distinct parts. It also strongly suggests the diagram is process-orientated. Secondly there is a one way arrow from each of these CAST diagrams to a graph. This suggests that the values shown on the CAST diagrams enable S3 to draw the correct graphs. The graphs can only be constructed from the information in the CAST diagrams and it is a one way process. In other words there is no indication that the graphs and the CAST diagrams are complementary and alternative representations of function but rather one process leads to another. The map gives every indication that S3 is firmly focussed on processes that are considered sequentially. It also suggests that S3 was still relying predominantly on memorising values. Overall the map suggests that S3's focus is on parts rather than the whole and those parts are processes and values.

The second exception was a student who used CAST diagram to find multiple angle solutions except when solving equations of the type cos θ = 0 when she used the graph. When asked about this she said she didn't like using CAST when 'it's on the lines (e.g. 90°, 180°, 270° etc)'. She explained:

> I use CAST normally because it is quicker to draw but when it's on the lines I use the graph because it does the same job and is less confusing.
>
> (S11)

This shows that S11 recognises the process value of the two representations and is able to flexibly alternate between either one. This allows her to choose a representation according to considerations of speed and difficulty.

This contrasts with S3's preference for a representation as S11appears to have choices at her disposal whilst S3 does not. S11 is showing signs of flexible thinking albeit to avoid potential pitfalls. In APOS terms this implies a process conception and by Sfard's theoretical framework it shows recognition of alternative operational images that could indicate partial reification of the instrumental processes into an object structure of the sub-concept.

Observations from the class lessons provided further evidence of students linking different sub-concepts together. Here is an example from the lesson on the double angle formulae for cos 2θ that shows a link between identities. By substituting θ for A and B into the formula cos (A + B) = cos A cos B – sin A sin B, the formula had developed to cos 2θ $=\cos^2\theta - \sin^2\theta$. The teacher then said:

> Teacher: Okay now this form is not really all that useful. We have both $\cos^2\theta$ and $\sin^2\theta$. Could we perhaps rewrite it in terms of just cos θ or sin θ?
>
> S7: We could use $\sin^2\theta + \cos^2\theta = 1$. So replace $\sin^2\theta$ with $1-\cos^2\theta$.

Here was evidence of knowledge being used creatively. The student had realised that one identity could be substituted into another identity to create yet further identities. This is an important and characteristic feature of trigonometry and here was an example of a student initiating the process naturally.

A second instance of this phenomenon was from the class discussions for solving specific problems that the teacher had written on the board. An example from module C4 was when the class were asked to find $\int \sec^2 x.dx$. One of the students suggested using the identity $\sec^2 x = \tan^2 x+1$ however another student said:

> But Sir, we know that when you differentiate tan x you get $\sec^2 x$ so isn't the integral just tan x?
>
> (S6)

This student showed signs of thinking beyond the material associated with trigonometry to creatively use the fundamental theorem of calculus, to short cut procedure and cut directly to a solution.

Another example is the discussion about the means to solve $\int \sin^2 x.\, dx$. The teacher posed the problem and asked the class for suggestions for methods to solve it. Prior to the course on trigonometry the class had covered procedures for (i) integrating a function of a function by substitution and (ii) integrating products by the process of integration by parts and though during the course on trigonometry the teacher had used the generic term “function” without reference to definition and properties, three suggestions were proposed by the class. The teacher allowed the class time to attempt the proposals as they were suggested. These were:

(1) Solving by substitution by letting $u = \sin x$. This proposed solution appears to be acceptable at first glance but the solution process very quickly requires complex algebra manipulation that is beyond the scope of this course and,

(2) Solving by parts since $\sin^2 x$ is the product $(\sin x)(\sin x)$. This proposal also appears sensible on initial inspection but the procedure for solving by parts travels full circle and we end up with $\int \sin x.\sin x.dx = \int \sin x.\sin x.dx$.

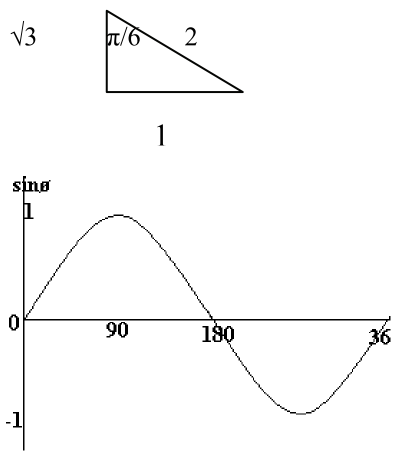
Both of these proposals linked recognition of a problem with specific conditions and a suitable procedure that may be implemented for these given conditions. At this point the possible instrumental procedures that could be applied by the students had been exhausted. There were cries from the class that it couldn’t be solved but the teacher insisted that it could be and urged them to “Think!” S8, (whose concept map featured predominantly the graphs), then suggested using the identity for Cos 2x. This is significant since the first two suggestions show an awareness of suitable procedures but the third suggestion indicates that S8 has interiorised the idea of a function having multiple algebraic representations that are interchangeable. This is the creative link the teacher was trying to encourage.

We may conclude therefore that the students in this group had made the link between the instrumental aspect of the graphs and the CAST circle. All but two of the students were able to use either spatial visual representation flexibly. However not all the students were able to do this all of the time. The second indication of students linking together different sub concepts was evidenced by the frequent insights students showed to problem solving as exemplified by S6's suggested use of the fundamental theorem of calculus. Finally throughout the course the teacher encouraged the students to think about alternative algebraic representations and there was evidence that from the beginning students were aware of the power of identities.

9.5 Is There any Evidence of Students Being Able to Curtail a Procedure or Change From One Representation to Another?

There was clear evidence that students were simultaneously thinking about trigonometry within two referential frameworks, algebraic and geometric, and switching between them. In addition most, but not all, students were switching between representations of an angle in degrees and radians. This is exemplified by S1's response to the first past paper question [§7.9] where he is required to find the coordinates of the y-intercept for y = 2 sin (2x+5/6π). The question is posed algebraically without a graph to refer to. Here is S1's answer:

Y=2sin (2x0+5/6 π)



x=0, Y=1

Figure 9.2 S1's answer to finding the co-ordinates of the y-intercept for the given function

S1's answer is brief and without explanation but we can see that he has linked sin 5/6π to sin π/6 as the special angles triangle has been drawn for π/6. The connection between the two angles was possibly aided by the symmetrical nature of the graph drawn but this is labelled in degrees. Whilst there is no clear explanation for his thinking it is clear from his solution that meaningful links between the different representations have been made that facilitate a solution that is simple and cognitively economic.

Another example of students accessing spatial visual representations is in the response of S1 and S2 to the question where they are asked to find the range of f(x) = 3 + 2sin (2x +30).

They both consider the function graphically, that is in spatial-visual terms. Both students use their knowledge of the maximum and minimum values of the function sin x and the graph transformations indicated by the expression to find the range [§7.9].

The evidence showed however that not all the students were accessing visual reasoning. S3 showed every indication of thinking only in terms of algebraic representations. For instance he connected the word ‘range’ to thoughts of maximum and minimum [§7.9] but he then connected these thoughts to a differential procedure. His subsequent action was to attempt to differentiate and equate the derivative to zero despite having drawn the graph. He proceeded to employ one instrumental process after another without thought of the inefficiency of method he was employing. When his method did not work out he had no alternative method at his disposal.

It is worth recalling the third concept map drawn by S2 and S3 and comparing them side by side (Figures 9.3 and 9.4):

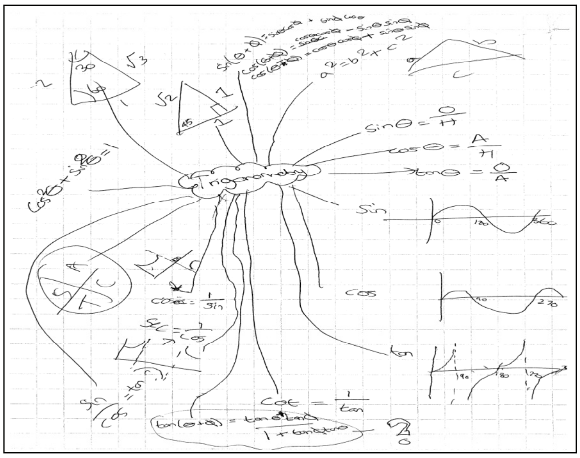


Figure 9.3 S2's Concept map 3

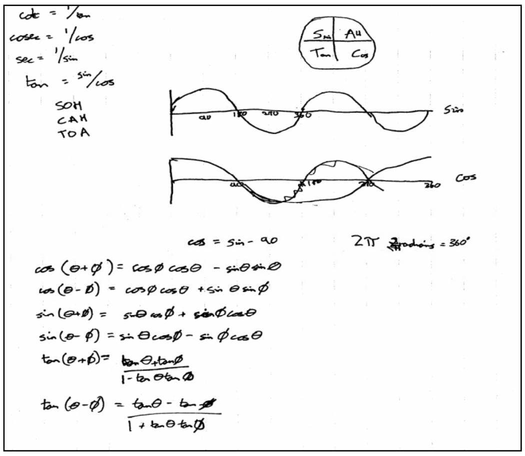


Figure 9.4 S3's Concept Map 3

S2 (Fig. 9.3) has indicated more spatial visual representations than S3 (Figure 9.4) but the more significant observation is that S2's map shows more aspects of the function with her representations. Another point that needs consideration is the qualitative nature of the spatial visual representations. This may be clearly seen in two examples from the pilot study P2 and P1 included spatial-visual representations but did not switch between representations. Closer examination of the manner in which these two students presented their spatial- visual representations is illuminating.

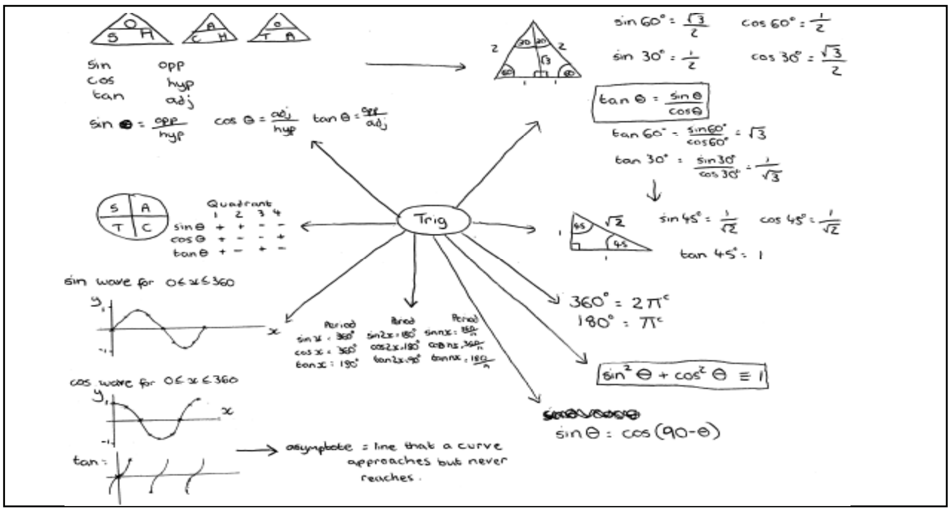


Figure 9.5 P1's second map

P1's second concept map (Fig. 9.5 and see also §5.8) retained the operational SOHCAHTOA triangles which are procedural aids to a process rather than procepts. She drew the special angle triangles but then listed the values of the sines, cosines and tangents beside the diagram as if to indicate that the triangles by themselves were not, for her, an immediate means to an end but were part of the process to find values. Similarly she drew the CAST circle but listed the values beside it and drew the graphs and listed their relevant periods beside them. This again indicates that the spatial-visual diagrams had not been interiorised in a meaningful way that provided an amalgam of information simultaneously but instead were a part of a procedural operation. P2 also included spatial-visual representations in her second concept map but the signs were that these also had not been interiorised or linked together in a meaningful way [§5.8]. She showed the graphs of y = sin x and cos x but both these were wrong. She also showed the graphs of: 2sin x, -sin x, sin (x-90) and sin 2x which were correct but since the graphs of sin x and cos x were wrong indicated that she had not interiorised the connection between these graphs and the graphs of y = sin x and cos x. The transformed graphs were being memorised as perceived objects. Like P1, P2 showed the special angle triangles but listed the values beside them which points to a process perception of the triangles.

There is clear evidence here of a qualitative element to spatial visual representations [Pitta & Gray (1999) §3.2.1]. Whilst this is best exemplified by the two students from the pilot study the third concept maps drawn by S2 and S3 also indicate signs of the same phenomenon. S2's map appears to describe a concept that has various interconnected aspects which can be represented algebraically or geometrically or both. The spatial visual representations (despite the cos graph being incorrectly drawn) have a proceptual element that complements the algebraic representations. S3's map shows a focus on selected parts of the concept when compared to S2's focus of the whole. The CAST diagram is shown and two graphs and as discussed earlier there had previously been evidence of a one dimensional dynamic between these two representations. There are algebraic representations of concept definitions in the top half but the bottom half is given over to various forms of the double angle formula. This gives every indication of being predominantly a memory aid and there appears to be little sense of a deeper conception that is holistic and unified. The maps are visual representations of each students developing mental schema and there is strong evidence of a qualitative difference between these two maps.

We may conclude that there appears to be a connection between the form of the concept map drawn by a student and the ability to curtail a procedure and oscillate between one representation and another.

9.6 Is There Any Evidence that Students Can De-Encapsulate Concept Images?

To 'de-encapsulate concept images' is used here in the specific sense of being able to recognise a graph of a complex function as the result of a base graph such as sin, cos or tan x that has had one or more geometric transformations applied to it.

The last question of the content uptake questionnaire asked students to identify which functions were identical [§7.7.5] and the past paper questions where the co-ordinates of where the graph y=2 sin (2x +5/6π) crosses the Y-axis have to be determined [§7.9.1] provided evidence of varying quality that some but not all students were able to de-encapsulate concept images.

For the question where students were asked to identify which functions were identical [§7.7.5] all the students identified congruence between sin x/cos x and tan x however this may have been 'learned' during the study of C2. It is however unlikely that the other functions chosen for this question, such as Sin (x+90), Cos (x-90) etc., were learned or their graphs specifically remembered. The method used by students to answer this question was to draw the graphs so the algebraic form of concept image, e.g. (sin x +90) was recognised as having a spatial visual representation which was the result of a process that translated horizontally the graph of sin x by 90° to the left. From this group 9 students identified congruence between cos x and sin (x+90) and 10 students recognised congruence between sin x and cos (x-90). The students who did not identify the correct solutions used the same method as those who did but made errors with their transformations such as a shift to the left instead of the right. Thus it may be concluded that the symbolic representations of these concept images had been de-encapsulated.

The importance of connecting a symbolic representation to a process or a series of processes is shown by the response given by S2 to the first part of the integrated questions where the co-ordinates of where the graph y=2 sin (2x +5/6π) crosses the Y-axis have to be determined [§7.9.1]. Although she was unable to find a means to the answer to the question directly, by de-encapsulating the expression y=2 sin (2x +5/6π) into a series of transformations of the graph Y =sin x, which she termed as 'a squash by a factor of 2' horizontally followed by 'a shift to the left of 5/6 π' then 'an upward stretch by a factor of 2' she was able to find a position of understanding from which she could then progress to the answer. By Dubinsky's (1991) definition of de-encapsulation she perceived the procept of $y=2 \sin(2x + 5/6\pi)$ in terms of a process or series of processes. (This was in sharp contrast to the pilot study group who were unable to de-encapsulate to processes [§5.9] and were therefore unable to find any means to an answer. Again a possible result of teaching emphasis).

Another point of interest is the contrast between the solution methods of S1 and S3. Both students immediately recognised that it was not necessary to de-encapsulate visually $y=2 \sin(2x + 5/6\pi)$ into its component transformations but that all that was required was to find the value of y when x = 0. However whereas S1 evaluated the answer by substituting 0 for x then using his calculator to find $\sin 5/6\pi$ then doubling the answer S3 did not do this, but instead attempted an algebraic expansion without consideration of a more economic means to the same end result. His problem solving approach was an automatic process stimulated by visual cues from presentation of the problem.

We can conclude therefore that after module C2 most of the students in this group were able to de-encapsulate algebraic representations of simple sub concepts by considering them in terms of geometric transformations applied to the graphs. At this point the de-encapsulation was a process which the students actually undertook by drawing out the base graphs and then either redrawing the result of a transformation or mentally imagining the result. The link between the different representations and applied transformations is one that was emphasised by the teacher and was the result of his philosophy that trigonometry is fundamentally a geometric subject.

Different students drew different visual cues from the expression $y=2 \sin(2x + 5/6\pi)$.

For S2 the strength of the link between the algebraic representations and the spatial visual graphs caused S2 to automatically begin to de- encapsulate it geometrically even though it was perhaps unnecessary to reach the solution. For S3 however, the visual cue of a form of double angle i.e. sin (A+B) stimulated him to undertake the action of expanding the expression without consideration of how this would progress the problem.

9.7 To What Extent do Students Think Their Own Perceptions of Trigonometry have Changed Since the Start of the Course?

One of the ways that students identified a difference between the trigonometry covered at GCSE and the trigonometry studied at AS/A-level was a shift from a set of operational procedures for solving problems involving triangles (or finding obtuse or reflex angle solutions to simple equations of the type cos x =0.6 etc.) to a more generalised study of properties and interconnected representations of function. It should be noted here that the definition of function was not considered during the observed lessons however the teacher explained that it had been covered previously. Specific comments made by the students highlighted this change of focus when comparing trigonometry at GCSE and AS/A level. They talk of trigonometry being 'more freely used', and there is now 'more than one answer to the same question' and the necessity of using identities and spatial visual representations 'all at the same time' as indicated here:

> Trig in general is a lot more freely used, for example, in quadratics or negative values (S8)

> At A-level you learn how to solve new types of trigonometric equations with identities and radians and these are used to get more than one answer to the same question (S2)

> At A-level there are identities, radians, the CAST diagram and you have to solve trigonometric equations and the graphs using them all at the same time (S5)

Another observation made by some of the students focused on a shift in their own 'understanding'. This is significant because the students are indicating an awareness of an understanding that is qualitatively different to the understanding that they previously had. This perception of a qualitative difference is indicated by phrases such as 'there is a lot more to think about' 'at GCSE I don't think I learnt it' 'you start to understand why' and 'you understand more about how it works' as evidenced in the following comments:

> I didn't find it difficult at GCSE but at A-level it takes more understanding. There is a lot more to think about. It's not as clear cut as 'use this formula' or 'to solve this you do that' (S12)

This comment indicates the student's recognition of different types of understanding. He is aware of a distinction between being able to apply a procedure 'to solve this you do that' and being able to use a flexible integrated solution process to solve a problem 'there is a lot more to think about'. This corresponds to Skemp's distinction between instrumental and relational understanding (Skemp, 1976 [§3.1.2]])

> At GCSE I don't think I learnt it. At GCSE I had to think 'Uh Oh Sin equals opposite over hypotenuse' yeah? and work it all out but now I don't have to. (S13)

This comment uses the word ‘learnt’ to distinguish between the time prior to the AS/A level course when S2 had to go through the step-by-step motions of a process and now when she does not need to.

> In the beginning its sort of you do this or this but then you start to understand why you do that and you don’t have to strain to remember it anymore
> (S5)

This student uses the term ‘understand’ to mark a shift from memorising. The student explicitly states the connection between ‘understanding’ and less strain on the memory. This observation and the previous one correspond to Sfard’s description of a process that has been interiorised and condensed (Sfard, 1991).

> GCSE mainly uses triangles but at A-level it relates more to graphs and the CAST diagram. You understand more about how it works (S6)

This comment shows the student’s awareness of his own change in perception of trigonometry from a set of instrumental processes to solve problems that involve triangles to an entity that can be variously represented. This corresponds to the formation of a structural object conception described by Sfard (1991) and APOS theory (Dubinsky, 1991). For these students therefore, their perceptions of trigonometry have changed from a subject that is predominantly a set of procedures for solving problems involving triangles to one that is an interconnected amalgam of graphs, spatial visual images and identities whose properties can then be flexibly used to solve abstract problems such as equations. This suggests the beginning of a more proceptual understanding.

Not all the students however had this development of understanding. The perceptions of a transitional change in understanding stated by the students above may be contrasted by this comment made by S3:

> I don’t really know its use aside from triangles but now we have a lot of formulas to remember like the double angle formula (S3)

This comment suggests that S3 perceives trigonometry instrumentally. To him it is a means to an end that has a ‘use’. This contrasts with the other students above who specifically comment on ‘understanding why you do that’ or as an entity that may be variously represented. S3 states explicitly that he relies on memorising formulae and the implication is that his knowledge is superficial in that he does not have any sense of why he undertaking a process.

S3's understanding of trigonometry is, in Skemp's terms, a set of rules without reasons. And the significance of these different views has consequences for later achievement.

9.8 What are Students Own Perceptions of Their Learning Experience?

The students specifically identified two types of learning that they have to undertake. One type of learning was described with phrases such as 'learn properly' and 'deeper learning'. The other type of learning was 'learning for the assessment'. They also distinguished between two types of teaching styles which, the students' perceived, was the effect of different teaching agendas. This distinction is exemplified by the following comments all made by students in the main study:

> There are different styles of teaching of course. With some teachers our understanding is deeper because those teachers are more concerned with our knowledge but some teachers, their teaching is really based on dictation of past papers and you learn more about what is going to be on the exam.
>
> (S5)

> We have to know two kinds of knowledge: One for depth and one for passing the exams
>
> (S8)

For the type of learning that was necessary to do well on the assessments the students identified the learning process as 'learn all the formulae' and 'practise past question papers'. The value of this learning process for improving assessment results was recognised by the students.

> 'If you keep doing practice papers then you refine it because you know the things that can come up and then you can get the top grades'
>
> (S5)

The students expressed frustration however of the time constraints placed on their learning.

> 'We don't get so much time to learn things properly because we have to start revising'
>
> (S7)

The students explained that both types of learning were necessary in order to get into the best universities. All the students said they thought modular assessment was to their advantage although as the comment above states the need to switch from 'learning things properly' to revising for the assessment was perceived as restriction on the time they had for ‘deep learning’. This time constraint on ‘deep learning’ was referred to by several of the students:

> When you learn doing practice papers it doesn’t actually reflect your knowledge, but it takes time to understand all the theories.
>
> (S5)

> Sometimes I don’t have the time to learn it properly so I just learn enough for the exam.
>
> (S3)

> The exam doesn’t always show who understands it better because it takes time to do that but on the exam they might not test that understanding.
>
> (S14)

The students’ distinction between the type of learning referred to as ‘deep learning’ or ‘learn things properly’ and learning for the assessments is significant. The choice of words used to describe the two types of learning show that the students clearly think that there is a qualitative dimension to these types of learning.

It is interesting to contrast the difference in teaching style between the newly qualified pilot group teacher and the experienced main study teacher. The pilot study teacher referred frequently to the upcoming assessment and his stated philosophy was to help as many students as possible get the best grades possible. To this end he often gave the class model answers to typical examination style questions and urged the students to learn them thoroughly. The main study teacher rarely mentioned the assessment but allowed the students class time to consider a variety of problems either on their own or with their neighbours and gave past papers for homework near to the assessments. The students commented that this was the best way as they resented spending deep learning time on learning for the assessments. He did not give any model answers but instead urged the students to think about the problem. When a student was really stuck he would remind them of a link that might prove profitable. Towards the time of the assessment he set past papers for homework. The students’ perception was that the best teachers were those who used both styles of teaching. The teacher of the main study group was identified by the students as a teacher of this type.

The importance of the teacher was specified by several of the students. Descriptions of his teaching style used words such as ‘simpler’ ‘links’ and ‘emphasis’ as shown by the comments below:

> 'When you have a good teacher they like make it all seem simpler somehow and you just seem to remember it.'
>
> (S8)

> 'At A-level there is an expansion of knowledge with harder questions than at GCSE but the teacher shows you how it all links up.'
>
> (S2)

> 'Mostly I find like it's quite logical. But with the hard things I look at specific things that the teacher emphasises'
>
> (S5)

Teaching styles that prioritise the strengthening of students understanding of the sub-concepts of a function and how they are interlinked make a huge contribution to the students overall development of the function concept [Bayazit, 2005]. However for the sub-concepts to be meaningfully understood they must be 'constructed' by the individual personally rather than perceived as another's construction. The process for this according to Sfard's (1991) theoretical framework is through interiorisation and condensation [§3.1.4]

9.9 Reflections on Theoretical Frameworks

This study was framed within the theoretical descriptions of conceptual development posed by Dubinsky's APOS theory (1991), Sfard's (1991) 3-step theory of reification and Nickerson's network theory. This section sets out how well the theories are supported by the empirical evidence.

9.9.1 The APOS Framework as a Description of Trigonometric Development

APOS theory describes the personal development of mathematical concepts through an Action that is automated to a Process that is perceived as having a beginning, an end and a purpose, to an Object which is the encapsulation of all processes and then to a Schema [§3.1.5]. There was evidence in this study that some of the students were stimulated to take Actions on seeing certain stimuli. For example in the pilot study P1 was stimulated to divide the opposite length of a triangle by the length of the hypotenuse without any understanding of what the result meant. However recognising when students had developed to the process stage was more problematic. APOS theory identifies a process as having the following characteristic:

> Once an individual has constructed a process, it may be reversed, or it maybe co-ordinated with other processes. In some cases, this co-ordination leads to a new process (as in the composition of functions).
>
> (Cottrill et al, 1996, p171)

However this could also be learned mechanistically as in the case of P2's graph transformations in the pilot study [Figure 5.11]. Thus whilst the Action part of the framework may be easily discerned in students the process part is less straightforward.

A specific example of the problem of distinguishing between Action and Process arose in the main study. In the integrated questions where the students were asked to find the range of f, S3 initially understood this in terms of determining the maximum and minimum and then undertook a procedure to evaluate these. He could be seen to have been stimulated by the words 'maximum' and 'minimum' to undergo a learned procedure of differentiation thus he was at the Action stage of development. However it could also be argued that S3 was employing a process that had a beginning, an end, a purpose and he was able to describe the steps without necessarily performing them:

> So I could differentiate it and then make it equal to zero and then see what x is....or not!
>
> (S3, §7.9.3)

This is the definition of a process. Even when a process conception is identified in the student it may not necessarily show useful development in the student's schema. The problem for S3 was not that he was undertaking the procedure mechanistically without any conscious control but that he had no other means to process the answer at his disposal. Unlike some of the other students he was not able to think simultaneously on a geometric plane and on an algebraic plane. Tall et al [2000] state that:

> A function as a process is determined as a whole by input-output, regardless of the internal procedure of computation. Thus the functions f(x) = 2x+2 and g(x) =2(x+1) are one and the same as processes.
>
> (Tall et al, p4)

The problem is not to make sure they are recognised as one and the same by a student but to ensure the student can work with either form and not, taking the above example, find it necessary to change 2(x+1) into 2x+2 for a deeper personal meaning. The teacher's instructional style can be pivotal in encouraging students to develop thinking in more then one plane. Thus for teaching purposes the emphasis is better placed not on the promotion of a student's development of a process but on developing parallel thinking that is richly connected to an already developed process. This was shown to be fundamental to the development of understanding trigonometry in this study. Those students who had developed the ability to perceive representations simultaneously algebraically and in spatial-visual terms appeared to have a qualitative advantage over those who did not [§9.3].

A further issue is that APOS says nothing about the duality in the meaning embedded within a mathematical symbol or image which can be interpreted as both process and object at the same time and thus has no explanation for the difference in understanding between, for example, P1's process representation of SOHCAHTOA within ratio triangles [§5.4.1] in the pilot study and S1's (and others) representation of sin x = opp/adj etc from the main study [§7.7.6 and §7.10]. The ratio representation is a procept that simultaneously represents both the means of processing a value and the concept.

Thus from a teaching perspective the stages of the APOS framework of development are difficult to specifically identify and the focus on developing a process may limit the promotion of connections to other processes that can lead to encapsulation.

9.9.2 Sfard's 3 Step Framework as a Description of Trigonometric Development

Sfard's theoretical development framework describes a progression from an operational understanding which is dynamic to a structural object conception [§3.1.4]. The result is a mathematical conception that has a dual nature: on the one hand, conceptual; and on the other, operational. The conceptual is defined as 'the whole cluster of internal representations and associations evoked by the concept' (Sfard, 1991, p3). The power of an object conception is that the conceptual and operational sides each provide information that is continually cross-referred between them which compares favourably with APOS. It would appear that Sfard sees the object conception as a mental formation that does not need to be associated with those processes through which the object was formed. In Dubinsky's terms the process and the object encapsulated and exist side by side. In one sense it might be said that the notion of procept contains the same idea except that the procept is a symbol that carries the ambiguity of process and concept. Dubinsky does not appear to indicate how the encapsulated object is represented.

The object conception is formed by the development of an operational procedure that is interiorised, condensed and then reified into the structural object.

Interiorisation is evident when a process no longer needs to be actually performed for the result to be considered. There was evidence of development fitting this description in this study in the graphs of y = sin x, cos x and tan x drawn by the students. Those students who were unable to draw the graphs correctly without plotting points or mentally plotting points may be described as not having interiorised the process whilst those who were able to draw the graphs correctly may have interiorised the process, or may have remembered the graphs correctly, but it is the use of the graphs for further processes such as transformations that implies an interiorised process. Another example was the special angle triangles [§7.7.2]. Those students who could be described as having interiorised the triangles did not need to actually carry out the process of dividing the opposite by the hypotenuse to evaluate the sine of the angle [§9.7: S13]. Those students like P1 and P2 who listed the values were indicating that the action still had to be undertaken and therefore the triangles had not been interiorised. We can conclude therefore that operational process and interiorised operation as described by Sfard was discernable in these students.

Sfard states that:

> An operational process that is condensed manifests itself in the growing easiness to alternate between different representations of the concept. It lasts as long as the entity remains tightly connected to a certain process.
>
> (Sfard, 1991, p19)

There was also evidence of this stage of development in this study. An example was the students' ability to use either the CAST diagram or the graph to find multiple angle solutions for a given equation [§9.4: S11]. Another example was the students' ability to alternate between radians and degrees. Thus this description of development was also discernable.

Reification is marked by 'an ontological shift whereby various representations of the concept become semantically unified by an imaginary abstracted construct' [§3.1.4]. Sfard suggests that reification is the product of a reorganisation of knowledge into a format that is hierarchical— typical representations of non organised and re-organised knowledge are shown in figure 9.1.

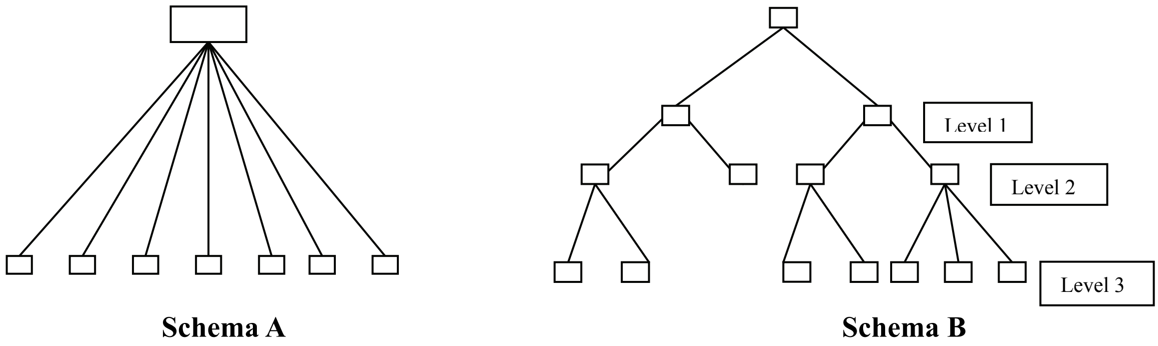


Figure 9.6 Representations of schema development

(Sfard & Linchevski, 1991)

Schema A illustrates an un-reified schema; the knowledge is sequential, shallow and wide. Schema B illustrates a reified schema that has a deeper narrower structure which allows for easier and faster retrieval and storing.

The evidence from this study shows that Schema A has strong similarities in structure to the initial concept maps drawn by the students. Each item of knowledge is given equal significance.

Schema B was not drawn in the manner described by Sfard above by any of the students but the salient features of the structure could be perceived in the third concept maps drawn by students. If the lowest level (Level 3) constitutes the perception of the importance of values (operational formulae and lists of identities etc) and the next level up (level 2) the role of spatial-visual diagrams such as the CAST diagram and the special-angle triangles and the higher levels (level 1) representing the significance of the graphs, identities and concept definitions then there could be said to be some evidence of reification having taken place. The third map drawn by S1 [§7.10] has shown the graphs, the identities for secant, cosecant, cotangent and the tangent identity along with the CAST and special angles triangles. This could be seen as showing the structure of trigonometry from a top down perspective with the operational levels not indicated as there are no specified identities or formulae. The third map drawn by S2 [§7.10] still maintains the 'spider' shape but the legs coming off the trigonometry 'body' show the graphs, identities, special angles triangles and definitions of sin θ, cos θ and tan θ. There are three legs that show operational formulae but the way they have been squeezed in amongst the other items indicate they were entered later which could also suggest a top down perspective. The third map drawn by S6 [§7.10] again shows graphs, identities, the CAST circle and the special angles triangles as well as the core definitions for sin cos and tan θ but S6 has used arrows to show the direction of his organisation of knowledge. There is an arrow from the definitions of sin, cos and tan θ past the special angles triangles to a list of values. Additionally there are arrows from the graphs to the CAST circle, to the definitions of cosec, sec and cot and from the sin, cos and tan θ definitions to the graphs. These arrows suggest that the sub concepts are being linked in a structure not unlike that of schema B in figure 9.1. Sfard states that the reified structure allows for faster storage and retrieval of knowledge and the items indicated and the arrows appear to show a cognitive structure that facilitates this.

The map drawn by S8 [§7.10] is interesting in that he pays particular attention to the graphs, placing them all on the same axes almost as if he is trying to get a picture of trigonometry as a whole. No other spatial-visual images are shown, neither are any definitions given. He itemises sin, cos and tan, cosec, sec, and cot and the sine rule and cosine rule. It could be that these have been subsumed within the concept image described by the graphs; either way, the map indicates that he has prioritised the content knowledge of trigonometry with the graphs being perceived as having more significance than the operational knowledge that is listed.

If these are maps that may indicate reified concepts of trigonometry then the maps of S3 and S7 [7.10] could show un-reified or partially reified concepts. the third map drawn by S3 shows the graphs of sin x and cos x and the CAST diagram, definitions and the tan identity but the inclusion of a list of formulae, 2π radians = 360° and ‘cos = sin -90’ indicates that that these have not been interiorised and his mental schema is more akin to schema A than schema B in figure 9.1. This analysis was borne out by S3’s attempts at the integrated questions where he was unable to retrieve or store knowledge efficiently. The third map drawn by S7 shows that he is grouping knowledge but again there does not appear to be any priority given to, for example, definitions over values indicating a schema that has not yet been reified but is working towards it.

Symbols, graphs and other spatial-visual representations are, Sfard claims, a frequent trigger for reification. In the context of trigonometry this appears to be supported by the empirical evidence but it is clear that the emphasis the teacher places on these images is pivotal to reification by the students. This emphasis can help the students’ perceptions develop from one that regards the symbolic images as a process to determine values to a procept that has both the process and concept embedded simultaneously.

9.9.3 Network Theory as a Description of Trigonometric Development

Nickerson's (1985) network theory [§3.1.7] describes understanding as a process that involves the active participation of the one who understands. The depth of understanding of a concept might be taken as the number of nodes of knowledge that are connected.

> It requires the connecting of facts, the relating of newly acquired information to what is already known, the weaving of bits of knowledge into an integrated and cohesive whole. In short it requires not only having knowledge but also doing something with it.
>
> (Nickerson, 1985, p234)

In other words, understanding requires both an operational component and a conceptual component. The conceptual component is described as a 'deep structure trace' which:

> When developed constitutes an explanation of, or reason for, the surface structure of the thing to be understood.
>
> (Ibid)

This 'deep structure' is evident by the degree of abstractedness that is conceptualised by the expert compared to the novice and by the ability to process problems through a 'top down' approach. In order to have a 'top down' approach however, an individual needs a mental conception that has a 'top' and a 'down'. This necessarily implies a qualitative organisation of knowledge. Nickerson goes on to say that experts tend to find solutions faster and suggests that this is because their relevant knowledge is stored in 'larger chunks'. If these 'larger chunks' are sets of nodes of knowledge, images and facts that are richly connected then this is possibly an alternative description of reified sub- concepts. This seems to indicate that there are many similarities between the frameworks of the development of understanding suggested by Sfard and Nickerson. The difference between the two frameworks is that Nickerson's framework concentrates on the linking of information and knowledge in order to achieve a 'deep structure' whilst Sfard's framework focuses primarily on the reorganisation of an individual's knowledge once the links have been made. This suggests the two theoretical frameworks are complementary. In this study it was found that the role of links between geometric representations and algebraic representations was fundamental to the development of a concept that is flexible and integrated. Those students who had sufficient links to be able to think simultaneously in both representations had the greatest cognitive flexibility [§7.9.2]. Nickerson says that representations play an important part in understanding and that experts create qualitative representations that require familiarity with concepts, principles and relationships. There was no evidence of any of the students in this study creating representations however some students did distinguish a qualitative dimension to the representations that they were presented with, such as the graphs, and developed a proceptual understanding of these representations.

In conclusion we can say that whilst Sfard's theory better describes the student's cognitive development of an Object construction, network theory is invaluable to teachers for its focus on the necessity for links between different representations to facilitate this cognitive development [§7.10].

We should at this point consider the two learning agendas that the students identified: One for deep learning and the other for doing well at assessments. The results of the A2 assessment show that the students whose concept maps indicated a variety of representations of sub-concepts or a top down perspective performed significantly better than those students whose maps focussed on processes and formulae [§7.11]. We may conclude therefore that the students who successfully made links to alternative representations had the advantage at both types of learning over those who did not and that a predictor of success in assessments is the number of links made and the strength of the links. The measure of the strength of the links is the extent to which they have been interiorised.

9.10 Limitations of the study

This study was undertaken at a selective school situated in the home counties of England. It cannot therefore be considered as a model for all school populations even within England. Most of the students had followed the National Curriculum to GCSE prior to starting the 2 year course at A-level and therefore the students' experience of the content of trigonometry and the learning orientation was similar. As such the sample in this study was only one of many different possible samples. The results of this sample therefore can not be generalised beyond this study but they can indicate information on quality of thinking and achievement.

Concept maps are by definition personal constructions of a phenomenon and therefore each individual's map is unique and cannot be cast as a model for a type. However the different styles of maps and in particular different representations that emphasise process or procept gave indications of future achievement.

The very activity of asking the student's to draw concept maps for trigonometry was likely to cause them to reflect on the topic in a manner that they may not otherwise have done and this may have influenced their perceptions of links and re-organisation of knowledge.
The teacher of the pilot study was new to teaching and the teacher of the main study was very experienced so a direct comparison between the two teaching styles should be considered within this context. The new teacher was understandably focussed on performing well in his first teaching role and felt responsible for his students' achievement in their final assessments. A proper comparative study of teaching styles would need to consider teachers who were comparable in experience.

The student interviews were more informative at the end of the course then at the beginning. There are several possible reasons for this: At the start of the course the students were wary and perhaps insecure about their knowledge or the 'true' reasons for the questions; at the start of the course the students were 16years old and lacked the confidence to talk to 'a teacher' in a familiar manner but by the end of the course they had more confidence when talking honestly to adults; at the start of the course the students were being asked to talk about their learning and knowledge in a manner that was unfamiliar to them.

The large size of the main study group precluded observing closely the progress of all the students. The students who were selected by the teacher for closer observation proved to be illuminating in their different perceptions of trigonometry but other students who were not interviewed such as S6 and S8 drew very interesting final concept maps. It was disappointing that all the students could not have been observed.

Many of these students chose to study mathematics-based subjects at university. The progress of the students as they embarked on their Higher Education courses was not observed and there is no indication from this study as to how well prepared these students were placed for the further study of trigonometry.

9.11 Suggestions for further investigations

This study found that the students in the pilot study and the main study had operational schemas initially and perceived trigonometry as a set of operational procedures that were used for problems with triangles. As the course developed the evidence showed that some students developed a perception of a qualitative dimension to the new information that they encountered. This seemed to be associated with teacher style (even though the students had a focus on the assessment results). In the pilot study both the students and the teacher were focussed on examination results however in the main study the teacher was more concerned with inter-relationships between the material and the development of understanding and this did not diminish student achievement. Sub-concepts were reified to provide a cognitive schema that had both conceptual and operational aspects. In this study it was found that the teacher's role in this development was crucial. The teacher concentrated on the students' development of parallel representational frameworks that could be simultaneously employed on presentation of a problem. The philosophy of this teacher was that trigonometry is primarily a geometric subject. Further investigation of other teaching styles that place the emphasis elsewhere: perhaps more centred on functions, or employing greater use of the available technology, would increase teachers' understanding of where their main focus should be placed for the development of concept in their students. The results of Blackett's (1990) research [§3.3.2] indicated that technology has a valuable role to play at GCSE level but further research into its potential at A-level would be useful. Delice & Monaghan (2005) concluded from their research of students learning trigonometry that 'what you teach is what you get.' [§3.3.5]. What sort of schemas would result from these different emphases and is there a qualitative distinction between them?

The students in this study identified two types of learning that they needed to undertake in order to be successful at A-level mathematics: one for a deep understanding and one for the assessments. Module assessments allow students to bank high marks on the easier AS modules of C1 and C2 that offset lower marks on the C3 and C4 modules. In this study there was evidence that some students with an operational schema of trigonometry obtained similar overall grades at A-level mathematics to those students who had developed a more flexible cognitive schema by achieving very high marks on the easier modules. The overall grade therefore could not distinguish between those students who had developed a 'deep understanding' and those who were adept at learning for assessments. This raises the question what is the emphasis of A-level achievement actually doing to the development of mathematical understanding? And further, given that A-level does not appear to discriminate between different types of knowledge what may be the consequences if students with different perceptions attempt a University course in mathematics? This suggests that an area for investigation could be whether a better marker of the quality of candidates is their C3 and C4 grades rather than their overall grade.

9.12 Concluding Remarks

This research has shown that concept maps have a useful role to play in identifying the types of schemas that students have constructed. Within this group there was evidence for three types of schema: (1) a process orientated schema (2) a schema that was an amalgam of complementary representations that had interiorised links and (3) a top-down schema that was represented by an overall concept image of the graphs. The students that had developed either the second or third type of schema had greater flexibility when problem solving and performed significantly better in the assessments.

The theoretical framework that best described the students' schema development was Sfard's (1991) 3-step model of process to static Object through interiorisation, condensation and reification. The resultant Object for trigonometry was an amalgam of axioms, identities, spatial visual and algebraic representations, alternative forms of answer and complementary processes. The processes were perceived as dynamic but finite entities. In addition it may be seen that terms such as sin θ were perceived as a process that involved dividing the opposite length by the hypotenuse but later brought images to the minds of students of graphs, circles and triangles. This indicates a proceptual perception of the term as described by Gray & Tall (1994)

Overall this research has shown that reification is not an automatic process for all students within the given time-frame of 2 years however when the students reflected on their learning of trigonometry they identified the teacher's instructional approaches as critical to the development of their understanding. The teacher's focus was on promoting deep structure understanding but prior to assessments he gave past papers as homework. The students were aware that their learning had a dual agenda: deep learning and learning for assessments; both of which were necessary for their future choices at Institutes of Higher Learning. They preferred the module system of assessment but resented the time away from deep learning for work for the assessments although recognised its worth. The grade for the Core 3 and Core 4 modules were better indicators of ability in trigonometry than the overall A-level grade since students were able to accrue high marks on the easier modules, re-sitting them if necessary, to offset low marks on the later more challenging modules.

Although this section of the A-level syllabus is entitled 'trigonometric functions' there appeared to be no specified link to the concept of function. It is possible that the students had made informal links but the syllabus had no specific requirement for trigonometry to be understood as a subset of functions.

This study has shown that within this group of students there was a qualitative difference between the trigonometric schemas that individuals constructed. Learning is a mental process that an individual develops at a personal rate through interaction with several variables that are external such as the teacher, the students and the difficulty of the course content. However this study has attempted to show that a major aspect of schema construction is psychological. By choosing a group who were proven to be capable in mathematics at the start of the course and who then studied the same course with the same teacher this research attempted to remove the external complications. Overall the research findings indicate that the qualitative distinction between individuals' trigonometric schemas depends largely on the focus of the individuals' attention when learning. Students who complemented algebraic processes with spatial representations had a qualitative advantage over those who concentrated their attention upon one aspect to the detriment of the other. The benefit of a schema that had recourse to both spatial and algebraic representations was that firstly it was more flexible and secondly it could be strengthened on two levels providing greater scope for understanding.

However, notwithstanding these conclusions, we have also seen from the pilot study, albeit from less detailed evidence, that teacher emphasis can be influential in student construction of a multi-representational approach to trigonometry and in a deeper-rooted perception of the nature of mathematics at A-level. It is also worth noting that though the trigonometrical components of the course make reference to 'trigonometrical functions' (Edexcel AS/A GCE in Mathematics: Specification p25.) it is not clear that either of the two groups forming the basis for this study had interiorised the notion that components of the course were subsets of the set of functions and had consequently developed an understanding of the properties of trigonometrical function in a formal way. Given these two caveats it would have been interesting to investigate the way that the two different cohorts within this study approached any further mathematical study.

Michele Challenger
February 2009

Appendices

Appendix 1: The Edexcel 2004 specification for A level mathematics for C1 to C4, S1 and M1.

2.4 Unit C1 - Core mathematics

1. Algebra and Functions.
    - Laws of indices for all rational exponents.
    - Use and manipulation of surds.
    - Quadratic functions and their graphs.
    - The discriminant of a quadratic function.
    - Completing the square. Solution of quadratic equations.
    - Simultaneous equations: Analytical solution by substitution.
    - Solution of linear and quadratic inequalities.
    - Algebraic manipulation of polynomials, including expanding brackets and collecting like terms, factorisation.
    - Graphs of functions; sketching curves defined by simple equations. Geometrical interpretation of algebraic solution of equations. Use of intersection points of graphs of functions to solve equations.
    - Knowledge of simple transformations on the graph of $y = f(x)$ as represented by $y = af(x)$, $y = f(x) + a$, $y=f(x+a)$, $y=f(ax)$.

2. Coordinate geometry in the (x,y) plane.
    - Equation of a straight line, including the forms

    $y – y_1= m(x - x_1)$ and $ax +by + c + 0$.
    - Conditions for two straight lines to be parallel or perpendicular to each other.

3. Sequences and Series.
    - Sequences, including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = f(x_n)$.
    - Arithmetic series, including the formula for the sum of the first *n* natural numbers.

4. Differentiation.
    - The derivative of f(x) as the gradient of the tangent to the graph of y=f(x) at a point; the gradient as a limit; interpretation as a rate of change; second order derivatives.
    - Differentiation of x^n, and related sums and differences.
    - Applications of differentiation to gradients, tangents and normals.
5. Integration.
    - Indefinite integration as the reverse of differentiation.
    - Integration of x^n.

2.5 Unit C2 - Core mathematics

1. Algebra and functions.
    - Simple algebraic division
    - Use of the Factor Theorem and the Remainder Theorem.

2. Coordinate geometry in the (x,y) plane.
    - Coordinate geometry of the circle using the equation of a circle in the form $(x-a)^2+(y-b)^2 = r^2$ and including use of the following circle properties:

- (i) the angle in a semi circle is a right angle;
- (ii) the perpendicular from the centre to a chord bisects the chord;
- (iii) the perpendicularity of radius and tangent.

3. Sequencies and series.
   - The sum of a finite geometric series;
   - the sum to infinity of a convergent series, including the use of $r<1$.
   - Binomial expansion of $(1+x)^n$ for positive integer n.
   - The notations n! and $[_r^n]$.

4. Trigonometry**.**
   - The sine and cosine rules,
   - the area of a triangle in the form ½ abSinC. Radian measure, including use for arc length and area of sector.
   - Sine Cosine and tangent functions. Their graphs, symmetries and periodicity.
   - Knowledge and use of $\tan\theta=\sin\theta/\cos\theta$, and $\sin^2\theta+\cos^2\theta=1$.
   - Solution of simple trigonometric equations in a given interval.

5. Exponentials and logarithms.
   - $Y=a^x$ and its graph.
   - Laws of logarithms.
   - The solution of equations of the form $a^x=b$.

6. Differentiation.
   - Applications of differentiation to maxima and minima and stationary points, increasing and decreasing functions.

7. Integration.
   - Evaluation of definite integrals.
   - Interpretation of the definite integral as the area under a curve.
   - Approximation of area under a curve using the trapezium rule.

2.6 <u>Unit M1 - Mechanics</u>
Prerequisites: Candidates are expected to have knowledge of Core 1 and of vectors in two dimensions.

1. Mathematical Models in Mechanics.
   - The basic ideas of mathematical modelling as applied in Mechanics.

2. Vectors in Mechanics.
   - Magnitude and direction of a vector. Resultant of vectors may also be required.
   - Application of vectors to displacements, velocities, accelerations and forces in a plane.

3. Kinematics of a particle moving in a straight line**.**
   - Motion in a straight line with constant acceleration.

4. Dynamics of a particle moving in a straight line or plane**.**
   - The concept of a force. Newton's laws of motion.

- Simple applications including the motion of two connected particles.
- Momentum and impulse. The impulse-momentum principle. The principle of conservation of momentum applied to two particles colliding directly.
- Coefficient of friction.

5. Statics of a particle.
    - Forces treated as vectors. Resolution of forces.
    - Equilibrium of a particle under coplanar forces. Weight, normal reaction, tension and thrust, friction.
    - Coefficient of friction.

6. Moments.
    - Moment of a force.

2.7 <u>Unit S1 - Statistics</u>

1. Mathematical models in probability and statistics**.**
    - The basic ideas of mathematical modelling as applied in probability and statistics.

2. Representations and summary of data.
    - Histograms, stem and leaf diagrams, box plots.
    - Measures of location –mean, median, mode.
    - Meaures of dispersion –variance, standard deviation, range and interpercentile ranges.
    - Skewness. Concepts of outliers.

3. Probability.
    - Elementary probability. Sample space. Exclusive and complementary events. Conditional probability.
    - Independence of two events.
    - Sum and product laws.

4. Correlation and regression.
    - Scatter diagrams. Linear regression.
    - Explanatory (independent) and response (dependent) variables. Applications and interpretations.
    - The product moment correlation coefficient, its use, interpretation and limitations.

5. Discrete random variables.
    - The concept of a discrete random variable.
    - The probability function and the cumulative distribution function for a discrete random variable.
    - Mean and variance of a discrete random variable.
    - The discrete uniform distribution.

6. The Normal distribution.
- The Normal distribution including the mean, variance and use of tables of the cumulative distribution function.

2.8 Unit C3 – Core mathematics.
Preamble. Methods of proof, including proof by contradiction and disproof by counter-example, are required. At least one question on the paper will require the use of proof.

Specification.

1. Algebra and functions.
   - Simplification of rational expressions including factorising and cancelling, and algebraic division.
   - Definition of a function.
   - Domain and range of functions.
   - Composition of functions.
   - Inverse functions and their graphs.
   - The modulus function.
   - Combinations of the transformations y=f(x) as represented by y=*a*f(x), y=f(x)+*a,* y=f(x+*a*), y=f(*a*x).

2. Trigonometry.
   - Knowledge of secant, cosecant and cotangent and of arcsin,arcos and arctan.
   - Their relationships to sine, cosine and tangent.
   - Understanding of their graphs and appropriate restricted domains.
   - Knowledge and use of $\sec^2\theta=1+\tan^2\theta$ and $\text{cosec}^2\theta=1+\cot^2\theta$.
   - Knowledge and use of the double angle formulae; use of formulae for sin (A ± B), cos(A±B) and tan(A±B) and of expressions for $a\cos\theta+b\sin\theta$ in the equivalent forms of $r\cos(\theta\pm\alpha)$ or $r\sin(\theta\pm\alpha)$.

3. Exponentials and logarithms**.**
   - The function e^x and its graph.
   - The function In x and its graph;
   - In x as the inverse function of e^x.

4. Differentiation.
   - Differentiation of e^x, In x, sin x, cos x, tan x and their sums and differences.
   - Differentiation using the product rule, the quotient rule and the chain rule.
   - The use of $\frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right)$

5. Numerical Methods.
   - Location of roots of f(x)=0 by considering changes of sign of f(x) in the interval of x in which f(x) is continuous.
   - Approximate solution of equations using simple iterative methods.

2.9 <u>Unit C4 – Core mathematics</u>
1. Algebra and Functions.
   - Rational functions. Partial fractions (denominators not more complicated than repeated linear terms).

2. Coordinate geometry in the (x,y) plane.

- Parametric equations of curves and conversion between Cartesian and parametric forms.

3. Sequences and Series.
   - Binomial series for any rational *n*.

4. Differentiation.
   - Differentiation of simple functions defined implicitly or parametrically.
   - Exponential growth and decay.
   - Formation of simple differential eequations.

5. Integration.
   - Integration of ex, $^1/_x$, sin x, cos x. tan x.
   - Evaluation of volume of revolution.
   - Simple cases of integration by substitution and integration by parts. These methods as the reverse processes of the chain and product rules respectively.
   - Simple cases of integration using partial fractions.
   - Analytical solution of simple first order differential equations with separable variables.
   - Numerical integration of functions.

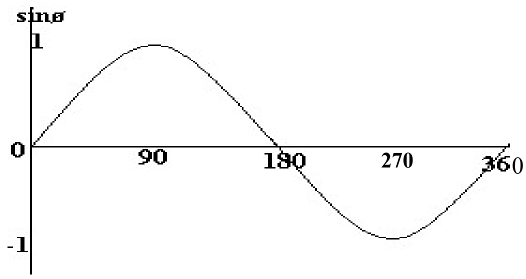
6. Vectors.
   - Vectors in two and three dimensions.
   - Magnitude of a vector.
   - Algebraic operations of vector addition and multiplication by scalars, and their geometrical interpretations.
   - Position vectors. The distance between two points.
   - Vector equations of lines.
   - The scalar product. Its use for calculating the angle between two lines.

<u>**Questionnaire for year 12 students at the start of the trigonometry component of the AS course.**</u>

Ability to Handle Pre- Requisite Skills

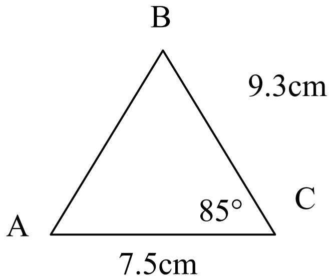
1. The following is a graph of Y =Sin x for 0<x<360. Using this graph can you find the answers to:

a) Sin 270
b) Sin 520
c) Sin -45



2. **Find all the solutions to $\tan^{-1}$ 1.75 in the ranges -180<x<180**

3. **Find length AB and hence determine angle ABC**



Interpretations of the Representations

4.Sin 30°
5. Sin 120°
6. Tan 90°

7. What does identity mean?

Interpretation of Spatial Images.

8. What comes to your mind at the mention of Sin x?

9. What is a Graph transformation?

10. What is an Asymptote?

11**. What is the Unit circle (Sometimes called the CAST diagram**).

Investigation of ideas of function

12. Cos^{-1} 0.5

13. Sin^{-1} 2.5

Functions Links

14. **What comes to mind at the word 'Function'?**

15. **What does Trigonometric function mean?**
16. **What do Radians mean?**
17. **What does dy/dx mean?**
18. **What would d/dx [sin x] mean?**

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