REVIEW OF COMPLEX ANALYSIS

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We discuss here some basic results in complex analysis concerning series and products (Section 1) as well as logarithms of analytic functions and the Gamma function (Section 2). It is assumed that the reader has already had a first course in complex analysis, so is familiar with terms like analytic, meromorphic, pole, and residue.

1. Infinite Series and Products

Given a sequence $\{z_1, z_2, \dots\}$ in $\mathbf{C}$, the series $\sum z_n = \sum_{n\geq 1} z_n$ is defined to be the limit of the partial sums $\sum_{n=1}^{N} z_n$, as $N \to \infty$. For a permutation π of the positive integers, we can consider the rearranged series $\sum z_{\pi(n)}$. Is there a difference between the series $\sum z_n$ and $\sum z_{\pi(n)}$ when both converge? Perhaps.

Consider the series

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2 = .693147\dots.$$

Write the nth term as z_n, so $z_n = (-1)^{n-1}/n$. Let's rearrange the terms, computing the sum in the following order:

$$z_1 + z_2 + z_4 + z_3 + z_6 + z_8 + z_5 + z_{10} + z_{12} + z_7 + \dots$$

So we place two even-indexed terms after an odd-indexed term. This sum looks like

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} + \cdots \leq .452$$

The general term tends to 0 and from the sign changes the sum converges to a value less than $\log 2$. Combining z_1 and z_2, z_3 and z_6, z_5 and z_{10}, etc., this sum equals

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{7} + \cdots = \frac{1}{2}\log 2 = .346573\dots.$$

We've rearranged the terms from a series for $\log 2$ and obtained half the original value.

Until the early 19th century, the evaluation of infinite series was not troubled by rearrangement issues as above, since there wasn't a clear distinction between two issues: defining convergence of a series and computing a series. An infinite series needs a precise defining algorithm, such as taking a limit of partial sums via an enumeration of the addends, upon which other summation methods may or *may not* be comparable.

This aspect of infinite series is at least historically tied up with zeta and L-functions, because it was resolved by Dirichlet in his (first) paper on infinitude of primes in arithmetic progressions, where he introduced the notion of absolutely convergent series to tame the rearrangement problem. Recall that a series $\sum z_n$ of complex numbers is called *absolutely convergent* if the series $\sum |z_n|$ of absolute values of the terms converges. This condition is usually first seen in a calculus course, as a "convergence test." Indeed, the partial sum differences satisfy

$$\left|\sum_{n=M}^{N} z_n\right| \leq \sum_{n=M}^{N} |z_n|$$

and the right side tends to 0 as $M, N \to \infty$ if $\sum |z_n|$ converges, so the left side tends to 0 and therefore $\sum z_n$ converges.

Absolutely convergent series are ubiquitous. For example, a power series $\sum c_n z^n$ centered at the origin that converges at $z_0 \neq 0$ converges absolutely at every z with $|z| < |z_0|$. (Proof: Let $r = |z/z_0| < 1$ and $|c_n z_0^n| \leq B$ for some bound B. Then $\sum |c_n z^n|$ is bounded by the geometric series $\sum Br^n < \infty$.) An analogous result applies to series centered at points other than the origin.

The behavior of absolutely convergent series is related to convergent series of nonnegative real numbers, and such series have very convenient properties, outlined in the following lemmas.

Lemma 1.1. *Let $\{a_n\}$ be a sequence of nonnegative real numbers. If the partial sums $\sum_{n=1}^{N} a_n$ are bounded, then the series $\sum_{n\geq 1} a_n$ converges. Otherwise it diverges to ∞.*

Proof. The partial sums are an increasing sequence (perhaps not strictly increasing, since some a_n may equal 0), so if they have an upper bound they converge, and if there is no upper bound they diverge to ∞. $\square$

Lemma 1.2 (Generalized Commutativity)**.** *Let $a_n \geq 0$ and assume the series $\sum_{n\geq 1} a_n$ converges, say to S. For every permutation π of the index set, the series $\sum a_{\pi(n)}$ also converges to S.*

Proof. Choose $\varepsilon > 0$. For all large N, say $N \geq M$ (where M depends on ε),

$$S - \varepsilon \leq \sum_{n=1}^{N} a_n \leq S + \varepsilon.$$

The permutation π takes on all values $1, 2, \ldots, M$ among some initial segment of the positive integers, say

$$\{1, 2, \ldots, M\} \subset \{\pi(1), \pi(2), \ldots, \pi(K)\}$$

for some K. For $N \geq K$, the set $\{a_{\pi(1)}, \ldots, a_{\pi(N)}\}$ contains $\{a_1, \ldots, a_M\}$. Let J be the maximal value of $\pi(n)$ for $n \leq N$. So for $N \geq K$,

$$S - \varepsilon \leq a_1 + a_2 + \cdots + a_M \leq \sum_{n=1}^{N} a_{\pi(n)} \leq a_1 + a_2 + \cdots + a_J \leq S + \varepsilon.$$

So for every ε, $\sum_{n=1}^{N} a_{\pi(n)}$ is within ε of S for all large N. Therefore $\sum a_{\pi(n)} = S$. $\square$

Because of Lemma 1.2, we can associate to a sequence $\{a_i\}$ of nonnegative real numbers indexed by a countable index set I the series $\sum_{i\in I} a_i$, by which we mean the limit of partial sums for any enumeration of the terms. If it converges in one enumeration it converges in all others, to the same value.

We apply the idea of a series running over a general countable index set right away in the next lemma.

Lemma 1.3 (Generalized Associativity)**.** *Let $\{a_i\}$ be a sequence of nonnegative real numbers with countable index set I. Let*

$$I = I_1 \cup I_2 \cup I_3 \cup \cdots$$

be a partition of the index set. If the series $\sum_{i \in I} a_i$ converges, then so does each series

$$s_j = \sum_{i \in I_j} a_i,$$

and

$$\sum_{i \in I} a_i = \sum_{j \geq 1} s_j = \sum_{j \geq 1} \left(\sum_{i \in I_j} a_i \right).$$

Conversely, if each s_j converges and the series $\sum_{j \geq 1} s_j$ converges, then the series $\sum_{i \in I} a_i$ converges to $\sum_{j \geq 1} s_j$.

Proof. Exercise. □

The importance of absolutely convergent series is that they satisfy the above convenient properties of series of nonnegative numbers.

Theorem 1.4. *Let a_i be a sequence of complex numbers. Assume $\sum |a_i|$ converges, i.e., $\sum a_i$ is absolutely convergent. Then Lemmas 1.2 and 1.3 apply to $\sum a_i$.*

Proof. Exercise. □

The definition of an absolutely convergent series of complex numbers makes sense with any countable indexing set, not only index set $\mathbf{Z}^+$. Allowing index sets other than $\mathbf{Z}^+$ is technically convenient in number theory. One might want to sum over numbers indexed by the ideals in a ring, for instance.

Theorem 1.4 justifies interchanging the order of a double summation $\sum_m \sum_n a_{mn}$, if it is absolutely convergent, *i.e.*, if $\sum_m \sum_n |a_{mn}|$ converges. The theorem is applied to zeta and L-functions to justify the rearrangements of certain double sums $\sum_p \sum_{k \geq 1} c_{p^k}$ over primes p and positive integers k into a single sum $\sum c_{p^k}$ over prime powers p^k in their usual linear ordering: $\sum c_{p^k} := \lim_{x \to \infty} \sum_{p^k \leq x} c_{p^k}$. (Note: Since $k \geq 1$ here, there is no term corresponding to the prime power 1.)

Series that are not absolutely convergent are nevertheless important. We just need to be careful in analytic manipulations with them. Examples of such series are values of a power series on the boundary of the disc of convergence. The following important theorem shows these boundary values, if they exist, are linked by continuity to the values inside the disc of convergence.

Theorem 1.5 (Abel, 1826)**.** *If $f(z) = \sum c_n z^n$ converges at the point z_0, then $f(z_0)$ is the limit of $f(z)$ as $z \to z_0$ along a radial path from the origin.*

In particular, if $\sum c_n$ converges, then

$$\lim_{x \to 1^-} \sum_{n \geq 0} c_n x^n = \sum_{n \geq 0} c_n.$$

Proof. The case of a series at z_0 is easily reduced to the case $z_0 = 1$ by a scaling and a rotation. So we now assume $z_0 = 1$.

Since $\sum c_n z^n$ converges at $z = 1$, the series converges on the open unit disc. Let $b_n = c_0 + \cdots + c_n$, $b = \lim_{n\to\infty} b_n$, $0 < x < 1$. Then (by partial summation)

$$\sum_{n=0}^{N} c_n x^n = \sum_{n=0}^{N} b_n x^n - x \sum_{n=0}^{N-1} b_n x^n = (1-x)\sum_{n=0}^{N-1} b_n x^n + b_N x^N.$$

Let $N \to \infty$. Since $b_N \to b$ and $x^N \to 0$, we get

$$\sum_{n\geq 0} c_n x^n = (1-x)\sum_{n\geq 0} b_n x^n.$$

Since $\sum c_n x^n - b = (1-x)\sum (b_n - b)x^n$, we choose $\varepsilon > 0$ and then M so that $|b_n - b| \leq \varepsilon$ for $n > M$. Then

$$\left|\sum_{n\geq 0} c_n x^n - b\right| \leq (1-x)\sum_{n=0}^{M} |b_n - b| x^n + \varepsilon \leq (1-x)\sum_{n=0}^{M} |b_n - b| + \varepsilon.$$

For $|x-1|$ small enough, the first term on the right side can be made $\leq \varepsilon$. □

The converse of Abel's theorem is not true without extra conditions, *e.g.*,$\sum_{n\geq 0}(-1)^n x^n = 1/(1+x)$ has a limit as $x \to 1^-$ but the series itself doesn't converge at $x = 1$.

The following convergence theorem was used by Dirichlet in his work on analytic number theory in lieu of a standard convergence theorem for Dirichlet series that was not available until much later.

Theorem 1.6 (Dirichlet's test)**.** *If the partial sums $\sum_{n\leq N} w_n$ are bounded and $c_1 \geq c_2 \geq \cdots \geq 0$ with $c_n \to 0$, then $\sum c_n w_n$ converges.*

Proof. Let $S_N = \sum_{n=1}^{N} c_n w_n$. We want to show $\{S_N\}$ is a Cauchy sequence.

Let $T_N = \sum_{n=1}^{N} w_n$, so for some $B > 0$, $|T_N| \leq B$ for all N. For $M < N$, by partial summation

$$\begin{aligned} S_N - S_M &= \sum_{n=M+1}^{N} c_n w_n \\ &= \sum_{n=M+1}^{N} c_n (T_n - T_{n-1}) \\ &= c_N T_N - c_{M-1} T_{M-1} - \sum_{n=M}^{N} T_{n-1}(c_n - c_{n-1}), \end{aligned}$$

and the absolute value of the final expression is at most

$$Bc_N + Bc_{M-1} + \sum_{n=M}^{N} B(c_n - c_{n-1}) = 2Bc_N,$$

which $\to 0$ as $M \to \infty$. □

Setting $w_n = (-1)^n$, this theorem is the alternating series test. Setting $w_n = z^n$ where $|z| = 1$, we see that a power series whose coefficients tend monotonically to 0 converges on the unit circle except possibly at $z = 1$. Setting $c_n = 1/\log n$, we see that if the partial sums $\sum_{n\leq x} w_n$ are bounded then $\sum w_n/\log n$ converges (omit the $n = 1$ term). The converse is not true.

We will want to work not only with series, but with infinite products. We generally work with infinite products only in cases where techniques related to absolutely convergent series play a role.

Theorem 1.7. *Let $\sum a_m$, $\sum b_n$ be two absolutely convergent series. The product of the sums of these two series is the sum of the absolutely convergent series $\sum a_m b_n$ over the index set $\mathbf{Z}^+ \times \mathbf{Z}^+$. More generally, a finite product of sums of absolutely convergent series is again the sum of an absolutely convergent series, whose terms are all the possible products of terms taken one from each of the original series.*

Proof. It suffices to treat the case of a product of two series.

Let $S = \sum a_m$, $T = \sum b_n$. Then

$$ST = \sum_n Sb_n = \sum_n \left(\sum_m a_m b_n \right).$$

The terms $a_m b_n$ give an absolutely convergent series if we think of them as being indexed by the countable index set $I = \mathbf{Z}^+ \times \mathbf{Z}^+$. Partitioning this index set into its rows or its columns and using generalized associativity equates the double series $\sum_n (\sum_m a_m b_n)$ and $\sum_m (\sum_n a_m b_n)$ with $\sum_{(m,n)\in \mathbf{Z}^+ \times \mathbf{Z}^+} a_m b_n$. □

The following theorem contains most of what we shall need concerning relations between infinite products and infinite series. Although we will review complex logarithms in the following section, for now we take for granted the basic property that the series $\log(1-z) := \sum_{m\geq 1} z^m/m$ for $|z| < 1$ is a right inverse to the exponential function: $\exp(\log(1-z)) = 1-z$ if $|z| < 1$.

Theorem 1.8. *Let $\{z_n\}$ be complex numbers with $|z_n| \leq 1 - \varepsilon$ for some positive ε (which is independent of n) and $\sum |z_n|$ convergent.*

a) The infinite product $\prod_{n\geq 1} \frac{1}{1-z_n} = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{1}{1-z_n}$ converges to a nonzero number.

b) This infinite product satisfies generalized commutativity (i.e., every rearrangement of factors gives the same product) and generalized associativity for products.

c) The product $\prod_{n\geq 1} \frac{1}{1-z_n} = \prod_{n\geq 1}(1 + z_n + z_n^2 + \dots)$ has a series expansion by collecting terms in the expected manner:

$$\prod_{n\geq 1} \frac{1}{1-z_n} = 1 + \sum_{r\geq 1} \sum_{\substack{k_1,\dots,k_r \geq 1 \\ 1 < i_1 < \cdots < i_r}} z_{i_1}^{k_1} \cdots z_{i_r}^{k_r},$$

and this series is absolutely convergent.

Proof. We consider the naive logarithm of the infinite product, which is, by definition,

$$-\sum_{n\geq 1} \log(1-z_n) := \sum_{n\geq 1} \sum_{m\geq 1} \frac{z_n^m}{m}.$$

(This equation is a definition of the left side; there is no claim that $\log(zw) = \log z + \log w$.)

Since

$$\sum_{n\geq 1} \sum_{m\geq 1} \frac{|z_n|^m}{m} \leq \sum_{n\geq 1} \sum_{m\geq 1} |z_n|^m = \sum_{n\geq 1} \frac{|z_n|}{1-|z_n|} \leq \frac{1}{\varepsilon} \sum_{n\geq 1} |z_n| < \infty,$$

the doubly indexed sequence $\{z_n^m/m\}$ is absolutely convergent, so it satisfies generalized commutativity and associativity for series. By continuity of the exponential,

$$\exp\left(\sum_{n\geq 1}\sum_{m\geq 1}\frac{z_n^m}{m}\right) = \prod_{n\geq 1}\exp\left(\sum_{m\geq 1}\frac{z_n^m}{m}\right) = \prod_{n\geq 1}\frac{1}{1-z_n},$$

so the product converges to a nonzero number (since it's a value of the exponential) and this product satisfies generalized commutativity and associativity since the double sum in the exponential does. We have taken care of a) and b).

For c), by absolute convergence of each $\sum_{j\geq 1} z_n^j$ the product $P_N := \prod_{n=1}^N \sum_{j\geq 0} z_n^j$ can be written as

$$\sum_{j_1,\ldots,j_N\geq 0} z_1^{j_1}\cdots z_N^{j_N} = 1 + \sum_{r=1}^N \sum_{\substack{k_1,\ldots,k_r\geq 1\\ 1\leq i_1<\cdots<i_r\leq N}} z_{i_1}^{k_1}\cdots z_{i_r}^{k_r}.$$

This equation follows from generalized associativity. Now let $N\to\infty$ to get convergence of the series in part c).

Replace each z_n with its absolute value $|z_n|$:

$$\begin{aligned}\prod_{n\geq 1}\frac{1}{1-|z_n|} &\geq \prod_{n=1}^N \frac{1}{1-|z_n|}\\ &= 1+\sum_{r=1}^N \sum_{\substack{k_1,\ldots,k_r\geq 1\\ 1\leq i_1<\cdots<i_r\leq N}} |z_{i_1}|^{k_1}\cdots|z_{i_r}|^{k_r}.\end{aligned}$$

Let $N\to\infty$ to see the series formed by $\{z_{i_1}^{k_1}\cdots z_{i_r}^{k_r}\}$ is absolutely convergent. □

2. Integration, Logarithms, and the Gamma function

In thie section we will review some facts about analytic functions and analytic singularities, and then focus specifically on logarithms and the Gamma function. The two most important theorems are: a limit of analytic functions converging uniformly on compact subsets is analytic (Theorem 2.3) and a nonvanishing analytic function on a simply connected domain has a logarithm (Theorem 2.16).

When we integrate along a contour γ, we assume γ is "nice," say a union of piecewise differentiable curves. If the endpoints of γ coincide, we call γ a loop, and if γ does not cross itself (except perhaps at the endpoints) we call γ simple.

Let Ω be an open set in $\mathbf{C}$ and $f\colon\Omega\to\mathbf{C}$ be a continuous function. The first miracle of complex analysis is that the property of f being analytic (we will also use the equivalent term holomorphic) can be described in several different ways: complex differentiability of f at each point in Ω, local power series expansions for f at each point in Ω, or that $\int_\gamma f(z)\,dz = 0$ for any (or any sufficiently small) contractible loop γ in Ω. That the real and imaginary parts of f satisfy the Cauchy–Riemann equations is another formulation of analyticity, important for links between complex analysis and partial differential equations. Notice in particular that the notion of analyticity is a *local* one, such as in the characterization by local power series expansions. Through the Cauchy integral formula these local conditions lead to global consequences, like the following.

Theorem 2.1. *Let f be an analytic function on $D(a,r)$, the open disc around a point a with radius r. Then the power series for f at the center a converges on $D(a,r)$.*

Proof. Let $r' < r$, and let $\gamma_{r'}(t) = a + r'e^{it}$ for $t \in [0, 2\pi]$ be the circular path around a of radius r', traversed once counterclockwise. For $z \in D(a, r')$,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(w)}{w-z}\, dw.$$

For w lying on $\gamma_{r'}$ (*i.e.*, $|w-a| = r'$) we have $|z-a| < |w-a|$ and so

$$\frac{1}{w-z} = \frac{1}{w-a}\cdot\frac{1}{1-(z-a)/(w-a)} = \sum_{n\geq 0}\frac{(z-a)^n}{(w-a)^{n+1}},$$

with the series converging uniformly in w. Multiplying both sides by $f(w)$ and integrating along $\gamma_{r'}$ we may interchange the sum and integral (by uniform convergence) and get

$$f(z) = \sum_{n\geq 0} c_n (z-a)^n, \tag{2.1}$$

where

$$c_n = \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(w)}{(w-a)^{n+1}}\, dw.$$

So (2.1) is a power series expansion for f around a that converges on the open disc $D(a, r')$. Thus $c_n = f^{(n)}(a)/n!$, so it is independent of r'. Now let $r' \to r^-$. The series (2.1) doesn't change, so it applies for all $z \in D(a,r)$. □

The analogue of this theorem in real analysis is *false*: $1/(1+x^2)$ as a real-valued function has local power series expansions at all points of $\mathbf{R}$ but its power series at the origin does not have an infinite radius of convergence. (As a complex-valued function, $1/(1+z^2)$ has a pole at $z = \pm i$, so it is not analytic on $\mathbf{C}$.)

Corollary 2.2. *Let Ω be an open set in the plane and f be an analytic function on Ω. If a disc D with radius r is in Ω, the power series of f at the center of D has radius of convergence at least r.*

Proof. Clear. □

Theorem 2.3. *Let Ω be open in the plane, f_n a sequence of analytic functions on Ω that converges uniformly to f on each compact subset of Ω. Then f is analytic and f_n' converges uniformly to f' on each compact subset.*

Proof. Choose $a \in \Omega$. Let $\overline{D} \subset \Omega$ be a closed disc of radius $R > 0$ containing a in its interior. So

$$\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-a}\, dz = f_n(a),$$

where γ traverses the boundary of $\overline{D}$ once counterclockwise. Since $f_n \to f$ uniformly on $\overline{D}$, f is continuous on $\overline{D}$, so $f(z)/(z-a)$ is integrable along γ. Letting the maximum value of a function g on $\overline{D}$ be written $||g||_{\overline{D}}$,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-a}\, dz - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\, dz\right| &\leq \frac{1}{2\pi}||f_n - f||_{\overline{D}}\left|\int_\gamma \frac{dz}{z-a}\right| \\ &= ||f_n - f||_{\overline{D}} \\ &\to 0.\end{aligned}$$

So $\frac{1}{2\pi i}\int_\gamma (f(z)/(z-a))\,dz = \lim_{n\to\infty} f_n(a) = f(a)$. Since a was arbitrary, f is analytic. To show $f_n' \to f'$ uniformly on compact subsets of Ω, it suffices to work with closed discs. Let $\overline{D}$ be a closed disc in Ω with radius $R > 0$. Choose a in the interior of $\overline{D}$. Then

$$f_n'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{(z-a)^2}\,dz, \quad f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\,dz.$$

So $|f_n'(a) - f'(a)| \le ||f_n - f||_{\overline{D}}/R \to 0$. □

While all analytic functions are (locally) expressible as series, some analytic functions are convenient to introduce in other ways, such as by an integral depending on a parameter. Here is a basic theorem in this direction.

Theorem 2.4. *Let $f(x,s)$ be a continuous function, where the real variable x runs over an interval from a to b and s varies over an open set U in the plane. Suppose, for each x, that $f(x,s)$ is analytic in s. Then $F(s) := \int_a^b f(x,s)\,dx$ is analytic in s and F may be differentiated under the integral sign: $F'(s) = \int_a^b (\partial_2 f)(x,s)\,dx$.*

Proof. See [1, Chap. XV, Lemma 1.1]. □

Theorem 2.5. *Let $f\colon \Omega \to \mathbf{C}$ be analytic on a region Ω, with $\overline{\Omega} = \{\overline{s} : s \in \Omega\}$ the conjugate region. Then $f^*(s) := \overline{f(\overline{s})}$ is analytic on $\overline{\Omega}$, with derivative $\overline{f'(\overline{s})}$.*

In practice this will be applied to self-conjugate regions, such as right half-planes.

Proof. Using local series expansions, the operation $f \mapsto f^*$ transforms a power series $\sum c_n(s-a)^n$ into $\sum \overline{c}_n(s-\overline{a})^n$. So analyticity of f^* and the formula for its derivative are obvious. □

Although analyticity is defined as a property of functions on open sets, it is convenient to have the notion available for functions on any set (especially closed sets). A function defined on any set A in the plane is called analytic if it has a local power series expansion around each point of A, or equivalently if it is the restriction to A of an analytic function defined on an open set containing A.

For example, an analytic function at a point is simply an analytic function on some open ball around the point. Since any open set containing a closed ball $B := \{z : |z| \le r\}$ will contain some open ball $\{z : |z| < r + \varepsilon\}$ (this is because B is compact), an analytic function on a closed ball is the restriction of an analytic function on some larger open ball (not merely larger open set). In contrast, half-planes are not compact, so an open set Ω containing a closed half-plane $H := \{z : \mathrm{Re}(z) \ge \sigma_0\}$ does *not* have to contain any open half-plane $\{z : \mathrm{Re}(z) > \sigma_0 - \varepsilon\}$. Indeed, the complement of H in Ω could become arbitrarily thin as we move far away from the real axis. So a function that is analytic on a closed half-plane is not guaranteed to be the restriction of an analytic function on a larger open half-plane. This causes quite a few technical difficulties with zeta and L-functions,

The reader should already be familiar with the two standard types of singularities for an analytic function f: poles and essential singularities. These concepts apply to isolated points, namely points a such that f is analytic on a punctured disc $0 < |z-a| < r$. Whether a is a pole or an essential singularity of f can be characterized by either the behavior of $|f(z)|$ as $z \to a$ or by the Laurent series expansion for f at a. In both cases $|f|$ is necessarily unbounded near a, by the following result of Riemann.

Theorem 2.6 (Riemann's Removable Singularity Theorem)**.** *Let f be holomorphic on an open set around the point a, except possibly at a. If f is bounded near a, then f extends to an analytic function at a.*

The converse of the theorem is trivial.

Proof. If $f(z)$ is bounded near a, then the function

$$g(z) = \begin{cases} (z-a)^2 f(z), & \text{if } z \neq a, \\ 0, & \text{if } z = a, \end{cases}$$

is certainly analytic in a punctured disc around a and is continuous at a, with $g(a) = 0$. It is also differentiable at a: for $z \neq a$,

$$\frac{g(z) - g(a)}{z - a} = \frac{g(z)}{z - a} = (z-a)f(z) \to 0$$

as $z \to a$. Therefore g is analytic in a neighborhood of a with $g(a) = 0$ and $g'(a) = 0$. The power series expansion of g around a therefore begins

$$g(z) = c_2(z-a)^2 + c_3(z-a)^3 + c_4(z-a)^4 + \dots,$$

which shows $f(z)$ extends to an analytic function at a, with power series $c_2 + c_3(z-a) + c_4(z-a)^2 + \dots$. □

It is crucial that we assume analyticity on a punctured neighborhood of a in the theorem. If a is not isolated in this way, the theorem is false. Of course, to make sense of this we need to have a notion of singularity that does not apply only to isolated points.

Definition 2.7. Let $f\colon \Omega \to \mathbf{C}$ be analytic on an open set Ω, $a \in \partial\Omega$ a point on the boundary of Ω. We call a an *analytic singularity* of f if there is no extension of f to an analytic function at a.

In other words, a is an analytic singularity of f if there is no power series centered at a that on the overlap of its disc of convergence and Ω coincides with f. (That is, f does not admit an analytic continuation to a neighborhood of a.) This is singular behavior from the viewpoint of complex analysis, but it does *not* mean f has to behave pathologically near a from the viewpoint of topology or real analysis. For instance, there are analytic functions on the open unit disc that extend continuously (even smoothly in the sense of infinite real-differentiability) to the unit circle, but at none of these boundary points is there a local power series expansion in a complex variable for the extended function. See [2, pp. 252–253] for an example.

Next we discuss some special functions: complex exponentials, logarithms, square roots, and the Gamma function.

The complex exponential is defined as $e^s := \sum_{n \geq 0} s^n/n!$ for all $s \in \mathbf{C}$.

Definition 2.8. For $u > 0$ and $s \in \mathbf{C}$, $u^s := e^{s \log u}$, where $\log u$ is the usual real logarithm of u.

Note that $u^{it} = e^{it \log u} = \cos(t \log u) + i \sin(t \log u)$, so $|u^{it}| = 1$. Clearly $|u^s| = u^{\mathrm{Re}(s)}$, so $u^s = 1 \Leftrightarrow s \in (2\pi i/\log u)\mathbf{Z}$. If $\mathrm{Re}(s) \geq 0$, then $\lim_{u \to 0^+} u^s = 0$.

Since the exponential function is not injective on $\mathbf{C}$, some care is needed to define logarithms. The usual remedy in a first course in complex analysis is to consider the *slit plane*

$$\{z \in \mathbf{C} : z \notin (-\infty, 0]\}$$

obtained by removing the negative real axis and 0. We can uniquely write each element of this slit plane in the form $z = re^{i\theta}$ with $-\pi < \theta < \pi$ and we define

$$\operatorname{Log} z := \log r + i\theta. \tag{2.2}$$

Here $\log r$ is the usual real logarithm. This function $\operatorname{Log} z$ is called the *principal value logarithm*, and specializes to the usual logarithm on the positive reals. The justification for calling $\operatorname{Log} z$ a logarithm is that it is a right inverse to the exponential function:

$$e^{\operatorname{Log} z} = e^{\log r + i\theta} = re^{i\theta} = z.$$

Notice that $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$ in general. For instance, if $z_1 = z_2 = i$, then $\operatorname{Log}(z_1 z_2)$ is not even defined. If $z_1 = z_2 = e^{3\pi i/4}$, then

$$z_1 z_2 = e^{3\pi i/2} = -i = e^{-\pi i/2},$$

so $\operatorname{Log}(z_1 z_2) = -\pi i/2 \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2) = 3\pi i/2$. However, on the slit plane we do have $\overline{\operatorname{Log}(z)} = \operatorname{Log}(\overline{z})$, $\operatorname{Log}(1/z) = -\operatorname{Log}(z)$, and $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$ if z_1, z_2 have positive real part (but perhaps $\operatorname{Log}(z_1 z_2 z_3) \neq \sum \operatorname{Log}(z_i)$ even if all z_i have positive real part).

We will need to use not only logarithms of numbers, but logarithms of analytic functions (for instance, logarithms of Dirichlet L-functions).

Definition 2.9. Let Ω be an open set in the complex plane and let f be an analytic function on Ω. A *logarithm* of f is an analytic function g on Ω whose exponential is f, *i.e.*, g satisfies $e^{g(z)} = f(z)$ for all $z \in \Omega$.

We could write $g = \log f$, but due to the multi-valued nature of logarithms such notation should be used with care. (The ambiguity is in the imaginary part, so the notation $\operatorname{Re} \log f$ is well-defined, being just $\log |f|$.)

Example 2.10. The principal value logarithm $\operatorname{Log} z$ defined in (2.2) is a logarithm of the identity function z on the slit plane.

Example 2.11. For an angle α, omit the ray $\{re^{i\alpha} : r \geq 0\}$ from the plane. Write each number not on this ray uniquely in the form $re^{i\theta}$ where $\alpha < \theta < \alpha + 2\pi$, and define

$$\log z := \log r + i\theta,$$

where $\log r$ is the usual real logarithm. This logarithm differs from the principal value logarithm by an integral multiple of 2π on their common domain of definition. We will call any logarithm of this type, on a plane slit by a ray from the origin, a "slit logarithm."

Example 2.12. If g is a logarithm of f, so is $g + 2\pi i k$ for each $k \in \mathbf{Z}$. There is never a unique logarithm of an analytic function.

Example 2.13. Let g_1 and g_2 be logarithms of the analytic functions f_1 and f_2. Then $g_1 + g_2$ is a logarithm of $f_1 f_2$. Indeed, $g_1 + g_2$ is analytic, and $e^{g_1(z)+g_2(z)} = e^{g_1(z)} e^{g_2(z)} = f_1(z) f_2(z)$.

Theorem 2.14. *Let $f \colon \Omega \to \mathbf{C}$ be analytic on the open set Ω. If g is a logarithm of f, then $g' = f'/f$.*

Proof. Informally, we can write $g = \log f$ and apply the chain rule.

More rigorously, differentiate the equation $f = e^g$ to get $f' = e^g g' = fg'$, so $g' = f'/f$. $\square$

The expression f'/f is called the logarithmic derivative of f, but note that one does *not* need f to have a logarithm in order to construct this ratio. Any meromorphic function has a logarithmic derivative. Writing the product rule as

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

shows that the logarithmic derivative of a product is the sum of the logarithmic derivatives.

The construction of logarithmic derivatives will be useful to us, because of the following calculation:

$$f(s) = c_m(s-a)^m + c_{m+1}(s-a)^{m+1} + \cdots \Longrightarrow \frac{f'(s)}{f(s)} = \frac{m}{s-a} + \ldots, \tag{2.3}$$

where $c_m \neq 0$, so the logarithmic derivative of f is holomorphic except at zeros and poles of f, where it has a simple pole with residue equal to the order of vanishing:

$$\mathrm{Res}_{s=a} \frac{f'(s)}{f(s)} = \mathrm{ord}_{s=a} f(s).$$

This suggests an analytic way of proving f is holomorphic at a and $f(a) \neq 0$: show $\mathrm{Res}_{s=a}(f'/f) = 0$. A meromorphic function on an open set is both holomorphic and nonvanishing if and only if its logarithmic derivative is holomorphic. In particular, a holomorphic function on an open set is nonvanishing if and only if its logarithmic derivative is holomorphic.

Because the multiplicity of a zero or pole is a residue of the logarithmic derivative, we can count zeros and poles inside a region by integration.

Theorem 2.15 (Argument Principle)**.** *Let f be meromorphic on a simple loop γ and its interior, and be holomorphic and nonvanishing on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(s)}{f(s)}\,ds = N - P,$$

where N is the number of zeros of $f(s)$ inside of γ and P is the number of poles. Each zero and pole is counted with its multiplicity.

In practice, γ will be a circle or rectangle, so we don't have to deal with subtle definitions of what the "inside" of γ means.

Proof. Apply the residue theorem. □

Since the integral is a real number, we only have to compute the imaginary part of the integral:

$$\frac{1}{2\pi}\,\mathrm{Im}\int_\gamma \frac{f'(s)}{f(s)}\,ds = N - P,$$

Theorem 2.16. *Let $f\colon \Omega \to \mathbf{C}$ be an analytic function on an open set $\Omega \subset \mathbf{C}$ that is simply connected. If f vanishes nowhere on Ω, then it has a logarithm.*

Proof. Intuitively, whatever "$\log f$" may turn out to be, its derivative ought to be

$$\frac{f'(z)}{f(z)}.$$

Since f is nonvanishing, this ratio always makes sense. Since we know what the derivative of the putative logarithm of f ought to be, we should obtain the logarithm itself by integrating

f'/f. That's the main point, and the proof amounts to a technical verification that this intuitive idea works.

By passing to a connected component of Ω, we can assume Ω is connected and open.

Pick $z_0 \in \Omega$. For $z \in \Omega$, let γ_z be a continuous piecewise differentiable path from z_0 to z. (Connected open sets in the plane are path-connected, so there is such a path. In practice, when Ω is a disc or half-plane, this is geometrically obvious.) Choose $w_0 \in \mathbf{C}$ such that $e^{w_0} = f(z_0)$. We set

$$L_f(\gamma_z, z) := w_0 + \int_{\gamma_z} \frac{f'(w)}{f(w)}\, dw.$$

This will turn out to be a logarithm for f.

Since Ω is simply connected, this definition does not depend on the path γ_z selected from z_0 to z, so we may write the function as $L_f(z)$. (It *does* depend on z_0, but that will remain fixed throughout the proof.) Note $L_f(z_0) = w_0$.

To show $L_f'(z) = f'(z)/f(z)$, we have for small h

$$\begin{aligned} L_f(z+h) - L_f(z) &= \int_{[z,z+h]} R(z)\, dw + \int_{[z,z+h]} (R(w) - R(z))\, dw \\ &= R(z)h + \int_{[z,z+h]} (R(w) - R(z))\, dw, \end{aligned}$$

where $R(z) = f'(z)/f(z)$ and $[z, z+h]$ denotes the straightline path between z and $z+h$. Now divide by h and let $h \to 0$.

To show L_f is a logarithm of f, *i.e.*, $e^{L_f(z)} = f(z)$, we show the ratio is constant:

$$\frac{d}{dz} e^{-L_f(z)} f(z) = e^{-L_f(z)} f'(z) - f(z) L_f'(z) e^{-L_f(z)} = e^{-L_f(z)} f'(z) - f'(z) e^{-L_f(z)},$$

which is 0. We're on a connected set, so it follows that $f(z) = ce^{L_f(z)}$ for some constant c. We want to show the constant c is 1. It suffices to check the equation at one point in Ω. At z_0 we have $f(z_0) = ce^{L_f(z_0)}$. Is $c = 1$? By our construction, $L_f(z_0) = w_0$ and $e^{w_0} = f(z_0)$, so $c = 1$. □

Example 2.17. Let D be an open disc in the plane that does not contain 0. The function z is nonvanishing on D, so there is a logarithm function $\ell(z)$ on D, *i.e.*, ℓ is analytic on D and $e^{\ell(z)} = z$. A construction of a logarithm is $\ell(z) = w_0 + \int_{r_z} dw/w$, where r_z is the radial path from the center of D to z and w_0 is a constant chosen so e^{w_0} is the center of D.

Example 2.18. Let $\Delta_+ = \{a + bi : a^2 + b^2 < 1, b > 0\}$ be the upper part of the unit disc and $f(z) = z^8$, which is nonvanishing on Δ_+. Since $f'(z)/f(z) = 8/z$, a logarithm of f on Δ_+ is $L_f(z) = w_0 + 8\int_{\gamma_z} dw/w$ where γ_z is a path in Δ_+ from (say) $i/2$ to z and w_0 is chosen so $e^{w_0} = i/2$, *e.g.*, $w_0 = \log(1/2) + i\pi/2$.

Notice that although Δ_+ is simply connected, the image of Δ_+ under f is the punctured unit disc $\{w : 0 < |w| < 1\}$, which has a hole. The principal value logarithm $\operatorname{Log} w$ is not defined on the whole punctured unit disc, so it is *false* that $L_f(z)$ equals the composite $\operatorname{Log}(f(z))$, as the latter is not always defined. It is true by our choice of w_0 that $L_f(z) = 8 \operatorname{Log} z$.

The lesson from Example 2.18 is that a logarithm of the function f is not typically a composite of the slit logarithm and the function f on the whole domain. If we are only concerned with an analytic function locally, then the situation is simpler:

Theorem 2.19. *Let $f\colon \Omega \to \mathbf{C}$ be an analytic function on the open set Ω. If $f(z_0) \neq 0$, then f has a local logarithm near z_0. That is, there is an analytic function g on some neighborhood of z_0 in Ω such that $e^g = f$ on this neighborhood.*

Proof. Since $f(z_0) \neq 0$, select a small disc D around $f(z_0)$ that doesn't contain the origin. There is a corresponding small disc $D_0 \subset \Omega$ around z_0 such that $f(D_0) \subset D$. There is a logarithm function defined on D, since D lies in some plane slit by a ray from the origin and we can easily write down logarithms on such domains. The composite of such a logarithm with f is clearly a logarithm of f, *i.e.*, it is analytic and its exponential is f. □

This proof is simple because we can just compose f and some slit logarithm function. To globalize the result from small discs to general Ω, as in Theorem 2.16, is technical precisely because the ordinary logarithm function is not a well-defined analytic function on $\mathbf{C}^\times$. Still, in almost all situations where we may want to apply Theorem 2.16, Theorem 2.19 will suffice.

Theorem 2.20. *Let $f\colon \Omega \to \mathbf{C}$ be an analytic function on a connected open set. Any two logarithms of f differ by an integral multiple of $2\pi i$.*

The theorem is vacuous if f doesn't admit a logarithm.

Proof. The difference of two logarithms of f is a function whose exponential is identically 1, so the difference takes values in the discrete set $2\pi i\mathbf{Z}$. By continuity of logarithms and connectedness of Ω, the difference is constant. □

In practice, we will only be discussing logarithms of analytic functions defined on regions, *i.e.*, connected open sets.

By Theorem 2.20, a logarithm of $f(z)$ is determined by its value at one point, or by its limit as z tends to ∞ in some fixed direction, if the function settles down to a constant value in that direction.

A consequence of the previous few theorems is that a continuous logarithm of an analytic function must be analytic.

Corollary 2.21. *Let $f\colon \Omega \to \mathbf{C}$ be an analytic function on a simply connected region (such as a disc or half-plane). Every continuous logarithm of f is an analytic logarithm. That is, every continuous function ℓ_f on Ω such that $e^{\ell_f(z)} = f(z)$ on Ω is analytic.*

Proof. Since $f(z) = e^{\ell_f(z)}$, f is nonvanishing, so f admits an analytic logarithm, say L_f. (Recall that this logarithm was constructed in Theorem 2.16, via path integrals of f'/f. Or, since analyticity is a local property, we may suppose Ω is a suitably small disc and appeal to the simpler Theorem 2.19.) The proof of Theorem 2.20 only used continuity of the two logarithms, not their analyticity. So that proof shows $\ell_f - L_f$ is a constant. Since L_f is analytic, ℓ_f must be analytic. □

Square roots should be expressible as $\sqrt{z} = e^{(1/2)\log z}$, so we can construct square roots of analytic functions if we can construct analytic logarithms.

Definition 2.22. Let $f\colon \Omega \to \mathbf{C}$ be analytic. A *square root* of f is an analytic function g on Ω such that $g^2 = f$.

For example, $e^{z/2}$ is a square root of e^z and z is a square root of z^2. Unlike logarithms of analytic functions, square roots of analytic functions can vanish. (The constant function 0 is a square root of itself; in no other case will a square root vanish except at isolated points.)

In practice we will be considering only nonvanishing functions (at least in the region where we may construct logarithms or square roots), so part i) of the following theorem covers the cases that will matter.

Theorem 2.23. *i) If ℓ_f is a logarithm of the analytic function f, then $g(z) = e^{(1/2)\ell_f(z)}$ is a square root of f. In particular, every nonvanishing analytic function on a simply connected region has a square root.*

ii) If g_1 and g_2 are square roots of f, then $g_1 = g_2$ or $g_1 = -g_2$.

iii) If $f\colon \Omega \to \mathbf{C}$ is an analytic function on an open set, every continuous square root is an analytic square root.

Proof. Exercise. $\square$

Example 2.24. On the upper half-plane $\mathfrak{H} = \{a + bi : b > 0\}$, with variable element τ, the function $f(\tau) = \tau/i$ is nonvanishing, so it has an analytic square root, in fact exactly two of them. Writing $\tau = re^{i\theta}$ where $0 < \theta < \pi$, we have $\tau/i = re^{i(\theta-\pi/2)}$. Set $\sqrt{\tau/i} = \sqrt{r}e^{i(\theta/2-\pi/4)}$. This is certainly a continuous square root function. It must be analytic by part iii) of Theorem 2.23. Notice this particular analytic square root is positive on the imaginary axis, namely if $\tau = bi$ for $b > 0$ then $\sqrt{\tau/i}$ is the real positive square root of b.

The next complex analytic topic is the Gamma function. This is an important function in both pure and applied mathematics. In number theory it arises in the functional equations of zeta and L-functions. It is not naturally defined by a power series expansion, but by an integral formula.

We begin with the equation

$$n! = \int_0^\infty x^n e^{-x}\,dx. \tag{2.4}$$

This is clear when $n = 0$, since $\int_0^\infty e^{-x}\,dx = 1$. Integrating by parts ($u = x^n$, $dv = e^{-x}dx$) gives (2.4) for larger n by induction. (Euler discovered this formula, but wrote it as $n! = \int_0^1 \log(1/y)^n\,dy$.) The right side of (2.4) makes sense for nonintegral values of n.

Definition 2.25. For complex numbers s with $\mathrm{Re}(s) > 0$, set

$$\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}\,dx = \int_0^\infty x^s e^{-x}\,\frac{dx}{x}.$$

We'll check below that this integral does converge. In addition to checking the behavior of the integrand $x^{s-1}e^{-x}$ near $x = \infty$, when $0 < \mathrm{Re}(s) < 1$ we have to pay attention to behavior near $x = 0$. This definition does not make sense if $\mathrm{Re}(s) \le 0$. But we will see that $\Gamma(s)$ can be continued to a meromorphic function on $\mathbf{C}$.

For a positive integer n, $\Gamma(n) = (n-1)!$. It may seem more reasonable to work with $\Pi(s) := \int_0^\infty x^s e^{-x}\,dx$, since $\Pi(n) = n!$, and this is the factorial generalization used by Riemann in his paper on the zeta-function. The function $\Gamma(s) = \Pi(s-1)$ was introduced by Legendre.

The Gamma function is an example of an integral of the form

$$\int_0^\infty f(x)\,\frac{dx}{x}. \tag{2.5}$$

Such integrals are invariant under multiplicative translations: for every $c > 0$,

$$\int_0^\infty f(cx)\,\frac{dx}{x} = \int_0^\infty f(x)\,\frac{dx}{x}.$$

Similarly, for $c > 0$

$$\int_{-\infty}^\infty f(cx)\,\frac{dx}{|x|} = \int_{-\infty}^\infty f(x)\,\frac{dx}{|x|}.$$

(The additive analogue is the more familiar $\int_{-\infty}^\infty g(x+c)\,dx = \int_{-\infty}^\infty g(x)\,dx$.)

To check the integral defining $\Gamma(s)$ makes sense, we take absolute values and reduce to the case of *real* $s > 0$. Behavior at 0 and ∞ are isolated by splitting up the integral into two pieces, from 0 to 1 and from 1 to ∞:

$$\int_0^1 x^{s-1}e^{-x}\,dx + \int_1^\infty x^{s-1}e^{-x}\,dx.$$

For $x > 0$, $x^{s-1}e^{-x} \le x^{s-1}$ and x^{s-1} is integrable on $[0,1]$ if $s > 0$, so the first integral converges. For the second integral, remember that exponentials grow much faster than powers. We write $e^{-x} = e^{-x/2}e^{-x/2}$ and then have

$$x^{s-1}e^{-x} \le C_s e^{-x/2}$$

on $[1,\infty)$ for some constant C_s, so the second integral converges. Actually, the second integral converges for *every* $s \in \mathbf{C}$, not just when $\mathrm{Re}(s) > 0$.

One important special value of Γ at a noninteger is at $s = 1/2$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-1/2}\,dx = 2\int_0^\infty e^{-u^2}\,du = \int_{-\infty}^\infty e^{-u^2}\,du.$$

A common method of calculating $\int_{-\infty}^\infty e^{-u^2}\,du$ is by squaring and then passing to polar coordinates. For $I = \int_{-\infty}^\infty e^{-u^2}\,du$,

$$\begin{aligned} I^2 &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\,dx\,dy \\ &= \int_0^\infty \int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta \\ &= \int_0^\infty re^{-r^2}\,dr \cdot \int_0^{2\pi} d\theta \\ &= \frac{1}{2}\cdot 2\pi \\ &= \pi, \end{aligned}$$

so (since $I > 0$) $I = \sqrt{\pi}$. Therefore

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

There is a method of computing I^2 without polar coordinates. The idea goes back to Laplace:

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\right)dy \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \operatorname{sign}(y)e^{-(y^2t^2+y^2)}y\,dt\right)dy \quad \text{where } x = ty \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |y|e^{-(t^2+1)y^2}\,dy\right)dt. \end{aligned}$$

Since $\int_{-\infty}^{\infty}|y|e^{-ay^2}\,dy = 1/a$ for $a > 0$, we have

$$I^2 = \int_{-\infty}^{\infty}\frac{dt}{t^2+1} = \pi,$$

so $I = \sqrt{\pi}$.

By Theorem 2.4, $\Gamma(s)$ is analytic on $\mathrm{Re}(s) > 0$. Now we extend Γ meromorphically to the entire complex plane.

For complex s with positive real part, an integration by parts ($u = x^s, dv = e^{-x}dx$) yields the *functional equation*

$$\Gamma(s+1) = s\Gamma(s). \tag{2.6}$$

This generalizes $n! = n(n-1)!$.

Theorem 2.26. *The function Γ extends to a meromorphic function on $\mathbf{C}$ whose only poles are at the integers $0, -1, -2, \ldots,$ where the poles are simple with residue at $-k$ equal to $(-1)^k/k!$.*

Proof. Use (2.6) to extend Γ step-by-step to a meromorphic function on $\mathbf{C}$. The residue calculation follows from the functional equation and induction. $\square$

There are a few important relations that Γ satisfies besides (2.6). For example, since $\Gamma(s)$ has simple poles at $0, -1, -2, \ldots$, $\Gamma(1-s)$ has simple poles at $1, 2, 3, \ldots$. The product $\Gamma(s)\Gamma(1-s)$ therefore has simple poles precisely at the integers, just like $1/\sin(\pi s)$. What is the relation between these functions? Also, $\Gamma(s/2)$ has simple poles at $0, -2, -4, -6, \ldots$ and $\Gamma((s+1)/2)$ has simple poles at $-1, -3, -5, \ldots$. Therefore $\Gamma(s/2)\Gamma((s+1)/2)$ has simple poles at $0, -1, -2, \ldots$, just like $\Gamma(s)$. What is the relation between $\Gamma(s)$ and the product $\Gamma(s/2)\Gamma((s+1)/2)$? In general, knowing where a functions has its poles (and zeros) is not enough to characterize it. One can always introduce an arbitrary multiplicative constant. That is the only purely algebraic modification, but we can also introduce a factor of an arbitrary nonvanishing entire function, say $e^{g(z)}$ where $g(z)$ is entire. For $g(z)$ linear, this factor looks like ae^{bz}. This is the type of factor we'll need to introduce to relate the above Gamma functions with each other.

To work with factorization questions, it is more convenient to use a different formula for $\Gamma(s)$ than the integral definition. The integral is an additive formula for $\Gamma(s)$. We now turn to two multiplicative formulas, valid in the whole complex plane. They are usually attributed to Gauss and Weierstrass, respectively, although neither one was the first to discover these formulas.

Lemma 2.27. *For every* $s \in \mathbf{C}$,

$$\Gamma(s) = \lim_{n\to\infty} \frac{n! n^s}{s(s+1)\cdots(s+n)} = \frac{e^{\gamma s}}{s} \prod_{n\geq 1} \left(1+\frac{s}{n}\right)^{-1} e^{s/n},$$

where $\gamma = \lim_{n\to\infty} 1 + 1/2 + \cdots + 1/n - \log n \approx .5772156649015\ldots$ *is Euler's constant.*

Proof. See [1, Chap. XV] or [2, Chap. 2]. For $s = 0, \pm 1, \pm 2, \ldots$ the products are set equal to ∞. ☐

The Weierstrass product can also be written as

$$\Gamma(s+1) = e^{\gamma s} \prod_{n\geq 1} \left(1+\frac{s}{n}\right)^{-1} e^{s/n}.$$

The exponent γ in the Weierstrass product has the characterizing role of guaranteeing that $\Gamma(s+1)/\Gamma(s)$ equals s rather than some other constant multiple of s.

Theorem 2.28. *For all complex numbers* s,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s).$$

The first equation is called the reflection formula, and the second is the duplication formula.

Proof. To get the first identity, write it as $\Gamma(s)\Gamma(-s) = -\pi/(s\sin(\pi s))$. Compute the left side using the product in Lemma 2.27 and the right side using the product for the sine function,

$$\sin s = s \prod_{n\geq 1} \left(1 - \frac{s^2}{n^2\pi^2}\right), \tag{2.7}$$

replacing s with πs.

To prove the second identity, use Gauss' limit formula in Lemma 2.27 to compute $\Gamma(s/2)\Gamma((s+1)/2)$. The result is $c\Gamma(s)/2^s$, and setting $s = 1$ shows $c = 2\sqrt{\pi}$. ☐

Corollary 2.29. *The function* $\Gamma(s)$ *does not vanish.*

Proof. The poles of $\Gamma(s)$ are at the integers ≤ 0. Since $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we see that $\Gamma(s) \neq 0$ for $s \notin \mathbf{Z}$. We already know the values of $\Gamma(s)$ at the integers, where it is not zero. ☐

In particular, while $\Gamma(s)$ is meromorphic, $1/\Gamma(s)$ is entire. So if $f(s)\Gamma(s)$ is an entire function, so is $f(s)$.

We conclude with a basic asymptotic formula for the Gamma function.

Theorem 2.30 (Stirling's Formula)**.** *Fix positive* $\varepsilon < \pi$. *As* $|s| \to \infty$ *in a sector* $\{s : |\mathrm{Arg}(s)| \leq \pi - \varepsilon\}$,

$$\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} e^{\mu(s)} = \sqrt{2\pi} e^{(s-1/2)\,\mathrm{Log}\, s - s + \mu(s)},$$

where the error term in the exponent satisfies $\mu(s) = O(1/|s|)$ *as* $|s| \to \infty$, *the constant in the* O*-symbol depending on the sector. In particular,* $\Gamma(s) \sim \sqrt{2\pi} s^{s-1/2} e^{-s}$ *as* $s \to \infty$ *in such a sector.*

Proof. See [1, Chap XV] or [2, Chap 2]. The proof involves producing an exact error estimate for $\log\Gamma(s)$, and leads to the additional formula

$$\frac{\Gamma'(s)}{\Gamma(s)} = \operatorname{Log} s - \frac{1}{2s} + O(1/|s|^2) = \operatorname{Log} s + O(1/|s|)$$

by differentiation of the exact error estimate. □

An important application of the complex Stirling's formula is to the growth of the Gamma function on vertical lines. (We call this vertical growth.) For $\sigma > 0$, the integral formula for the Gamma function shows $|\Gamma(\sigma+it)| \leq |\Gamma(\sigma)|$ for any real t, so vertical growth is bounded. But in fact it is exponentially decaying, as follows.

Corollary 2.31. *For fixed σ, $|\Gamma(\sigma+it)| \sim \sqrt{2\pi}e^{-(\pi/2)|t|}|t|^{\sigma-1/2}$ as $|t| \to \infty$. More generally, this estimate applies in any vertical strip of the complex plane, and is uniform with respect to σ in that strip.*

Proof. By iterating the functional equation $\Gamma(s+1) = s\Gamma(s)$, we can reduce to the case of a closed vertical strip in the half-plane $\operatorname{Re}(s) > 0$. We leave this reduction step to the reader, and for simplicity we only treat the case of a single vertical line rather than a strip.

Since $|\Gamma(\sigma+it)| = |\Gamma(\sigma-it)|$, we only need to consider $t > 0$, $t \to \infty$.

For $\sigma > 0$ fixed and $t \to \infty$, Stirling's formula (in the form of Theorem 2.30) gives

$$\begin{aligned} |\Gamma(\sigma+it)| &\sim \sqrt{2\pi}e^{\operatorname{Re}((\sigma-1/2+it)\operatorname{Log}(\sigma+it))}e^{-\sigma} \\ &= \sqrt{2\pi}e^{(\sigma-1/2)(1/2)\log(\sigma^2+t^2)-t\arctan(t/\sigma)-\sigma} \\ &\sim \sqrt{2\pi}e^{(\sigma-1/2)\log t-(\pi/2)t} \end{aligned}$$

since $t(\pi/2-\arctan(t/\sigma)) \to \sigma$ as $t \to \infty$. □

As a particular example, note that for real a, b, and c with $a \neq 0$, the exponential factors in the Stirling estimates for $|\Gamma(as+b)|$ and $|\Gamma(-as+c)|$ (when $t = \operatorname{Im}(s) \to \infty$) are identical, so

$$\frac{\Gamma(as+b)}{\Gamma(-as+c)} = O(t^M) \tag{2.8}$$

as $t \to \infty$, where M depends on the parameters a, b, and c. This polynomial upper bound on growth is convenient when estimating the growth of zeta and L-functions in vertical strips.

References

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