MATHematics
illuminated

UNIT 02

COMBINATORICS COUNTS
FACILITATOR GUIDE

ACTIVITIES

NOTE: At many points in the activities for Mathematics Illuminated, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

NOTE: Instructions, answers, and explanations that are meant for the facilitator only and not the participant are in grey boxes for easy identification.

NOTE to Facilitator: the following activities may be done in any combination and any order. Activities 3 and 4 may be most appropriate for a group that is more advanced in this subject.
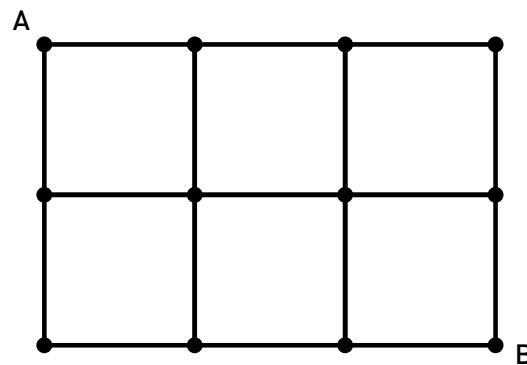
ACTIVITY 1

(40 minutes)

EXPLORING PASCAL'S TRIANGLE

In <u>Combinatorics Counts</u> you saw the connection between the counting function $C(n,k)$, the number of ways to choose a subset of k objects out of a set of n objects, and Pascal's Triangle, one of the most famous ideas in mathematics. Pascal's Triangle is a source of many mathematical mysteries, and in this activity you will explore some of the more surprising connections.

A (15 minutes)

1. How many different shortest paths are there from point A to point B if you must stay on the grid lines?

Answer: For the 2 x 3 grid, there are C(5, 3) = 10 paths from A to B.
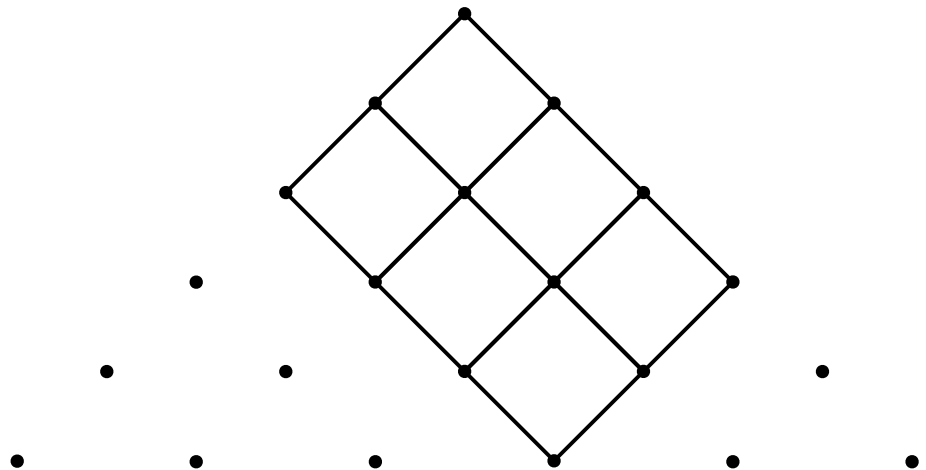
On the above diagram, label the points of the triangle with numbers according to the pattern of Pascal's Triangle.
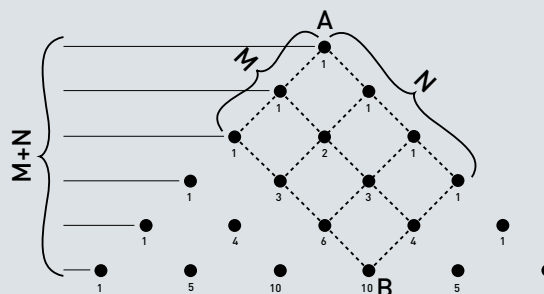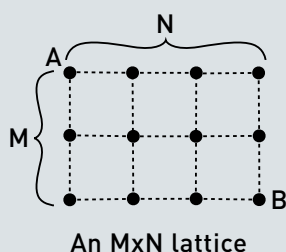
**EXPLORING PASCAL'S
TRIANGLE**
CONTINUED

2. How does the number of shortest paths to point B relate to the number you found on the previous page? How do the numbers in Pascal's Triangle relate to the number of shortest paths between two points on a lattice?
Answer: They're the same. If you arrange the lattice on Pascal's Triangle, as shown above, the number corresponding to the terminal point (point B) is equal to the number of shortest paths from A to B.

3. Use the formula for *C(n,k)* found in the text along with the information you just acquired to write a general expression for the number of shortest paths from A to B on an M x N lattice.
Answer: For an M x N grid, there are C(M+N, N) = (M+N)!/(N!(N-(M+N))!) possible shortest paths between opposite corners. Connection to Pascal's Triangle:

An MxN lattice

A skewed MxN lattice shown in the context of Pascal's Triangle

Item 2622 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.1 (2008).
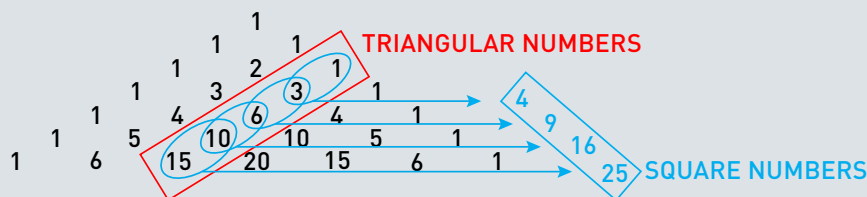Courtesy of Oregon Public Broadcasting.

ACTIVITY 1

EXPLORING PASCAL'S TRIANGLE
CONTINUED

B (5 minutes)

Find the square and triangular numbers in Pascal's Triangle. Explain as precisely as possible.

Answer: The triangular numbers, familiar from unit 1 (1, 3, 6, 10, 15, …), lay along the third diagonal of Pascal's Triangle. The square numbers can be generated by taking the sum of every two consecutive triangular numbers.

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C (10 minutes)

Find the Fibonacci sequence (0, 1, 1, 2, 3, 5, 8, 13, 21, …) in Pascal's Triangle. Explain.

Hint 1: recall that to find a number in the Fibonacci sequence, you just add together the two numbers that precede it. For example: 0 + 1 = **1**, 1 + 1 = **2**, 1 + 2 = **3**, 2 + 3 = **5**, etc.

NOTE: This one may require some facilitator prompting.
Answer: The sums of upward "shallow" diagonals on Pascal's Triangle are the Fibonacci numbers. Convene the large group and discuss answers to B and C.
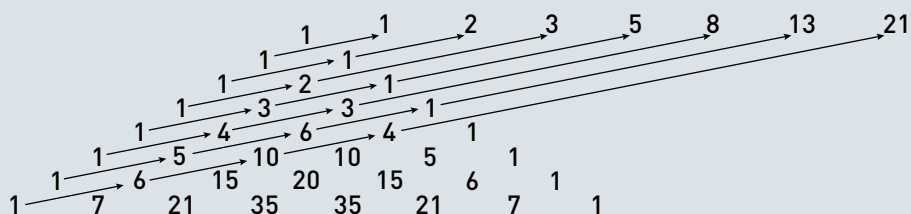
Item 2624 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.3 (2008). Courtesy of Oregon Public Broadcasting.

ACTIVITY 1

EXPLORING PASCAL'S TRIANGLE
CONTINUED

Convene the large group and discuss answers to B and C.

D (5 minutes)

Look at the prime-numbered rows (remember that the top of the triangle is considered to be row zero). What do you notice?

Answer: All non-one entries of the prime-numbered rows are multiples of the number of the row. For example, the unique entries of row 7 are 7, 21, and 35, all multiples of 7.

IF TIME ALLOWS:

E (5 minutes)

Compute the first 5 powers of 11, starting with the 0th power. What does this have to do with Pascal's Triangle?

Answer: The first 7 powers of 11 are: 1, 11, 121, 1331, 14641. Compare these to the numbers you get when you take the entries of a given row and make them into a number. For example: row three is 1 3 3 1, and 11³ = 1,331.

Convene the group to discuss the answers to D and E and review and summarize the patterns found in Pascal's Triangle.

ACTIVITY	2

THE HEXI GAME

(50-60 minutes)

Use the given K_6 graphs to play the game "Hexi."

MATERIALS

- Hexi game sheets
- Colored pencils, crayons, or markers

RULES

- This is a two-player game; each player should have a different color of pen, crayon, or marker.
- Decide who goes first.
- The first player colors one edge of the graph with her color.
- The next player colors a different edge of the graph with his color.
- The players alternate turns until all edges are colored. (Note: Players must color an edge when it is their turn.)
- The object is to force one's opponent to create a triangle of her own color. The first player to create a triangle of his own color loses.

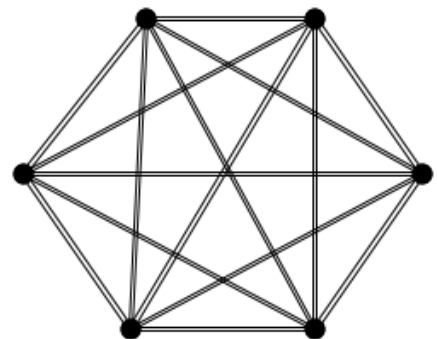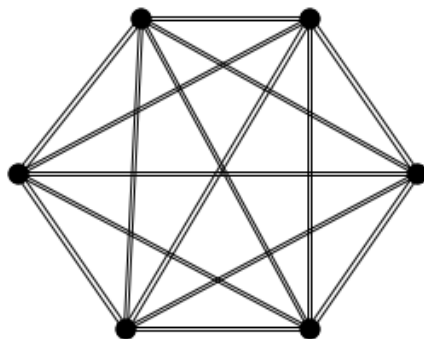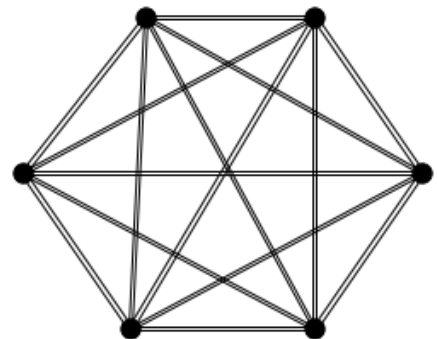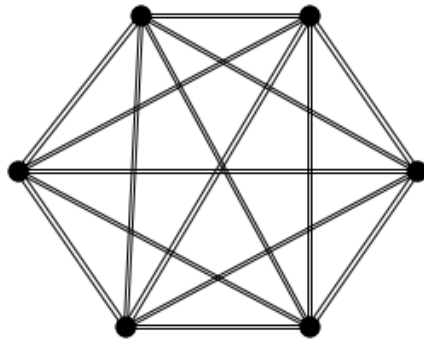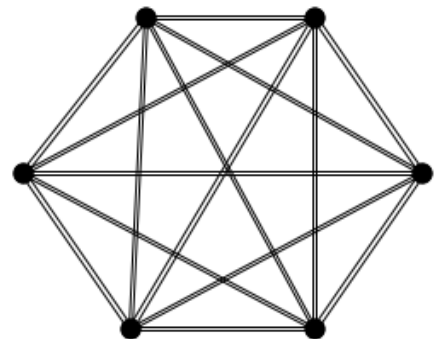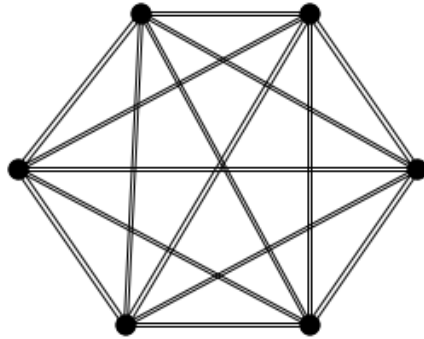**A** Play this game with a partner from your small group. Play "best of three" before switching partners. Play until you figure out a winning strategy or until time runs out.

Hexi Game Sheet - Complete K_6 Graph

THE HEXI GAME
CONTINUED

UNIT 2

ACTIVITY **2**

THE HEXI GAME
CONTINUED

In small groups, discuss the following questions: (5-10 minutes)

1. What does this game have to do with the combinatorics unit?
Answer: This is a version of the "Party Problem," the fact that in any group of six people, at least three will be either mutual acquaintances or mutual strangers. It is also a utilization of the Ramsey Number R(3,3) = 6. One color represents "friendship," and the other represents "unfamiliarity."

2. Can there be a tie? Why not?
Answer: No. This is a direct result of the pigeonhole principle; every edge must be either one color or the other, so a homogeneous triangle of one color or the other is inevitable.

After small group discussion of the above questions, convene the large group to discuss the connections of this game to the Party Problem. Steer the discussion toward the method of proof given in the text. Participants will need to have this proof on hand to complete the next section.

B (15 minutes)

What is R(3,4)?

Have small groups consider the following questions:

1. Draw K_3. How many possible colorings of K_3 are there using two colors?
Answer: This is a triangle. For K_n, there are $C(n,2)$ edges, each of which can be one of the two colors, which is just like the binary strings bijection from the textbook (replacing 1 and 0 with one color and the other). This implies that there are $2^{C(n,2)}$ possible colorings of K_n. In this instance, there are eight possible colorings.

ACTIVITY 2

THE HEXI GAME
CONTINUED

2. Draw K_4. How many possible colorings of K_4 are there using two colors?

Answer: By the reasoning above, there are 64 possible colorings of K_4.
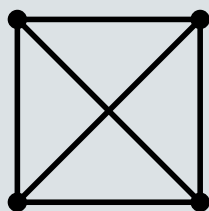
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3. In words, what does R(3,4) represent?

Answer: This expression represents the minimum number of people needed to guarantee that at least three are friends or at least four are strangers—OR—the minimum size of the complete graph required so that any coloring of its edges with two colors has either a triangle (K_3) of one color or a K_4 of the other color.

4. Say that the two colors in R(3,4) are red and blue respectively. Can you find a coloring of K6 that has neither a red K_3 nor a blue K_4? Do the same for K_3, K_4, and K_5.

Answer: R(3,4) is the minimum number required to force a red K_3 or a blue K_4. Any coloring that has neither a red K_3 nor a blue K_4 serves as a counterexample that shows that the graph does not force either of the desired complete subgraphs. The K_6 counterexample is the most difficult to find, because there are a lot of possible colorings. Here are some counterexamples for K_3 through K_6.
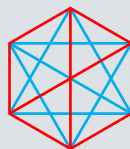
This coloring of K_6 has neither a red K_3 nor a blue K_4 although it does have 2 blue $K_3$5
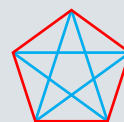
The K_5 counter example

The K_4 counter example

The K_3 counter example

THE HEXI GAME
CONTINUED

5. What have you just shown about the lower bound of R(3,4)?
Answer: R(3,4) is at least seven.

Convene the large group and discuss the answers to the above questions to make sure that everyone is on the same page before delving into the next section. (5 minutes)

Using the methods outlined in the textbook, show that K10 forces either a red K3 or a blue K_4. (20 minutes)

6. First, look at the edges coming out of one vertex of K_{10}. If four of them are red, show that this guarantees either a red K_3 or a blue K_4.
Answer:

If any of these are red, then a red K_3 exists, if none are red, then a blue K_4 exists

If 4 or more of a node's nine incident edges are red, then either a red K_3 or a blue K_4 is forced.
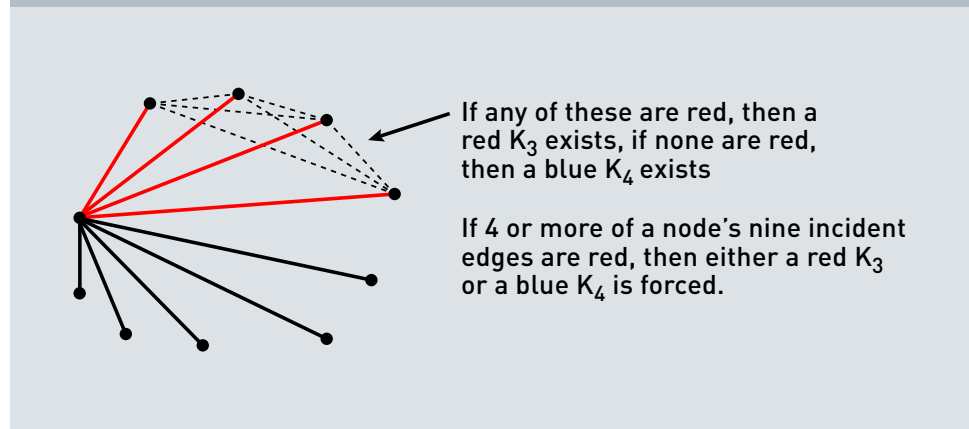
Item 2627 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.6 (2008).
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7. After checking the possibility that four edges coming out of one vertex are red, what is the other scenario that we need to check?
Answer: The alternative scenario is six or more blue edges coming out of one vertex.

8. Show that if six or more edges extending from one vertex are blue, a red K_3 or a blue K_4 is forced.

Answer: The vertices at the end of the six blue edges form a K_6. Since $R(3,3) = 6$, any coloring of the edges of the K_6 forces a red or blue K_3. A blue K^3, together with three of the original six blue edges, forms a K_4.

The potential connections between the 6 blue-edge nodes form a K_6.

From playing the hexi game, we know that there must be a red K_3 or a blue K_3. Any blue K_3 will combine with three blue edges to create a blue K_4.

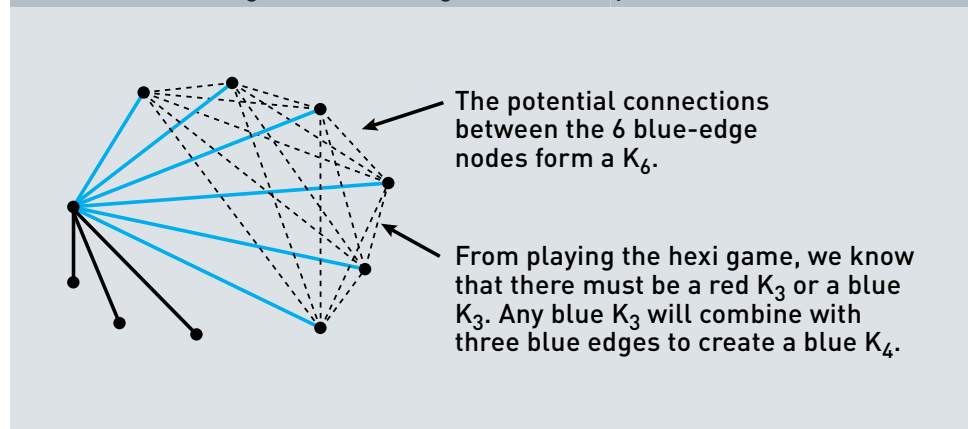Item 3110 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.7 (2008).
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9. What does this say about the upper bound of $R(3,4)$?

Answer: Because K_{10} forces either a red K_3 or a blue K_4, $R(3,4)$ can be no larger than 10.

10. What are the possible values for $R(3,4)$?

Answer: 7, 8, 9, or 10

11. How would you determine which of these is the correct value? What is your opinion of the difficulty of this task? (Hint: it might help to think about the number of possible colorings that have to be checked for counterexamples.)

Answer: You need the smallest number that guarantees the conditions, so you have to work your way up, finding counterexamples on K_7 and K_8 to show that $R(3,4)$ can't be 7 or 8 and thus has to be 9. There are $2^{c(7,2)} = 2^{21}$ colorings that could possibly have to be checked for K_7 and $2^{c(8,2)} = 2^{28}$ colorings that could possibly have to be checked for K_8. Opinions will vary, but anyone who says this is easy needs to be able to back that up with some good explanation!

12. Let's say that it turns out that $R(3,4)$ is 9 (which it is). What is $R(4,3)$? Why?

Answer: $R(4,3)$ is also 9, by symmetry. Color assignations are arbitrary, so flipping the two colors presents the same exact scenario.

ACTIVITY **2**

THE HEXI GAME
CONTINUED

IF TIME ALLOWS:

Have the participants find counterexamples for K_7 and K_8.

ACTIVITY   3

FINDING DE BRUIJN SEQUENCES

(30 minutes)

NOTE: This entire activity can be done in small groups, with periodic large-group check-ins to be held at the facilitator's discretion.

Use the techniques from the text to find a sequence that contains all possible three-letter "words" that can be made with an alphabet consisting of just the letters A and B.

1. How many such three-letter "words" can be made?
Answer: 8 You can get this by direct counting or by using the formula (explained in the textbook) 2^n, with n = 3

2. What is the shortest length of a sequence that contains all possible three-letter words from above?
Answer: Assuming that an overlap of two can be achieved, the shortest sequence would be 10 (3 letters for the first word, with each additional letter creating another word for 3 + 7 = 10).

3. How many de Bruijn sequences are there (assuming an overlap of two)?
Answer: 2

**FINDING DE BRUIJN
SEQUENCES**
CONTINUED

4. Describe your process.

Answer: Encourage the participants to refer to the text for a general method. First, they should create a directed graph out of the eight possible "words," as shown here:
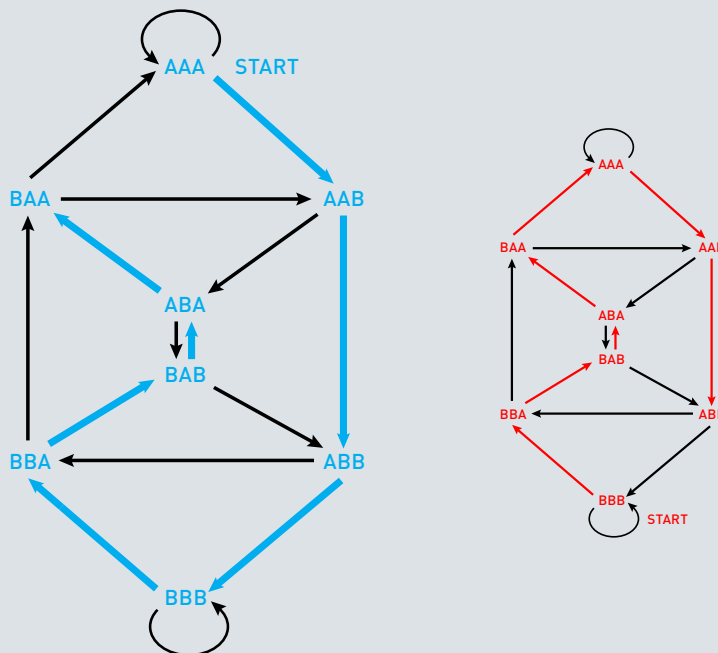
Item 2628 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.11 (2008).
Courtesy of Oregon Public Broadcasting.

Participants should then find as many Hamilton paths as they can on this graph. This gives AAABBBABAA and BBBABAAABB as the two unique de Bruijn sequences.

①		②	
	AAA		BBB
	AAB		BBA
	ABB		BAB
	BBB		ABA
	BBA		BAA
	BAB		AAA
	ABA		AAB
	BAA		ABB
AAABBBABAA		**BBBABAAABB**	
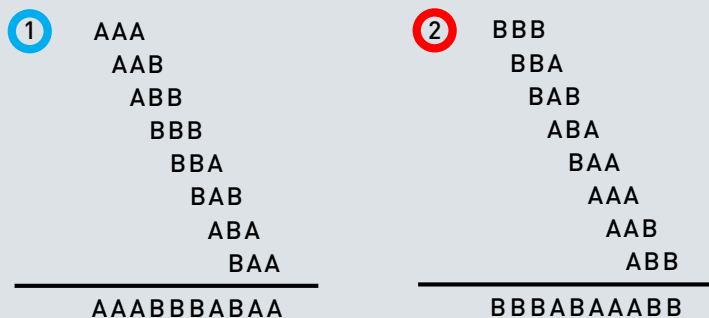
Item 2629 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.9 (2008).
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ACTIVITY **3**

**FINDING DE BRUIJN
SEQUENCES**
CONTINUED

5. Show that your sequence does indeed contain every three-letter "word."
Answer: Answers will vary, but they should somehow show that the two de Bruijn sequences contain all of the possible three-letter "words": AAA, AAB, ABB, BBB, BBA, BAB, ABA, BAA.

6. If you take an Eulerian path instead of a Hamilton cycle, what does the resulting sequence represent?

Hint 1: When finding a de Bruijn sequence via the Hamilton cycle method, each vertex represents one of the "words." What does each edge represent?
Answer: Each edge corresponds to a four-letter sequence. For example, the edge connecting AAA to AAB can be thought of as representing the "word" AAAB. NOTE: It is important to include the self-referencing edges connecting AAA to itself and BBB to itself.

Hint 2: How many edges are there on the directed graph? How many four-letter words can be made from the given alphabet?
Answer: There are 16 edges, and there are 2^4 (16) possible four-letter "words." Therefore, the Eulerian cycle on this graph corresponds to a de Bruijn sequence that contains every four-letter "word."

ACTIVITY 4

MATERIALS

(35 minutes)

• Graph paper

One of the most famous sequences of numbers in mathematics is the Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, ... That this series appears in Pascal's Triangle, as you discovered in one of the mini-activities, is surely no coincidence. In this activity you will explore Fibonacci numbers from a combinatorial standpoint and then find out why there is a connection between the Fibonacci numbers and Pascal's Triangle.

A   (10 minutes)

Do this activity in small groups.

1. How many ways can you express the number 4 as a sum of 1's and 2's?
Answer: Five ways (1 + 1 + 1 + 1, 1 + 1 + 2, 2 + 1 + 1, 1 + 2 + 1, and 2 + 2)

A "4-board" is a string of four cells like so:

A 4-BOARD   2 POSSIBLE TILINGS
                   OF A 4-BOARD

2. Show the sums you found in the first question as tilings of 4-boards.
Note: we'll refer to single squares as just "squares" and double squares as "dominoes."

Answer:

1 + 1 + 1 + 1   2   + 1 + 1   2   +   2

1 + 1   + 2   1   + 2   + 1
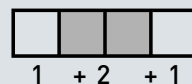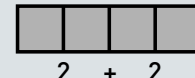
Item 2631 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.10 (2008).
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Let's call the number of tilings of a 4-board using squares and dominoes f_4.

ACTIVITY **4**

3. Use 3-boards to find f_3.

Answer:

1 + 1 + 1

2 + 1

1 + 2

Item 2637 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.11 (2008). Courtesy of Oregon Public Broadcasting.

4. Use 5-boards to find f_5.

Answer:

1 + 1 + 1 + 1 + 1

1 + 1 + 1 + 2

1 + 1 + 2 + 1

1 + 2 + 1 + 1

2 + 1 + 1 + 1

1 + 2 + 2

2 + 1 + 2

2 + 2 + 1

Item 2636 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.12 (2008). Courtesy of Oregon Public Broadcasting.

5. Write an expression for f_5 using f_3 and f_4.
Answer: $f_5 = f_4 + f_3$

6. If f_5 is replaced with fn, what would the above expression become?
Answer: $f_n = f_{n-1} + f_{n-2}$

Convene the large group to discuss the answers to the above questions, One thing to discuss is whether or not the generalization done in the last question is valid. In the next section, participants will see a more-valid generalization. (5 minutes)

ACTIVITY 4

B (10 minutes)
Do this activity in small groups.

1. On an n-board, if the first tile is a square, how many ways are there to complete the board?
Answer: f_{n-1}

2. On an n-board, if the first tile is a domino, how many ways are there to complete the board?
Answer: f_{n-2}

3. Write an expression for the number of tilings of an n-board. (Hint: think of it as the sum of the n-boards with a square as the first tile and the n-boards with a domino as the first tile.
Answer: $f_n = f_{n-1} + f_{n-2}$

4. If we define f-1 = 0 and f0 = 1, start with f1 and write the first ten fn's.
Answer: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

5. Does this series look familiar?
Answer: It is the Fibonacci series.

Convene the group and discuss the answers so far. Then remind them how to find the Fibonacci numbers in Pascal's Triangle. The following diagram will help: (5 minutes)
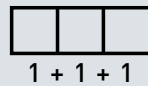
For Facilitator:

Item 2624 / HUB Collective LTD., created for Mathematics Illuminated, IMAGE 2.13 (2008).
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ACTIVITY **4**

C   (10 minutes)

We can use the fact that upward diagonal sums in Pascal's Triangle correspond to the Fibonacci numbers to write an expression for the nth Fibonacci number.

1. How many 12-tilings use two dominoes and eight squares? three dominoes and six squares? k dominoes and 12-2k squares?
Answer: $C(10,2)$, $C(9,3)$, and $C(12-k, k)$, respectively.
The idea is to count the number of tiles (i.e., the number of squares and dominoes) in the particular tiling. If there are nine tiles used, think of them as being labeled 1, 2, 3, ..., 9. Then select which of the nine tiles are to be the dominoes.

2. How many n-tilings use one domino? two dominoes? k dominoes?
Answer: $C(n-1,1) = n-1$, $C(n-2,2)$, and $C(n-k,k)$.

3. Write an expression for f_n in terms of the binomial coefficient, $C(n, k)$.

Hint 1: Count the number of n-tilings by considering the number of dominoes used.
Answer: The first few terms of an upward diagonal sum are $C(n,0) + C(n-1,1) + C(n-2, 2) + ... + C(n-k, k)$. This is more efficiently expressed as $f_n = \text{SUM } k=0 \rightarrow n \ C(n-k,k)$

4. Compare and contrast the expressions for f_n found earlier in parts A and B.
Answer: The summation expression does not rely on recursion; we only need to know n, not f_{n-1} and f_{n-2}. In other words, we can find the nth Fibonacci number without having to know the $(n-1)$st and $(n-2)$nd Fibonacci numbers.

ACTIVITY **4**

IF TIME ALLOWS:

For n greater than or equal to zero, show that $f_0 + f_1 + f_2 + f_3 + ... + f_n = f_{n+2} - 1$.

Hint 1: In terms of n-boards, what does f_{n+2} represent?
Answer: The tilings of an (n+2)-sized board.

Hint 2: How many (n+2)-boards use at least one domino? It might help to think about how many tilings use no dominoes.
Answer: f_{n+2} -1 There's only one tiling that uses no dominoes—the one with all squares.

Hint 3: Answer the question in Hint 2 another way: let the last domino fall in cells k+1 and k+2—all the squares after k+2, up to cell n, are then squares, but the cells before k+1 can be a mix of squares and dominoes. How many tilings are there? Perhaps start with the last domino in cells 1 and 2, then 2 and 3, then 3 and 4, etc.
Answer: The last domino in cells 1 and 2 means there are f_1 tilings; the last domino in cells 2 and 3 means there are f_2 tilings; the last domino in cells 3 and 4 means there are f_3 tilings. Extending this thinking leads to the conclusion that the sum of the number of tilings of the various last-domino positions should be equal to the number of tilings that use at least one domino, which is what was found in Hint 2.

CONCLUSION | **4**

(30 minutes)

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Facilitator's note:
Have copies of national, state, or district mathematics standards available.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 2, Combinatorics Counts could be related and brought into your classroom.

Questions to consider:

1. Which parts of this unit seem accessible to my students with no "frontloading?"

2. Which parts would be interesting, but might require some amount of preparation?

3. Which parts seem as if they would be overwhelming or intimidating to students?

4. How does the material in this unit compare to state or national standards? Are there any overlaps?

5. How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS (30 minutes)

Please use the last 30 minutes of class to watch the video for the next unit: How Big Is Infinity? Workshop participants are expected to read the accompanying text for How Big Is Infinity? before the next session.

NOTES