Solving Integer Programming with Branch-and-Bound Technique

This is the divide and conquer method. We divide a large problem into a few smaller ones. (This is the "branch" part.) The conquering part is done by estimate how good a solution we can get for each smaller problems (to do this, we may have to divide the problem further, until we get a problem that we can handle), that is the "bound" part.

We will use the *linear programming relaxation* to estimate the optimal solution of an integer programming.

* For an integer programming model P, the linear programming model we get by dropping the requirement that all variables must be integers is called the linear programming relaxation of P.

The steps are:

- Divide a problem into subproblems
- Calculate the LP relaxation of a subproblem
  - The LP problem has no feasible solution, done;
  - The LP problem has an integer optimal solution; done. Compare the optimal solution with the best solution we know (the incumbent).
  - The LP problem has an optimal solution that is worse than the incumbent, done.

  In all the cases above, we know all we need to know about that subproblem. We say that subproblem is fathomed.

  - The LP problem has an optimal solution that are not all integer, better than the incumbent. In this case we would have to divide this subproblem further and repeat.

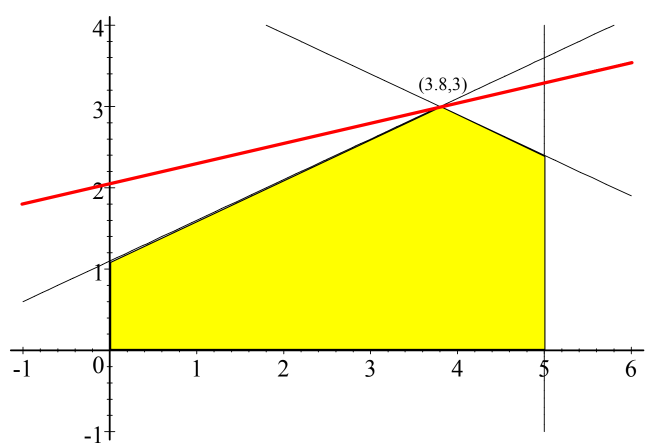
Example 1

$$Max\ Z = -x_1 + 4x_2$$

Subject to

$$\begin{aligned} -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 5 \end{aligned}$$

$$x_i \geq 0,\ x_i\text{'s are integers}$$



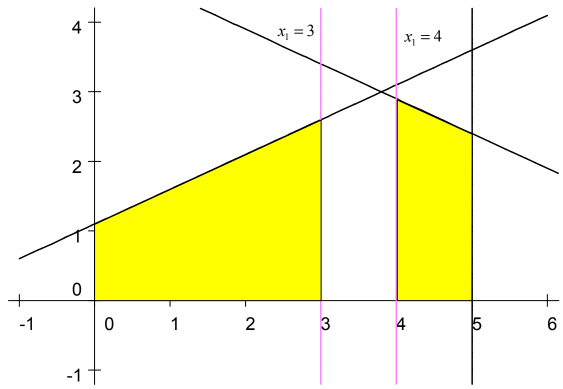
For the LP relaxation

$$MaxZ = -x_1 + 4x_2$$

$$\begin{aligned} \text{s. t. } -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 5 \\ x_i &\geq 0 \end{aligned}$$

Optimal solution of the relaxation is $(3.8, 3)$ with $z = 8.2$. Then we consider two cases: $x_1 \geq 4$ and $x_1 \leq 3$.

$$-10x_1 + 20x_2 = 22$$

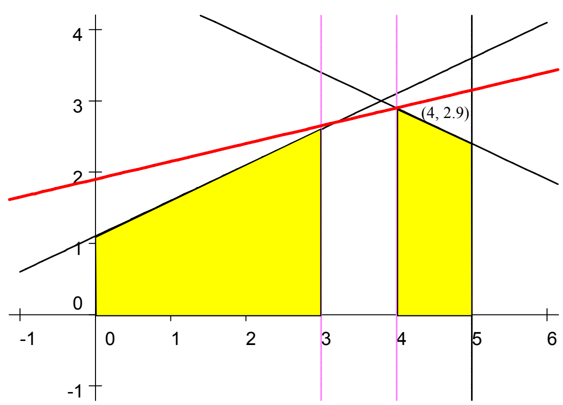


The linear programming relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t. } -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 5 \\ x_1 &\geq 4 \\ x_2 &\geq 0 \end{aligned}$$

has optimal solution at $(4, 2.9)$ with $Z = 7.6$.



The linear programming relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

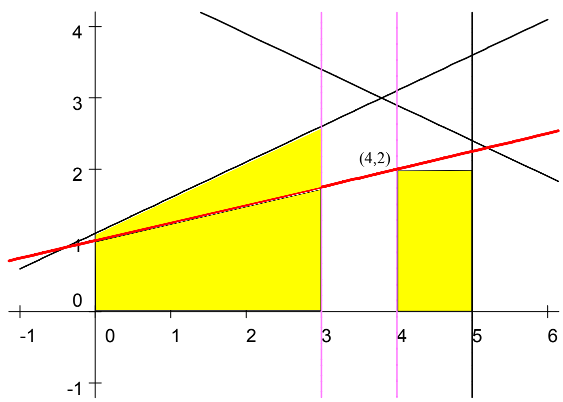
$$\begin{aligned} \text{s.t. } -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 4 &\leq x_1 \leq 5 \\ x_2 &\geq 3 \end{aligned}$$

has no feasible solution ($5x_1 + 10x_2 \geq 50$) so the IP has no feasible solution either.
The linear programming relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t. } -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 4 &\leq x_1 \leq 5 \\ 0 &\leq x_2 \leq 2 \end{aligned}$$

has an optimal solution at $(4, 2)$ with $Z = 4$. This is the optimal solution of the IP as well. Currently, the best value of Z for the original IP is $Z = 4$.

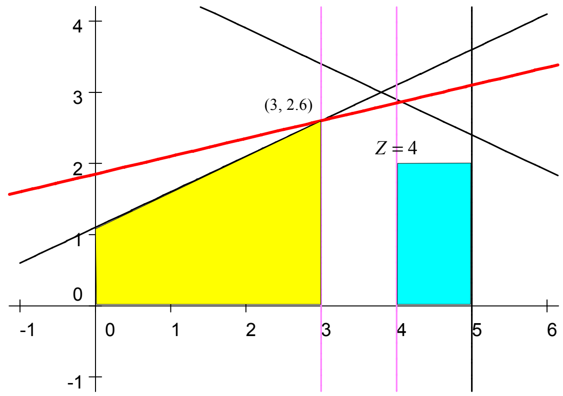


Now we consider the branch of $0 \leq x_1 \leq 3$. The LP relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t. } -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 3 \\ 0 &\leq x_i \end{aligned}$$

has an optimal solution at $(3, 2.6)$ with $Z = 7.4$. We branch out further to two cases: $x_2 \leq 2$ and $x_2 \geq 3$.



The LP relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

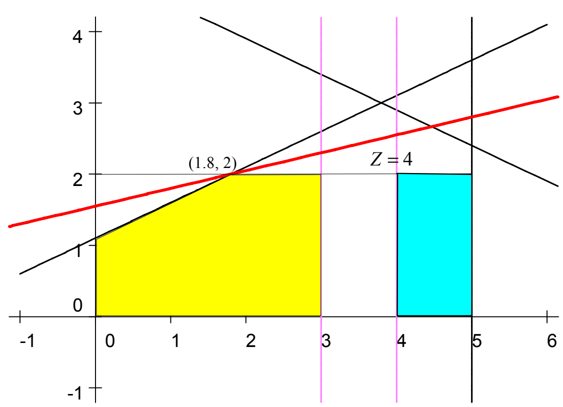
$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 3 \\ x_2 &\geq 3 \end{aligned}$$

has no feasible solution ($-10x_1 + 20x_2 \geq 30$). The IP has no solution either. The LP relaxation

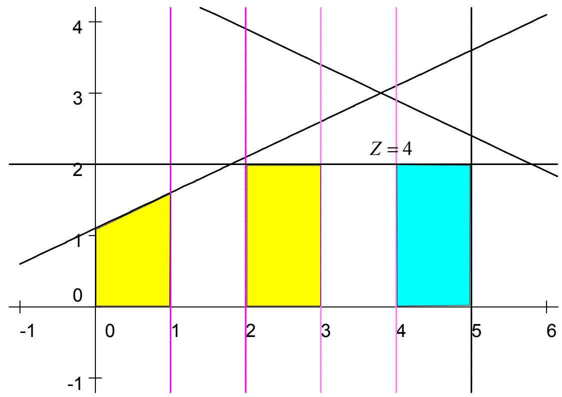
$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 0 &\leq x_1 \leq 3 \\ 0 &\leq x_2 \leq 2 \end{aligned}$$

has an optimal at $(1.8, 2)$ with $Z = 6.2$.



We branch further with two cases: $x_1 \geq 2$ or $x_1 \leq 1$ (we still have $0 \leq x_2 \leq 2$).

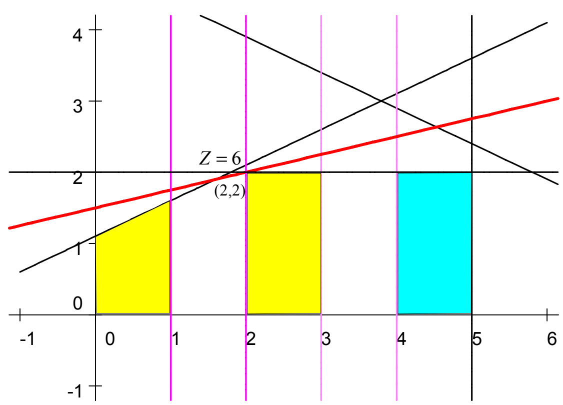


The LP relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t. } -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 2 &\leq x_1 \leq 3 \\ 0 &\leq x_2 \leq 2 \end{aligned}$$

has an optimal at $(2,2)$, with $Z = 6$. Since this is better than the incumbent $Z = 4$ at $(4,2)$, this new integer solution is our current best solution.

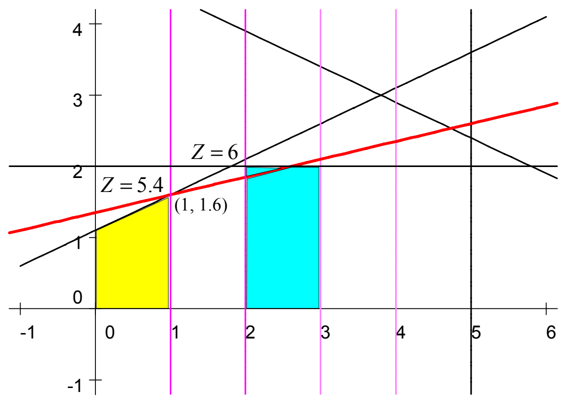


The LP relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t. } -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 0 &\leq x_1 \leq 1 \\ 0 &\leq x_2 \leq 2 \end{aligned}$$

has an optimal at $(1, 1.6)$ with $Z = 5.4$. Then any integer solution in this region can not give us a solution with the value of Z greater than 5.4. This branch is fathomed.



Example 2 *Consider the following BIP problem:*

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$\begin{aligned} \text{s.t. } 6x_1 + 3x_2 + 5x_3 + 2x_4 &\leq 10 \\ x_3 + x_4 &\leq 1 \\ -x_1 + x_3 &\leq 0 \\ -x_2 + x_4 &\leq 0 \end{aligned}$$

$$x_i \text{ are binary}$$

The optimal solution of the LP relaxation

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$\begin{aligned} \text{s.t. } 6x_1 + 3x_2 + 5x_3 + 2x_4 &\leq 10 \\ x_3 + x_4 &\leq 1 \\ -x_1 + x_3 &\leq 0 \\ -x_2 + x_4 &\leq 0 \\ x_i &\leq 1 \text{ for } 1 \leq i \leq 4 \\ x_i &\geq 0 \end{aligned}$$

has optimal solution at $(5/6, 1, 0, 1)$ with $Z = 16.5$.
Branch 1: $x_1 = 0$ or $x_1 = 1$.
1. $x_1 = 0$. The problem becomes

$$\text{Max } Z = 5x_2 + 4x_4$$

Subject to

$$\begin{aligned} 3x_2 + 2x_4 &\leq 10 \\ -x_2 + x_4 &\leq 0 \end{aligned}$$

$$x_i \text{ are binary}$$

The optimal solution of the LP relaxation is at $(1,1)$ with $Z=9$. (Current best solution.)
2. $x_1 = 1$. The LP relaxation

$$\text{Max } Z = 9 + 5x_2 + 6x_3 + 4x_4$$

$$\begin{aligned} \text{s.t. } 3x_2 + 5x_3 + 2x_4 &\le 4 \\ x_3 + x_4 &\le 1 \\ x_3 &\le 1 \\ -x_2 + x_4 &\le 0 \\ x_i &\le 1 \text{ for } 2 \le i \le 4 \\ x_i &\ge 0 \end{aligned}$$

has optimal solution at $(1, 0.8, 0, 0.8)$ with $Z = 16.2$.
Branch: $x_2 = 0$ or $x_2 = 1$.
2.1 $x_2 = 0$. The LP relaxation (in this case we have $x_4 = 0$ as well)

$$\text{Max } Z = 9 + 6x_3$$

$$\begin{aligned} \text{s.t. } 5x_3 &\le 4 \\ x_3 &\le 1 \\ x_3 &\ge 0 \end{aligned}$$

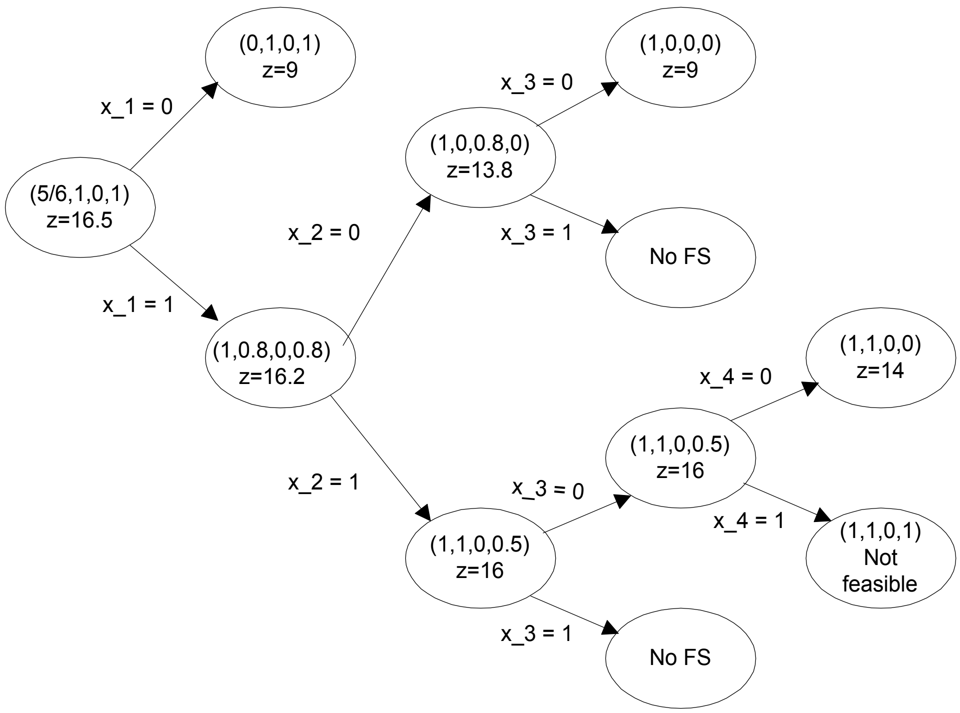
has optimal solution at $(1, 0, 0.8, 0)$ with $Z = 13.8$.
Branch: $x_3 = 0$ or $x_3 = 1$.
2.1.1 $x_3 = 0$. The optimal solution is $(1,0,0,0)$ with $Z = 9$ (not better than the current best solution).
2.1.2 $x_3 = 1$. Not feasible.
2.2 $x_2 = 1$. The optimal solution of the LP relaxation

$$\text{Max } Z = 14 + 6x_3 + 4x_4$$

$$\begin{aligned} \text{s.t. } 5x_3 + 2x_4 &\le 1 \\ x_3 + x_4 &\le 1 \\ x_3 &\le 1 \\ x_4 &\le 1 \\ x_i &\ge 0 \end{aligned}$$

is at $(1, 1, 0, 0.5)$ with $Z = 16$.
2.2.1 $x_3 = 0$. The previous optimal solution $(1, 1, 0, 0.5)$ is still feasible therefore still ooptimal.
2.2.1.1 $x_4 = 0$. $(1,1,0,0)$ is feasible. $Z = 14$. (It becomes the current best solution.)
2.2.1.2 $x_4 = 1$. $(1,1,0,1)$ is not feasible.
2.2.2 $x_3 = 1$. No feasible solution.

Therefore the current best solution $(1,1,0,0)$ with $Z = 14$ is the optimal solution.



Rule of Fathoming

A subproblem is fathomed

1. The relaxation of the subproblem has an optimal solution with $z < z^*$ where z^* is the current best solution;
2. The relaxation of the subproblem has no feasible solution;
3. The relaxation of the subproblem has an optimal solution that has all integer values (or all binary if it is an BIP).