In this Lecture…

Frequency domain analysis

Discrete-Time Fourier Transform

Discrete Fourier Series or Discrete Fourier Transform

Introduction

- In most cases we want to know the frequency content of our signal
  - Why?
- Most popular analysis in frequency domain is based on work of Joseph Fourier
- We will cover two methods of frequency domain analysis of discrete-time signals
  - Discrete-Time Fourier Transform
  - Discrete Fourier Series or Discrete Fourier Transform

Spectrum of a signal

Fourier		Spectrum: Continuous	Spectrum: Discrete
Signal	Continuous	Fourier transform	Fourier series
Signal	Discrete	Discrete-time Fourier transform	Discrete Fourier transform Or Discrete Fourier series

Aperiodic Signal

Periodic Signal

Discrete Time Fourier Transform

- Applies to a-periodic discrete-time signal
- Mathematically calculated as,

$$X(\omega) = DTFT\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad -\pi < \omega < \pi$$

- The Inverse DTFT is calculated as

$$x[n] = IDTFT\{X(\omega)\} = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega \qquad -\infty < n < \infty$$

Periodicity?

- The Fourier transform of a Discrete-time signal

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$e^{-j2\pi kn} = \cos(2\pi kn) - j\sin(2\pi kn) = 1$$

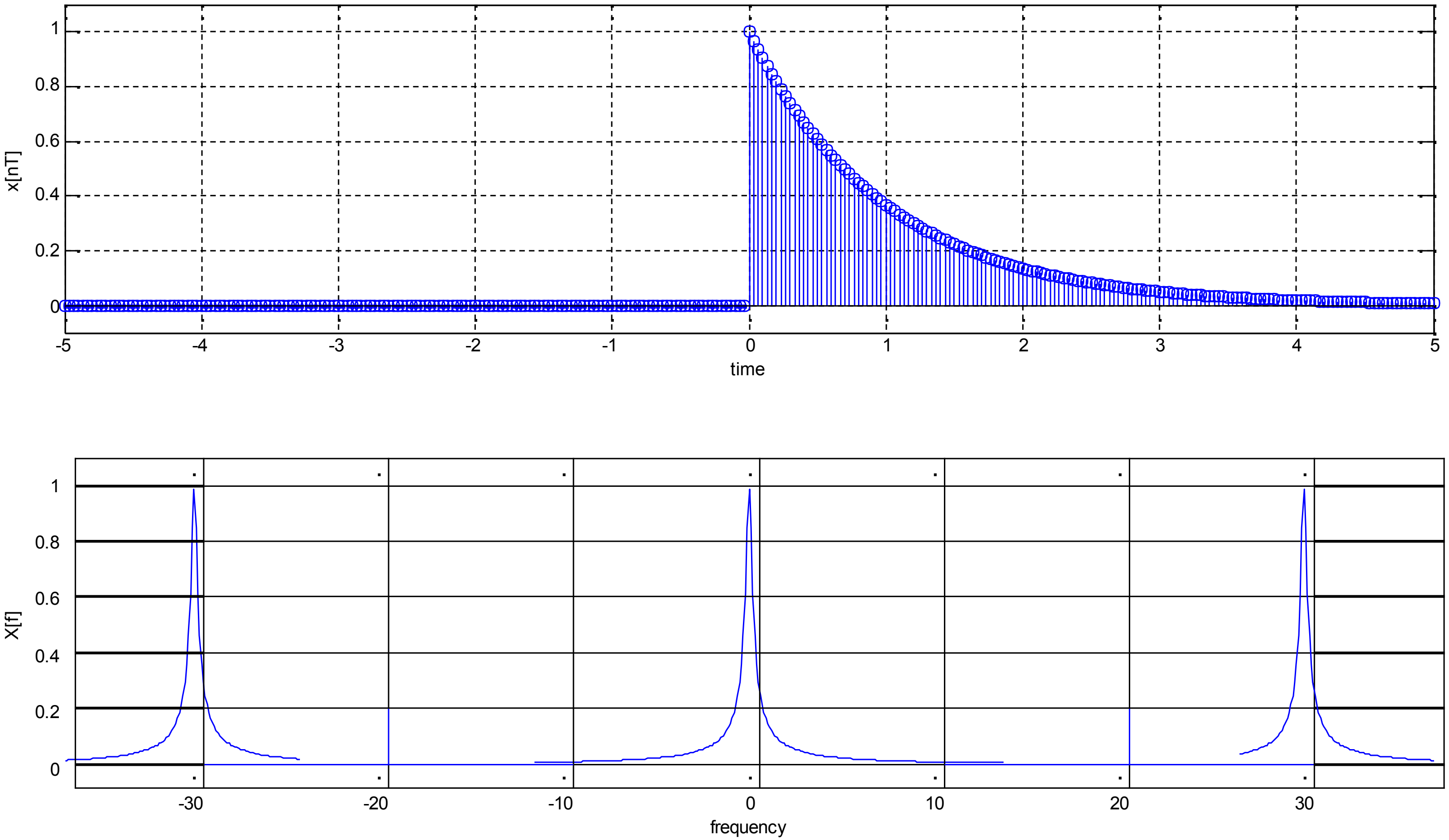
$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n]\; e^{-j(\omega + 2\pi k)n}$$

$$X(\omega + 2\pi k) = \sum x[n]\; e^{-j\omega n} e^{-j2\pi kn}$$

$e^{-j2\pi kn}$

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n]\; e^{-j\omega n}$$

$$X(\omega + 2\pi k) = X(\omega)$$



x[nT]
time
-5
-4
-3
-2
-1
0
1
2
3
4
5
0
0.2
0.4
0.6
0.8
1
X[f]
frequency
-30
-20
-10
0
10
20
30

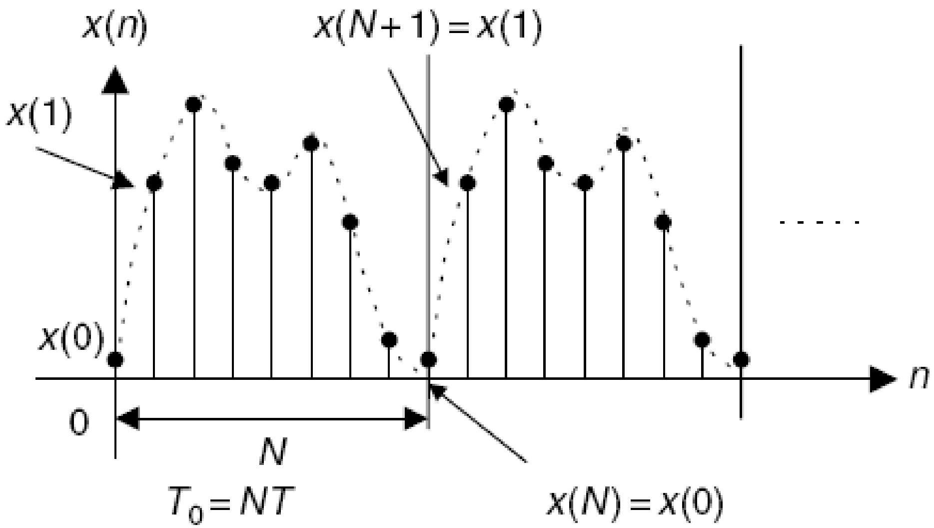
Points to ponder…

- The signal is discrete in time domain but continuous in frequency domain
  - The integration sign gives it away!
- Usually used for signals with infinite period
- DTFT gives exact frequency concentration for w (omega)[1]
  - But requires sequence to last $-\infty < n < \infty$
  - Requires infinite memory
- IDTFT also requires a infinite amount of memory to perform integration $-\pi < \omega < \pi$ and infinite processing time!

- In practice, we ***don't*** have infinite memory, we ***can't*** afford to have infinite processing time and the signal we want to analyze ***does not*** have infinite duration
- Besides, spectrum usually changes with time!
  - Music… changing notes
- Question for you
  - DTFT/IDTFT fails?
- DTFT/IDTFT is only good for simple analytical expressions like u[n]

Discrete Fourier Series

- Applies to periodic discrete-time signal
- Explanation by example!
- Suppose one wants to estimate the spectrum of discrete-time periodic signal



x(t) is sampled at fs Sampling period $T = 1/fs$

Fundamental period $T_0 = NT$

Fourier series for analog signal is given as

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt \qquad -\infty < k < \infty$$

Substituting $T_0 = NT$ and $w_0 = 2\pi/T_0$ and approximating the integration over one period by summation over one period, also t by nT, we get

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}} \qquad -\infty < k < \infty$$

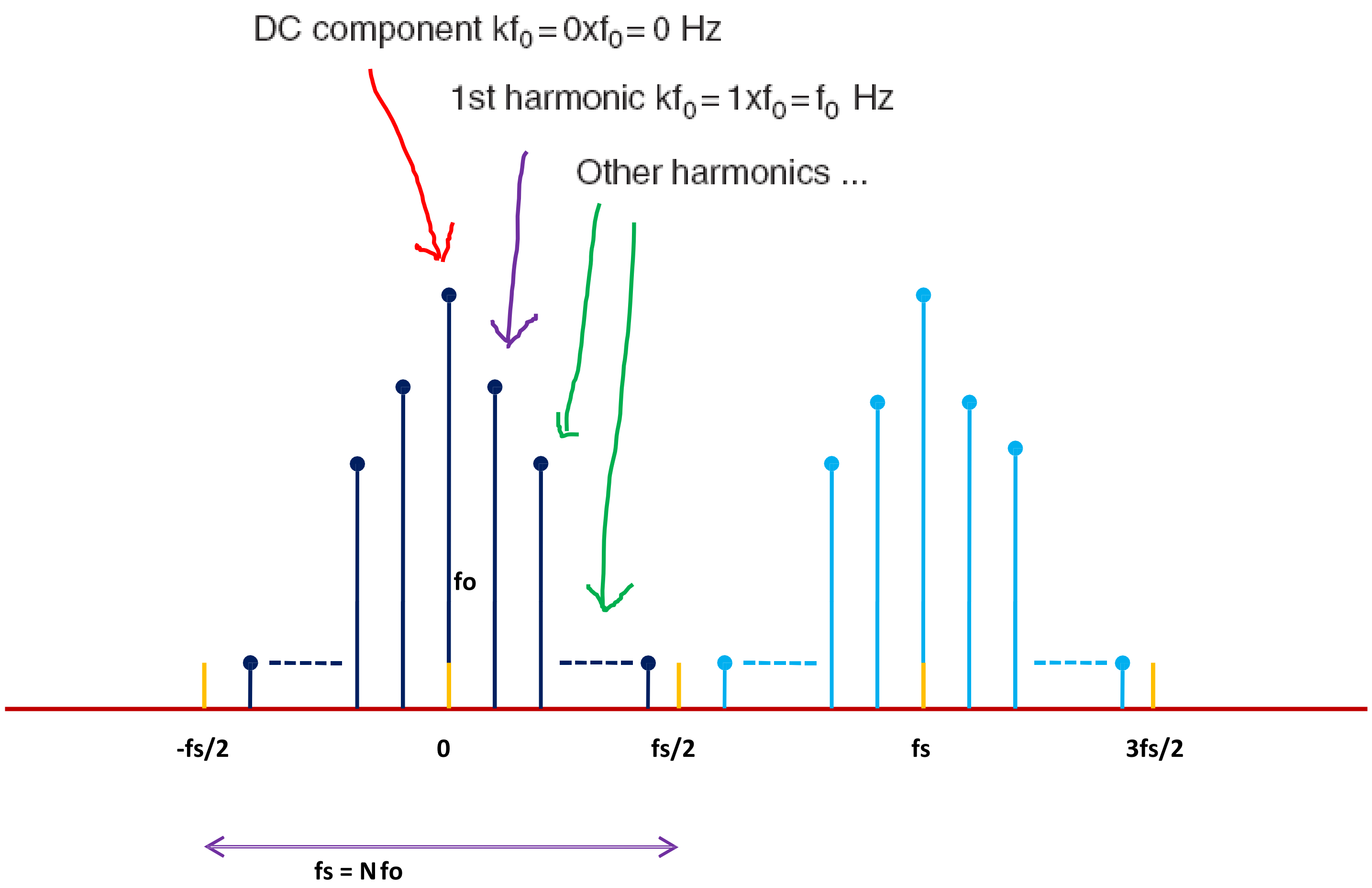
Analysis

$$\tilde{x}[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi kn}{N}}$$

Synthesis

It can be easily proved that c_k is periodic with N (just replace k by N+K)

So that, $c_k = c_{k+N}$



Amplitude spectrum for c_k

Points to ponder…

- As displayed in Figure, only the line spectral portion between the frequency -fs/2 and frequency fs/2 (folding frequency) represents the frequency information of the periodic signal

- The spectral portion from fs/2 to fs is a copy of the spectrum in the negative frequency range from -fs/2 to 0 Hz due to the spectrum being periodic for every Nf_0 Hz

- Amplitude spectral components from fs/2 to fs are really the folded version of the amplitude spectral components from 0 to fs/2 in terms of fs-f
  - So we usually only calculate for coeffs for positive frequencies and use periodic property to find coeffs for negative frequency

    i.e.

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\ e^{-j\frac{2\pi kn}{N}} \qquad k = 0,1,\ldots,N-1$$

- For the kth harmonic the frequency is f = kfo Hz

Example 1: The periodic signal $x(t) = \sin(2\pi)t$, is sampled at 4 Hz

(a) Compute the spectrum c_k

(b) Plot the two sided spectrum

From the analog signal we can find the fundamental frequency

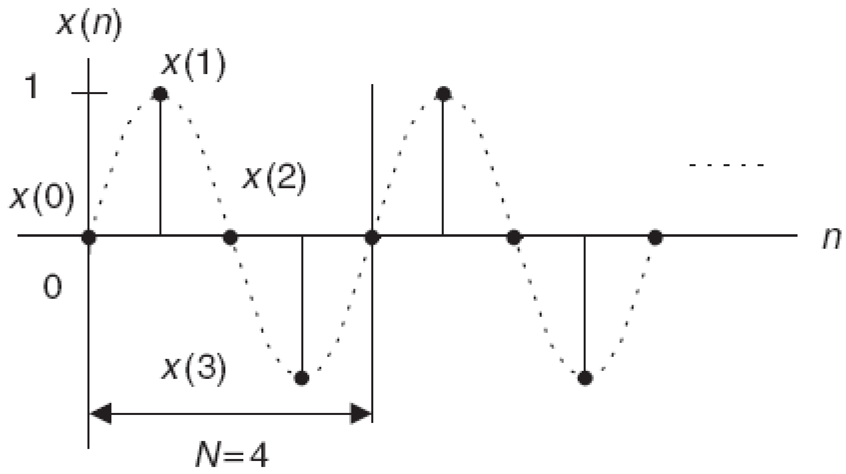
$$\omega_o = 2\pi \text{ rad/sec} \Rightarrow f_o = \frac{\omega_o}{2\pi} = \frac{2\pi}{2\pi} = 1 \text{ Hz}$$

And the fundamental period is $T_o = 1\sec$

Using the sampling interval $T = \frac{1}{f_s} = \frac{1}{4} = 0.25\sec$, we get the

Sampled signal as

$$x[n] = \sin(2\pi\ nT) = \sin(0.5\pi\ n)$$



Choosing the duration of one period we take the values of x[n]

$$x[0]=0,\ \ x[1]=1,\ \ x[2]=0,\ \ x[3]=-1$$

Solving the equation for DFS

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\ e^{-j\frac{2\pi kn}{N}}$$

$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = \frac{1}{4}\{x[0]+x[1]+x[2]+x[3]\} = \frac{1}{4}\{0+1+0-1\} = 0$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x[n]\ e^{-j\,2\pi\,k\,\frac{n}{4}} = \frac{1}{4}\left\{x[0]+x[1]e^{-j\pi/2}+x[2]e^{-j\pi}+x[3]e^{-j3\pi/2}\right\}$$

$$= \frac{1}{4}\{x[0]+x[1](-j)+x[2](-1)+x[3](j)\}$$

Using Euler's expansion of $e^x = \cos(x)+j\sin(x)$

$$= \frac{1}{4}\{0+1(-j)+0(-1)-1(j)\} = \frac{1}{4}\{-2j\} = -j0.5$$

Similarly, after solving c_2 and c_3, we'll get

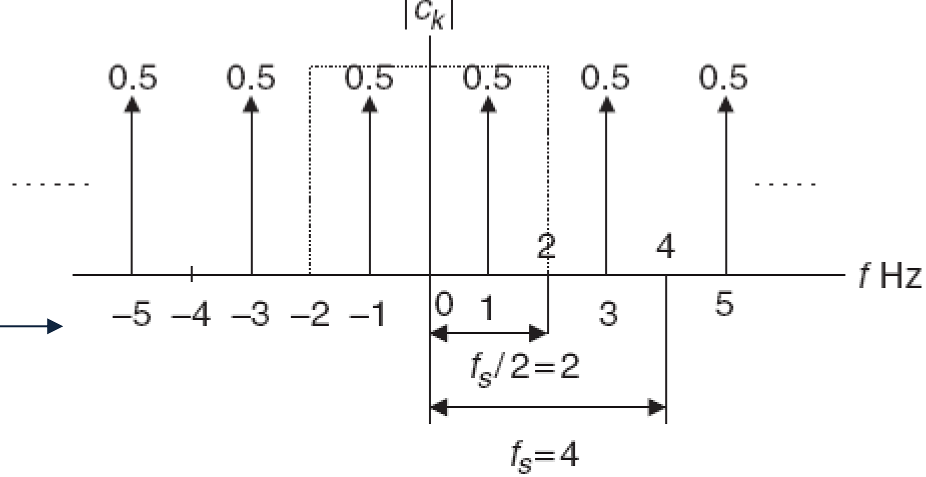
$$c_2 = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-j2\pi \times \frac{2n}{4}} = 0$$

$$c_3 = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-j2\pi \times \frac{3n}{4}} = j0.5$$

Using Periodicity property $c_k = c_{k+N}$, we get

$$c_{-1} = c_3 = j0.5 \qquad c_{-2} = c_2 = 0$$

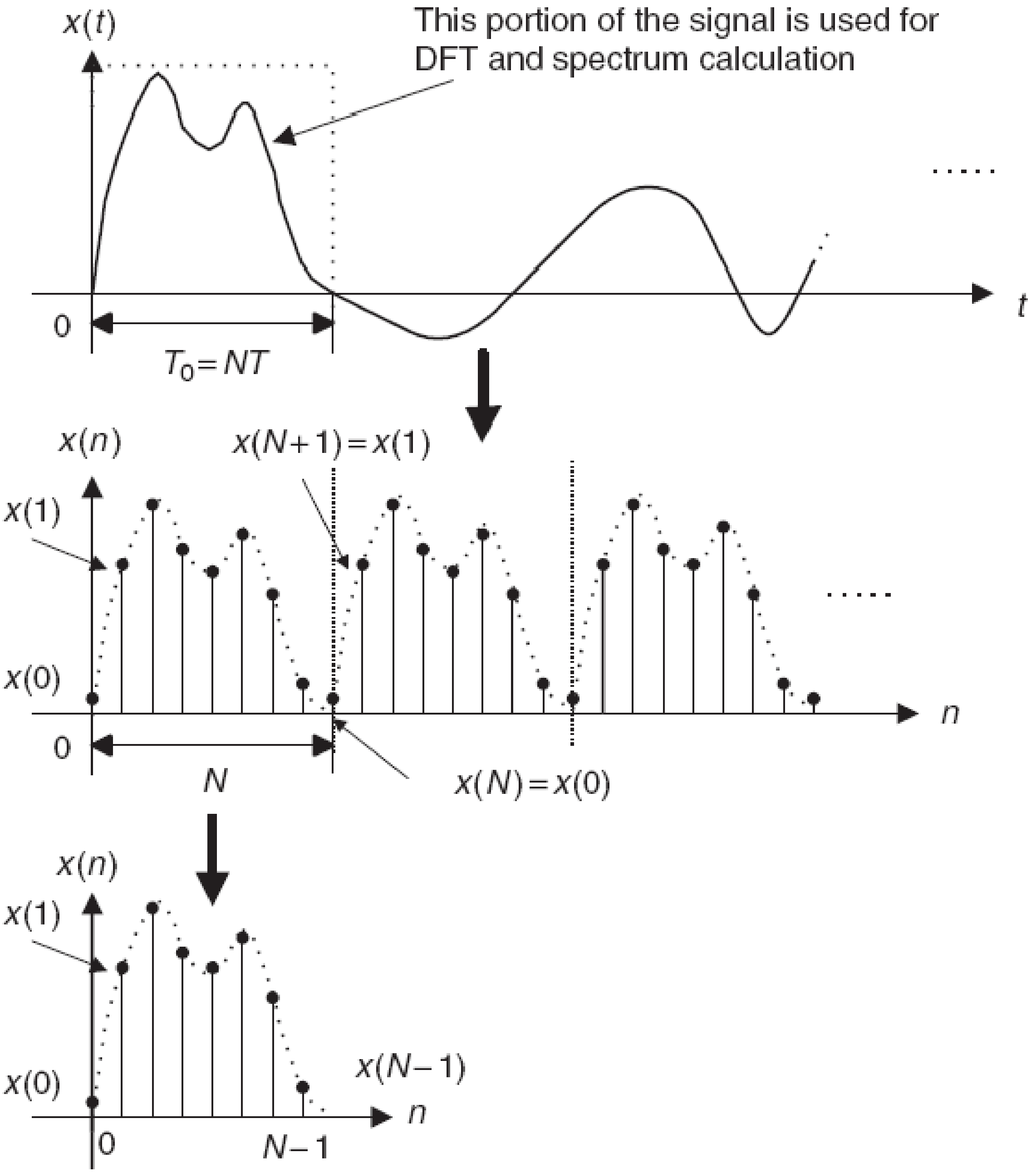
Values of k →



Example 2: Given a sequence x[n] for $0 \leq n \leq 3$ where $x[n] = [1\ 2\ 3\ 4]$, find the Discrete Fourier Series

Discrete Fourier Transform

- Applies to periodic discrete-time sequences
  - We can also make an a-periodic sequence seem periodic
- Lets concentrate on development of DFT
  - First, we assume that the process acquires data samples from digitizing the interested continuous signal for a duration of To seconds
  - Next, we assume that a periodic signal x[n] is obtained by copying the acquired N data samples with the duration of To to itself repetitively



x(t)
This portion of the signal is used for DFT and spectrum calculation
t
0
T0 = NT
x(n)
x(N+1) = x(1)
x(1)
x(0)
n
0
N
x(N) = x(0)
x(n)
x(1)
x(0)
x(N−1)
n
0
N−1

- Continued… developing DFT
  - We determine the Fourier series coefficients using one-period N data samples (using equation for DFS)
  - Multiply the coeffs by N to obtain i.e. $X(k) = Nc_k$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\ e^{-j\frac{2\pi kn}{N}} \qquad k = 0,1,\ldots,N-1 \rightarrow \textbf{DFS}$$

$$X(k) = \sum_{n=0}^{N-1} x[n]\ e^{-j\frac{2\pi kn}{N}} \qquad k = 0,1,\ldots,N-1 \rightarrow \textbf{DFT}$$

where n is the time-index

and k is the frequency index, also known as 'bins'

Lets simply the equation by substituting,

$$W_N = e^{-j2\pi/N} = \cos(\tfrac{2\pi}{N}) - j\sin(\tfrac{2\pi}{N})$$

Therefore the Discrete Fourier Transform of x[n] becomes

$$X(k) = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad k = 0,1,\ldots, N-1$$

and the Inverse Discrete Fourier Transform of X[k] becomes

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad n = 0,1,\ldots, N-1$$

Example 3: Given a sequence x[n] for 0≤n≤3 where x[n] = [1 2 3 4], find the Discrete Fourier Transform

Since N = 4, therefore $W_N^{kn} = W_4^{kn} = e^{-j2\pi kn/4} = e^{-j\pi kn/2}$

so now we can use the simplified formula

$$X(k) = \sum_{n=0}^{3} x[n]\ W_4^{kn} = \sum_{n=0}^{3} x[n]\ e^{-j\pi kn/2} \qquad k = 0,1,2,3$$

Solving for k = 0

$$X(0) = \sum_{n=0}^{3} x[n]\ e^{-j0} = x[0]e^{-j0} + x[1]e^{-j0} + x[2]e^{-j0} + x[3]e^{-j0}$$

$$= 1 + 2 + 3 + 4 = 10$$

Solving for k = 1

$$X(1) = \sum_{n=0}^{3} x[n]\ e^{-j\frac{\pi n}{2}} = x[0]e^{-j0} + x[1]e^{-j\frac{\pi}{2}} + x[2]e^{-j\pi} + x[3]e^{-j\frac{3\pi}{2}}$$

$$= x[0](1) + x[1](-j) + x[2](1) + x[3](-1)$$

$$= 1 - j2 - 3 - j4 = -2 + j2$$

Similarly, solving for k = 2 and k = 3 we get

$$X(2) = -2$$

$$X(3) = -2 - j2$$

You can even verify this result yourself using Matlab, try this

```
fft([1 2 3 4])
ans = 10          -2+2i          -2          -2-2i
```

Example 4: Find the Inverse Discrete Fourier Transform of the DFT found in example 3

In example 2, we found out the following values in freq. domain

X(0) = 10 ; X(1) = -2+j2 ; X(2) = -2 ; X(3) = -2-j2

Since N = 4, $W_4^{-1} = e^{-\pi/2}$, therefore,

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} X(n)\ W_4^{-kn} = \frac{1}{4}\sum_{k=0}^{3} X(n)\ e^{j\pi kn/2} \qquad n = 0,1,2,3$$

Solving for n = 0

$$x[0] = \frac{1}{4}\sum_{k=0}^{3} X(k)\ e^{j0} = X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0}$$

$$= \frac{1}{4}\{10 + (-2 + j2) - 2 + (-2 - j2)\} = 1$$

Solving for n = 1

$$x[1]=\frac{1}{4}\sum_{k=0}^{3} X(k)\ e^{j\frac{\pi k}{2}} = X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}$$

$$=\frac{1}{4}[X(0)\{1\}+X(1)\{j\}+X(1)\{-1\}+X(1)\{-1\}]$$

$$=\frac{1}{4}\{10+j(-2+j2)-(-2)-j(-2-j2)\}=2$$

Similarly, solving for n = 2 and n = 3 we get

$$x[2]=3$$

$$x[3]=4$$

You can even verify this result yourself using Matlab, try this

```
ifft([10, -2+2i, -2, -2-2i])

ans = 1     2     3     4
```

Relationship between **k** and its associated frequency

The N-point DFT expresses frequency content from -fs/2 Hz to fs/2 Hz

So can we map the frequency bin 'k' to its corresponding frequency as

$$f = \frac{kf_s}{N} \quad \text{Hz}$$

The frequency resolution, the frequency step between two DFT points, is given as

$$\Delta f = \frac{f_s}{N} \quad \text{Hz}$$

Example 4: In example 3, we found the DFT of the given sequence x[n]
If the sampling rate is 10 Hz, determine

(a) The sampling period, time index, and sampling time instant for a digital sample x[3] in time domain

(b) The frequency resolution, frequency bin number, and mapped frequency for each of the DFT coefficients X(1) and X(3) in frequency domain

(a) In time-domain, the sampling period can be calculated as

$$T = \frac{1}{f_s} = \frac{1}{10} = 0.1 \quad \text{sec}$$

For x[3] the time index is n = 3 and the sampling time instant is

$$t = nT = 3 \times 0.1 = 0.3 \quad \text{sec}$$

(b) In frequency-domain, since the total number of DFT points is N = 4

$$\Delta f = \frac{f_s}{Hz} = \frac{10}{N4} = 2.5$$

The frequency bin number for X(1) is k = 1, so its associated freq is

$$\Delta f = \frac{kf_s}{N} = \frac{1 \times 10}{4} = 2.5 \quad Hz$$

Similarly, for X(3) is k = 3, so its associated freq is

$$\Delta f = \frac{kf_s}{N} = \frac{3 \times 10}{4} = 7.5 \quad Hz$$

Amplitude & Phase response

Once we have X(k), we can find its amplitude and phase response

$$X(k) = R(k) + \mathrm{j}I(k)$$

$$\left|X(k)\right| = \sqrt{\{R(k)\}^2 + \{I(k)\}^2}$$

$$\varphi_k = \tan^{-1}\left(\frac{I(k)}{R(k)}\right)$$

Example 5: Plot the magnitude and phase response for the following discrete-time signal x[n]=[1 0 01]