new elements of discord. Hence, the principle of harmony tends to become more effective with the advance of knowledge and to dominate the energies of the investigator more powerfully as he attacks problems of a more fundamental nature.

[1] Speiser, A., "Die Theorie der Gruppen von Endlicher Ordung," **1927,** p. 77.

[2] Klein, F., "Vorlesungen über die Enlivicklung der Mathematik im 19. Jahrhundert," **1926,** p. 97.

[3] Poincaré, H., "La Valeur de la Science," **1908,** p. 7.

NOTE ON THE BEHAVIOR OF CERTAIN POWER SERIES ON THE CIRCLE OF CONVERGENCE WITH APPLICATION TO A PROBLEM OF CARLEMAN

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1. The first example of a continuous periodic function whose associated Fourier series $\sum_{-\infty}^{+\infty} c_\nu e^{i\nu x}$ is such that $\sum |c_\nu|^\rho$ converges for no $\rho < 2$ was given by Carleman.[1] Landau,[2] who simplified the example, called attention to a power series studied by Fabry[3] and Hardy[4] which series is continuous on the unit circle, the series $\sum |c_\nu|^{2-\delta}$ being divergent for a fixed but arbitrarily small $\delta > 0$. Hardy had to use a powerful machinery in his study of the singularities of this function; if one is satisfied, however, with determining merely the properties of convergence of the series on the unit-circle, a simpler argument can be used which applies to a much larger class of series. The method which I use for this purpose is based on some recent applications of Weyl's ideas regarding equi-distributed point-sets to number-theoretic questions due to van der Corput.[5] A fairly simple solution of Carleman's problem is obtained in this manner.

2. We shall study a class of power series

$$\sum_{n_0}^{\infty} f(n) \exp[2\pi i a(n)]\, z^n \qquad (1)$$

where $f(n)$ and $a(n)$ are subjected to one of the following three sets of conditions:

I. (*i*) $a(u)$ is a real differentiable function when $u \geqq n_0$ and $\lim a(u) = +\infty$;

(*ii*) $a'(u)$ is positive, never increasing and $\lim a'(u) = 0$;

(*iii*) $\sum |f(n) - f(n+1)|$ converges and $\lim f(n) = 0$.

II. (*i*) and (*ii*) as in I;

(*iii*) $\sum |f(n) - f(n+1)| [a'(n)]^{-1}$ converges and $\lim f(n)[a'(n)]^{-1} = 0$.

III. (*i*) and (*ii*) as in I;

(*iii*) $a''(u)$ exists and is never decreasing but remains <0;

(*iv*) $\sum |f(n) - f(n+1)| |a''(n)|^{-1/2}$ converges and $\lim f(n) |a''(n)|^{-1/2} = 0$.

We shall say that case N occurs when our series satisfies the set of conditions N.

THEOREM. *In case I the series* (1) *converges when* $|z| \leqq 1$, $z \neq +1$. *The convergence is uniform when* $|z| \leqq 1$, $|1 - z| \geqq \epsilon$. *In case II the series converges when* $|z| \leqq 1$ *and the convergence is uniform when* $|z| \leqq 1$, *arg* $(1 - z) \leqq \pi/2 - \epsilon$. *In case III the series converges uniformly for* $|z| \leqq 1$.

3. Writing

$$z = \exp(2\pi i t),\ P_{m,n}(t) = \sum_{k=m}^{n} f(k) \exp\{2\pi i[a(k) + tk]\},$$

$$S_{m,n}(t) = \sum_{k=m}^{n} \exp\{2\pi i[a(k) + tk]\},$$

we get

$$P_{m,n}(t) = \sum_{k=m}^{n} [f(k) - f(k+1)]\ S_{m,k}(t) + f(n)\ S_{m,n}(t)$$

where $m \geqq n_0$ is to be suitably chosen. Thus in order to prove the theorem we have merely to show the existence of constants K_ϵ, B and C, independent of k, such that

$$|S_{m,k}(t)| \leqq K_\epsilon \quad \text{, when } 0 < \epsilon \leqq t \leqq 1 - \epsilon, \tag{2}$$

$$|S_{m,k}(t)| \leqq B[a'(k)]^{-1} \quad \text{, when } 0 \leqq t \leqq 1/2, \tag{3}$$

$$|S_{m,k}(t)| \leqq C|a''(k)|^{-1/2}, \text{ when } 0 \leqq t \leqq 1, \tag{4}$$

in cases I, II and III, respectively.[6]

4. The proof of these inequalities is based on the following three lemmas:

LEMMA 1. Let $F(u)$ be real and differentiable in the interval $m \leqq u \leqq n$, let $F'(u)$ be monotone and $|F'(u)| \leqq 1 - \delta$, $\delta > 0$ fixed. Then there exists a constant A_δ such that

$$\left| \sum_{m}^{n} \exp[2\pi i\, F(\nu)] - \int_{m}^{n} \exp[2\pi i\, F(u)]du \right| < A_\delta.$$

This lemma is due to van der Corput[7] who considered the case $\delta = 1/2$. It follows from his proof in this case that

$$A_\delta < 1 + \frac{1-\delta}{2} \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu - 1 + \delta)}.$$

LEMMA 2. Let $F'(u)$ be monotone decreasing but remain >0, then

$$\left|\int_m^n \exp[2\pi i\, F(u)]du\right| \leqq \frac{1+\sqrt{2}}{2\pi}\,[F'(n)]^{-1}.$$

This follows from the identity

$$\int_m^n \exp[2\pi i\, F(u)]du = (2\pi i)^{-1}\int_m^n [F'(u)]^{-1}d\exp[2\pi i\, F(u)]$$

by applying Bonnet's form of the second mean value theorem.

LEMMA 3. Let $F'(u)$ be monotone decreasing and suppose that $F''(u)$ exists and is monotone increasing but remains <0. Then

$$\left|\int_m^n \exp[2\pi i\, F(u)]du\right| \leqq {}^5/_2[-F''(u)]^{-1/2}.$$

This lemma, save for the value of the numerical constant, is a special case of van der Corput's Hilfsatz 2 (loc. cit., p. 62). Its proof follows readily from lemma 2.

5. We can now set

$$F(u) = a(u) + t\,u.$$

Assuming $-1+\epsilon \leqq t \leqq 1-\epsilon$, we can find an $m \geqq n_0$ and a $\delta > 0$ such that

$$|F'(u)| = |a'(u) + t| \leqq 1 - \delta, \text{ when } u \geqq m.$$

Hence, we get by lemma 1

$$|S_{m,n}(t)| \leqq A_\delta + \left|\int_m^n \exp[2\pi i\, F(u)]du\right|,$$

and by lemma 2 the integral is less than

$${}^2/_5[a'(n) + t]^{-1}$$

provided $0 \leqq t \leqq 1 - \epsilon$. This estimate is sufficient to prove formulas (2) and (3). In case III we can apply lemma 3 for the estimating of the integral, obtaining

$$|S_{m,n}(t)| \leqq A_\delta + {}^5/_2[-a''(n)]^{-1/2}$$

which proves formula (4).

6. As an example to illustrate our theorem we can choose

$$a(n) = n^\alpha,\ f(n) = n^{-\beta},\ 0 < \alpha < 1,\ 0 < \beta.$$

The corresponding series

$$P(z, \alpha, \beta) = \sum_{n=1}^{\infty} n^{-\beta}\exp(2\pi i\, n^\alpha)\, z^n \qquad (5)$$

is the series of Fabry and Hardy mentioned in §1. Applying our theorem, we find that the series converges everywhere on the unit circle when

$\alpha + \beta > 1$, and uniformly if $\alpha/2 + \beta > 1$. These are the precise limits found by Hardy.[8] Thus the function $P(z, \alpha, \beta)$ is continuous on the unit circle when $\beta > 1 - \alpha/2$. This quantity can be made as close as we please to $^1/_2$ by choosing α near to 1. On the other hand, the series $\sum_{n=1}^{\infty} (n^{-\beta})^{\rho}$ converges only for $\rho > 1/\beta$ which can be made as close as we please to 2. This is the essence of Landau's observation.

As a second example we take

$$a(n) = n(\log n)^{-\alpha},\ f(n) = n^{-1/2}(\log n)^{-\beta},\ 0 < \alpha,\ 0 < \beta.$$

The corresponding series

$$P_{\alpha,\beta}(z) = \sum_{n=2}^{\infty} n^{-1/2}(\log n)^{-\beta} \exp[2\pi i\, n(\log n)^{-\alpha}]\, z^n \tag{6}$$

converges everywhere on the unit circle and the convergence is uniform whenever $\beta > {}^1/_2(\alpha + 3)$. The series $\sum [f(n)]^{\rho}$ is evidently never convergent for any $\rho < 2$. *Thus the function $P_{\alpha,\beta}(z)$ is a solution of Carleman's problem whenever $\beta > {}^1/_2(\alpha + 3)$.* The functions $P(z, \alpha, \beta)$ and $P_{\alpha,\beta}(z)$ are both infinitely many-valued and their only singularities are 0, 1 and ∞.[9]

A more general solution of Carleman's problem can be found as follows. Let $a(n)$ and $f(n)$ satisfy conditions III and the additional restriction that $\sum [f(n)]^{\rho}$ shall be convergent for $\rho \geqq 2$ and divergent for $\rho < 2$. The corresponding series (1) will be a solution of Carleman's problem.

[1] T. Carleman, *Acta Math.*, **41**, 1918 (377–389).

[2] E. Landau, *Math. Zeitschrift*, **5**, 1919 (147–153). See also O. Szász, *Ibid.*, **8**, 1920 (222–236) and T. H. Gronwall, *Bull. Amer. Math. Soc.*, **27**, 1921 (320–321).

[3] E. Fabry, *Acta Math.*, **36**, 1913 (69–104).

[4] G. H. Hardy, *Quart. Journ.*, **44**, 1913 (147–160). Cf. G. H. Hardy and J. E. Littlewood, these PROCEEDINGS, **2**, 1916 (583–586).

[5] J. G. van der Corput, *Math. Annalen*, **84**, 1921 (53–79). Extensions are to be found *Ibid.*, **87**, 1922 (39–65) and **89**, 1923 (215–254). Cf. Landau, *Acta Math.*, **48**, 1926 (217–263).

[6] The uniform convergence of the series in the regions stated in the theorem follows readily from the uniform convergence on the various arcs of the unit circle implied by the estimates (2)–(4).

[7] First paper quoted in note 5, p. 58, Satz 1. A more general theorem is to be found on p. 40 of the second paper.

[8] Loc. cit., p. 157. That the series (5) diverges when $\alpha + \beta < 1$ has been shown by Hardy in *Proc. London Math. Soc.* (2) **9**, 1911 (126–144).

[9] Proved by Hardy for series (5) who gives a thorough discussion of the nature of the singularity at $z = +1$. The location and possibly also the nature of the singularities can be determined for both series with the aid of a theorem due to E. Le Roy and extended by E. Lindelöf. See the latter's *Calcul des residus*, chapter V, especially pp. 109 and 124. This method can be used for an attack on the problem proposed by Hardy on p. 160 of his paper; it cannot give a complete solution of the problem, but such a solution is scarcely to be expected from the nature of the problem.