An Introduction to Maps Between Surgery Obstruction Groups

by

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In order to apply surgery theory we need methods for determining whether a given surgery obstruction $\sigma(f : N \to M, \hat{f}) \in L_n(\mathbb{Z}G, \omega)$ is trivial. Notice that it is more important to have invariants which detect L-groups than to be able to compute the L-group themselves.

One approach is to use numerical invariants such as Arf invariants, multisignatures, or the new "semi-invariants" [M1], [Da], [H-Mad], [P]. Another approach is to use transfer maps. For example,

(i) Dress [D] has shown that when G is a finite group, $L_n(\mathbb{Z}G)$ is detected under the transfer by using all subgroups of G which are hyper-elementary.

(ii) Wall [W9] has shown that when M is closed and G is finite, then image $(\sigma : [M, G/TOP] \to L_n(\mathbb{Z}G, \omega))$ is detected by $L_n(\mathbb{Z}G_2, \omega)$, where G_2 is the 2-Sylow subgroup of G.

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(iii) If G is a finite 2-group, then $L_n^p(\mathbb{Z}G)$ is detected under transfer and projection by using subquotients of G which are dihedral, quaternionc, semi-dihedral, and cyclic (see [T-W]).

The goal of this paper is to give a systematic procedure (when G is a finite 2-group) for computing transfer maps and the "twisted" transfer maps arising from codimension 1 surgery theory. Recall that if H is any subgroup of a finite 2-group G, then there exists a sequence of subgroups $H = H_0 \subset H_1 \subset H_2 \subset \dots \subset H_e = G$ such that H_i is an index 2 subgroup of H_{i+1} foı $i = 0, \dots, e-1$. Thus we might as well assume that H is an index 2 subgroup.

Suppose H is an index 2 subgroup of an arbitrary group G. Then we get the "push forward" exact sequence

$$\dots \to L_n(\mathbb{Z}H,\omega) \xrightarrow{f_!} L_n(\mathbb{Z}G,\omega) \to L_n(f_!) \to \dots$$

(see [R1],§2), and the transfer exact sequence

$$\dots \to L_n(\mathbb{Z}G,\omega) \xrightarrow{f^!} L_n(\mathbb{Z}H,\omega) \to L_n(f^!) \to \dots$$

(see [R1], §7.6).

One can view $S = \mathbb{Z}G$ as a <u>twisted quadratic extension</u> of $R = \mathbb{Z}H$. More precisely, suppose we choose $t \in G - H$. We let $a = t^2 \in H$, and we let $\rho : R \to R$ be conjugation by t. Then,

$$S = R_\rho[\sqrt{a}] = R_\rho[t]/(t^2 - a),$$

where t is viewed as an indeterminate over R such that $tx = \rho(x)t$ for all $x \in R$.

Let γ denote the Galois automorphism of S over R given by

$$\gamma : S \to S;\ x + yt \to x - yt \qquad (x,y \in R).$$

We want to extend the classical results in [L] chap. 7 and [M-H] appendix 2 where $f : R \to S$ is a quadratic extension of fields.

Recall that Wall [W2] defined groups $L_n(R,\alpha,u)$ for any ring with anti-structure, i.e. α is an anti-automorphism, u is a unit, $\alpha^2(r) = uru^{-1}$ for all $r \in R$, and $\alpha(u) = u^{-1}$. For example, $L_n(\mathbb{Z}G,\omega) = L_n(\mathbb{Z}G,\alpha_\omega,1)$, where $\alpha_\omega(\Sigma\, n_g g) = \Sigma\, n_g \omega(g) g^{-1}$.

Suppose we have a map of rings with anti-structure

$$f : (R,\alpha_0,u) \to (S,\alpha,u)$$

where S is a twisted quatratic extension $R_\rho[\sqrt{a}]$ with Galois automorphism γ. Then we also have the following γ-conjugate map

$${}^{\gamma}f : (R,\alpha_0,u) \to (S,\gamma\alpha,u).$$

Moreover, we can "twist" (α,u) to get $\widetilde{(\alpha,u)} = (\tilde{\alpha},\tilde{u})$, where $\tilde{\alpha}(s) = \sqrt{a}\ \gamma\alpha(s)\sqrt{a}^{\,-1}$ for all $s \in S$ and $\tilde{u} = \sqrt{a}\ \gamma\alpha(\sqrt{a}^{\,-1})u$. This yields a map

$$\tilde{f} : (R,\tilde{\alpha}_0,\tilde{u}) \to (S,\tilde{\alpha},\tilde{u}),$$

where $\tilde{\alpha}_0$ is the restriction of $\tilde{\alpha}$.

If we twist $(\gamma\alpha,u)$ we get that $\widetilde{(\gamma\alpha,u)} = (\gamma\tilde{\alpha}, -\tilde{u})$, and we get a map

$$\widetilde{{}^{\gamma}f} : (R,\tilde{\alpha}_0, -\tilde{u}) \to (S,\gamma\tilde{\alpha}, -\tilde{u}).$$

Then we get the following amazing isomorphisms.

$$\Gamma_! : L_{n-1}(\tilde{f}_!) \xrightarrow{\simeq} L_n(f_!) \qquad \Gamma_! : L_{n-1}(\widetilde{{}^{\gamma}f_!}) \xrightarrow{\simeq} L_n({}^{\gamma}f_!)$$

$$\Gamma^! : L_n(f^!) \xrightarrow{\simeq} L_{n+1}(\tilde{f}^!) \qquad \Gamma^! : L_n({}^{\gamma}f^!) \xrightarrow{\simeq} L_{n+1}(\widetilde{{}^{\gamma}f^!})$$

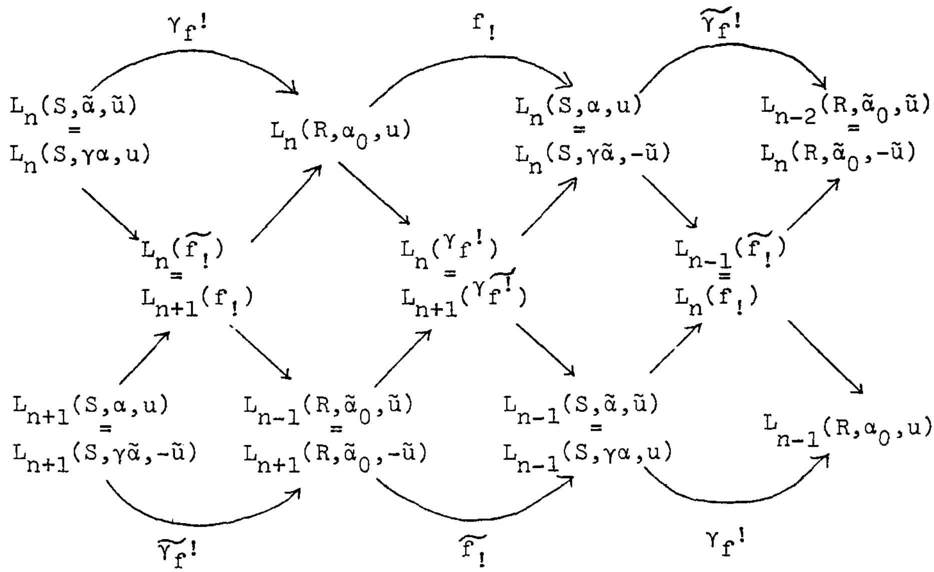
The maps $\Gamma_!$ and $\Gamma^!$ are defined using an algebraic version of integration along the fibre for line bundles. In the case of group rings the isomorphism $\Gamma_!$ is implicit in [W1] chap. 12.C, [C-S1] and explicit in [H]. The general case is due to Ranicki (after some prodding by us). (See [R1], §7.6 and the appendix by Ranicki in [H-T-W]).

By combining $\Gamma_!$ for f, $\Gamma^!$ for ${}^{\gamma}f$, and scaling isomorphisms

$$\sigma^{\sqrt{a}} : L_n(S,\alpha,u) \to L_n(S,\gamma\tilde{\alpha}, -\tilde{u}) = L_n(S,\widetilde{\gamma\alpha,u}) \quad \text{(see 2.5.5)}$$

Ranicki constructed the following commutative braid of exact sequences

(0.1) Twisting Diagram for $f : (R,\alpha_0,u) \to (S,\alpha,u)$

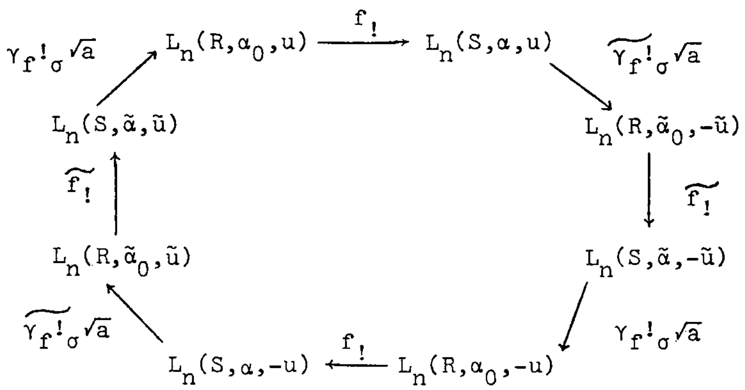


Thus the problem of computing $f_!$ and $\gamma_f{}^!$ is intimately related to the problem of computing the "twisted" maps $\widetilde{f_!}$ and $\widetilde{\gamma_f{}^!}$.

Examples:

1. Suppose $L_n = L_n^p$, n is even, plus R and S are semi-simple rings. Recall that L_{odd}^p is trivial for semi-simple rings (see [R2]). Thus, all of the groups along the bottom of (0.1) are trivial and the groups along the top form the following exact octagon (see also [War] and [Le]).

(0.2) Semi-simple 8-Fold Way



(a) Suppose we have

$$f : (F,\mathrm{id},1) \to (K,\mathrm{id},1)$$

where $F \to K$ is a quadratic extension of fields. If $n = 0$, then $L_n(F,\mathrm{id},-1) \simeq L_n(K,\mathrm{id},-1) \simeq (0)$; and we get the following exact sequence

(0.3) $0 \to L_0(K,\gamma) \to L_0(F) \to L_0(K) \to L_0(F) \to L_0(K,\gamma) \to 0$

which extends the exact sequences of Lam [L], chap. 7 and Milnor-Husemoller [M-H], appendix 2.

(b) Suppose we have

$$f : (K,\mathrm{id},1) \to (D,\alpha,1)$$

where D is a quaternionic division algebra and K is a maximal subfield of D. Since $\alpha|_K = \mathrm{id}$, α must be an involution of type O. If we let $L_0(D;O) = L_0(D,\alpha,1)$, $L_0(D;S_p) = L_0(D,\alpha,-1)$, and $L_0(K,\rho) = L_0(K,\tilde{\alpha}_0,\pm 1)$, then we get the following exact sequence

$$(0.4)\quad 0 \to L_0(D;S_p) \to L_0(K,\rho) \to L_0(D;O) \to L_0(K) \to L_0(D;O) \to L_0(K,\rho) \to L_0(D;S_p) \to 0$$

(c) Suppose D is a division ring with center F. Let K be a quadratic extension of F such that $D\otimes_F K$ is still a division ring. Then for any (anti) involution α_0 on D, we get another example

$$(D,\alpha_0,1) \to (D\otimes_F K,\alpha_0\otimes id,1)$$

(d) (trivial quadratic extension) Suppose we have

$$(d : (R,\alpha_0,u) \to (R \times R,\ \alpha_0 \times \alpha_0,u \times u)$$

where d is the diagonal map. Then $L_n(S,\tilde{\alpha},\pm\tilde{u})$ is trivial and (0.2) breaks into the short exact sequences of the form

$$1 \to L_n(R,\alpha_0,u) \xrightarrow{d_!} L_n(R,\alpha_0,u) \times L_n(R,\alpha_0,u) \to L_n(R,\alpha_0,u) \to 1$$

2. Codimension 1 Surgery (see [B-L], [Me], [W1] Chap. 12C, [C-S1], [C-S2], [H], and [R1] Chap. 7)

Suppose we have

$$f : (\mathbb{Z}H,\alpha_\omega,1) \to (\mathbb{Z}G,\alpha_\omega,1)$$

where H is an index 2 subgroup of G. Also, suppose X^n is a closed manifold with $(\pi_1 X,\omega_1 X) = (G,\omega)$. Let Y^{n-1} be a connected submanifold such that $\omega_1(\nu(Y \to X))$ induces the map $\psi : G \to G/H = \{\pm 1\}$. Then by combining results of Wall

[W1], chap. 12C, Cappell-Shaneson [C-S1], and Hambleton [H], we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 & & & & L_n(\mathbb{Z}H,\omega) & & \\
 & & & & \downarrow f_! & & \\
\cdots \to S(X) & \longrightarrow & [X,G/TOP] & \xrightarrow{\sigma_*} & L_n(\mathbb{Z}G,\omega) & \searrow^{\widetilde{\gamma_{f^!}\sigma^t}} & \\
\downarrow \tilde{\sigma}_* & & \downarrow & & \downarrow & & \\
 & & [Y,G/TOP] & & & & \\
 & & \downarrow & & & & \\
 & & L_{n-1}(\mathbb{Z}G,\omega\psi) & & L_n(f_!) & & L_n(\mathbb{Z}H,\tilde{\alpha}_\omega,-\tilde{1}) \\
 & & \downarrow\wr\, \sigma^t & & \uparrow\wr\, \Gamma_! & & \| \\
L_{n-1}(\mathbb{Z}H,\tilde{\alpha}_\omega,\tilde{1}) & \xrightarrow{\widetilde{f_!}} & L_{n-1}(\mathbb{Z}G,\tilde{\alpha}_\omega,\tilde{1}) & \longrightarrow & L_{n-1}(\widetilde{f_!}) & \longrightarrow & L_{n-2}(\mathbb{Z}H,\tilde{\alpha}_\omega,\tilde{1})
\end{array}$$

where $t \in G - H$ and $L_n = L_n^s$.

Assume $\pi_1 Y \doteq \pi_1 X$ and $n \geq 5$. If $f : M \to X$ represents an element in $S(X)$, then $\tilde{\sigma}_*(f)$ is trivial if and only if f is homotopic to a map f_1 such that $f_1^{-1}(Y) \to Y$ and $f_1^{-1}(X - Y) \to X - Y$ are simple homotopy equivalences.

Cappell-Shaneson [C-S1], [C-S2] and Hambleton [H] have observed that since

$$\text{image}(\sigma_* : [M,G/TOP] \to L_n(\mathbb{Z}G,\omega)) \subset \ker(\widetilde{\gamma_{f^!}\sigma^t})$$

$\widetilde{\gamma_{f^!}\sigma^t}(x)$ can be viewed as the <u>primary obstruction</u> to an element $x \in L_n(\mathbb{Z}G,\omega)$ arising from surgery on <u>closed</u> manifolds.

In this paper we compute the twisting diagram (0.1) where

$$f : (\mathbb{Z}H,\alpha_\omega,1) \to (\mathbb{Z}G,\alpha_\omega,1),$$

G is a finite 2-group, and $L_n = L_n^p$. Our motivation is that we have used these results to compute $\sigma_*^p : [M,G/TOP] \to L_n^p(\mathbb{Z}G,\omega)$ (see [H], [T-W], and [H-T-W]).

Roughly speaking, we proceed as follows

(i) We show that the Twisting Diagram (0.1) for f decomposes into a sum of diagrams indexed by the irreducible $\mathbb{Q}$-representations of G.

(ii) We use quadratic Morita theory to construct an isomorphism between each component diagram and the twisting diagrams associated to integral versions of Examples 1. (a),(b),(c),(d) i.e. maximal orders in division rings.

(iii) By using classical results on quadratic forms over division rings and localization sequences we are able to finish the calculation.

In Part I we carry out this program for the groups along the top and bottom of the twisting diagram (0.1). In Part II we compute the actual diagrams.

This paper is a preliminary version of [H-T-W] where we give details, compute the twisting diagrams for other L-groups in addition to L_n^p, and give geometric applications.

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PART I: Computation of the L^p-groups

Let G be a finite 2-group and H an index 2 subgroup.

In the oriented case, the groups $L_n^p(\mathbb{Z}G)$ have been computed by Bak-Kolster [K1], [K2], [B-K], Pardon [P], and Carlson-Milgram [C-M]. These results are nicely summarized by Theorem A in [H-M] where they give a decomposition of $L_n^p(\mathbb{Z}G)$ indexed by the irreducible $\mathbb{Q}$-representation of G. Besides extending their computations to the unoriented L-groups, $L_n^p(\mathbb{Z}G,\omega)$ and the codimension 1 surgery groups $L_n^p(\mathbb{Z}G,\tilde{\alpha}_\omega,\tilde{1})$, we also have to overcome the following problem. All of the above computations were based upon choosing a maximal involution invariant order, M_G which contains $\mathbb{Z}G$. Unfortunately, it is not always true that $M_G \cap \mathbb{Q}H$ is a maximal involution invariant order in $\mathbb{Q}H$. (Bruce Magurn has observed that this can happen even when G is the dihedral group of order 8). Thus it is not clear that the above computations and decompositions are functorial and we have had to modify their method somewhat. We have attempted to keep Part I fairly self-contained, but we would like to emphasize that Part I is based upon the work of the above authors and Wall's fundamental sequence of papers [W1] - [W8]. In Section (2.5) we try to clarify certain questions involving quadratic Morita theory.

§1. Basic Definitions and Overview

(1.1) Intermediate L-group

The use of arithmetic squares forces us to use L-groups other than L_n^p . Thus we shall start by recalling the relationships between the various L-groups.

A <u>ring with antistructure</u> (R,α,u) is an associative ring R, an anti-automorphism $\alpha : R \to R$ and a unit $u \in R$ such that $\alpha^2(a) = uau^{-1}$ for all $a \in R$, and such that $\alpha(u) = u^{-1}$. For any right R-module M, $D^\alpha(M) = \mathrm{Hom}_R(M,R)$ is the right R-module where

$(f \cdot r)(m) = \alpha(r) \cdot f(m)$ where $f \in D^\alpha M$, $m \in M$, and $r \in R$.

Since inner automorphisms act trivially on K-theory, D^α induces involutions on $K_i(R)$ and $\tilde{K}_i(R) = \mathrm{coker}(K_i(\mathbb{Z}) \to K_i(R))$ which we also denote by α.

If Y is an α-invariant subgroup of $\tilde{K}_i(R)$, $i=0$ or 1, then $L_n^Y(R,\alpha,u)$ denotes the <u>standard L-group</u> defined in [Ca], [R3] §9, and [R1] p. 688.

If $R = \mathbb{Z}G$ and $\alpha_\omega : \Sigma\, n_g g \to \Sigma\, n_g \omega(g) g^{-1}$, then we get the following <u>geometric L-groups</u>,

$$L_n^s(\mathbb{Z}G,\omega) = L_n^{\{\pi^{ab}\}\subset\tilde{K}_1}(\mathbb{Z}G,\alpha_\omega,1) \qquad \text{(see [W1])}$$

$$L_n^h(\mathbb{Z}G,\omega) = L_n^{\tilde{K}_1}(\mathbb{Z}G,\alpha_\omega,1) \qquad \text{(see [Sh])}$$

$$L_n^p(\mathbb{Z}G,\omega) = L_n^{\tilde{K}_0}(\mathbb{Z}G,\alpha_\omega,1) \qquad \text{(see [Ma], [P-R])}$$

If $Y_2 \subset Y_1 \subset \tilde{K}_i(R)$, $i = 0$ or 1 are both α-invariant subgroups, then we get the following Rothenberg exact sequence (see [R3] 9.1)

(1.1.1) $$\cdots \to L_n^{Y_2}(R,\alpha,u) \to L_n^{Y_1}(R,\alpha,u) \to H_\alpha^n(Y_1/Y_2) \to \cdots$$

where $H_\alpha^n(Y_1/Y_2)$ is the Tate cohomology group $\hat{H}^n(\mathbb{Z}/2, Y_1/Y_2)$ associated to the action of $\mathbb{Z}/2$ on Y_1/Y_2 via α.

Also,

(1.1.2) $$L_n^{\tilde{K}_1(R)}(R,\alpha,u) = L_n^{0\subset\tilde{K}_0(R)}(R,\alpha,u)$$

If $Y = \tilde{K}_0$, then

(1.1.3)

$$L_n^Y(R_1 \times R_2, \alpha_1 \times \alpha_2, u_1 \times u_2) \simeq L_n^Y(R_1,\alpha_1,u_1) \times L_n^Y(R_2,\alpha_2,u_2),$$

and

(1.1.4) $L_n^Y(R_1,\alpha_1,u_1) \simeq L_n^Y(R_2,\alpha_2,u_2)$ whenever (R_1,α_1,u_1) and (R_2,α_2,u_2) are quadratic Morita equivalent (see Section 2 for definition)

Since $\tilde{K}_i(R_1 \times R_2) \neq \tilde{K}_i(R_1) \times \tilde{K}_i(R_2)$ and since $\tilde{K}_i$ is not a Morita invariant, (1.3) and (1.4) are false for most Y : e.g. $Y = 0 \subset \tilde{K}_0$. This problem is overcome by introducing the following <u>variant L-groups</u>.

If X is an α-invariant subgroup of $K_i(R)$, then we get L-groups, $L_n^X(R,\alpha,u)$. (See [W3] for $i = 1$ and [B-W] for any $i \geq 0$).

If $\text{image}K_0(\mathbb{Z}) \subset X \subset K_0(R)$, then

(1.1.5) $$L_n^X(R,\alpha,u) \simeq L_n^{\tilde{X}}(R,\alpha,u),$$

where $\tilde{X}$ = image of X in $\tilde{K}_0(R)$.

If $\text{image}K_1(\mathbb{Z}) \subset X \subset K_1(R)$, then we get an exact sequence

(1.1.6) $\quad \cdots \to L_n^X(R,\alpha,u) \to L_n^{\tilde{X}}(R,\alpha,u) \to H_\alpha^n(\text{image}K_0(\mathbb{Z})) \to \cdots,$

where $\tilde{X}$ = image of X in $\tilde{K}_1(R)$.

Again we get Rothenberg sequences as in (1.1.1), and

(1.1.7) $$L_n^{K_i(R)}(R,\alpha,u) = L_n^{0 \subset K_{i-1}(R)}(R,\alpha,u).$$

Furthermore,

(1.1.8)

$$L_n^{Y_1 \times Y_2}(R_1 \times R_2, \alpha_1 \times \alpha_2, u_1 \times u_2) \simeq L_n^{Y_1}(R_1,\alpha_1,u_1) \times L_n^{Y_2}(R_2,\alpha_2,u_2)$$

and

(1.1.9) $L_n^Y(R_1,\alpha_1,u_1) \simeq L_n^{\phi(Y)}(R_2,\alpha_2,u_2)$ whenever (R_1,α_1,u_1) and (R_2,α_2,u_2) are quadratic Morita equivalent. ($\phi : K_i(R_1) \xrightarrow{\simeq} K_i(R_2)$ is the isomorphism induced by the Morita equivalence)

(1.1.10) <u>Convention</u>: Henceforth $L_n(R,\alpha,u)$ will denote $L_n^{K_1}(R,\alpha,u) \xrightarrow{\simeq} L^{0 \subset K_0}(R,\alpha,u)$.

Our first goal is to compute $L_n^p(\mathbb{Z}G,\alpha,u)$ for G any finite 2-group and (α,u) any anti-structure.

First consider the following long exact sequence

(1.1.11)

$$\cdots \to L_n^p(\mathbb{Z}G,\alpha,u) \to L_n^p(\hat{\mathbb{Z}}_2G,\alpha,u) \to L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \to \cdots$$

(1.2) Computation of $L_n^p(\hat{\mathbb{Z}}_2G,\alpha,u)$

(1.2.1) Theorem: For any finite 2-group and any anti-structure (α,u) on $\hat{\mathbb{Z}}_2G$, we get

$$L_n^p(\hat{\mathbb{Z}}_2G,\alpha,u) \cong L_n^p(\mathbb{Z}/2,\mathrm{id},1) = \begin{cases} \mathbb{Z}/2 & \text{if } n \equiv 0(2) \\ 0 & \text{if } n \equiv 1(2) \end{cases}$$

Theorem 1.2.1 follows from the following two results.

(1.2.2) Reduction Theorem: If R is a complete local ring then for any 2-sided ideal I,

(i) $K_0(R) \cong K_0(R/I)$, and

(ii) $L_n^p(R,\alpha,u) \cong L_n^p(R/I,\alpha,u)$, (assuming $\alpha(I) = I$)

Proof: See [W5], [B].

(1.2.3) Lemma: If G is a finite p-group, then $\ker\,(\mathbb{Z}/p)G \to \mathbb{Z}/p$ is nilpotent.

Proof: See [SE], p. 57.

Notice (1.2.3) implies that kernel $(\hat{\mathbb{Z}}_pG \to (\mathbb{Z}/p)G \to \mathbb{Z}/p)$ is complete.

(1.3) Computation of $L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$

It is well known that

$$\mathbb{Q}G = \pi A_\phi, \qquad \text{(see [S1] or [Y])}$$

where the product is taken over the set of isomorphism classes of irreducible $\mathbb{Q}$-representations. Each A_ϕ is a simple ring,

and $A_\phi = a_\phi \mathbb{Q}G$ for some central idempotent a_ϕ which can not be expresses as a sum of nontrivial central idempotents.
Since α^2 is an inner automorphism $\alpha(a_\phi)$ is either a_ϕ or $a_{\alpha(\phi)}$ where $\alpha(\phi)$ is another irreducible $\mathbb{Q}$-representation. In fact $a_\phi \in \mathbb{Z}[\frac{1}{2}]G$ (see [Y], p. 4) and $\Lambda_\phi = a_\phi(\mathbb{Z}[\frac{1}{2}]G)$ is a $\mathbb{Z}[\frac{1}{2}]$-maximal order in A_ϕ (see [Re], p. 379). Restriction gives a decomposition of rings with anti-structure.

(1.3.1) $(\mathbb{Z}[\frac{1}{2}]G,\alpha,u) =$

$$\prod_{\phi=\alpha(\phi)} (\Lambda_\phi,\alpha_\phi,u_\phi) \times \prod_{\phi\neq\alpha(\phi)} (\Lambda_\phi \times \Lambda_{\alpha(\phi)},\alpha_{\phi\times\alpha(\phi)},u_\phi \times u_{\alpha(\phi)}) .$$

The $\Lambda_\phi \times \Lambda_{\alpha(\phi)}$ are called type GL factors and make no contribution to any Wall group.

We prove the following result in Section 2.

(1.3.2) <u>Decomposition Theorem</u>: For any finite 2-group G and any anti-structure on $\mathbb{Z}G$, we get the following canonical isomorphisms.

$$L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \simeq L_n(\mathbb{Z}[\tfrac{1}{2}]G \to \hat{\mathbb{Q}}_2G,\alpha,u)$$

$$\simeq \prod_{\substack{\phi \\ \phi=\alpha(\phi)}} L_n(\Lambda_\phi \to \hat{\Lambda}_{\phi(2)},\alpha_\phi,u_\phi)$$

(Recall that L_n denotes $L_n^{K_1} \simeq L_n^{0\subset K_0}$)

Consider the following $\mathbb{Z}[\frac{1}{2}]$-algebras, where ζ_j is a primitive 2^j-th root of 1 and $-$ denotes complex conjugation.

(1.3.3)

1) $\Gamma_N = \mathbb{Z}[\frac{1}{2}][\zeta_{N+1}]$

2) $R_N = \mathbb{Z}[\frac{1}{2}][\zeta_{N+2} + \overline{\zeta}_{N+2}]$

3) $F_N = \mathbb{Z}[\frac{1}{2}][\zeta_{N+2} - \overline{\zeta}_{N+2}]$

4) $H_N = (\frac{-1,-1}{\mathbb{Z}}) \otimes R_{N-2}$, where

$(\frac{-1,-1}{\mathbb{Z}}) = \{\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k \mid ij = -ji = k,\ i^2 = j^2 = -1\}$

<u>Remark</u>: Each of these $\mathbb{Z}[\frac{1}{2}]$-algebras is a free $\mathbb{Z}[\frac{1}{2}]$-module of rank 2^N.

Now, consider the following anti-structures.

(1.3.4)

1) On Γ_N, $(Id,1)$, $(-,1)$, $(\tau,1)$, $(\overline{\tau},1)$

Id is the identity; $-$ is complex conjugation;
$\tau(\zeta_{N+1}) = -\zeta_{N+1}$ or, equivalently, Γ_{N-1} is the fixed field of τ; $\overline{\tau}$ has fixed field F_{N-1}.

2) On R_N, $(Id,1)$, $(\tau,1)$

3) On F_N, $(Id,1)$, $(-,1)$

4) On H_N, $(\alpha_1,1)$, $(\hat{\alpha},1)$ where $\alpha_1(i) = \hat{\alpha}(i) = i$

$\alpha_1(j) = \hat{\alpha}(j) = j$

and $\alpha_1|_{R_{N-2}} = Id$, $\hat{\alpha}|_{R_{N-2}} = \tau$.

In Section 2 we also prove the following result.

(1.3.5) Identification Theorem: If G is a finite 2-group, and (α,u) is any anti-structure on $\mathbb{Z}G$; then for any irreducible $\mathbb{Q}$-representation ϕ with $\alpha(\phi) = \phi$, we get that

$$L_n(\Lambda_\phi \to \hat{\Lambda}_{\phi(2)},\alpha_\phi,u_\phi) \cong L_n(\Delta_\phi \to \hat{\Delta}_{\phi(2)},\beta_\phi,v_\phi)$$

where $\Lambda_\phi = a_\phi \mathbb{Z}[\frac{1}{2}]G$ and $(\Delta_\phi,\beta_\phi,\pm v_\phi)$ is one of the rings with anti-structure in list (1.3.4). Recall that $L_n(\Delta_\phi,\beta_\phi,v_\phi) \cong L_{n+2}(\Delta_\phi,\beta_\phi,-v_\phi)$.

In Section 3 we compute $L_n(\Delta \to \Delta_{(2)},\beta,v)$ for all of the rings with anti-structure in List (1.3.4) and tabulate the results in Table 1.

Theoretically we could then calculate $L_n^p(\mathbb{Z}G,\alpha,u)$, but we restrict the anti-structure slightly at the start of Appendix I in order to easily identify $(\Delta_\phi,\beta_\phi,v_\phi)$ on List (1.3.4) (see Appendix I, part 1). In part 2 we settle the remaining questions involved in using 1.1.11.

§2. Proofs of the Decomposition Theorem (1.3.2) and of the Identification Theorem (1.3.5)

(2.1) Excision in Arithmetic Squares

Suppose S is a multiplicative subset of a ring A. Then $S^{-1}A$ is the localization of R away from S, $\hat{A} = \lim\limits_{s\in S} A/sA$ is the S-adic completion of A, and

(2.1.1)
$$\begin{array}{ccc} A & \longrightarrow & \hat{A} \\ \downarrow & & \downarrow \\ S^{-1}A & \to & S^{-1}\hat{A} \end{array}$$

is the arithmetic square associated to (A,S).

(2.1.2) K-theory Excision Theorem: For any integer i,

$$K_i(A \to \hat{A}) \cong K_i(S^{-1}A \to S^{-1}\hat{A})$$

(2.1.3) Corollary: For any finite 2-group G and any integer i,

$$K_i(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G) \cong K_i(\mathbb{Z}[\tfrac{1}{2}]G \to \hat{\mathbb{Q}}_2G)$$

$$\cong \prod_{\substack{\phi \\ \text{irred. } \mathbb{Q}\text{-rep.}}} K_i(\Lambda_\phi \to \hat{\Lambda}_{\phi(2)})$$

(same notation as in (1.3))

(2.1.4) L-theory Excision Theorem: (See [R1]. Also [B], [B-W], [C-M], [P], and [W7].)

We assume that A in (2.1.1) is equipped with an anti-structure (α,u) such that $\alpha|_S$ is the identity. Localization and completion then induce anti-structures on the other rings in (2.1.1). Let $X \subset K_i(S^{-1}A)$ and $Y \subset K_i(\hat{A})$ be α-invariant subgroups. Let C = kernel of $K_i(A) \to K_i(S^{-1}A)/X \oplus K_i(\hat{A})/Y$, and let I = image of $X \oplus Y \to K_i(S^{-1}A) \oplus K_i(\hat{A}) \to K_i(S^{-1}\hat{A})$. Then,

$$L_n^{C\to Y}(A \to \hat{A}) \cong L_n^{X\to I}(S^{-1}A \to S^{-1}\hat{A}).$$

(2.1.5) Corollary: For any finite 2-group G and any anti-structure (α,u) on $\mathbb{Z}G$, letting X and Y be trivial we get

$$L_n^{C_i(G)}(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \cong L_n^{K_{i+1}}(\mathbb{Z}[\tfrac{1}{2}]G \to \hat{\mathbb{Q}}_2G,\alpha,u)$$

$$\cong \prod_{\substack{\phi \\ \alpha(\phi)=\phi}} L_n^{K_{i+1}}(\Lambda_\phi \to \hat{\Lambda}_{\phi(2)},\alpha_\phi,u_\phi)$$

where $C_i(G) = \ker K_i(\mathbb{Z}G) \to K_i(\mathbb{Z}[\frac{1}{2}]G) \oplus K_i(\hat{\mathbb{Q}}_2G)$ and the rest of the notation is the same as in (1.3).

If $i = 1$, then the $L_n^{C_i(G)}$-groups are the L_n'-groups which were computed by Wall [W8].

Notice that if we can show that $L_n^{C_0(G)}(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G) \cong L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G)$, then (2.1.5) would imply the Decomposition Theorem (1.3.2).

(2.2) <u>Representation Theory for Finite 2-groups</u>

<u>Definition</u>: A finite 2-group π is <u>special</u> if it has no noncyclic, normal abelian subgroups.

(2.2.1) <u>Proposition</u>: A group π is special if and only if it is one of the following groups

(i) cyclic, $C_N = \langle x \mid x^{2^N} = 1 \rangle$

(ii) dihedral, $D_N = \langle x,y \mid x^{2^{N-1}} = y^2 = 1,\ yxy = x^{-1} \rangle$, $N > 3$

(iii) semi-dihedral, $SD_N = \langle x,y \mid x^{2^{N-1}} = y^2 = 1,\ yxy^{-1} = x^{2^{N-2}-1} \rangle$, $N > 3$

(iv) quaternionic, $Q_N = \langle x,y \mid x^{2^{N-1}} = 1,\ y^2 = x^{2^{N-2}},\ yxy^{-1} = x^{-1} \rangle$, $N \geq 3$.

Each special group π has a unique faithful, irreducible, $\mathbb{Q}$-representation $\psi(\pi)$.

For any irreducible $\mathbb{Q}$-representation of a group G $\rho : G \to GL(V_\rho)$, we let

$$D_\rho = \mathrm{End}_{\mathbb{Q}G}(V_\rho).$$

Schur's lemma implies that D_ρ is a division ring.

(2.2.2) Theorem: For any irreducible $\mathbb{Q}$-representation ϕ on a finite 2-group G, there exists a subgroup H with normal subgroup N such that

(i) H/N is special

(ii) If we pull $\psi(H/N)$ back to H and then induce up to G, we get ϕ.

(iii) $D_\phi \simeq D_\psi$, where $\psi = \psi(H/N)$.

Proof: (See [F])

(2.2.3) Table

π	$D_{\psi(\pi)}$
C_N	$\Gamma_{N-1} \otimes \mathbb{Q}$
D_N	$R_{N-3} \otimes \mathbb{Q}$
SD_N	$F_{N-3} \otimes \mathbb{Q}$
Q_N	$H_{N-1} \otimes \mathbb{Q}$

Thus the rings from list (1.3.3) are $\mathbb{Z}[\frac{1}{2}]$-maximal orders in the division rings $D_{\psi(\pi)}$. Notice that the centers of these division rings are precisely the fields which are subfields of $\mathbb{Q}(\zeta_j)$ for some j, namely fields of the form $\mathbb{Q}(\zeta_i)$, $\mathbb{Q}(\zeta_i + \overline{\zeta}_i)$, and $\mathbb{Q}(\zeta_i - \overline{\zeta}_i)$. (Recall ζ_j is a primitive 2^j-th root of 1.)

(2.2.4) Weber's Theorem: Suppose K is a subfield of $\mathbb{Q}(\zeta_j)$ for some j. Let $\mathcal{O}$ be the ring of algebraic integers in

K, and let $R = \mathcal{O}[\frac{1}{2}]$. Then

(i) K/Q is unramified over all odd primes. Over 2, it is totally ramified, and the unique dyadic prime d is principal.

(ii) The ideal class group $\Gamma(K) \simeq \tilde{K}_0(\mathcal{O}) = \tilde{K}_0(R)$ has odd order

(iii) The narrow ideal class group

$$\Gamma^*(K) \simeq \frac{(\text{group of ideals})}{\left(\begin{array}{l}\text{principal ideals } (x)\\ \text{such that } x > 0\\ \text{for all real places}\end{array}\right)}$$

also has odd order

Proof: For (ii), see Theorem 10.4 in [Was]. Class field theory implies that if $K = \mathbb{Q}(\zeta_i + \bar{\zeta}_i)$, and $\Gamma^*(K)$ does not have odd order; then K has a quadratic entension E/K which is unramified at all finite primes. But, then $E \otimes_K \mathbb{Q}(\zeta_i)$ would be an unramified, quadratic extension of $\mathbb{Q}(\zeta_i)$. Thus (ii) for $K = \mathbb{Q}(\zeta_i)$ implies (iii) for $K = \mathbb{Q}(\zeta_i + \bar{\zeta}_i)$.

(2.2.5) Corollary: For any N, $\tilde{K}_0(\Gamma_N)$, $\tilde{K}_0(R_N)$, $\tilde{K}_0(F_N)$, and $\tilde{K}_0(H_N)$ have odd order.

Proof: Notice that if $R = \Gamma_N$, R_N, or F_N , then $R = \mathcal{O}[\frac{1}{2}]$ where $\mathcal{O}$ = ring of algebraic rings in a subfield of $\mathbb{Q}(\zeta_i)$ for some i. Since H_N is a maximal order in the division algebra $H_N \otimes \mathbb{Q}$, (36.3) in [Re] implies that

$$\tilde{K}_0(H_N) \simeq \Gamma^*(\mathbb{Q}(\zeta_N + \bar{\zeta}_N)).$$

(2.3) (Linear) - Morita Theory: (see [Bass 1] and [Re] for details).

Definition: A Morita equivalence between two rings A and B is a 4-tuple (M,N,u,τ) where M and N are bimodules ${}_BM_A$ and ${}_AN_B$; $u : M\otimes_A N \to B$ and $\tau : N\otimes_B M \to A$ are bimodule isomorphisms such that

$$\tau(n \otimes m) \cdot n' = n \cdot u(m \otimes n'),$$

and

$$u(m \otimes n) \cdot m' = m \cdot \tau(n \otimes m')$$

for all $n, n' \in N$, and all $m, m' \in M$.

For any ring A, we let P_A denote the category of finitely generated projective right R-modules.

(2.3.1) Theorem: Assume (M,N,u,τ) is a Morita equivalence between A and B. Then, we get an equivalence of categories

$$P_A \underset{\otimes_B M}{\overset{\otimes_A N}{\rightleftarrows}} P_B$$

and an isomorphism

$$K_i(A) \xrightarrow{\sim} K_i(B).$$

Furthermore, center(A) = B - A - bimodule endomorphisms of M = center(B).

Examples

(2.3.2) Derived Morita equivalence

Suppose $M \in Ob\ (P_A)$ and A is a direct summand of M^n for some $n > 0$, i.e. M is a progenerator. Then

A and $B = \mathrm{End}_A(M)$ are Morita equivalent via $(M, N = \mathrm{Hom}_A(M,A), u, \tau)$ where $u(m \otimes n) \cdot m' = m \cdot n(m')$ and τ is the evaluation map.

If $\phi : G \to GL_n(\mathbb{Q})$ is a $\mathbb{Q}$-irreducible representation of a finite group G, then we let V_ϕ denote the simple module of the simple component $A_\phi \subset \mathbb{Q}G$. Thus A_ϕ and the division ring $D_\phi = \mathrm{End}_{A_\phi}(V_\phi)$ are Morita equivalent. Furthermore,

$$K_i(\mathbb{Q}G) \simeq \prod_\phi K_i(D_\phi).$$

(2.3.3) If R is a commutative ring, then a R-algebra Λ is <u>Azumaya</u> if there is a R-algebra B and a progenerator M of P_R such that $\Lambda \otimes_R B \cong \mathrm{End}_R(M)$ as R-algebras. (See [K-O].) If Λ is an Azumaya R-algebra, then Λ is central i.e. center $\Lambda = R$. Assume R is a Dedekind domain with field of fractions K. Then, whenever Λ is an Azumaya R-algebra, Λ is also a R-maximal order in $\Lambda \otimes_R K$. Conversely, if Λ is a R-maximal order in a simple K-algebra A with center R, then Λ is Azumaya if and only if $\hat{A}_\rho \simeq M_n(\hat{K}_\rho)$ for all finite prime ideals in R. (See [Rog].)

Suppose $\phi : G \to GL(V_\phi)$ is a irreducible $\mathbb{Q}$-representation of a finite group of order m. Then $\Lambda_\phi = a_\phi \cdot \mathbb{Z}[\frac{1}{m}]G$ is an Azumaya R_ϕ-algebra where $R_\phi = \mathrm{center}(\Lambda_\phi)$. (See [F], Corollaire 1 of Prop. 8.1.)

<u>Definition</u>: For any commutative ring R, $Br(R)$ is the set of Morita equivalence classes of Azumaya R-algebras. It becomes an abelian group under tensor product over R.

Suppose R is a Dedekind domain with quotient field K a finite extension of $\mathbb{Q}$ or $\hat{\mathbb{Q}}_p$ for some prime p.

(2.3.4) Theorem: Let $\Lambda_j \subset A_j$ for $j = 1,2$ be R-maximal orders in simple K-algebras A_j, $j = 1,2$. Then Λ_1 and Λ_2 are Morita equivalent if and only if A_1 and A_2 are Morita equivalent.

Proof: See [Re], Theorem (21.6).

(2.3.5) Corollary: The map $Br(R) \to Br(K)$ is a momomorphism.

(2.3.6) Theorem: Suppose that G is a finite 2-group. Then, for any i,

$$K_i(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G) \to \prod_{\phi} K_i(\Delta_\phi \to \hat{\Delta}_{\phi(2)})$$

where ϕ runs over the irreducible rational representations of G and Δ_ϕ is one of the rings on list (1.3.3).

Proof: First apply corollary (2.1.3). The result follows from (2.3.4) after consulting paragraph two of (2.3.2); (2.2.2) (iii); and Table 2.2.3.

(2.4) Proof of the Decomposition Theorem (1.3.2)

(2.4.1) Theorem (Swan): If G is a finite group, then $\tilde{K}_0(\mathbb{Z}G)$ is a finite group.

Then (2.3.6), (2.2.5), and (1.2.2 (i)) imply that $C_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}G)$ becomes an isomorphism when we localize at 2. A Rothenberg sequence argument then implies that

$$L_n^{C_0(\mathbb{Z}G)}(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G) \to L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G) .$$

Thus (1.3.2) is a special case of (2.1.5).

(2.5) Quadratic Morita Theory (Compare with [F-Mc] and [F-W])

Definition: A quadratic Morita equivalence between two rings with anti-structure (A,α,u) and (B,β,v) is given by a Morita equivalence (M,N,μ,τ) plus a B-A-bimodule isomorphism $h : M \to N$ where we make N into a B-A bimodule using α and β. We also require that

$$\alpha\tau(h(m_1 u) \otimes m_2) = \tau(h(m_2) \otimes m_1 v) \qquad \text{for all } m_1, m_2 \in M$$

(2.5.1) Theorem: A quadratic Morita equivalence between (A,α,u) and (B,β,v) induces isomorphisms

$$\phi : K_i(A) \cong K_i(B), \quad \text{(equivariant with respect to } \alpha_* \text{ and } \beta_*)$$

$$H^n_\alpha(K_i(A)) \cong H^n_\beta(K_i(B)),$$

and

$$L_n^X(A,\alpha,u) \cong L_n^{\phi(X)}(B,\beta,v) \quad \text{where } X \text{ is any } \alpha_*\text{-invariant subgroup of } K_i(A).$$

(2.5.2) Derived Quadratic Morita Equivalence Theorem: Suppose (A,α,u) is a ring with anti-structure and (M,N,μ,τ) is a (linear) Morita equivalence between A and B. Let R = center A = center B. Assume $h : M \to N$ is a right A-module isomorphism, where we use α to make N into a right A-module. Then,

(i) B admits a unique anti-automorphism β such that h becomes a B-A-bimodule isomorphism when we use β to make N a left S-module. $(\beta|R = \alpha|R)$; and

(ii) there exists a unique unit $v \in B$ such that (M,N,μ,τ,h) is a quadratic Morita equivalence between (A,α,u) and (B,β,v).

(2.5.3) <u>Corollary</u>: Suppose (A,α,u) is a ring with anti-structure where A is a simple algebra over a field K. Let V = simple right A-module and let D = the division algebra $\mathrm{End}_A(V)$. Then (A,α,u) is quadratic Morita equivalent to (D,β,v) for some anti-structure (β,v).

<u>Proof</u>: Since V and $V^\alpha = \mathrm{Hom}_A(V,A)$ are both simple right A-modules, there exists a right A-module isomorphism $h : V \to V^\alpha$.

(2.5.4) <u>Corollary</u>: Suppose (α,u) is an anti-structure on $\mathbb{Q}G$ for some finite group G. Then,

$$L_n(\mathbb{Q}G,\alpha,u) \doteq \prod_{\substack{\phi \\ \phi=\alpha(\phi)}} L_n(D_\phi,\beta_\phi,v_\phi).$$

(2.5.5) <u>Definition</u>: If (R,α,u) is a ring with anti-structure and w is a unit in R, then the <u>scaling</u> of (α,u) by w is the new anti-structure

$$(\alpha,u)^w = (\beta,v)$$

where $\beta(r) = w\alpha(r)w^{-1}$ for all $r \in R$, and $v = w\alpha(w^{-1})u$.

For any R-module M, there exists an isomorphism

$$D^\alpha M \to D^\beta M;\ f \to (f^w : x \to wf(x)).$$

Thus we get an isomorphism

$$\sigma^w : L_n(R,\alpha,u) \to L_n(R,(\alpha,u)^w)$$

Alternatively, σ^w can be gotten by applying (2.5.2) with

$M_R = R$ and $h : R \to \mathrm{Hom}_R(R,R) = R$ the map that sends r to rw^{-1}.

(2.5.6) <u>Definition</u>: Suppose R is a commutative ring with involution α_0. Then <u>$Br(R,\alpha_0)$</u> is the set of quadratic Morita equivalence classes of rings with anti-structure (Λ,α,u), where Λ is a Azumaya R-algebra and $\alpha|_R = \alpha_0$. $Br(R,\alpha_0)$ is an abelian group under tensor product.

<u>Warning</u>: We shall see in (2.5.9) that the quadratic analogue of (2.3.4) is <u>not</u> true in general.

Let $Br_0(R,\alpha_0)$ be the kernel of the forgetful map $Br(R,\alpha_0) \to Br(R)$.

Assume that R is a Dedekind domain with quotient field K. Let I = the group of R-fractional ideals in K, and let $g : K^* \to I$ be the map that sends $x \in K^*$ to the ideal (x). By sending elements and fractional ideals to their images under the map $\alpha_0 : K \to K$ we get an action of $\mathbb{Z}/2$ on K^* and I.

<u>Warning</u>: The map $I \to K_0(R)$ which sends a fractional ideal $\mathfrak{a}$ to the underlying module $[\mathfrak{a}]$ is not equivariant. Indeed, $[\alpha_0(\mathfrak{a}^{-1})] \simeq [\mathfrak{a}]^{\alpha_0} = \mathrm{Hom}_R(\mathfrak{a},R)$ made into a right R-module via α_0.

(2.5.7) <u>Theorem</u>: There exists an isomorphism

$$\Psi : Br_0(R,\alpha_0) \to \hat{H}^0(\mathbb{Z}/2, K^* \to I)$$

$$\cong \frac{\{(x,\mathfrak{a}) \in K^* \oplus I \mid \alpha_0(x) = x^{-1}, (x)\mathfrak{a} = \alpha_0(\mathfrak{a})\}}{\{(y\alpha_0(y^{-1}),\ (y)\beta\alpha_0(\beta) \mid (y,\beta) \in K^* \oplus I\}}$$

The map Ψ is defined as follows. Suppose (Λ,α,u) represents an element in $Br_0(R,\alpha_0)$. Choose M so that $\Lambda = End_R(M)$. Let $V = M\otimes_R K$ and $A = \Lambda\otimes_R K$. as in (2.5.3) we can choose a right A-module isomorphism $h : V \to V^\alpha$ which yields an anti-structure (β,v) on $End_A(V)$. Notice that $K = End_A(V)$ and $\beta = \alpha_0$. Let $ad(h) : V \times V \to K$ be the adjoint of $h : V \to V^\alpha = Hom_K(V,K)$. Then $\Psi(\Lambda,\alpha,u)$ is represented by $(v,\mathfrak{a})$ where $\mathfrak{a}$ is the fractional ideal generated by $h(M \times M)$.

The map Ψ has the following interpretation. Assume h is choosen so that $\mathfrak{a} \subset R$. Then the (linear) Morita equivalence derived from M and the pairing $h : M \times M \to \mathfrak{a}$ determines an equivalence of categories $Sesq(\Lambda,\alpha,u) \to Sesq(R,\alpha_0,v)$. But, nonsingular forms are sent to $\mathfrak{a}$-valued modular forms.

The following result was suggested to us by Karoubi.

(2.5.8) Proposition: Any ring with anti-structure (A,α,u) is quadratic Morita equivalent to $(M_2(A),\beta,1)$ where

$$\beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha(d) & \alpha(b)u \\ u^{-1}\alpha(c) & u^{-1}\alpha(a)u \end{pmatrix}$$

Proof: Let $\{e_1,e_2\}$ be the standard basis for $M = A \oplus A$ and $\{e_1^*,e_2^*\}$ be the dual basis for $N = M^*$. Then we let $h : M \to N$ be given by $h(e_1) = ue_2^*$ and $h(e_2) = e_1^*$ and we apply (2.5.2) to the derived Morita equivalence.

This implies that $Br_0(R,\alpha_0)$ is isomorphic to $B_0(R,\mathbb{Z}/2)$ in the sense of Frohlich-Wall [F-W]. Thus (2.5.7) is at least implicit in [F-W].

(2.5.9) Sample Calculations:

(i) $R = K$

If $\mathrm{char}K \neq 2$, then $Br_0(K,id)$ has two elements which are represented by $(K,\alpha_0,1)$ and $(K,\alpha_0,-1)$.

$Br_0(K,\alpha_0) = (1)$ when $\alpha_0 \neq id$ or $\mathrm{char}K = 2$.

(ii) R = finite extension of $\hat{\mathbb{Z}}_p$.

$$Br_0(R,id) \tilde{\to} Br_o(K,id) \simeq \{\pm 1\}$$

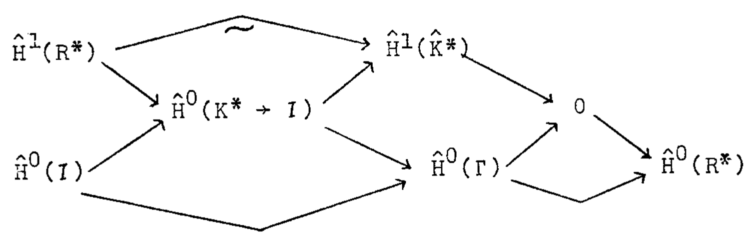
If $\alpha_0 \neq id$, we let r be the fixed subring of α_0 . Then, $Br_0(R,\alpha_0) = (1)$, when R is inert over r, and $Br_0(R,\alpha_0)$ has order 2 when R is ramified over r. Notice that $Br(R,\alpha_0) \to Br(K,\alpha_0)$ is not an injection.

In cases (iii) and (iv) we let $R = \mathcal{O}_{(\Sigma)}$ where $\mathcal{O}$ is a ring of algebraic integers and Σ is a set of prime ideals in $\mathcal{O}$, e.g. the center of the rings in List (1.3.3).

(iii) $Br_0(R,id) = {}_2R^* \oplus \Gamma/\Gamma^2$

where $\Gamma = \mathrm{coker}(K^* \to I)$, and ${}_2R^* = \{x \in R^* \mid x^2 = 1\}$.

The isomorphism comes from the following braid



which is induced by the following exact sequence of $\mathbb{Z}\ \mathbb{Z}/2$ - modules

$$1 \to R^* \to K^* \to I^* \to \Gamma \to 1$$

Remark: The localization sequence (see [R1], §4.2) implies that

$$L_3^p(R,id,1) \simeq \text{coker}(L_0^p(K,id,1) \to \bigoplus_\rho L_0^p(\hat{R}_\rho \to \hat{K}_\rho,id,1)) .$$

Similarly, if (Λ,α,u) represents an element $a \in \Gamma/\Gamma^2 \subset Br_0(R,id)$ and $A = \Lambda \otimes_R K$, then $L_3^p(\Lambda,\alpha,u) \simeq \text{coker}(L_0^p(A,id,1) \to \bigoplus_\rho L_0^p(\hat{\Lambda}_\rho \to \hat{A}_\rho,id,1))$, where $L_0^p(A,id,1) \simeq L_0^p(K,id,1)$ and where $L_0^p(\hat{\Lambda}_\rho \to \hat{A}_\rho,id,1) \simeq L_0^R(\hat{R}_\rho \to \hat{K}_\rho,id,1)$. But it is not true in general that $L_3^p(R,id,1) \simeq L_3^p(\Lambda,\alpha,u)$. For example if a is nontrivial and $\frac{1}{2} \in R$, then

$$\text{order } L_3^p(R,id,1) = 2 \times \text{order } L_3^p(\Lambda,\alpha,u).$$

(iv) Assume $\alpha_0 \neq id$.

Case 1: K is unramified over the fixed field of α_0 and Σ = the set of all prime ideals in $\mathcal{O}$. Then $Br_0(R,\alpha_0)$ has order 2, but the map $Br_0(R,\alpha_0) \to Br_0(K,\alpha_0) \oplus \pi_\rho Br_0(\hat{R}_\rho,\alpha_0)$ is trivial. Furthermore, if (Λ,α,u) represents the nontrivial element in $Br_0(R,\alpha_0)$. Then

$$L^p(R,\alpha_0,1) \not\simeq L_0^p(\Lambda,\alpha,u).$$

Case 2: Otherwise,

$$Br_0(R,\alpha_0) \cong \oplus\, Br_0(\hat{R}_\rho,\alpha_0),$$

where we can sum all finite primes ρ in Σ which are ramified over the fixed field of α_0.

These results are proven by using the isomorphism

$$\hat{H}^0(\mathbb{Z}/2; K^* \to I) \cong \hat{H}^0(\mathbb{Z}/2; \prod_{\rho\notin\Sigma} \hat{R}^*_\rho \times \prod_{\rho\in\Sigma} \hat{K}^*_\rho \times \prod_{\substack{v \\ arch}} \hat{K}^*_v \to \mathcal{C}(K)),$$

where $\mathcal{C}(K)$ is the idele class group of K.

Remark: If $R = \mathcal{O}$, then case (iv) is related to Connor's book [C]. In fact,

$$\hat{H}^0(\mathbb{Z}/2; K^* \to I) \simeq Gen(K/K^{\alpha_0})$$

(see chap. I in [C]), and if $\Psi(\Lambda,\alpha,u) = [(x,\mathfrak{a})]$, then $L_0^p(\Lambda,\alpha,u) \cong H_x(\mathfrak{a})$, where $H_x(\mathfrak{a})$ is the Witt group of x-symmetric, $\mathfrak{a}$-modular forms studied in chap. IV of [C].

(2.6) Proof of the Identification Theorem (1.3.5)

Theorem (1.3.5) will follow from (2.5.1) (or rather its relative version) if we can prove that $(\Lambda_\phi,\alpha_\phi,u_\phi)$ is quadratic Morita equivalent to $(\Delta_\phi,\beta_\phi,\pm 1)$ where $(\Delta_\phi,\beta_\phi,1)$ is one of the rings with anti-structure in List (1.3.4)

From the proof of Theorem (2.3.6), we know that Λ_ϕ is linearly Morita equivalent to Γ_N, F_N, R_N, or H_N for some N. Let R denote the center of Λ_ϕ. Then $(\Lambda_\phi,\alpha_\phi,u_\phi) \in Br(R,\alpha_0)$ for some α_0.

The proof divides into three cases.

1) D_ϕ is commutative, $\alpha_0 = Id$.

Then $(\Lambda_\phi,\alpha_\phi,u_\phi) \in Br_0(R,Id)$. From (2.5.9) (iii) and (2.2.4) (ii), $Br_0(R,Id) \cong \mathbb{Z}/2\mathbb{Z}$ and from (2.5.9) (i) we see that (R,Id,1) and (R,Id,-1) are the two elements.

2) D_ϕ is non-commutative, $\alpha_0 = \mathrm{Id}$.

The calculation in 1) shows that $(H_N,\alpha_1,1)$ and $(H_N,\alpha_1,-1)$ are the two distinct elements in Br(R,Id) which map to $[H_N] \in$ Br(R). From (2.3.6) we know that Λ_ϕ is linearly Morita equivalent to H_N, so done.

3) $\alpha_0 \neq \mathrm{Id}$.

First notice that R with each non-trivial involution occurs on List (1.3.4). From (2.2.4) (i) and (2.5.9) (iv) case 2, we see $Br_0(R,\alpha_0) \cong (1)$. Using (2.3.6) we are finished.

<u>Remark</u>: Notice that if $R = \Gamma_N$, F_N, or R_N, then, for any α_0,

$$Br(R,\alpha_0) \to Br(K,\alpha_0) \quad \text{is one to one.}$$

§3. Localization Sequence

The goal of this section is to compute $L_n(\Delta \to \hat{\Delta}_2,\beta,1)$ where Δ is any of the rings from List (1.3.3) and β is any (anti)-involution on Δ. Recall that L_n denotes $L_n^{O\subset K_0}$. Henceforth, we shall suppress writing the 1 in $(\beta,1)$.

The results are summarized in Table 1.

(3.1) General Background

Suppose K is an algebraic number field with ring of algebraic integers $\mathcal{O}$. Let $R = \mathcal{O}[\frac{1}{2}]$; D = central, simple, K-division algebra; Δ = R-maximal order in D; and β any (anti)-involution of Δ. We assume $\widetilde{K_0}(\Delta)$ has odd order.

Consider the following arithmetic square

$$\begin{array}{ccc} \Delta & \to & D \\ \downarrow & & \downarrow \\ \hat{\Delta} & \to & \hat{D} \end{array} .$$

Then,

(3.1.1)

$L_n(\Delta \to D,\beta) \simeq L_n(\hat{\Delta} \to \hat{D},\beta)$ (by L-theory Excision Theorem (2.1.4))

$\simeq \oplus\, L_n(\hat{\Delta}_\rho \to \hat{D}_\rho,\beta)$ (by 4.1.2 and 4.1.5 in [R1]),

where we sum over all maximal ideals ρ in R such that $\beta(\rho) = \rho$.

(3.1.2) Local Quadratic Morita Theorem: Suppose ρ is a maximal ideal in R such that $\beta(\rho) = \rho$ and such that $\hat{D}_\rho \simeq M_k(\hat{K}_\rho)$ for some k.

If $\beta|_R = \mathrm{id}$, we assume that $(\Delta,\beta,1) \in Br(R,\mathrm{id})$ maps to the trivial element in $Br(\bar{K},\mathrm{id}) \simeq \{\pm 1\}$, where $\bar{K}$ is the algebraic closure of K.

If $\beta|_R \neq \mathrm{id}$, we assume that ρ is unramified over the fixed field for $\beta|_K$.

Then,

$$L_n(\hat{\Delta}_\rho \to \hat{D}_\rho,\beta) \simeq L_n(\hat{R}_\rho \to \hat{K}_\rho,\beta)$$

Proof: Apply (2.5.1) and (2.5.9) (ii).

(3.1.3) Divissage Theorem: Suppose ρ is a maximal ideal in R such that $\beta(\rho) = \rho$. If $\beta|_R \neq \mathrm{id}$, we also assume that ρ is unramified over the fixed field for $\beta|_K$. Then, since $\frac{1}{2} \in \hat{R}_\rho$, we get

$$L_n(\hat{R}_\rho \to \hat{K}_\rho,\beta) \simeq L^p_n(k_\rho,\beta),$$

where k_ρ is the residue field R/ρ.

<u>Proof</u>: See 4.2.1 in [R1].

If (Δ,β) satisfies the assumptions in (3.1.2) and (3.1.3), then we get the following <u>localization diagram</u> with exact rows and columns

(3.1.4)

$$\begin{array}{ccccccc}
\vdots & & & & \vdots & & \vdots \\
\downarrow & & & & \downarrow & & \downarrow \\
L_{i+1}(\Delta\to\hat{\Delta}_2,\beta) & \to & L_i(D,\beta) & \xrightarrow{\psi} & L_i(\hat{\Delta}_2,\beta)\oplus\oplus L^p_i(k_\rho,\beta) & \to & L_i(\Delta\to\hat{\Delta}_2,\beta)\to\cdots \\
\downarrow & & \Vert & & \downarrow & & \downarrow \\
L_i(\Delta,\beta) & \longrightarrow & L_i(D,\beta) & \longrightarrow & \oplus L^p_i(k_\rho,\beta) & \to & L_{i-1}(\Delta,\beta)\to\cdots \\
\downarrow & & & & \downarrow{\scriptstyle \text{0-map}} & & \downarrow \\
L_i(\hat{\Delta}_2,\beta) & & & & L_{i-1}(\hat{\Delta}_2,\beta) & = & L_{i-1}(\hat{\Delta}_2,\beta) \\
\downarrow & & & & & & \\
\vdots & & & & & &
\end{array}$$

Since $\frac{1}{2} \in \hat{\Delta}_2$, we get that $\hat{\Delta}_2 \simeq \hat{D}_2$. If $\mathcal{O}$ contains a unique dyadic prime, then $\hat{D}_2$ is a simple ring. Recall that in $\oplus\, L^p_i(k_\rho,\beta)$ we are summing over the set of maximal ideals in R such that $\beta(\rho) = \rho$. Notice that this is the same as summing over the β-invariant, N.D. maximal ideals in $\mathcal{O}$ (where N.D. stands for nondyadic).

We shall compute $L_*(\Delta \to \hat{\Delta}_2,\beta)$ by computing the map ψ. Notice that the domain and range of ψ is expressed in terms of L-groups of semi-simple rings.

(3.1.5) Semi-simple Theorem: If A is a semi-simple ring, then $L^p_{2i+1}(A,\beta) = 0$ for any involution β.

Proof: See [R2].

(3.1.6) Reduction Theorem: $L_i(\hat{R}_\rho,\beta) = L_i(k_\rho,\beta)$, where $k_\rho = R_\rho/\rho$.

For any abelian group G, ${}_2G = \{g \in G | g^2 = 1\}$.

(3.2) Type O-Commutative Case: (Γ_N, id), (R_N, id), (F_N, id),

We assume $\beta = id$ which we suppress writing.

Then for any field K with char$K \neq 2$, $L^p_0(K) = W(K)$, the classical Witt ring of symmetric bilinear pairings over K (see [L], [M-H], [O'M], and [W4], p. 135). Multiplication in W(K) comes from the tensor product of pairings. Let $I(K) = \text{kernel } r : W(K) \to \mathbb{Z}/2$ where r is the rank map.

The group $L^p_2(K) \simeq (1)$ because any skew-symmetric non-singular pairing b has a symplectic basis, i.e. b is hyperbolic (see [M-H], 3.5).

The Rothenberg sequence plus (3.1.5) then imply that

$$L_n(K) \simeq 0,0,{}_2K^*,I(K) \quad \text{for } n \equiv 3,2,1,0(4)$$

(3.2.1) Examples

(i) If k is a finite field with char$k \neq 2$, then

$$\text{disc}: I(k) \cong k^*/k^{*2} \text{ has order } 2.$$

(ii) If $\hat{K}_\rho/\hat{\mathbb{Q}}_p$ with $[\hat{K}_\rho,\hat{\mathbb{Q}}_p] = \ell$, then

$$\text{disc} : I(\hat{K}_\rho)/I^2(\hat{K}_\rho) \tilde{\to} \hat{K}^*_\rho/\hat{K}^{*2}_\rho ;$$

Hasse-Witt: $I^2(\hat{K}_\rho) \tilde{\to} {}_2\mathrm{Br}(\hat{K}_2) \simeq \{\pm 1\}$. If ρ is N.D., then the map $L_0(\hat{K}_\rho) \to L_0(\hat{R}_\rho \to \hat{K}_\rho) \tilde{\to} L_0^p(k_\rho)$, sends $I^i(\hat{K}_\rho)$ onto $I^{i-1}(k_\rho)$ (see [M-H], IV, 1.4). Thus $I^1(\hat{K}_\rho) \to k^*_\rho/k^{*2}_\rho$ can be identified with the Hasse-Witt invariant. We also get the following exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \hat{O}^*_\rho/\hat{O}^{*2}_\rho & \to & \hat{K}^*_\rho/\hat{K}^{*2}_\rho & \xrightarrow{v} & \mathbb{Z}/2 & \to & 1 \\ & & & & \wr\uparrow \text{disc} & & \wr\uparrow r & & \\ & & & & I(\hat{K}_\rho)/I^2(\hat{K}_\rho) & \to & L_0(k_\rho)/I(k_\rho), & & \end{array}$$

where $\hat{O}_\rho$ is the integral closure of $\hat{\mathbb{Z}}_p$ in $\hat{K}_\rho$.

For any ρ, $\hat{O}^*_\rho \simeq \mu(\hat{K}_\rho) \times \hat{\mathbb{Z}}^\ell_\rho$ (see [S2], XIV, §4), where $\mu(\hat{K}_\rho)$ = roots of unity, Thus

$$\hat{O}^*_\rho/\hat{O}^{*2}_\rho \simeq \begin{cases} \mathbb{Z}/2 & \text{if } \rho \text{ is N.D.} \\ \mathbb{Z}/2^{\ell+1} & \text{if } \rho \text{ is dyadic} \end{cases}$$

(iii) $I(\mathbb{C}) \simeq (0)$ and $\text{sig} : I(\mathbb{R}) \tilde{\to} 2\mathbb{Z}$.

(iv) If $K/\mathbb{Q}$ with $[K,\mathbb{Q}] = r_1 + 2r_2$ where r_1 is the number of embeddings of K into $\mathbb{R}$, then

$$\text{disc} : I(K)/I^2(K) \tilde{\to} K^*/K^{*2},$$

Hasse-Witt: $I^2(K)/I^3(K) \tilde{\to} {}_2\mathrm{Br}(K)$, and

$\text{sig} : I^3(K) \tilde{\to} \bigoplus_v I^3(\hat{K}_v) \tilde{\to} (8\mathbb{Z})^{r_1}$, where v varies over the real embeddings of K.

Suppose K is a field on 2.2.3 , $\mathcal{O}$ is the ring of algebraic integers in K, and $R = \mathcal{O}[\frac{1}{2}]$. Since $\mathcal{O}$ has a unique prime over 2, $\hat{K}_2$ is a field and the Localization sequence (3.1.4) implies that $L_3(R \to \hat{R}_2)$ and $L_2(R \to \hat{R}_2)$ are trivial. We also get the following commutative diagram with exact rows and columns.

(3.2.2)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & I^2(K) & \xrightarrow{\psi_2} & I^2(\hat{K}_2) \oplus \bigoplus I(k_\rho) & & \\
 & & & & \downarrow & & \downarrow \rho_{ND} & & \\
0 & \to & L_1(R \to \hat{R}_2) & \to & L_0(K) & \xrightarrow{\psi} & L_0(\hat{K}_2) \oplus \bigoplus\limits_{\substack{\rho \\ N.D.}} L_0^\rho(k_\rho) & \to & L_0(R \to \hat{R}_2) \to 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K^*/K^{*2} & \xrightarrow{\psi_1} & \hat{K}_2^*/\hat{K}_2^{*2} \oplus \bigoplus\limits_{\rho} \mathbb{Z}/2 & & \\
 & & & & \downarrow & & \downarrow N.D. & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The snake lemma then yields the following exact sequence

(3.2.3)

$$0 \to \ker\psi_2 \to L_1(R \to \hat{R}_2) \to \ker\psi_1 \xrightarrow{\partial} \operatorname{coker}\psi_2 \to L_0(R \to \hat{R}_2) \to \operatorname{coker}\psi_1 \to 0$$

<u>Computation of ψ_1:</u>

If $\mathcal{I}$ = group of $\mathcal{O}$-fractional ideals in K and Γ = ideal class group, then we get the following exact sequence

$$1 \to \mathcal{O}^* \to K^* \to \mathcal{I} \to \Gamma \to 1$$

Since Γ has odd order by Weber's Theorem (2.2.4), we get the following short exact sequence

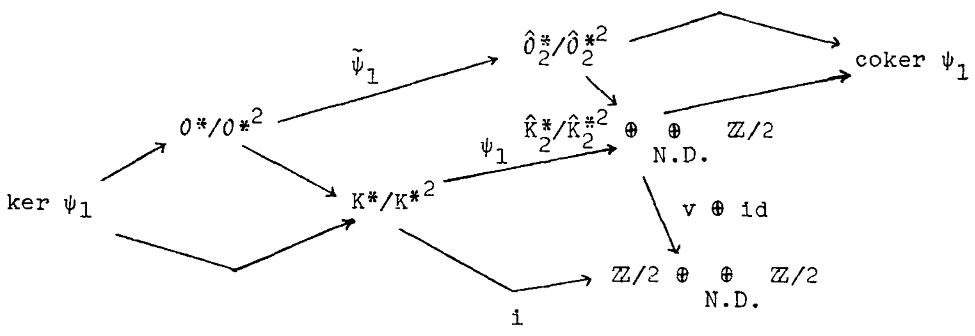
$$1 \to \mathcal{O}^*/\mathcal{O}^{*2} \to K^*/K^{*2} \xrightarrow{i} I/I^2 \to 1$$

Since any $\mathcal{O}$-fractional ideal can be expressed uniquely as a product of prime ideals, we can identify I with the free abelian group generated by the maximal ideals in $\mathcal{O}$. Thus,

$$I/I^2 \cong \bigoplus_{\rho \subset \mathcal{O}} \mathbb{Z}/2 .$$

Consider the following commutative braid of exact sequences

(3.2.4)



(3.2.6) Lemma: Assume $L/\hat{\mathbb{Q}}_p$ is a finite extension and p is odd. Then for any element $x \in L^*$, $L(\sqrt{x})/L$ is unramified if and only if $v_\rho(x)$ is even, where $v_\rho : L^* \to \mathbb{Z}$ is the valuation map.

Proof: Recall that the extension $L(\sqrt{x})/L$ is determined by $\bar{x}$, the image of x in L^*/L^{*2}. Since $v_\rho(x)$ is even, $\bar{x} \in A^*/A^{*2}$; where A is the integral closure of $\hat{\mathbb{Z}}_p$ in L.

Since p is odd, $A^*/A^{*2} \simeq \ell^*/\ell^{*2} \simeq \mathbb{Z}/2$, where ℓ = residue field. Thus we get that either $\bar{x} = 1$ and $L\sqrt{x}$ is a product of two fields i.e. split or $\ell(\sqrt{x})/\ell$ is quadratic and $L(\sqrt{x})/L$ is inert.

(3.2.7) <u>Corollary</u>: $\text{Kernel}(\tilde{\psi}_1) = \text{Kernel}(\psi_1) = (1)$

<u>Proof</u>: Suppose $x \in \mathcal{O}^*$ represents a nontrivial element $\bar{x}$ in the kernel of $\tilde{\psi}_1$. Since $\tilde{\psi}_1(x) = 1$, $K\sqrt{x}/K$ is split over the unique dyadic prime in K. Since $x \in \mathcal{O}^*$, $v_\rho(x) = 0$ for all prime ideals in $\mathcal{O}$, and (3.2.6) implies that $K\sqrt{x}/K$ is split at all N.D. primes. Global class field theory implies that $\text{Gal}(K\sqrt{x}/K) \simeq \mathbb{Z}/2$ is a quotient group of $\Gamma^*(K)$ the narrow class group. But this is impossible by Weber's Theorem (2.2.4).

Let $[K,\mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of embeddings of K into $\mathbb{R}$. Then,

$$\mathcal{O}^* = \mu(K) \oplus \mathbb{Z}^{r_1+r_2-1} \qquad \text{(Dirichlet Unit Theorem)}$$

and

$$\hat{\mathcal{O}}_2^* = \mu(\hat{K}_2) \oplus \hat{\mathbb{Z}}_2^{r_1+2r_2} \qquad \text{(see [Se], XIV, §4, Prop. 10)}$$

Thus $\text{coker}\ \psi_1 \simeq \text{coker}\ \tilde{\psi}_1 = (\mathbb{Z}/2)^{r_2+1}$.

<u>Computation of ψ_2</u>

Recall the reciprocity sequence (see [C-F]).

$$1 \to Br(K) \to \bigoplus_{\substack{\text{all} \\ \rho}} Br(\hat{K}_\rho) \oplus \bigoplus_{\substack{\text{real} \\ v}} Br(\hat{K}_v) \xrightarrow{R} \mathbb{Q}/\mathbb{Z} \to 1$$

where R restricted to $Br(\hat{K}_\rho)$ is an isomorphism for any ρ and R restricted to $Br(\hat{K}_v)$ maps isomorphically onto $2\mathbb{Z}/\mathbb{Z}$. Thus we get the following commutative diagram.

$$\begin{array}{ccccccc}
1 \to I^2(K)/I^3(K) & \overset{\tilde{\psi}_2}{\to} & I^2(\hat{K}_2) \oplus \oplus I(k_\rho) \oplus \bigoplus_v Br(\hat{K}_v) & \to & \mathbb{Z}/2 \to 1 \\
\uparrow \wr & & \wr\| & & \| \\
1 \to {}_2Br(K) & \longrightarrow & \bigoplus_{\text{all } \rho} {}_2Br(\hat{K}_\rho) \oplus \bigoplus_v Br(\hat{K}_v) & \longrightarrow & \mathbb{Z}/2 \to 1
\end{array}$$

<u>Case 1</u>: $r_2 = 0$ i.e. $R = R_N$ with $N = r_1$.

Then,

$$\text{coker } \psi_2 = 0, \quad L_0(R \to \hat{R}_2) \simeq \mathbb{Z}/2,$$

and we get the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I^3(K) & \longrightarrow & \ker \psi_2 & \to & \ker \psi_2/I^3(K) & \to 1 & \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
1 & \to & \bigoplus_v I^3(\hat{K}_v) & \to & \bigoplus_v I^2(\hat{K}_v) & \to & \bigoplus_v I^2(\hat{K}_v)/I^3(\hat{K}_v) & \simeq & \bigoplus_v Br(\hat{K}_v) \to 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}/2 & = & \mathbb{Z}/2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Thus $\ker \psi_2 \simeq L_1(R \to \hat{R}_2) \simeq \mathbb{Z}^{r_1}$.

<u>Case 2</u>: $r_1 = 0$ i.e. $R = \Gamma_N$ or F_N, with $N = 2r_2$.

Then, $\ker \psi_2 \simeq L_1(R \to \hat{R}_2) \simeq 0$, and we get the following diagram

$$\begin{array}{ccccccccc} & & \mathbb{Z}/2 & & & & & & \\ & & \wr\| & & & & & & \\ 1 & \longrightarrow & I^2(\hat{K}_2) & \longrightarrow & L_0(\hat{K}_2) & \longrightarrow & \hat{K}_2^*/\hat{K}_2^{*2} & \to & 1 \\ & & \downarrow\wr & & \downarrow & & \downarrow & & \\ 1 & \to & \operatorname{coker}\psi_2 & \to & L_0(R \to \hat{R}_2) & \to & \operatorname{coker}\psi_1 & \to & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array}$$

<u>Case 2(a)</u>: $R = \Gamma_{2r_2}$

Then Theorem 2.29 in [L] implies the top sequence splits. Since $K_2^*/K_2^{*2} \to \operatorname{coker}\psi_1$ splits, we can conclude the bottom sequence also splits. Thus,

$$L_0(\Gamma_{2r_2} \to \hat{\Gamma}_{2r_{2(2)}}) \simeq \mathbb{Z}/2^{r_2+2}.$$

<u>Case 2(b)</u>: $R \simeq F_{2r_2}$.

Then Theorem 2.29 in [L] implies the top sequence does not split. Thus,

$$L_0(F_{2r_2} \to \hat{F}_{2r_{2(2)}}) \simeq \mathbb{Z}/2^{r_2} \oplus \mathbb{Z}/4.$$

(3.3) <u>Type U-Commutative Case</u>: $(\Gamma_N,-)$, (Γ_N,τ), $(\Gamma_N,\bar{\tau})$, (R_N,τ), $(F_N,-)$.

If β is nontrivial, $Br_0(K,\beta) \simeq (1)$. Thus $(K,\beta,1)$ and $(K,\beta,-1)$ are quadratic Morita equivalent and $L_i(K,\beta) \xrightarrow{\simeq} L_{i+2}(K,\beta)$ for any i. For any field K, $L_0^p(K,\beta) = W_\beta(K)$ the classical Witt ring of hermitian pairings over K (see [C] and [W], p. 135). Again, let $I_\beta(K) = \operatorname{kernel} r : W_\beta(K) \to \mathbb{Z}/2$, where r is the rank map.

The Rothenberg sequence plus (3.1.5) imply that $L_n(K,\beta) \simeq 0, I_\beta(K)$ for $n = 1,0(2)$.

(3.3.1) Examples

(i) If k is a finite field, then $I_\beta(k) \simeq (1)$.

(ii) If $\hat{K}_\rho$ is a finite extension of $\hat{\mathbb{Q}}_p$, then $\text{disc}: I(\hat{K}_\rho) \stackrel{\sim}{\to} \hat{F}^*_{\rho_0}/N_{\hat{K}_\rho/\hat{F}_{\rho_0}}\hat{K}^*_\rho$, where $\hat{F}_{\rho_0}$ is the fixed field for β. Local class field theory ([S2]) implies that $\hat{F}^*_{\rho_0}/N_{\hat{K}_\rho/\hat{F}_{\rho_0}}\hat{K}^*_\rho \simeq \text{Gal}(\hat{K}_\rho/\hat{F}_{\rho_0}) \simeq \mathbb{Z}/2$.

If $\hat{K}_\rho/\hat{F}_{\rho_0}$ is unramified, then the Divissage Theorem (3.1.3) implies that

$$L_{2i}(\hat{K}_\rho,\beta) \stackrel{\sim}{\to} L_{2i}(\hat{\mathcal{O}}_\rho \to \hat{K}_\rho,\beta) \stackrel{\sim}{\to} L^p_{2i}(k_\rho,\beta) \stackrel{\sim}{\to} \mathbb{Z}/2,$$

where $\hat{\mathcal{O}}_\rho$ is the integral closure of $\hat{\mathbb{Z}}_p$ in $\hat{K}_\rho$

(iii) The signature map yields an isomorphism

$$\text{sig} : I_-(\mathbb{C}) \to 2\mathbb{Z}$$

(iv) If $K/\mathbb{Q}$ with $[K,\mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of embeddings of K into $\mathbb{R}$; then

$$\text{disc}: I_\beta(K)/I^2_\beta(K) \to F^*/N_{K/F}K^*,$$

where F is the fixed field for β. If $[F,\mathbb{Q}] = s_1 + s_2$ where s_1 is the number of embeddings of K into $\mathbb{R}$, then

$$\text{sig} : I^2_\beta(K) \stackrel{\sim}{\to} \bigoplus_v I^2_\beta(K_v) \stackrel{\sim}{\to} (4\mathbb{Z})^{s_1 - \frac{r_1}{2}},$$

where we sum over conjugate pairs of embeddings
$v : K \to \mathbb{C}$ such that $v(F) \subset \mathbb{R}$, but $v(K) \not\subset \mathbb{R}$,
i.e. the ramified archimedian places
for K/F.

Suppose $K \subseteq \mathbb{Q}(\zeta_j)$ for some j, $\mathcal{O}$ is the ring of integers in K, and $R = \mathcal{O}[\frac{1}{2}]$. Then we get the following commutative diagram with exact rows and columns. (Recall that K/F is unramified over N.D. primes.)

(3.3.2)

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & I^2_\beta(K) & = & I^2_\beta(K) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 \to & & L_{2i+1}(R\to\hat{R}_2,\beta) & \to & L_{2i}(K,\beta) & \xrightarrow{\psi} & L_{2i}(\hat{K}_2,\beta) \oplus \bigoplus_{\substack{\beta(\rho)=\rho \\ N.D.}} L^p_{2i}(k_\rho,\beta) & \to & L_{2i}(R\to\hat{R}_2,\beta) \to 1 \\
 & & \downarrow & & \downarrow & & \wr\downarrow & & \wr\downarrow \\
1 \longrightarrow & & \ker\psi_1 & \longrightarrow & F^*/NK^* & \xrightarrow{\psi_1} & \hat{F}^*_2/N\hat{K}^*_2 \oplus \bigoplus \hat{F}^*_{\rho_0}/N\hat{K}^*_\rho & \longrightarrow & \operatorname{coker}\psi_1 \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Global Class Field Theory (see [C-F]) yields the following short exact sequence

(3.3.3)

$$1 \to F^*/NK^* \to \bigoplus_{\beta(\rho)=\rho} \hat{F}^*_{\rho_0}/N\hat{K}^*_\rho \oplus \bigoplus_{v} \hat{F}^*_{v_0}/NK^*_v \xrightarrow{\mathcal{R}} \mathbb{Z}/2 \to 0$$

where v varies over the ramified archimedian places for K/F. Furthermore, $\mathcal{R}$ becomes an isomorphism when restricted to $\hat{F}^*_{\rho_0}/N\hat{K}^*_\rho$ for any ρ (N.D. or dyadic) or $\hat{F}^*_{v_0}/N\hat{K}^*_v$ for any v.

<u>Type UI</u>: $r_1 = 0$ and $s_2 = 0$, i.e. K is totally nonreal and F is totally real. $((\Gamma_{2r_2},-)$ or $(F_{2r_2},-))$

Then ψ_1 is onto, and

$$L_{2i}(R \to \hat{R}_2,\beta) \simeq \operatorname{coker} \psi_1 \simeq (0).$$

We also get the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I^2_\beta(K) & \longrightarrow & L_{2i+1}(R \to \hat{R}_2,\beta) & \longrightarrow & \ker \psi_1 & \to & 1 \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \bigoplus\limits_{\substack{\text{arch.}\\ \text{unram.}}} I^2_\beta(\hat{K}_v) & \longrightarrow & \bigoplus\limits_{\substack{\text{arch.}\\ \text{unram.}}} L_{2i}(\hat{K}_v,\beta) & \to & \bigoplus F^*_{v_0}/NK^*_v & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}/2 & \xrightarrow{\sim} & \mathbb{Z}/2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Thus $L_{2i+1}(R \to \hat{R}_2,\beta) \simeq \mathbb{Z}^{r_2}$.

<u>Type UII</u>: Otherwise, $((\Gamma_N,\tau), (\Gamma_N,\bar{\tau})$, or $(R_N,\tau))$. Then $L_{2i+1}(R \to \hat{R}_2,\beta) = 0$ and $L_{2i}(R \to \hat{R}_2,\beta) \simeq \mathbb{Z}/2$.

(3.4) <u>Type O - Noncommutative</u>: $(H_N,\alpha_1,1)$

Let $D = H_N \otimes \mathbb{Q}$. Then D is a quaternionic division ring over $K = \mathbb{Q}(\zeta_N + \bar{\zeta}_N)$. If ρ is a N.D. prime, then $\hat{D}_\rho \simeq M_2(\hat{K}_\rho)$. Furthermore, for any real embedding v of K, $\hat{D}_v$ is a division ring. If $N = 2$, then $\hat{D}_2$ is a division ring; but if $N > 2$, then $\hat{D}_2 = M_2(\hat{K}_2)$.

The (anti) involution α_1 is such that $\alpha_1|K = \mathrm{id}$ and $(D,\alpha_1,1) \in \mathrm{Br}(K,\mathrm{id})$ maps to the trivial element in $\mathrm{Br}(\bar{K},\mathrm{id}) \simeq \{\pm 1\}$.

Then $L_0^p(D,\alpha_1)$ is the classical Witt group of Hermitian pairings over (D,α_1), i.e. what Wall calls Type O_D; and $L_2^p(D,\alpha_1)$ is the classical Witt group of skew-Hermitian pairings over (D,α_1), i.e. what Wall calls type Sp_D. For background see [W4], p. 135 and [K].

Examples:

(i) If $N=2$, then $L_n(\hat{D}_2,\alpha_1) \simeq 0,0,0,\ \hat{\mathbb{Q}}_2^*/\hat{\mathbb{Q}}_2^{*2}$ for $n \equiv 3,2,1,0(4)$.

(ii) For any real embedding v of K,
$L_n(\hat{D}_v,\alpha_1) \simeq 0\ ,2\mathbb{Z},0,0$ for $n \equiv 3,2,1,0(4)$.

(iii) For any i, $L_{2i+1}(D,\alpha_1) = 0$ (apply the Semi-simple Theorem (3.1.5) and the Rothenberg sequence).

We also get $L_2(D,\alpha_1) \xrightarrow{\sim} \underset{v}{\oplus} L_2(\hat{D}_v,\alpha_1) \simeq (2\mathbb{Z})^{2^{N-2}}$.

The discriminate yields an onto map

disc: $L_0(D,\alpha_1) \to K^+/K^{*2}$, where

$K^+ = \{x \in K^* | v(x) \in \mathbb{R}^+$ for all real embeddings $v\}$.

Let $I_2(D) = \ker$ disc, and let $I_3(D)$ be the kernel of the onto map

$$I_2(D) \to \oplus\, I^2(\hat{K}_\rho) \simeq \oplus\, \mathbb{Z}/2$$

where we sum over all finite primes ρ (dyadic or N.D.) such $\hat{D}_\rho \simeq M_2(\hat{K}_\rho)$. Then

$$I_3(D) \simeq \begin{cases} \mathbb{Z}/2^{2^{N-2}-2} & \text{if } N > 2 \\ 0 & \text{if } N = 2. \end{cases}$$

From (3.1.4) we get the following exact sequence

(3.4.1)

$$\cdots \to L_{i+1}(H_N \to \hat{H}_{N(2)},\alpha_1) \to L_i(D,\alpha_1) \overset{\psi}{\to} L_i(\hat{D}_2,\alpha_1) \oplus \bigoplus_{N.D.} L_i^p(k_\rho) \to \cdots$$

When $i = 2$ we get,

$$L_3(H_N \to \hat{H}_{N(2)},\alpha_1) \simeq L_2(D,\alpha_1) \simeq (2\mathbb{Z})^{2^{N-2}},$$

and

$$L_2(H_N \to \hat{H}_{N(2)},\alpha_1) \simeq (0).$$

<u>Case 1</u>: $(N = 2)$ Then, when $i = 0$, (3.4.1) yields the following commutative diagram with exact rows and columns

(3.4.2)

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & I_2(D) & \xrightarrow{\sim} & 0 \oplus \bigoplus_{N.D.} I^2(\hat{\mathbb{Q}}_p) & & \\
 & & & & \downarrow & & \downarrow & & \\
1 \to & L_1(H_2 \to \hat{H}_{2(2)},\alpha_1) & \to & L_0(D,\alpha_1) & \overset{\psi}{\to} & L_0(\hat{D}_2,\alpha_1) \oplus \bigoplus_{N.D.} L_0^p(\mathbb{F}_p) & \to & L_0(H_2 \to \hat{H}_{(2)},\alpha_1) & \to 1 \\
 & \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr & \\
1 \longrightarrow & \ker \overline{\Psi}_1 & \longrightarrow & \mathbb{Q}^+/\mathbb{Q}^{*2} & \xrightarrow{\overline{\Psi}_1} & \hat{\mathbb{Q}}_2^*/\hat{\mathbb{Q}}_2^{*2} \oplus \bigoplus_{N.D.} \mathbb{Z}/2 & \longrightarrow & \operatorname{coker} \overline{\Psi}_1 & \longrightarrow 1 \\
 & & & \downarrow & & & & & \\
 & & & 1 & & & & &
\end{array}$$

<u>Case 2</u>: $(N > 2)$ Then $\hat{D}_2 \simeq M_2(\hat{K}_2)$, and $L_i(\hat{D}_2,\alpha_1) \simeq L_i(\hat{K}_2)$. When $i = 0$, (3.4.1) yields the following commutative diagram with exact rows and columns

(3.4.3)

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & I_2(D) & \longrightarrow & I^2(\hat{K}_2)\oplus \bigoplus\limits_{N.D.} I^2(\hat{K}_\rho) & & & & \\
 & {}_2K^* & & & & & & & & & \\
 & \wr\| & & & \downarrow & & \downarrow & & & & \\
0\to & L_1(\hat{K}_2) & \to & L_1(H_N\to\hat{H}_{N(2)},\alpha_1) \to & L_0(D,\alpha_1) & \to & L_0(\hat{K}_2,\alpha_1)\oplus \bigoplus\limits_{N.D.} L_0^p(k_\rho) & \to & L_0(H_N\to\hat{H}_{N(2)},\alpha_1) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \wr\| & & \\
 & & & & K^+/K^{*2} & \xrightarrow{\overline{\psi}_1} & K_2^*/K_2^{*2}\oplus \bigoplus\limits_{N.D.} \mathbb{Z}/2 & \longrightarrow & \operatorname{coker}\overline{\psi}_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 1 & & 1 & & & &
\end{array}
$$

Consider the following commutative braid of exact sequences (compare with (3.2.4)).

(3.4.4)

$$
\begin{array}{ccccccccc}
 & & & \nearrow & \bigoplus\limits_{v} \hat{K}_v^*/\hat{K}_v^{*2} \simeq \mathbb{Z}/2^{2^{N-2}} & \searrow & & & \\
 & & K^*/K^{*2} & & & & \operatorname{coker}\overline{\psi}_1 & & \\
 & \nearrow & & \searrow^{\psi_1} & & \nearrow & & \searrow & \\
K^+/K^{*2} & & & & \hat{K}_2^*/\hat{K}_2^{*2} \oplus \bigoplus \mathbb{Z}/2 & & & & \mathbb{Z}/2 \quad (r_2 = 0) \\
 & \searrow & \underset{\overline{\psi}_1}{\longrightarrow} & \nearrow & & \searrow & \longrightarrow & \nearrow &
\end{array}
$$

Since (3.2.7) implies that ψ_1 is injective, we get that $\overline{\psi}_1$ is also injective. Also, in both Case 1 and Case 2, we get $L_0(H_2 \to \hat{H}_{(2)},\alpha) \simeq \operatorname{coker}\overline{\psi}_1 \simeq \mathbb{Z}/2^{2^{N-2}+1}$. In Case 1, we get that $L_1(H_2 \to \hat{H}_{2(2)},\alpha) \simeq 0$. In Case 2, we get the following short exact sequence

$$
(3.4.6) \quad 1 \to {}_2K^* \to L_1(H_N \to \hat{H}_{H(2)},\alpha) \to \underset{\underset{\mathbb{Z}/2^{2^{N-2}-2}}{\wr\|}}{I_3(K)} \to 1 \quad (N > 2).
$$

In Part II, (4.5.6) we show that a twisting braid argument implies that this sequence splits.

(3.5) Type U - Noncommutative Case $(H_N,\hat{\alpha})$, $N > 2$

Again, let $D = H_N \otimes Q$ with center K. Since $\hat{\alpha}|K \neq id$, $Br_0(K,\hat{\alpha}) \simeq (1)$ (see (2.5.9) (i)) and $L_i(D,\hat{\alpha}) = L_{i+2}(D,\hat{\alpha})$. Furthermore, $L^p_{2i}(D,\hat{\alpha})$ is the classical Witt group of Hermitian pairings over $(D,\hat{\alpha})$, i.e. what Wall calls Type U_D. For background see [W4], p. 135.

For any i, $L_{2i+1}(D,\hat{\alpha}) = 0$ (apply the Semi-simple Theorem (3.1.1) and the Rothenberg sequence). The discriminate map yields an isomorphism

$$\text{disc}: L_{2i}(D,\hat{\alpha}) \xrightarrow{\sim} F^+/F^+ \cap N_{K/F}K^*,$$

where F = fixed field for $\hat{\alpha}|K$ and

$$F^+ = \{x \in F^* | w(x) > 0 \text{ for all real embeddings } w \text{ of } F\}.$$

(3.5.1) Lemma: $\phi : F^+/F^+ \cap N_{K/F}K^* \to F^*/N_{K/F}K^*$ is an isomorphism.

Proof: Clearly ϕ is injective and the cokernel of ϕ is isomorphic to the cokernel of

$$K^* \xrightarrow{N_{K/F}} F^* \to F^*/F^+.$$

Consider the following commutative diagram

$$\begin{array}{ccc} K^* & \xrightarrow{N_{K/F}} & F^* \\ \downarrow & & \downarrow \\ K^*/K^+ & & F^*/F^+ \\ \downarrow s_K & & \downarrow s_F \\ \bigoplus_{v \text{ real}} K_v^*/K_v^+ & \xrightarrow{N} & \bigoplus_{w \text{ real}} F_w^*/F_w^+ \end{array}$$

The Weak Approximation Theorem (see [C-F]) implies that s_K and s_F are isomorphisms. Since K and F are both totally real, N is onto.

We then get the following localization sequence

$$\begin{array}{ccccccccc} 0 \to L_{2i+1}(\Delta\to\hat{\Delta}_2,\hat{\alpha}) & \to & L_{2i}(D) & \xrightarrow{\psi} & L_{2i}(\hat{K}_2,\hat{\alpha}) \oplus \bigoplus_{\substack{\hat{\alpha}(\rho)=\rho \\ \text{N.D.}}} L^p_{2i}(k_\rho,\hat{\alpha}) & \to & L_{2i}(\Delta\to\hat{\Delta}_2,\hat{\alpha}) \to 0 \\ \wr\| & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ 0 \longrightarrow \ker\psi_1 & \longrightarrow & F^*/NK^* & \xrightarrow{\psi_1} & \hat{F}_2^*/N\hat{K}_2^* \oplus \oplus \hat{F}^*_{\rho_0}/N\hat{K}^*_\rho & \longrightarrow & \operatorname{coker}\psi_1 \longrightarrow 0 \end{array}$$

Since both F and K are totally real, (3.3.3) implies that

$$L_{2i+1}(\Delta \to \hat{\Delta}_2,\hat{\alpha}) = 0, \quad \text{and} \quad L_{2i}(\Delta \to \hat{\Delta}_2,\alpha) \simeq \mathbb{Z}/2.$$

(3.6) <u>Summary</u>: Let $\Delta = \Gamma_N$, F_N, R_N, or H_N, R = center of Δ, K = the quotient field for R, and $\overline{K}$ = the algebraic closure of K. Suppose (α,u) is an anti-structure on Δ. Let F be the quotient field for r, where r is the fixed ring for $\alpha|R$.

<u>Definition</u>: Assume $\alpha|_R = \text{id}$. Then

$$(\alpha,u) \text{ has type } \begin{cases} 0 & \text{if } (\alpha,u) \text{ maps to the trivial element in } Br(\overline{K},\text{id}), \\ S_p & \text{otherwise} \end{cases}$$

Assume $\alpha|_R \neq id$. Then

$$(\alpha,u) \text{ has type } \begin{cases} UI & \text{if } K \text{ is \underline{fake} i.e. has no real places and } F \\ & \text{is totally real,} \\ UII & \text{otherwise} \end{cases}$$

By combining (2.6) with the computations in this chapter we get the following result.

(3.6.1) Theorem: Assume (α,u) is any anti-structure on $\Delta = \Gamma_N, F_N, R_N$, or H_N.

If $\alpha|_R = id$, then $L_i(\Delta \to \hat{\Delta}_2,\alpha,u)$ is determined by Δ and the type of (α,u). Furthermore, if (α,u) has type 0 and (α',u') has type S_p, then $L_i(\Delta \to \hat{\Delta}_2,\alpha,u) \cong L_{i+2}(\Delta \to \hat{\Delta}_2,\alpha',u')$.

If $\alpha|_R \neq id$, then $L_i(\Delta \to \hat{\Delta}_2,\alpha,u)$ is determined by just the type of (α,u). Thus, there exist the following isomorphisms.

UI: $L_i(\Gamma_N,-,1) \cong L_i(F_N,-,1)$

UII: $L_i(\Gamma_N,\tau,1) \cong L_i(\Gamma_N,\bar{\tau},1) \cong L_i(R_N,\tau,1) \cong L_i(H_N,\alpha,1)$.

Furthermore, $L_i(\Delta,\alpha,u) \cong L_{i+2}(\Delta,\alpha,u)$.

PART II: Maps Between L-groups

§4. Basic definitions for transfers and twisted quadratic extensions

(4.1) Transfer maps in Algebraic K-Theory

Suppose $f : R \to S$ is any ring homomorphism. Then we get the "push forward" map

$$f_! : K_n(R) \to K_n(S);\ M \to M \otimes_R S.$$

If the map f makes S into a finitely generated, projective right R-module map, then restriction of scalers induces a transfer map

$$f^! : K_n(S) \to K_n(R).$$

If S is a progenerator as a right R-module, then $f^!$ also has the following alternative description. Let

$$T(f) : S \to \mathrm{End}_R(S)$$

be the map given by left-multiplication. Then the Morita equivalence derived from S viewed as a right R-module yields an isomorphism $\phi : K_nR \to K_n\mathrm{End}_R(S)$ such that the following diagram commutes

(4.1.1)
$$\begin{array}{ccc} K_n(S) & \xrightarrow{f^!} & K_n(R) \\ & {\scriptstyle T(f)_!}\searrow & \downarrow \phi \\ & & K_n(\mathrm{End}_R(S)) \end{array}$$

If ${}_RS_S$ is isomorphic to ${}_R\mathrm{Hom}(S,R)_S$, then we also get that the following diagram commutes

(4.1.2)
$$\begin{array}{ccc} K_n(R) & \xrightarrow{f_!} & K_n(S) \\ \phi \searrow\nwarrow & & \nearrow T(f)^! \\ & K_n(\mathrm{End}_R(S)) & \end{array}$$

(4.1.3) Examples:

(i) If $\Delta : R \to R \times R$ is the diagonal map, then $T(\Delta)$ can be identified with the map $R \times R \to M_2(R)$ which sends (r_1, r_2) to

$$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

(ii) Suppose $f : K \to D$ is the inclusion map of a maximal subfield in a division ring where $F = \text{center}(D)$ and $m^2 = [D,F]$. Then $T(f)$ can be identified with the map $D \to D\otimes_F K \simeq M_m(K)$.

(4.2) <u>Relative (linear) Morita Theory</u>

Suppose M is a progenerator for P_R. Let $R_1 = \text{End}_R(M)$ and $S_1 = \text{End}_S(M\otimes_R S)$. Then we get the following commutative diagrams

(4.2.1)

$$\begin{array}{ccc} K_n(R) & \xrightarrow{f_!} & K_n(S) \\ \wr\big\downarrow \phi_M & & \wr\big\downarrow \phi_{M\otimes_R S} \\ K_n(R_1) & \xrightarrow{f_{1!}} & K_n(S_1) \end{array} \qquad \begin{array}{ccc} K_n(S) & \xrightarrow{f^!} & K_n(R) \\ \wr\big\downarrow \phi_{M\otimes_R S} & & \wr\big\downarrow \phi_M \\ K_n(S_1) & \xrightarrow{f_1^!} & K_n(R_1) \end{array}$$

where $f_1 : R_1 \to S_1$ is given by tensoring with 1_S, and the maps ϕ_M and $\phi_{M\otimes_R S}$ come from derived Morita equivalences.

(4.2.2) <u>Examples</u>:

Suppose H is an index 2 subgroup of a finite 2-group G. Then t, the nontrivial element in G/H acts on $\{a_\rho\}$ = the set of primitive central idempotents in H. Furthermore, the map $\mathbb{Q}H \to \mathbb{Q}G$ decomposes as a product of maps

<u>Case 1</u>: $a_\rho\mathbb{Q}H \to a_\rho\mathbb{Q}G$ (for $t(\rho) = \rho$), and

<u>Case 2</u>: $a_\rho\mathbb{Q}H \times a_{t(\rho)}\mathbb{Q}H \to (a_\rho + a_{t(\rho)}) \cdot \mathbb{Q}G$ (for $t(\rho) \neq \rho$)

For any ρ we let V_ρ be the simple module for $a_\rho\mathbb{Q}H$. In Case 1 we let M be V_ρ, and in Case 2 we let M be $V_\rho \times V_{t(\rho)}$. In both cases we get that $f_1 : R_1 \to S_1$ is either one of the following maps or T applied to one of the following maps

(a) $F \subset K$, where F and K are subfields of $\mathbb{Q}(\zeta_N)$ for some N; and K is a quadratic extension of F.

(b) $K \to D_N = \left(\frac{-1,-1}{\mathbb{Q}(\zeta_N + \overline{\zeta}_N)}\right)$, where K is either $\mathbb{Q}(\zeta_N)$ or $\mathbb{Q}(\zeta_{N+1} - \overline{\zeta}_{N+1})$.

(c) $D_N \to D_{N+1}$

or

(d) $\Delta : A \to A \times A$, where A is either a subfield of $\mathbb{Q}(\zeta_N)$ or $\left(\frac{-1,-1}{\mathbb{Q}(\zeta_N + \overline{\zeta}_N)}\right)$ for some N.

(Compare with Example 1 in the Introduction.)

Thus the problem of computing

$$f_! : K_n(\mathbb{Q}H) \to K_n(\mathbb{Q}G) \quad \text{and} \quad f^! : K_n(\mathbb{Q}G) \to K_n(\mathbb{Q}H)$$

can be reduced to the problem of computing the push forward and transfer maps associated to the maps in (a), (b), (c), and (d).

(4.3) Transfer maps in L-Theory

Suppose $f : (R,\alpha_0,u) \to (S,\alpha,u)$ is a map of rings with anti-structure. Then we get the "push forward" map

$$f_! : L_n(R,\alpha_0,u) \to L_n(S,\alpha,u)$$

(4.3.1) <u>Definition</u>: <u>A trace for f</u> is a map $X : S \to R$ such that

(i) X is a right R-linear map where we use f to make S a right R-module.

(ii) $X(\alpha(s)) = \alpha_0 X(s)$ for all $s \in S$.

(iii) if $\lambda^X : S \times S \to R$ sends (s_1, s_2) to $X(\alpha(s_1)s_2)$, then $ad(\lambda^X) : S \to Hom_R(S,R)$ is onto.

and

(iv) S is a finitely-generated projective right R-module.

Notice that a choice of trace X for f (assuming one exists) determines a functor

$Sesq(S,\alpha,u) \to Sesq(R,\alpha_0,u)$; $(b : N \times N \to S) \to (X \cdot b : N \times N \to R)$,

and a transfer map

$$f^X : L_n(S,\alpha,u) \to L_n(R,\alpha_0,u)$$

(4.3.2) <u>Example</u>: Suppose $f : (\mathbb{Z}H,\alpha_\omega,1) \to (\mathbb{Z}G,\alpha_\omega,1)$ is induced by an inclusion of groups $H \subset G$. The $\mathbb{Z}$-linear map $X : \mathbb{Z}G \to \mathbb{Z}H$ such that

$$X(g) = \begin{matrix} g & \text{if } g \in H \\ 0 & \text{if } g \in G - H \end{matrix}$$

is a trace. Furthermore, the induced transfer map is the same as the geometric transfer defined using covering spaces.

Consider the map $T(f) : S \to End_R(S)$. By the Derived Quadratic Morita Equivalence Theorem (2.5.2) we get that $ad(\lambda^X)$ determines an anti-structure (β,v) on $End_R(S)$, such that (R,α_0,u) and $(\mathrm{End}_R(S),\beta,v)$ are quadratic Morita equivalent.

(4.3.3) Proposition: We get a map of rings with anti-structure

$$T(f) : (S,\alpha,u) \to (\mathrm{End}_R(S),\beta,v),$$

and the following diagram commutes

$$\begin{array}{ccc} L_n(S,\alpha,u) & \xrightarrow{f^X} & L_n(R,\alpha_0,u) \\ & \searrow^{T(f)_!} & \downarrow\wr \\ & & L_n(\mathrm{End}_R(S),\beta,v) \end{array}$$

(4.4) Twisted quadratic extensions

Recall that in the Introduction we considered the notion of a twisted quadratic extension.

$$f: R \to R_\rho[\sqrt{a}] = S, \quad \text{with Galois automorphism } \gamma.$$

Notice that the examples in (4.2.2) can all be viewed as twisted quadratic extensions. We are particularly interested in the following examples where we pass to $\mathbb{Z}[\frac{1}{2}]$-maximal orders.

(4.4.1) List: (see (1.3.5) for notation)

$f : R \to R_\rho[\sqrt{a}]$	ρ	$t = \sqrt{a}$
$R_{N-1} \to R_N$	Id	$\zeta_{N+2} + \bar{\zeta}_{N+2}$
$\Gamma_{N-1} \to \Gamma_N$	Id	ζ_{N+1}
$F_{N-1} \to \Gamma_N$	Id	i
$R_{N-1} \to \Gamma_N$	Id	i
$R_{N-1} \to F_N$	Id	$\zeta_{N+2} - \bar{\zeta}_{N+2}$
$f_{\pm} : \Gamma_{N-1} \to H_N$, where $f_+(i) = i$ and $f_-(i) = k$	-	j
$f : F_{N-1} \to H_N$, where $f(\zeta_{N+1} - \bar{\zeta}_{N+1}) = k(1 - \zeta_N)$	-	j
$H_{N-1} \to H_N$	Id	$\zeta_N + \bar{\zeta}_N$
$d : \Delta \to \Delta \times \Delta$, diagonal map, where $\Delta = \Gamma_N, R_N, F_N$, or H_N	Id	$(1,-1)$

(4.4.2) Proposition: Assume $\frac{1}{2} \in R$, a is a unit in R, and

$$f : R \to R_\rho[\sqrt{a}] = S$$

is a twisted quadratic extension with Galois automorphism γ. Then

(i) $$T(f) : S \to \mathrm{End}_R(S)$$

is also a twisted quadratic extension. More precisely, there exists a ring isomorphism $G : S_\gamma[\sqrt{1}] \to \mathrm{End}_R(S)$ such that the following diagram commutes

$$\begin{array}{ccc} S & \longrightarrow & S_\gamma[\sqrt{1}] \\ & {\scriptstyle T(f)} \searrow & \downarrow {\scriptstyle G} \\ & & \mathrm{End}_R(S). \end{array}$$

For any $s_1 + s_2\sqrt{1} \in S_\gamma(\sqrt{1})$, $G(s_1 + s_2\sqrt{1})$ is the endomorphism of S (as a right R-module) which sends $z \in S$ to $s_1 z + s_2\gamma(z)$, and

(ii) ${}_R S_S$ is isomorphic to ${}_R\mathrm{Hom}(S,R)_S$.

(4.4.3) <u>Theorem</u>: If H is an index 2 subgroup of a finite 2-group, then $\mathbb{Z}[\frac{1}{2}]H \to \mathbb{Z}[\frac{1}{2}]G$ can be expressed as a product of maps such that each component map is either in List (4.4.1) or it is T of a map in List (4.4.1) (up to Morita equivalence).

Thus the problem of computing the K-theory push forward and transfer maps for $\mathbb{Z}[\frac{1}{2}]H \to \mathbb{Z}[\frac{1}{2}]G$ is reduced to the analogous problem for the maps in (4.4.1).

(4.5) L-Theory for twisted quadratic extensions

Suppose we have a map of rings with anti-structure

$$f : (R,\alpha_0,u) \to (S,\alpha,u)$$

where $f : R \to R_\rho[\sqrt{a}] = S$ is a twisted quadratic extension with Galois automorphism γ.

Then a trace for f is given by

$$X : R_\rho[\sqrt{a}] \to R;\ X(x + yt) = x, \quad \text{for all } x,y \in R.$$

Since our X is fixed, we also denote f^X by $f^!$.

As in the Introduction we get a twisting diagram for $f : (R,\alpha_0,u) \to (S,\alpha,u)$. Notice that the twisting diagram for $\tilde{f} : (R,\tilde{\alpha}_0,\tilde{u}) \to (S,\tilde{\alpha},\tilde{u})$ is the same as the twisting diagram for f (up to reindexing).

If $\frac{1}{2} \in R$ and a is a unit, then (4.3.3) and (4.4.2) imply that

$$T(f) : (S,\alpha,u) \to (\mathrm{End}_R(S),\beta,v)$$

is a map of rings with anti-structure where $\mathrm{End}_R(S) \simeq S_\gamma[\sqrt{1}]$ and $T(f)$ is a twisted quadratic extension.

(4.5.1) <u>Proposition</u>: The twisting diagram for $T(f)$ is isomorphic to the twisting diagram for ${}^{\gamma}f$ (up to reindexing).

Suppose we equip one of the twisted quadratic extensions in (4.4.1) with anti-structure $f: (\alpha_0,u) \to (\alpha,u)$. Then as in (3.6.1) one can show that the twisting diagram for f is determined by the rings, type (α_0,u), and type (α,u).

(4.5.2) List: Twisted quadratic extensions with anti-structure

	$f:R\to R_\rho[\sqrt{a}]$	Type (f)	Type ($\tilde{f}$)	Type ($^\gamma f$)	Type ($\widetilde{^\gamma f}$)
(1)	$\Gamma_{N-1}\to\Gamma_N$	$0\to 0$	$S_p\to UII$	$0\to UII$	$0\to 0$
(2)		$UI\to UI$	$UI\to UII$	$UI\to UII$	$UI\to UI$
(3)	$R_{N-1}\to R_N$	$0\to 0$	$S_p\to UII$	$0\to UII$	$0\to 0$
(4)	$R_{N-1}\to\Gamma_N$	$0\to 0$	$S_p\to UI$	$0\to UI$	$0\to 0$
(5)		$UII\to UII$	$UII\to UII$	$UII\to UII$	$UII\to UII$
(6)	$R_{N-1}\to F_N$	$0\to 0$	$S_p\to UI$	$0\to UI$	$0\to 0$
(7)	$F_{N-1}\to\Gamma_N$	$0\to 0$	$S_p\to UII$	$0\to UII$	$0\to 0$
(8)		$UI\to UI$	$UI\to UII$	$UI\to UII$	$UI\to UI$
(9)	$\Gamma_{N-1}\to H_N$	$0\to 0$	$UI\to 0$	$0\to 0$	$UI\to 0$
(10)		$UII\to UII$	$UII\to UII$	$UII\to UII$	$UII\to UII$
(11)	$F_{N-1}\to H_N$	$0\to 0$	$UI\to 0$	$0\to 0$	$UI\to 0$
(12)	$H_{N-1}\to H_N$	$0\to 0$	$S_p\to UII$	$0\to UII$	$0\to 0$
(13)	$\Delta\xrightarrow{d}\Delta\times\Delta$	$(\alpha_0,1)\to(\alpha_0,1)\times(\alpha_0,1)$	$(\alpha_0,-1)\to GL$	$(\alpha_0,1)\to GL$	$(\alpha_0,-1)\to(\alpha_0,-1)\times(\alpha_0,-1)$

In fact we get isomorphisms of twisting diagrams between Cases (2) and (8), and also between Cases (5) and T of (10).

(4.5.3) Theorem: Suppose we have a map of rings with anti-structure $f : (\mathbb{Z}[\frac{1}{2}]H,\alpha_0,u) \to (\mathbb{Z}[\frac{1}{2}]G,\alpha,u)$ where G is a finite 2-group and H is an index 2 subgroup. Then the L^p-twist diagram for f decomposes into a direct sum of diagrams such that each component diagram is isomorphic (up to reindexing) to the L^p-twist diagram for one of the twisted quadratic extensions with anti-structure in List (4.5.2).

(4.5.4) Definition: For any ring with anti-structure (S,α,u) we let

$$(S,\alpha,u)_n = L_n^{0\subset K_0}(S \to \hat{S}_2,\alpha,u)$$

If $f : (R,\alpha_0,u) \to (R_\rho[\sqrt{a}],\alpha,u)$ is a twisted quadratic extension of rings with anti-structure; then we get a "push forward" exact sequence

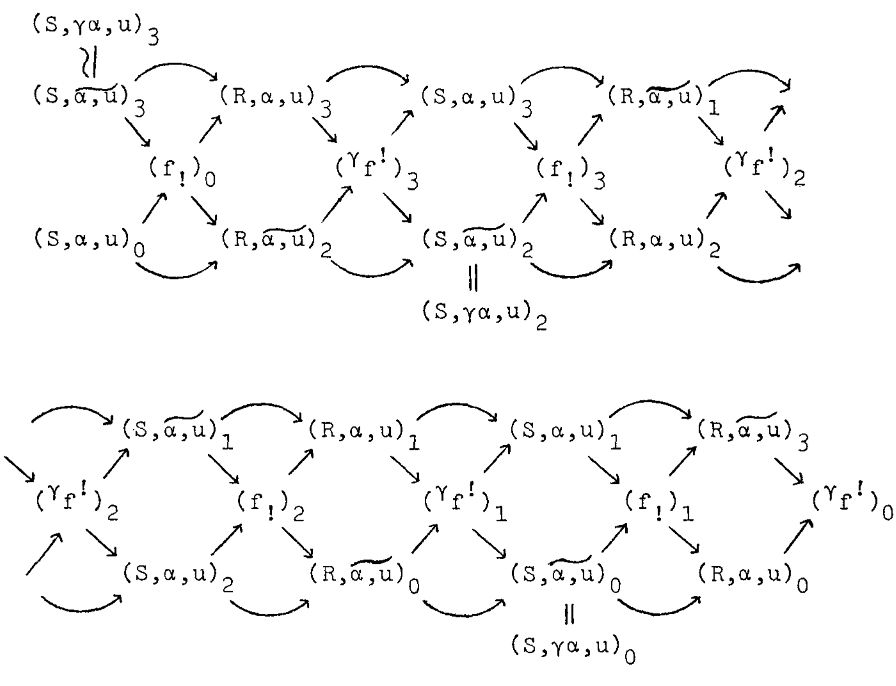
$$\cdots \to (R,\alpha_0,u)_n \xrightarrow{f_!} (S,\alpha,u)_n \to (f_!)_n \to \cdots,$$

a transfer exact sequence

$$\cdots \to (S,\alpha,u)_n \xrightarrow{f^!} (R,\alpha_0,u)_n \to (f^!)_n \to \cdots,$$

and a

(4.5.5) Relative Twist Diagram



Furthermore, we get a relative version of (4.5.3).

At the end of the paper there are tables giving the relative push forward and transfer exact sequences for all cases in (4.5.2) except cases (10) and (13). The twist diagram for (13) is easy: the one for (10) is T of the one for (5). In particular, the push forward map for $\Gamma_{N-1} \to H_N$, type UII $\to$ type UII is read off Table 3 not Table 2 !

Each relative twist diagram from (4.5.2) is determined by the groups along the top and bottom rows of the diagram except in cases (5) and (10). These are determined by using

$f_!: (R_{N-1}, \tau)_0 \to (\Gamma_N, \tau)_0$ is trivial, and

$\gamma_f^!: (H_N, \hat{\alpha})_0 \to (\Gamma_{N-1}, \tau)_0$ is trivial.

Both these facts can be derived from the other diagrams.

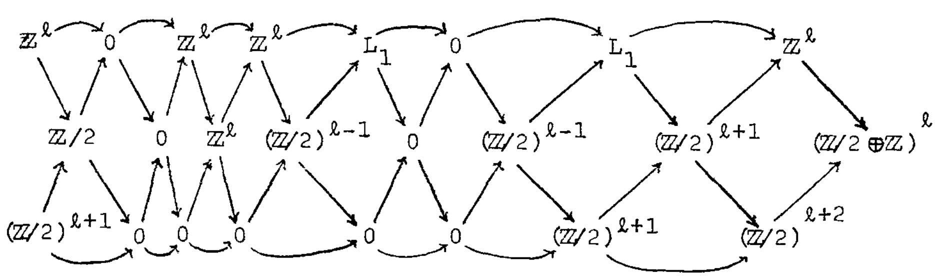
Recall from (3.4.6) the short exact sequence

$$1 \to {}_2K^* \to L_1(H_N \to \hat{H}_{N(2)}, \alpha) \to \mathbb{Z}/2^{2^{N-2}-2} \to 1.$$

We write out the twisting diagram below to show that this sequence splits.

(4.5.6) $(f_+: \Gamma_{N-1} \to H_N)$, Type $0 \to 0$, $L_1 = L_1(H_N \to \hat{H}_{N(2)}, \alpha)$

$\ell = 2^{N-2}$



APPENDIX I: Computing $L^p_r(\mathbb{Z}G,\alpha,u)$

To compute $L^p_r(\mathbb{Z}G,\alpha,u)$ we shall use the sequence

$$\dots \to L^p_r(\mathbb{Z}G,\alpha,u) \to L^p_r(\hat{\mathbb{Z}}_2G,\alpha,u) \xrightarrow{\Psi_r} L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \quad \dots$$

Since $L^p_r(\hat{\mathbb{Z}}_2G,\alpha,u) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & r \text{ even} \\ 0 & r \text{ odd} \end{cases}$ (1.2.1), most of the work comes in computing $L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$.

All of the antistructures encountered in surgery theoretic applications have the following description. We are given a homomorphism $\omega\colon G \to \pm 1$; an automorphism $\theta\colon G \to G$; and an element $b \in G$. We require $\omega\circ\theta = \omega$; $\theta\circ\theta(g) = bgb^{-1}$ for all $g \in G$; $\omega(b) = 1$; and $\theta(b) = b$. We define two associated antistructures (α,u) by

$$\alpha(g) = \omega(g)\theta(g^{-1}) \quad \text{for all } g \in G\text{:} \qquad u = \pm b.$$

We call such an antistructure a geometric antistructure.

Given any anti-automorphism $\alpha\colon \mathbb{Z}G \to \mathbb{Z}G$ which takes G to $\pm G$, there are θ and ω so that $\alpha(g) = \omega(g)\theta(g^{-1})$ for all $g \in G$. No integral group ring is known to have units of finite order other than $\pm G$, so it is conceivable that all anti-automorphisms have the above form. One can produce units which are not of the form $\pm b$ (scale by some strange unit in the group ring).

Any geometric antistructure can arise in the codimension 1 surgery diagram. The small group, H, is our G and the G is

$$\pi = G * \mathbb{Z} \,/\, tgt^{-1} = \theta(g);\ t^2 = b$$

where t generates $\mathbb{Z}$. There are two extensions of ω to π and the correct choice yields α for $\tilde{\alpha}_\omega$ and u for $\tilde{1}$.

In Part 1 we compute $L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$ for any geometric antistructure. In Part 2 we compute Ψ_{2r} and settle the extension questions which arise.

Part 1: Compute $L_r^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$

The goal of this section is to explain how to use Table 1 to compute $L_r^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$ for any geometric antistructure given the characters of the irreducible rational representations. Henceforth, χ denotes such a character.

The subtables of Table 1 are labeled by a type, U or O: we assign a type (GL, U, O, or S_p) to each χ. The columns of these subtables are labeled by a symbol Γ_N, F_N, R_N, or H_N or by a symbol UI_N or UII.

In steps 1 and 2 below we show how to determine Type χ. In steps 2 and 3 we show how to assign a symbol $E\chi = \Gamma_N$, F_N, R_N, or H_N or a symbol $U\chi = UI_N$ or UII.

Step 1: Initial crucial remarks.

The type of χ really depends on χ and (α,u) but as the antistructure is fixed during one of these calculations we suppress it.

We first determine if χ has type GL or not:

Type χ is GL iff $\chi(g) \neq \omega(g)\chi(\theta(g^{-1}))$ for some $g\varepsilon G$.

Define a character χ^α by $\chi^\alpha(g) = \omega(g)\chi(\theta(g^{-1}))$ for all $g\varepsilon G$.

If χ has type GL, it makes no contribution to any L theory. If χ does not have type GL, we let $L_r(\chi)$ denote the contribution of χ to $L_r^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$. In the remaining steps we assume that the type of χ is not GL.

Step 2: Type and initial symbol calculations.

Compute the two numbers

$$T_\chi = \frac{1}{|G|} \sum_{g\varepsilon G}\omega(g)\chi(g\theta(g)u) \quad ; \quad S_\chi = \frac{1}{|G|} \sum_{g\varepsilon G} \chi(g^2)$$

From T_χ and S_χ we find the type of χ; partial information about $E\chi$; and define a number m_χ for later use. Explicitly

T_χ	Type		S_χ	$E\chi$	m_χ
positive	0		positive	R_N	1
zero	U		zero	Γ_N or F_N	1
negative	S_p		negative	H_N	2

where N is defined as follows: $2^N = m_\chi \delta_\chi$ and we can always find δ_χ from

$$\delta_\chi = \frac{1}{|G|} \sum_{g \in G} (\chi(g))^2$$

<u>However</u>, if $T_\chi \neq 0$, $\delta_\chi = |T_\chi|$; if $S_\chi \neq 0$, $\delta_\chi = |S_\chi|$

If $T_\chi \neq 0$ and $S_\chi \neq 0$, go directly to step 4.

If $T_\chi = 0$ and $S_\chi \neq 0$, we have $U\chi = UII$: go to step 4.

<u>Step 3</u>: Unresolved issues and a pairing.

If $S_\chi = 0$, we must determine a symbol. If the type of χ is U, we use AI.1.1 below to decide if $U\chi = UI_N$ or UII: if the type of χ is 0 or S_p, we use AI.1.2 below to decide if $E\chi = \Gamma_N$ or F_N.

We will determine these symbols by using a pairing

$$\Lambda : \overline{QG} \times \overline{QG} \to \mathbb{Q}$$

where $\overline{QG}$ is the rational vector space based on the conjugacy classes of G, and

$$\Lambda(C_1, C_2) = \sum_{\substack{g \in C_1 \\ h \in C_2}} \chi(gh)$$

We shall need some related pairings which we proceed to define.

For each N there is an operation, λ_N, on $\overline{QG}$ which sends a conjugacy class, C, to $C^{-5^{2^{N-1}}}$. Define $T_N(C_1, C_2) = \Lambda(C_1, \lambda_N(C_2))$.

There is an operation, α, on $\overline{QG}$ which sends a conjugacy class, C, to $\omega(C)\theta(C^{-1}) \in \overline{QG}$. Define $A(C_1, C_2) = \Lambda(C_1, \alpha(C_2))$.

These pairings are used in the following results.

(AI.1.1) Assume that $S_\chi = T_\chi = 0$ and let $2^N = m_\chi \delta_\chi$. Then $U\chi = UI_N$ or UII:

$U\chi = UI_N$ iff $T_1(C,C) = A(C,C)$ for every conjugacy class C of G.

(AI.1.2) Assume $S_\chi = 0$ and let $2^N = m_\chi \delta_\chi$. Then $E\chi = \Gamma_N$ or F_N:

$E\chi = \Gamma_N$ iff $T_1(C,C) = T_N(C,C)$ for every conjugacy class C of G.

Remark: Of course the symbol $E\chi$ is just the name for a $\mathbb{Z}[\frac{1}{2}]$-maximal order in the division algebra associated to χ (see section 2.2) and hence $E\chi$ is independent of the antistructure. We could use S_χ, N, and AI.1.2 to find $E\chi$ for any χ we wanted. Working through the steps as outlined above only computes $E\chi$ if it is needed to read Table 1.

Step 4: Find the contribution of χ to $L_r^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G,\alpha,u)$

If χ has type U, we use subtable U: $L_r(\chi)$ is found on column $U\chi$ on the row "odd" if r is odd or on the row "even" if r is even.

If χ has type O or S_p, we use subtable O: $L_r(\chi)$ is found in column $E\chi$ on the row k = 3, 2, 1, or 0: $k \equiv r \pmod 4$ if Type χ is O; $k \equiv r+2 \pmod 4$ if Type χ is S_p.

Part 2: Compute $L_r^p(\mathbb{Z}G, \alpha,u)$

We have reduced this problem to understanding a pair of exact sequences

$$0 \to L_{2r+1}^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G,\alpha,u) \xrightarrow{\partial_{2r+1}} L_{2r}^p(\mathbb{Z}G,\alpha,u) \xrightarrow{\kappa_{2r}} L_{2r}^p(\hat{\mathbb{Z}}_2 G,\alpha,u) \xrightarrow{\Psi_{2r}}$$

$$L_{2r}^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G,\alpha,u) \xrightarrow{\partial_{2r}} L_{2r-1}^p(\mathbb{Z}G,\alpha,u) \to 0$$

(for r = 0, 1). Some terminology will be useful.

A representation (or its character χ) is called cyclic if it can be obtained by pulling back the faithful irreducible rational representation of C_N along some epimorphism $\gamma: G \to C_N$, $N \geq 0$. A representation (or its character) is called dihedral if it can be obtained by pulling back the faithful irreducible rational representation of D_N along some epimorphism $\gamma: G \to D_N$, $N \geq 3$.

The epimorphism γ determines χ but not vice versa. The kernel of χ determines the kernel of γ but χ only determines γ up to an automorphism of the quoient.

Next we determine Ψ_{2r}. Since $L^p_{2r}(\hat{\mathbb{Z}}_2 G,\alpha,u) \simeq \mathbb{Z}/2\mathbb{Z}$, Ψ_{2r} is either trivial or one to one.

Theorem AI.2.1: Ψ_{2r} is determined by:

Ψ_0 is one to one iff there is a type 0 cyclic representation;

Ψ_2 is one to one iff there is a type S_p cyclic representation.

It is easy to describe the right-hand extension.

Theorem AI.2.2: ∂_{2r} is a split epimorphism.

The left-hand extension is more difficult to describe, since even if κ_{2r} is onto, two different things can happen.

Associated to a dihedral representation $\gamma: G \to D_N$ there are two other maps; $\gamma_1: G \to \pm 1 = D_N/C_{N-1}$ and $\gamma_2: \ker \gamma_1 \to C_{N-1} \twoheadrightarrow \pm 1$. A dihedral representation is twisted (with respect to the geometric antistructure θ,ω,b) iff γ_1 is θ invariant and $\gamma_2(y^{-1}\theta(y)) = -1$ for any (and hence every) $y \in G-\ker \gamma_1$.

A cyclic representation is twisted iff the composite $G \xrightarrow{\gamma} C_N \twoheadrightarrow \pm 1$ sends b to -1.

We have

Theorem AI.2.3: If κ_0 is onto, then it is split unless there is a type UI twisted cyclic or a type 0 twisted dihedral representation. Then κ_0 is not split.

If κ_2 is onto, then it is split unless there is a type UI twisted cyclic or a type S_p twisted dihedral representation. Then κ_2 is not split.

If κ_{2r} is not split, then any $x \in L^p_{2r}(\mathbb{Z} G,\alpha,u)$ with $\kappa_{2r}(x) = -1$ has infinite order.

To apply the above results it is desirable to be able to find cyclic and dihedral representations. A return to the theory behind the results in Part I yields the critera below.

A character χ is cyclic iff $E\chi = \Gamma_N$ or R_0 and $\chi(e) = 2^N$ or 1. A character χ is dihedral iff $E\chi = R_N$ and $\chi(e) = 2^{N+2}$. A cyclic character χ is twisted iff $(b^{2^N}) = -\chi(e)$; a dihedral character χ is twisted iff

$\chi((g^{-1}\theta(g))^{2^{N+1}}) = -\chi(e)$ for at least one $g \in G$.

Another way to give such representations is to give the epimorphism γ directly. In this case there is a quicker way to find the type than by using step 2 of Part 1.

In the cyclic case, extend $\gamma: G \to C_N$ to $\hat{\gamma}: \pm G \to C_N$ by defining $\hat{\gamma}(-g) = -\gamma(g)$. Recall that α induces a map from G to $\pm G$.

χ has type UI iff $\hat{\gamma}(g^{-1}) = \hat{\gamma}(\alpha(g))$ for all $g \in G$

χ has type O or S_p iff $\hat{\gamma}(g) = \hat{\gamma}(\alpha(g))$ for all $g \in G$

the type is O if $\hat{\gamma}(u) = 1$; the type is S_p if $\hat{\gamma}(u) = -1$.

Any twisted dihedral representation has type O or S_p. For any $g \in G$, define $\tau_g = \omega(g)\chi(g\theta(g)u)$: τ_g is either 0, $\chi(e)$, or $-\chi(e)$. We have type O if there exists a $g \in G$ with $\tau_g = \chi(e)$: we have type S_p if there exists a $g \in G$ with $\tau_g = -\chi(e)$.

Any cyclic character with $\chi(e) = 1$ is called linear: any cyclic character with $\chi(e) = 2$ is called quadratic. A linear cyclic character is either the trivial character or a cyclic character with $\gamma: G \to C_1$. The quadratic characters are the cyclic characters with $\gamma: G \to C_2$. Notice that the linear characters are in one to one correspondence with $H^1(G; \mathbb{Z}/2\mathbb{Z})$.

Examples:

1) $(\mathbb{Z}G, \alpha, e)$ $\alpha(g) = g^{-1}$: any non-linear cyclic representation is type U; all the linear ones are type O. Therefore Ψ_2 is trivial: Ψ_0 is one to one. There are no twisted cyclic or twisted dihedral representations, so κ_2 is split.

2) $(\mathbb{Z}G, \alpha_\omega, e)$ $\alpha(g) = \omega(g)g^{-1}$; ω non-trivial: any non-quadratic cyclic

representation has type GL or U. A quadratic representation $\gamma: G \to C_2$ has either type GL or O. The type is O iff the composite $G \xrightarrow{\gamma} \mathbb{Z}/4\mathbb{Z} \to \pm 1$ is ω. If we consider $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$, there are type O quadratic representations iff $\omega^2 = 0 \in H^2(G; \mathbb{Z}/2\mathbb{Z})$. Therefore Ψ_2 is always trivial: Ψ_0 is trivial iff $\omega^2 \neq 0$. There are no twisted cyclic or twisted dihedral representations, so κ_{2r} is split whenever it is onto.

3) $(\mathbb{Z}C_N, \alpha, x)$ $\alpha(g) = g^{-1}$; $x \in C_N$ a generator: the non-linear representations have type U. The trivial representation has type O; the other linear representation has type S_p. Therefore both Ψ_0 and Ψ_2 are one to one.

4) $(\mathbb{Z}G, \alpha, u)$: $G = C_N \times \mathbb{Z}/2\mathbb{Z}$ generated by $x \in C_N$ and $t \in \mathbb{Z}/2\mathbb{Z}$; $\omega(x) = 1 = -\omega(t)$; $\theta(x) = x$; $\theta(t) = tx^{2^{N-1}}$; $u = x$. There is a type UI twisted cyclic representation and no type O or S_p cyclic representations. Hence both κ_0 and κ_2 are onto but neither is split.

APPENDIX II: Computing push forward maps and transfers

We wish to describe how to compute the push forward and transfer maps associated to an index 2 inclusion of groups, say $H \subset G$. We will assume that we have a map of rings with antistructure and that the antistructures are geometric, but we begin by describing the "simple pieces" of the map $\mathbb{Q}H \to \mathbb{Q}G$.

To do this requires some notation. If χ_0 is the character of an irreducible rational representation of H, define

$$\chi_0^t(h) = \chi_0(tht^{-1}) \qquad \text{for all } h \in H; \quad t \in G-H \text{ is a fixed element.}$$

If χ is the character of an irreducible rational representation of G, define

$$\chi^\psi(g) = \psi(g)\chi(g) \qquad \text{for all } g \in G; \quad \text{where } \psi: G \to \pm 1 \text{ has kernel } H.$$

Recall (1.3) that $\mathbb{Q}H$ is a product of simple rings indexed by the characters, χ_0, of the irreducible rational representations of H: $\mathbb{Q}G$ has a similar description. The map $\mathbb{Q}H \to \mathbb{Q}G$ is a product of the following three sorts of maps. In the three descriptions below, χ_0 is a constituant of χ restricted to H:

Case I: $\chi_0^t = \chi_0$; $\chi \neq \chi^\psi$: $A_{\chi_0} \to A_\chi \times A_{\chi^\psi}$

Case II: $\chi_0^t \neq \chi_0$; $\chi = \chi^\psi$: $A_{\chi_0} \times A_{\chi_0^t} \to A_\chi$

Case III: $\chi_0^t = \chi_0$; $\chi = \chi^\psi$: $A_{\chi_0} \to A_\chi$

When we add the antistructures to the picture, we need to refine this decomposition further into types. We proceed to describe the various cases which occur. Recall $\chi^\alpha(g) = \omega(g)\chi(\theta(g^{-1}))$ (Appendix I, step 1).

The easiest to describe is the GL type. Here, two pieces of the same sort (I, II, III) are interchanged by the antistructure. A GL type makes no contribution to the L theory and so can be ignored.

In case I there are two types in addition to the GL type discussed above. These further types are denoted IX_0GL and $IX_0\Delta$. In IX_0GL, $\chi_0^\alpha = \chi_0$ and $\chi^\alpha = \chi^\psi$: in $IX_0\Delta$, $\chi_0^\alpha = \chi_0$ and $\chi^\alpha = \chi$.

There are similar types in case II: denoted IIGLX and IIAX. We have type IIGLX if $\chi_0^\alpha = \chi_0^t$ and $\chi^\alpha = \chi$: we have type IIAX if $\chi_0^\alpha = \chi_0$ and $\chi^\alpha = \chi$.

In case III the type is either GL or $\chi_0^\alpha = \chi_0$ and $\chi^\alpha = \chi$. This time we divide into type III2 and III3. To describe these two types compute $d = 2^N$ and $m = m_\chi$ for χ. (This was probably done in computing the L group, but, if not, step 2 in Appendix I will do it.) Compute the corresponding numbers d_0 and m_0 for χ_0. Finally, decide if χ and χ_0 both have type UII or not.

Assume either that $m_0 = m$ or that not both χ and χ_0 have type UII:

we have type III2 iff $2d_0 = d$
we have type III3 iff $d_0 = 2d$

Assume that $m_0 = m$ and that both χ and χ_0 have type UII:

we have type III2 iff $m_0 = 2$

we have type III3 iff $m = 2$

Some further definitions will be useful. To describe the maps which come up in cases I and II, define the following kinds of maps:

a Δ-map is a map $A \to B_0 \times B_1$ so that the two composites $A \to B_0 \times B_1 \to B_i$ are isomorphisms;

an A-map is a map $A_0 \times A_1 \to B$ so that the two composites $A_i \to A_0 \times A_1 \to B$ are isomorphisms.

In case III, we introduce the notion of subtype:

if Type χ_0 = Type χ is 0, the subtype is 0;

if Type χ_0 = Type χ is S_p, the subtype is S_p;

if Type χ_0 = Type χ is U, the subtype is U:

if Type $\chi_0 \neq$ Type χ we have a mixed subtype.

There are four cases of mixed subtype denoted

$0 \to U$ $\qquad U \to 0$

$S_p \to U$ $\qquad U \to S_p$

Part 1: Relative push forward maps

Our goal is to describe

(AII.1.1) $\ldots \to L_r^p(\mathbb{Z}H \to \hat{\mathbb{Z}}_2H,\alpha,u) \xrightarrow{i_!} L_r^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \to L_r^p(i_!) \to \ldots$

This sequence decomposes into a product of exact sequences where the product is taken over the types in the decomposition of the map $\mathbb{Q}G \to \mathbb{Q}H$. Since GL types make no contribution we need only describe what happens in the remaining cases. We begin with cases I and II: in the four cases below we list the contribution of the type to AII.1.1.

IX_0GL: $\ldots \to L_r(\chi_0) \to 0 \to L_r(IX_0GL_!) \to \ldots$

IX_0 : $\ldots \to L_r(\chi_0) \to L_r(\chi) \times L_r(\chi^{\psi}) \to L_r(LX_0\Delta_!) \to \ldots$

where $i_!$ is a Δ-map

IIGLX: $\ldots \to 0 \to L_r(\chi) \to L_r(\text{IIGLX}_!) \to \ldots$

IIAX: $\ldots \to L_r(\chi_0) \times L_r(\chi_0^t) \to L_r(\chi) \to L_r(\text{IIAX}_!) \to \ldots$

where $i_!$ is an A-map.

In case III we either use Table 2 or Table 3. We must decide which subtable to use; which row of that subtable; and which columns to use. An integer $\mathcal{k}$ mod 4 (or mod 2 on the U → U subtable) determines a sequence of three groups on each row: this sequence will be isomorphic to the contribution of this factor of the map to sequence AII.1.1.

If the type is III2 we use Table 2. If the subtype is mixed it is Type χ_0 → Type χ.

(A.II.1.2)

subtype	subtable	row	$\mathcal{k}$
0	$0 \to 0$	$E\chi_0 \subset E\chi$	$\mathcal{k} \equiv r \pmod 4$
S_p	$0 \to 0$	$E\chi_0 \subset E\chi$	$\mathcal{k} \equiv r+2 \pmod 4$
U	$U \to U$	$U\chi_0 \to U\chi$	$\mathcal{k} \equiv r \pmod 2$
$0 \to U$	$0 \to U\chi$	$E\chi_0 \subset E\chi$	$\mathcal{k} \equiv r \pmod 4$
$S_p \to U$	$0 \to U\chi$	$E\chi_0 \subset E\chi$	$\mathcal{k} \equiv r+2 \pmod 4$
$U \to 0$	$U\chi \to 0$	—	$\mathcal{k} \equiv r \pmod 4$
$U \to S_p$	$U\chi \to 0$	—	$\mathcal{k} \equiv r+2 \pmod 4$

Remarks: The — in the row column means that the subtable in question has only 1 row. In the $0 \to U$ (or $S_p \to U$) case we may need to go back to steps 2 and 3 in Appendix I to compute $E\chi$. Note that we do not need $E\chi$ if we are using subtable $0 \to$ UII, $E\chi_0$ suffices.

If the type is III3 we use Table 3. If the subtype is mixed, it is Type χ → Type χ_0.

(AII.1.3)

subtype	subtable	row	$\mathcal{k}$
0	$0 \to 0$	$E\chi \subset E\chi_0$	$\mathcal{k} \equiv r \pmod 4$
S_p	$0 \to 0$	$E\chi \subset E\chi_0$	$\mathcal{k} \equiv r+2 \pmod 4$
U	$U \to U$	$U\chi \to U\chi_0$	$\mathcal{k} \equiv r \pmod 2$
$0 \to U$	$0 \to U\chi$	$E\chi \subset E\chi_0$	$\mathcal{k} \equiv r \pmod 4$
$S_p \to U$	$0 \to U\chi$	$E\chi \subset E\chi_0$	$\mathcal{k} \equiv r+2 \pmod 4$
$U \to 0$	$U\chi \to 0$	—	$\mathcal{k} \equiv r \pmod 4$
$U \to S_p$	$U\chi \to 0$	—	$\mathcal{k} \equiv r+2 \pmod 4$

Remark: The only visable difference between AII.1.2 and AII.1.3 is that in the row column we have interchanged the role of χ_0 and χ. A closer study shows that on the $0 \to UI_N$ subtable we need EX to use Table 2 but that on Table 3, this subtable has only one row.

Part 2: Relative transfer maps

This time our goal is to describe

$$\text{(AII.2.1)} \quad \dots \to L_r^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \xrightarrow{i^!} L_r^p(\mathbb{Z}H \to \hat{\mathbb{Z}}_2H,\alpha,u) \to L_r^p(i^!) \to \dots$$

As in part 1 of Appendix II, we get that AII.2.1 is a sum of exact sequences. We describe the contribution from each of the non-GL types.

IX_0GL: $\dots \to 0 \to L_r(\chi_0) \to L_r(IX_0GL^!) \to \dots$

$IX_0\Delta$: $\dots \to L_r(\chi) \times L_r(\chi^\psi) \to L_r(\chi_0) \to L_r(IX_0\Delta^!) \to \dots$

where $i^!$ is an A-map

IIGLX: $\dots \to L_r(\chi) \to 0 \to L_r(IIGLX^!) \to \dots$

IIAX: $\dots \to L_r(\chi) \to L_r(\chi_0) \times L_r(\chi_0^t) \to L_r(IIAX^!) \to \dots$

where $i^!$ is a Δ-map.

If the type is III2, we use Table 3: if the subtype is mixed it is Type $\chi_0 \to$ Type χ. The subtable-row-$\mathcal{R}$ data is read off chart AII.1.2.

If the type is III3, we use Table 2: if the subtype is mixed it is Type $\chi \to$ Type χ_0. The subtable-row-$\mathcal{R}$ data is read off chart AII.1.3.

Part 3: Push forward and transfer maps

We want to describe

$$\text{(AII.3.1)} \quad \dots \to L_r^p(\mathbb{Z}H,\alpha,u) \xrightarrow{i_!} L_r^p(\mathbb{Z}G,\alpha,u) \to L_r^p(i_!) \to \dots$$

and

$$\text{(AII.3.2)} \quad \dots \to L_r^p(\mathbb{Z}G,\alpha,u) \xrightarrow{i^!} L_r^p(\mathbb{Z}H,\alpha,u) \to L_r^p(i^!) \to \dots$$

The map $L_r^p(\hat{\mathbb{Z}}_2H,\alpha,u) \to L_r^p(\hat{\mathbb{Z}}_2G,\alpha,u)$ is an isomorphism, so $L_r^p(i_!)$ is isomorphic to the relative group computed in part 1 of Appendix II. The map $L_r^p(\hat{\mathbb{Z}}_2G,\alpha,u) \to L_r^p(\hat{\mathbb{Z}}_2H,\alpha,u)$ is always the zero map so we have not yet computed $L_r^p(i^!)$. We leave this for [H-T-W].

The maps in AII.3.1 and AII.3.2 are almost completely determined by the corresponding maps in the relative sequences, AII.1.1 and AII.2.1. In some cases the fate of elements which map non-zero into the 2-adic terms is ambiguous. In one case we can give a complete description.

Define the notion of a <u>twisted quaternionic representation</u> by replacing D_N, $N \geq 3$ with Q_N, $N \geq 4$ everywhere. We say that $(\mathbb{Z}G,\alpha,u)$ satisfies condition $\underline{ARF_0}$ iff there are <u>no</u> UI twisted cyclic; 0 twisted dihedral; 0 twisted quaternionic; or 0 cyclic representations: $(\mathbb{Z}G,\alpha,u)$ satisfies condition $\underline{ARF_2}$ iff there are <u>no</u> UI twisted cyclic; S_p twisted dihedral; S_p twisted quaternionic; or S_p cyclic representations.

If $(\mathbb{Z}G,\alpha,u)$ satisfies condition ARF_{2r}, we can define an element $A_{2r} \in L^p_{2r}(\mathbb{Z}G,\alpha,u)$ such that A_{2r} has order 2; $\kappa_{2r}(A_{2r}) = -1$; and the following theorem holds.

<u>Theorem AII.3.3</u>: Let $i:(\mathbb{Z}H,\alpha,u) \to (\mathbb{Z}G,\alpha,u)$ be the usual map. If $(\mathbb{Z}H,\alpha,u)$ satisfies condition ARF_{2r} then so does $(\mathbb{Z}G,\alpha,u)$. Moreover

$$i_!(A_{2r}) = A_{2r} \qquad ; \qquad i^!(A_{2r}) = 0.$$

The antistructures which arise in ordinary surgery theory (the ones with $\alpha = \alpha_\omega$ and $u = e$) never have any twisted representations. Hence they satisfy condition ARF_{2r} iff κ_{2r} is onto.

Our proofs of these results must wait for [H-T-W], but perhaps a word is in order as to how they go.

The first step is to use representation theory to show that all problems can be resolved by studying a short list of groups (e.g. Theorem 2.2.2).

To do the necessary calculations for these groups involves the explicit calculations in Section 3 and the work of C. T. C. Wall [W4-W8]. Finally, whenever the going gets tough, we resort to a twisting diagram (e.g. 4.5.6). Twisting diagrams seem to be a new tool of some power in the long history of these sorts of calculations.

TABLE 1

Type O	Γ_N	F_N	R_N	H_N
3	0	0	0	$\mathbb{Z}^{2^{N-2}}$
2	0	0	0	0
1	0	0	$\mathbb{Z}^{2^N}$	$\mathbb{Z}/2^{2^{N-2}-1}$
0	$\mathbb{Z}/2^{2^{N-1}+2}$	$\mathbb{Z}/2^{2^{N-1}} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2^{2^{N-2}+1}$

Type U	UI_N	UII
odd	$\mathbb{Z}^{2^{N-1}}$	0
even	0	$\mathbb{Z}/2$

TABLE 2

$$\cdots \to L^p_k(R \to \hat{R}_2, \alpha_0, 1) \longrightarrow L^p_k(S \to \hat{S}_2, \alpha, 1) \longrightarrow L^p_k(f_!) \to \cdots$$

$$L^p_k(R \to \hat{R}_2, \alpha_0, 1) = R_k, \quad L^p_k(S \to \hat{S}_2, \alpha, 1) = S_k, \quad L^p_k(f_!) = L^p_k(R \to S, \alpha, 1) = P_k$$

$R \subset S$

$O \to O$	R_3	S_3	P_3	R_2	S_2	P_2	R_1	S_1	P_1	R_0	S_0	P_0
$R_{N-1} \subset R_N$	0	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^N}$	$\mathbb{Z}^{2^{N-1}} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$H_{N-1} \subset H_N$	$\mathbb{Z}^{2^{N-3}}$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-3}}$	0	0	0	$(\mathbb{Z}/2)^{2^{N-3}-1}$	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$(\mathbb{Z}/2)^{2^{N-3}}$	$(\mathbb{Z}/2)^{2^{N-3}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-3}}$
$\Gamma_{N-1}, F_{N-1} \subset \Gamma_N$	0	0	0	0	0	0	0	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ $(\mathbb{Z}/2)^{2^{N-2}} \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^{2^{N-1}+2}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$
$R_{N-1} \subset \Gamma_N, F_N$	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	0	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-1}+2}$ $(\mathbb{Z}/2)^{2^{N-1}} \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^{2^{N-1}+1}$ $(\mathbb{Z}/2)^{2^{N-1}-1} \oplus \mathbb{Z}/4$
$\Gamma_{N-1}, F_{N-1} \subset H_N$	0	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-2}}$	0	0	0	0	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ $(\mathbb{Z}/2)^{2^{N-2}} \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$\mathbb{Z}/2$
$O \to UI_N$												
$R_{N-1} \subset \Gamma_N, F_N$	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	$(\mathbb{Z}/2)^{2^{N-1}+1}$ $(\mathbb{Z}/2)^{2^{N-1}-1} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$	0	0
$UI_{N-1} \to O$												
Fake $\subset H_N$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-1}}$	$(\mathbb{Z}/2)^{2^{N-2}-1}$	0	0	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$
$O \to UII$												
$R_{N-1} \subset R_N$	0	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-1}} \oplus \mathbb{Z}/2$	$\mathbb{Z}^{2^{N-1}}$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$H_{N-1} \subset H_N$	$\mathbb{Z}^{2^{N-3}}$	0	0	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-3}}$	$(\mathbb{Z}/2)^{2^{N-3}-1}$	0	$(\mathbb{Z}/2)^{2^{N-3}}$	$(\mathbb{Z}/2)^{2^{N-3}+1}$	$\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-3}}$
$\Gamma_{N-1}, F_{N-1} \subset \Gamma_N$	0	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ $(\mathbb{Z}/2)^{2^{N-2}} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$	0

$U \to U$	R_{odd}	S_{odd}	P_{odd}	R_{even}	S_{even}	P_{even}
$UI_{N-1} \to UI_N$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-2}} \oplus \mathbb{Z}/2$	0	0	0
$UI_{N-1} \to UII$	$\mathbb{Z}^{2^{N-2}}$	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-2}} \oplus \mathbb{Z}/2$
$UII \to UII$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

TABLE 3

$$\cdots \to L^p_k(S \to \hat{S}_2, \alpha, 1) \to L^p_k(R \to \hat{R}_2, \alpha_0, 1) \to L^p_k(f^!) \to \cdots$$
$$\quad\; = S_k \qquad\qquad\qquad\quad = R_k \qquad\qquad\quad = Tr_k$$

$R \subset S$

$0 \to 0$	S_3	R_3	Tr_3	S_2	R_2	Tr_2	S_1	R_1	Tr_1	S_0	R_0	Tr_0
$R_{N-1} \subset R_N$	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^N}$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_{N-1} \subset H_N$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-3}}$	0	0	0	$(\mathbb{Z}/2)^{2^{N-3}}$	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$(\mathbb{Z}/2)^{2^{N-3}-1}$	$(\mathbb{Z}/2)^{2^{N-3}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-3}+1}$	$\mathbb{Z}^{2^{N-3}} \oplus \mathbb{Z}/2$
$\Gamma_{N-1}, F_{N-1} \subset \Gamma_N$	0	0	0	0	0	0	0	0	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-1}+2}$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ / $(\mathbb{Z}/2)^{2^{N-2}} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$
$R_{N-1} \subset \Gamma_N, F_N$	0	0	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-1}+2}$ / $(\mathbb{Z}/2)^{2^{N-1}} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$	0
$\Gamma_{N-1}, F_{N-1} \subset H_N$	$\mathbb{Z}^{2^{N-2}}$	0	0	0	0	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$(\mathbb{Z}/2)^{2^{N-2}-1}$	0	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ / $(\mathbb{Z}/2)^{2^{N-2}} \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2 \oplus \mathbb{Z})^{2^{N-2}}$

$0 \to UI_N$

$R_{N-1} \subset$ Fake	$\mathbb{Z}^{2^{N-1}}$	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-1}} \oplus \mathbb{Z}/2$

$UI_{N-1} \to 0$

Fake $\subset H_N$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-2}}$	$(\mathbb{Z}/2)^{2^{N-2}-1}$	0	0	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$\mathbb{Z}^{2^{N-2}}$	$(\mathbb{Z}/2 \oplus \mathbb{Z})^{2^{N-2}}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	0	0

$0 \to UII$

$R_{N-1} \subset R_N$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_{N-1} \subset H_N$	0	$\mathbb{Z}^{2^{N-3}}$	$\mathbb{Z}^{2^{N-3}} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	$(\mathbb{Z}/2)^{2^{N-3}-1}$	$(\mathbb{Z}/2)^{2^{N-3}}$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-3}+1}$	$(\mathbb{Z}/2)^{2^{N-3}+1}$
$\Gamma_{N-1}, F_{N-1} \subset \Gamma_N$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	0	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ / $(\mathbb{Z}/2)^{2^{N-2}} \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^{2^{N-2}+1}$

$U \to U$	S_{odd}	R_{odd}	Tr_{odd}	S_{even}	R_{even}	Tr_{even}
$UI_{N-1} \to UI_N$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-2}}$	0	0	0	$\mathbb{Z}^{2^{N-2}}$
$UII_{N-1} \to UII$	0	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}/2$	0	0
$UII \to UII$	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

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