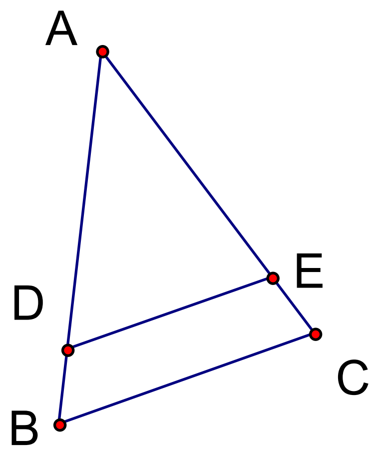
Ratios in Triangles and Trapezoids

Consider this figure of a triangle ABC and a segment DE from line AB to line AC.



This figure can be the basis of a number of problems and theorems. First, consider a couple of familiar scenarios:

Start with equal ratios of triangle sides

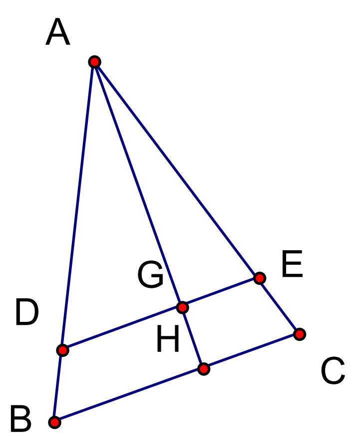
- Given that AD/AB = AE/AC = k, then triangle ABC is similar to triangle ADE, so DE/BC = k also and the corresponding angles are equal. In addition, line DE is parallel to line BC.

Start with parallels: either of these

- Given that DE is parallel to BC, if we denote AD/AB by k, then also AE/AC = k and DE/BC = k.
- Given that DE is parallel to BC, if we denote DE/BC by k, then also AE/AC = k and AD/AB = k.

Numerical examples

In any of these cases, let k = 4/5. Then if we know AB = x, then AD = (4/5)x. If we know AD = u, then AB = (5/4)u. Similar reasoning holds for AC and AE.



Altitudes

The ratio of altitudes of ABC and ADE is the same as for the sides. Let H be on line BC so that AH is perpendicular to BC, so AH is the altitude of ABC through A. Also let H intersect line DE at G, so AG is the altitude of ADE through A. Then show that triangle ABH is similar to triangle ADG, so AG/AH = AD/AB = k/

A new element: What about DB or EC?

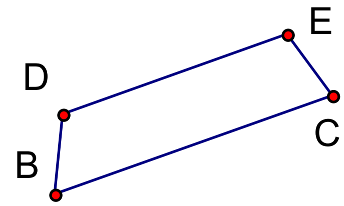
- In either case, if AD/AB = k, then since AD + DB = AB, then (AD + DB)/AB = 1, so k + (DB/AB) = 1 and DB/AB = 1 – k. Likewise EC/AC = 1 – k.

Numerical examples

- In any of these cases, if k = 4/5, then AD = (4/5)AB, so the rest of the segment = (1/5)AB. This illustrates DB/AB = 1 – k = 1 – (4/5) = 1/5.
- This reasoning even when AD > AB, so k > 1. (In this case B is on segment AD.) For example, suppose AD/AB = 4/3. Then DB/AB = 1 – (4/3) = -1/3. The sign is correct, for then AB and DB have opposite direction.

New variant: Start with the figure BCDE and work up!

Suppose you start with the bottom of the previous figure: the trapezoid BCED. Given this figure, it is easy to construct A with the straightedge. Just draw lines BD and CE and intersect them. If A is determined, it must be possible to compute the distance BA from B to A if we know some information about BCED.



How can this be done? Just reverse some of the arguments before. We begin this time with the numerical examples.

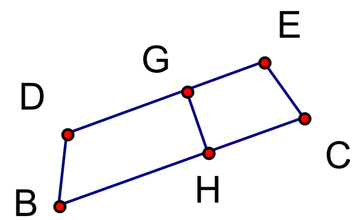
- Assume that DE is parallel to BC and also DE/BC = 4/5. This is the same k as before. Suppose we know DB = p, what is AB? In this case we have already seen that DB/AD = 1 – k = 1/5. So AD = 5 DB = 5p..
- What is AD? AD = AB – DB = 5p – p = 4p. This is pretty easy to see intuitively. Since AD = (4/5) AB and DB = (1/5) AB, then AD/DB = (4/5)/(1/5) = 4!

So the conclusion is that **we can figure out exactly where A is located on line BD**. A is the unique point on line BD with signed ratio BA/BD = 5

Altitudes

Suppose we know the height of trapezoid BCED and the ratios of the parallel sides. Can we find the height to triangle ABC? Yes!

- By the same reasoning with ratios we have AH = 5GH, so the height of triangle ABC is 5 times the height of the trapezoid BCED!



IMPORTANT. This says that if the height of the trapezoid GH = x, then the height of ABC (and the distance from A to line BC) is 5x regardless of the position ("left or right") of DE or the angles at B and C. The only determining factors are the height GH and the ratio DE/BC.

If we make this figure with Sketchpad and drag DE so that the length is constant and the height x is constant, then A will move at a distance of 5x from BC and thus along a line parallel to BC.

General, non-numerical solution

For clarity, we assumed above that k = 4/5. Instead, if we just assume that DE/BC is some ratio k, then we saw before that DB/AB = (1-k). So you can deduce AB/DB from this. The same ratio will apply to AH/GH.

Test Your Understanding: Draw a figure BCDE on graph paper with some simple K. Predict the height of A and then check by the coordinates on the graph paper.