Lecture 8: Integrals of Trigonometric Functions

8.1 Powers of sine and cosine

Example Using the substitution $u = \sin(x)$, we are able to integrate

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)dx = \int_0^1 u^2 du = \frac{1}{3}.$$

In the previous example, it was the factor of $\cos(x)$ which made the substitution possible. That is the motivation behind the algebraic and trigonometric manipulations in the next example.

Example To evaluate $\int \sin^3(2x)dx$, we first note that

$$\int \sin^3(2x)dx = \int \sin^2(2x)\sin(2x)dx = \int (1 - \cos^2(2x))\sin(x)dx.$$

Now we may use the substitution $u = \cos(2x)$ to obtain

$$\int \sin^3(2x)dx = -\frac{1}{2}\int (1-u^2)du = -\frac{1}{2}u + \frac{1}{6}u^3 + c = -\frac{1}{2}\cos(2x) + \frac{1}{6}\cos^3(2x) + c.$$

The technique of the previous example will work for any integral of the form $\int \sin^n(x)dx$ or $\int \cos^n(x)dx$ when n is a positive odd integer.

Example Using the half-angle formula for sine, we have

$$\int \sin^2(x)dx = \frac{1}{2}\int (1 - \cos(2x)dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + c.$$

Example Using the half-angle formula for cosine, we have

$$\int \cos^2(2x)dx = \frac{1}{2}\int (1 + \cos(4x)dx = \frac{1}{2}x - \frac{1}{8}\sin(4x) + c.$$

The general technique for evaluating integrals of the form $\int \sin^n(x)dx$ or $\int \cos^n(x)dx$ when n is a positive even integer is to use the half-angle formulas to reduce the powers.

Example For $n = 4$, we have

$$\begin{aligned}\int \sin^4(x)dx &= \int \left(\frac{1}{2}(1-\cos(2x))\right)^2 dx \\ &= \frac{1}{4}\int (1 - 2\cos(2x) + \cos^2(2x))dx \\ &= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{4}\int \cos^2(2x)dx \\ &= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}\int (1+\cos(4x))dx \\ &= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + c \\ &= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c.\end{aligned}$$

For higher even powers of cosine and sine, you may wish to use a reduction formula, or check out the web site http://integrals.wolfram.com.

Example Note that

$$\int \sin^2(x)\cos^2(x)dx = \int \sin^2(x)(1-\sin^2(x))dx = \int \sin^2(x)dx - \int \sin^4(x)dx,$$

which we could now evaluate as above. However, the double-angle formula for sine is of use in this situation since

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int (\sin(x)\cos(x))^2 dx \\ &= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx \\ &= \frac{1}{4}\int \sin^2(2x)dx \\ &= \frac{1}{8}\int (1-\cos(4x)dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c.\end{aligned}$$

8.2 Powers of secant and tangent

Recall that the secant and tangent functions are related by the identity $\tan^2(x) + 1 = \sec^2(x)$.

Example We have

$$\int \tan^2(3x)dx = \int (\sec^2(3x) - 1)dx = \frac{1}{3}\tan(3x) - x + c.$$

Example Using the substitution $u = \sec(x)$, we have

$$\begin{aligned}\int \tan^3(x)\sec(x)dx &= \int (\sec^2(x) - 1)\sec(x)\tan(x)dx \\ &= \int (u^2 - 1)du \\ &= \frac{1}{3}u^3 - u + c \\ &= \frac{1}{3}\sec^3(x) - \sec(x) + c.\end{aligned}$$

Example Using the substitution $u = \tan(4x)$, we have

$$\begin{aligned}\int \tan^4(4x)dx &= \int \tan^2(4x)(\sec^2(4x) - 1)dx \\ &= \int \tan^2(4x)\sec^2(4x)dx - \int \tan^2(4x)dx \\ &= \frac{1}{4}\int u^2 du - \int (\sec^2(4x) - 1)dx \\ &= \frac{1}{12}u^3 - \frac{1}{4}\tan(4x) + x \\ &= \frac{1}{12}\tan^3(4x) - \frac{1}{4}\tan(4x) + x + c\end{aligned}$$

Example Using the substitution $u = \sec(x) + \tan(x)$, we have

$$\begin{aligned}\int \sec(x)dx &= \int \sec(x)\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\,dx \\ &= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\,dx \\ &= \int \frac{1}{u}\,du \\ &= \log|u| + c \\ &= \log|\sec(x) + \tan(x)| + c.\end{aligned}$$

Example Using the substitution $u = \tan(5x)$, we have

$$\begin{aligned}\int \sec^4(5x)dx &= \int \sec^2(5x)(\tan^2(5x)+1)dx \\ &= \int \tan^2(5x)\sec^2(5x)dx + \int \sec^2(5x)dx \\ &= \frac{1}{5}\int u^2 du + \frac{1}{5}\tan(5x) \\ &= \frac{1}{15}u^3 + \frac{1}{5}\tan(5x) + c \\ &= \frac{1}{15}\tan^3(5x) + \frac{1}{5}\tan(5x) + c.\end{aligned}$$

Example To evaluate $\int \sec^3(x)dx$, we use integration by parts with

$$\begin{array}{rr} u = \sec(x) & dv = \sec^2(x) \\ du = \sec(x)\tan(x)dx & v = \tan(x) \end{array}.$$

Then we have

$$\begin{aligned}\int \sec^3(x)dx &= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx \\ &= \sec(x)\tan(x) - \int (\sec^2(x)-1)\sec(x)dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx.\end{aligned}$$

Hence

$$2\int \sec^3(x)dx = \sec(x)\tan(x) + \int \sec(x)dx.$$

Thus

$$\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\log|\sec(x)+\tan(x)| + c.$$