Mathematica (4)

Mathematica can be used to find power series expansions of functions. For example, to find the first five non-zero terms in the power series expansion of Sin[x] about the point x = 0 we use the built in Mathematica function Series[] which has two arguments, the first of which is the function definition, and the second of which is a list which gives the expansion variable, the value of the point x_0 about which the expansion will be expanded, and the number of terms in the series (in this case 9 some of which are 0). This is a Maclaurin series expansion.

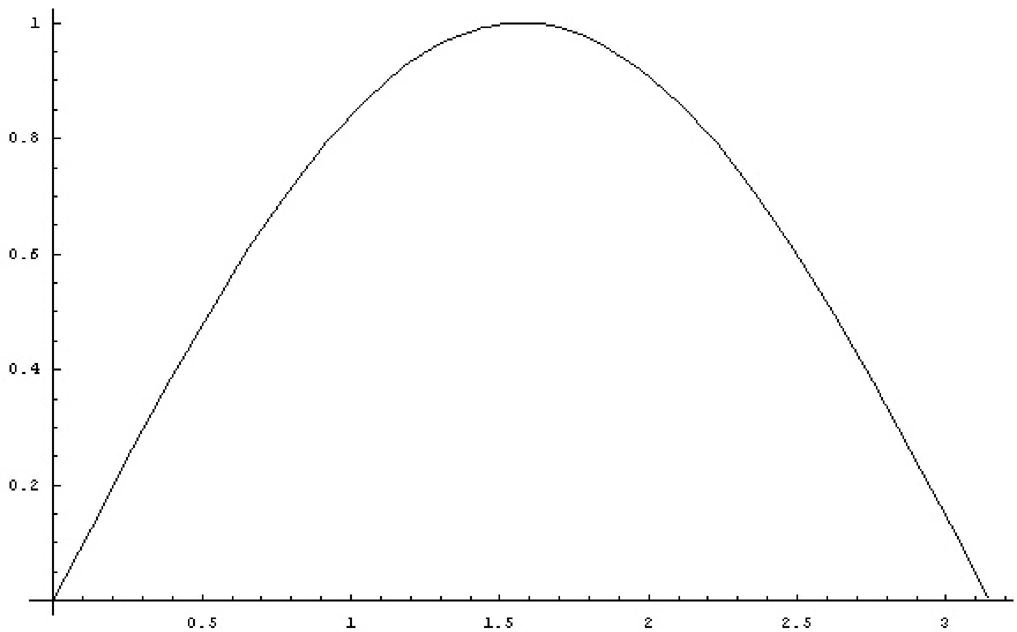
```
In[6]:= Series[Sin[x], {x, 0, 9}]
```

Out[6]= $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} + O[x]^{10}$

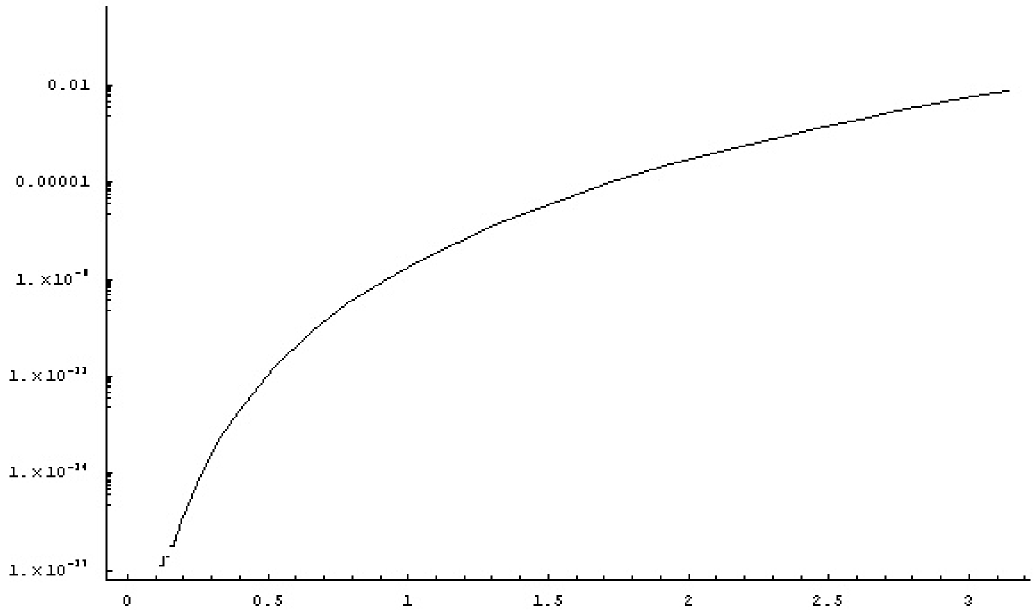
```
In[7]:= f[x_] = x - x^3/6 + x^5/120 - x^7/5040 + x^9/362880
```

Out[7]= $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880}$

```
In[14]:= <<Graphics`
         Plot[f[x], {x, 0, Pi}]
         LogPlot[(f[x] - Sin[x]), {x, 0, Pi}]
```



The command <<Graphics` loads the graphics package that contains Log and LogLog plots. The LogPlot command gives the following Log[y] vs x plot of the difference between f[x], the five term series representation of Sin[x] and Sin[x]. Even in the least accurate part of the plot when x is near Pi, the series approximation is well within 1% of the actual value of Sin[x].



As another example, we can find the first three non-zero terms in the series expansion of the function e^x sin(x):

```
In[1]:= Series[Exp[x] Sin[x], {x, 0, 3}]
```

Out[1]= $x + x^2 + \frac{x^3}{3} + O[x]^4$

We can expand this function in a Taylor's series expansion about the point $x_0 = 1$. This gives the much more complicated looking expansion:

```
In[1]:= Series[ Exp[x] Sin[x], {x, 1, 3}]
```

Out[1]= $e\,\text{Sin}[1] + (e\,\text{Cos}[1] + e\,\text{Sin}[1])\,(x-1) + e\,\text{Cos}[1]\,(x-1)^2 + \left(\frac{1}{3}\,e\,\text{Cos}[1] - \frac{1}{3}\,e\,\text{Sin}[1]\right)(x-1)^3 + O[x-1]^4$

Note that the last term, $O[x - 1]^4$ is simply a statement that the first neglected term in the series is of order $(x - 1)^4$.

We can also expand Sin[x] about the point x_0 = Pi/2, i.e. a Taylor series expansion. Now we get:

```
Out[16]= - Graphics -

In[17]:= Series[Sin[x], {x, Pi / 2, 5}]
```

Out[17]= $1-\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{24}\left(x-\frac{\pi}{2}\right)^4+O\left[x-\frac{\pi}{2}\right]^6$

```
In[18]:= g[x_] = 1 - 1/2 (x - Pi / 2)^2 + 1/24 (x - Pi / 2)^4
```

Out[18]= $1-\frac{1}{2}\left(-\frac{\pi}{2}+x\right)^2+\frac{1}{24}\left(-\frac{\pi}{2}+x\right)^4$

```
In[19]:= g[y + Pi / 2]
```

Out[19]= $1-\frac{y^2}{2}+\frac{y^4}{24}$

```
In[20]:= Simplify[Sin[y + Pi / 2]]

Out[20]= Cos[y]
```

We see that Sin[y + Pi/2] = Cos[y] and the first three terms in the series for Cos[y] are $1 - y^2/2 + y^4/24$ …

As another example, lets find the value of e by summing terms in the power series for Exp[x] with x = 1. The general term is $a_n = 1/n!$ We proceed as follows:

```
In[1]:= a[n_] = 1. / n! ;
        Sum[a[n], {n, 0, 6}]

Out[2]= 2.71806
```

Notice that we again define the function a[n] and then use the built in Mathematica function Sum[] to evaluate the sum of a[n] terms as n goes from 0 to 6. The value of is accurate to three decimal places after 7 terms in the power series of e. Mathematica can also calculate the mathematical value of e. To do this we use the built in function N[] as follows:

```
In[1]:= N[Exp[1]]

Out[1]= 2.71828

In[2]:= N[Exp[1], 10]

Out[2]= 2.718281828
```

In the second example, we use the optional second argument of the N (Numerical) function to get 10 decimal place accuracy rather than the default 5 decimal place accuracy.