5.4 Multiple-Angle Identities

What you'll learn about

- Double-Angle Identities
- Power-Reducing Identities
- Half-Angle Identities
- Solving Trigonometric Equations

. . . and why

These identities are useful in calculus courses.

Double-Angle Identities

The formulas that result from letting $u = v$ in the angle sum identities are called the *double-angle identities.* We will state them all and prove one, leaving the rest of the proofs as exercises. (See Exercises 1–4.)

Double-Angle Identities

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \begin{cases} \cos^2 u - \sin^2 u \\ 2\cos^2 u - 1 \\ 1 - 2\sin^2 u \end{cases}$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

There are three identities for $\cos 2u$. This is not unusual; indeed, there are plenty of other identities one could supply for $\sin 2u$ as well, such as $2 \sin u \sin(\pi/2 - u)$. We list the three identities for $\cos 2u$ because they are all *useful* in various contexts and therefore worth memorizing.

EXAMPLE 1 Proving a Double-Angle Identity

Prove the identity: $\sin 2u = 2 \sin u \cos u$.

SOLUTION

$$\sin 2u = \sin(u + u)$$

$$= \sin u \cos u + \cos u \sin u \quad \text{Sine of a sum } (v = u)$$

$$= 2 \sin u \cos u$$

Now try Exercise 1.

Power-Reducing Identities

One immediate use for two of the three formulas for $\cos 2u$ is to derive the *power-reducing identities.* Some simple-looking functions like $y = \sin^2 u$ would be quite difficult to handle in certain calculus contexts were it not for the existence of these identities.

Power-Reducing Identities

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

We will also leave the proofs of these identities as exercises. (See Exercises 37 and 38.)

EXAMPLE 2 Proving an Identity

Prove the identity: $\cos^4\theta - \sin^4\theta = \cos 2\theta$.

SOLUTION

$$\begin{aligned}\cos^4\theta - \sin^4\theta &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)\\ &= 1\cdot(\cos^2\theta - \sin^2\theta) && \text{Pythagorean identity}\\ &= \cos 2\theta && \text{Double-angle identity}\end{aligned}$$

Now try Exercise 15.

EXAMPLE 3 Reducing a Power of 4

Rewrite $\cos^4 x$ in terms of trigonometric functions with no power greater than 1.

SOLUTION

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2\\ &= \left(\frac{1+\cos 2x}{2}\right)^2 && \text{Power-reducing identity}\\ &= \left(\frac{1+2\cos 2x+\cos^2 2x}{4}\right)\\ &= \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\left(\frac{1+\cos 4x}{2}\right) && \text{Power-reducing identity}\\ &= \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x\\ &= \frac{1}{8}(3+4\cos 2x+\cos 4x)\end{aligned}$$

Now try Exercise 39.

Half-Angle Identities

The power-reducing identities can be used to extend our stock of "special" angles whose trigonometric ratios can be found without a calculator. As usual, we are not suggesting that this algebraic procedure is any more practical than using a calculator, but we are suggesting that this sort of exercise helps you to understand how the functions behave. In Exploration 1, for example, we use a power-reducing formula to find the exact values of $\sin(\pi/8)$ and $\sin(9\pi/8)$ without a calculator.

EXPLORATION 1 Finding the Sine of Half an Angle

Recall the power-reducing formula $\sin^2 u = (1 - \cos 2u)/2$.

1. Use the power-reducing formula to show that $\sin^2(\pi/8) = (2 - \sqrt{2})/4$.
2. Solve for $\sin(\pi/8)$. Do you take the positive or negative square root? Why?
3. Use the power-reducing formula to show that $\sin^2(9\pi/8) = (2 - \sqrt{2})/4$.
4. Solve for $\sin(9\pi/8)$. Do you take the positive or negative square root? Why?

A little alteration of the power-reducing identities results in the *half-angle identities*, which can be used directly to find trigonometric functions of $u/2$ in terms of trigonometric functions of u. As Exploration 1 suggests, there is an unavoidable ambiguity of sign involved with the square root that must be resolved in particular cases by checking the quadrant in which $u/2$ lies.

Did We Miss Two ± Signs?

You might have noticed that all of the half-angle identities have unresolved ± signs except for the last two. The fact that we can omit them on the last two identities for tan $u/2$ is a fortunate consequence of two facts: (1) $\sin u$ and $\tan(u/2)$ always have the same sign (easily observed from the graphs of the two functions in Figure 5.10), and (2) $1 \pm \cos u$ is never negative.

Half-Angle Identities

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \begin{cases} \pm\sqrt{\dfrac{1-\cos u}{1+\cos u}} \\ \dfrac{1-\cos u}{\sin u} \\ \dfrac{\sin u}{1+\cos u} \end{cases}$$

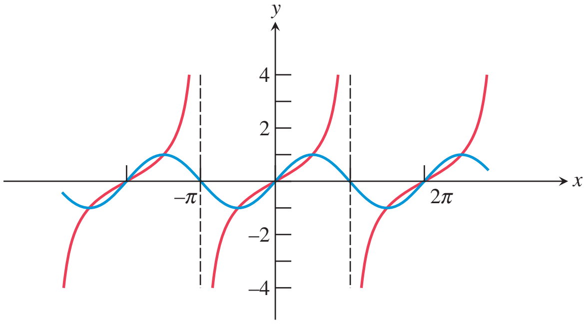


FIGURE 5.10 The functions $\sin u$ and $\tan(u/2)$ always have the same sign.

Solving Trigonometric Equations

New identities always provide new tools for solving trigonometric equations algebraically. Under the right conditions, they even lead to exact solutions. We assert again that we are not presenting these algebraic solutions for their practical value (as the calculator solutions are certainly sufficient for most applications and unquestionably much quicker to obtain), but rather as ways to observe the behavior of the trigonometric functions and their interwoven tapestry of identities.

EXAMPLE 4 Using a Double-Angle Identity

Solve algebraically in the interval $[0, 2\pi)$: $\sin 2x = \cos x$.

SOLUTION

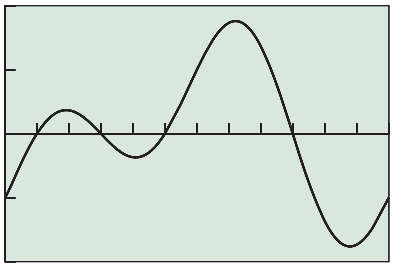
$$\sin 2x = \cos x$$
$$2\sin x\cos x = \cos x$$
$$2\sin x\cos x - \cos x = 0$$
$$\cos x\,(2\sin x - 1) = 0$$
$$\cos x = 0 \quad \text{or} \quad 2\sin x - 1 = 0$$
$$\cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$$

The two solutions of $\cos x = 0$ are $x = \pi/2$ and $x = 3\pi/2$. The two solutions of $\sin x = 1/2$ are $x = \pi/6$ and $x = 5\pi/6$. Therefore, the solutions of $\sin 2x = \cos x$ are

$$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}.$$

We can **support** this result **graphically** by verifying the four x-intercepts of the function $y = \sin 2x - \cos x$ in the interval $[0, 2\pi)$ (Figure 5.11).

Now try Exercise 23.



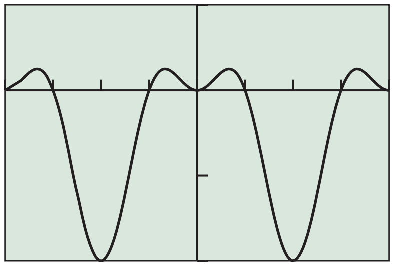
$[0, 2\pi]$ by $[-2, 2]$

FIGURE 5.11 The function $y = \sin 2x - \cos x$ for $0 \le x \le 2\pi$. The scale on the x-axis shows intervals of length $\pi/6$. This graph supports the solution found algebraically in Example 4.

EXAMPLE 5 Using Half-Angle Identities

Solve $\sin^2 x = 2\sin^2(x/2)$.

SOLUTION The graph of $y = \sin^2 x - 2\sin^2(x/2)$ in Figure 5.12 suggests that this function is periodic with period 2π and that the equation $\sin^2 x = 2\sin^2(x/2)$ has three solutions in $[0, 2\pi)$.



$[-2\pi, 2\pi]$ by $[-2, 1]$

FIGURE 5.12 The graph of $y = \sin^2 x - 2\sin^2(x/2)$ suggests that $\sin^2 x = 2\sin^2(x/2)$ has three solutions in $[0, 2\pi)$. (Example 5)

Solve Algebraically

$$\sin^2 x = 2\sin^2\frac{x}{2}$$

$$\sin^2 x = 2\left(\frac{1-\cos x}{2}\right) \qquad \text{Half-angle identity}$$

$$1 - \cos^2 x = 1 - \cos x \qquad \text{Convert to all cosines.}$$

$$\cos x - \cos^2 x = 0$$

$$\cos x(1 - \cos x) = 0$$

$$\cos x = 0 \quad \text{or} \quad \cos x = 1$$

$$x = \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} \quad \text{or} \quad 0$$

The rest of the solutions are obtained by periodicity:

$$x = 2n\pi, \quad x = \frac{\pi}{2} + 2n\pi, \quad x = \frac{3\pi}{2} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now try Exercise 43.

QUICK REVIEW 5.4 *(For help, go to Section 5.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–8, find the general solution of the equation.

1. $\tan x - 1 = 0$

2. $\tan x + 1 = 0$

3. $(\cos x)(1 - \sin x) = 0$

4. $(\sin x)(1 + \cos x) = 0$

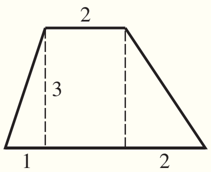
5. $\sin x + \cos x = 0$

6. $\sin x - \cos x = 0$

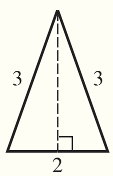
7. $(2\sin x - 1)(2\cos x + 1) = 0$

8. $(\sin x + 1)(2\cos x - \sqrt{2}) = 0$

9. Find the area of the trapezoid.



10. Find the height of the isosceles triangle.



SECTION 5.4 EXERCISES

In Exercises 1–4, use the appropriate sum or difference identity to prove the double-angle identity.

1. $\cos 2u = \cos^2 u - \sin^2 u$ **2.** $\cos 2u = 2\cos^2 u - 1$

3. $\cos 2u = 1 - 2\sin^2 u$ **4.** $\tan 2u = \dfrac{2\tan u}{1 - \tan^2 u}$

In Exercises 5–10, find all solutions to the equation in the interval $[0, 2\pi)$.

5. $\sin 2x = 2\sin x$ **6.** $\sin 2x = \sin x$

7. $\cos 2x = \sin x$ **8.** $\cos 2x = \cos x$

9. $\sin 2x - \tan x = 0$ **10.** $2\cos^2 x + \cos x = \cos 2x$

In Exercises 11–14, write the expression as one involving only $\sin\theta$ and $\cos\theta$.

11. $\sin 2\theta + \cos\theta$ **12.** $\sin 2\theta + \cos 2\theta$

13. $\sin 2\theta + \cos 3\theta$ **14.** $\sin 3\theta + \cos 2\theta$

In Exercises 15–22, prove the identity.

15. $\sin 4x = 2\sin 2x\cos 2x$ **16.** $\cos 6x = 2\cos^2 3x - 1$

17. $2\csc 2x = \csc^2 x\tan x$ **18.** $2\cot 2x = \cot x - \tan x$

19. $\sin 3x = (\sin x)(4\cos^2 x - 1)$

20. $\sin 3x = (\sin x)(3 - 4\sin^2 x)$

21. $\cos 4x = 1 - 8\sin^2 x\cos^2 x$

22. $\sin 4x = (4\sin x\cos x)(2\cos^2 x - 1)$

In Exercises 23–30, solve algebraically for exact solutions in the interval $[0, 2\pi)$. Use your grapher only to support your algebraic work.

23. $\cos 2x + \cos x = 0$ **24.** $\cos 2x + \sin x = 0$

25. $\cos x + \cos 3x = 0$ **26.** $\sin x + \sin 3x = 0$

27. $\sin 2x + \sin 4x = 0$ **28.** $\cos 2x + \cos 4x = 0$

In Exercises 29 and 30, use a graphing calculator to find all exact solutions in the interval $[0, \pi)$. [*Hint*: All solutions are rational multiples of π.]

29. $\sin 2x - \cos 3x = 0$ **30.** $\sin 3x + \cos 2x = 0$

In Exercises 31–36, use half-angle identities to find an exact value without a calculator.

31. $\sin 15°$ **32.** $\tan 195°$

33. $\cos 75°$ **34.** $\sin(5\pi/12)$

35. $\tan(7\pi/12)$ **36.** $\cos(\pi/8)$

37. Prove the power-reducing identities:

(a) $\sin^2 u = \dfrac{1 - \cos 2u}{2}$ **(b)** $\cos^2 u = \dfrac{1 + \cos 2u}{2}$

38. **(a)** Use the identities in Exercise 37 to prove the power-reducing identity $\tan^2 u = \dfrac{1 - \cos 2u}{1 + \cos 2u}$.

(b) Writing to Learn Explain why the identity in part (a) does not imply that $\tan u = \sqrt{\dfrac{1 - \cos 2u}{1 + \cos 2u}}$.

In Exercises 39–42, use the power-reducing identities to prove the identity.

39. $\sin^4 x = \dfrac{1}{8}(3 - 4\cos 2x + \cos 4x)$

40. $\cos^3 x = \left(\dfrac{1}{2}\cos x\right)(1 + \cos 2x)$

41. $\sin^3 2x = \left(\dfrac{1}{2}\sin 2x\right)(1 - \cos 4x)$

42. $\sin^5 x = \left(\dfrac{1}{8}\sin x\right)(3 - 4\cos 2x + \cos 4x)$

In Exercises 43–46, use the half-angle identities to find all solutions in the interval $[0, 2\pi)$. Then find the general solution.

43. $\cos^2 x = \sin^2\left(\dfrac{x}{2}\right)$ **44.** $\sin^2 x = \cos^2\left(\dfrac{x}{2}\right)$

45. $\tan\left(\dfrac{x}{2}\right) = \dfrac{1 - \cos x}{1 + \cos x}$ **46.** $\sin^2\left(\dfrac{x}{2}\right) = \cos x - 1$

Standardized Test Questions

47. True or False The product of two functions with period 2π has period 2π. Justify your answer.

48. True or False The function $f(x) = \cos^2 x$ is a sinusoid. Justify your answer.

You should answer these questions without using a calculator.

49. Multiple Choice If $f(x) = \sin x$ and $g(x) = \cos x$, then $f(2x) =$

(A) $2f(x)$. **(B)** $f(2)f(x)$. **(C)** $f(x)g(x)$.

(D) $2f(x)g(x)$. **(E)** $f(2)g(x) + g(2)f(x)$.

50. Multiple Choice $\sin 22.5° =$

(A) $\dfrac{\sqrt{2}}{4}$. **(B)** $\dfrac{\sqrt{3}}{4}$. **(C)** $\dfrac{\sqrt{6} - \sqrt{2}}{4}$.

(D) $\sqrt{\dfrac{2 - \sqrt{2}}{2}}$. **(E)** $\sqrt{\dfrac{2 - \sqrt{2}}{2}}$.

51. Multiple Choice How many numbers between 0 and 2π satisfy the equation $\sin 2x = \cos x$?

(A) None **(B)** One **(C)** Two **(D)** Three **(E)** Four

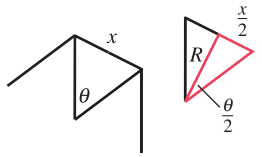
52. Multiple Choice The period of the function $\sin^2 x - \cos^2 x$ is

(A) $\dfrac{\pi}{4}$. **(B)** $\dfrac{\pi}{2}$. **(C)** π. **(D)** 2π. **(E)** 4π.

Explorations

53. Connecting Trigonometry and Geometry In a regular polygon all sides are the same length and all angles are equal in measure.

(a) If the perpendicular distance from the center of the polygon with n sides to the midpoint of a side is R, and if the length of the side of the polygon is x, show that



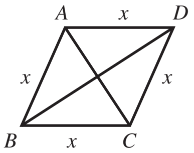
Regular polygon with n sides

$$x = 2R\tan\frac{\theta}{2}$$

where $\theta = 2\pi/n$ is the central angle subtended by one side.

(b) If the length of one side of a regular 11-sided polygon is approximately 5.87 and R is a whole number, what is the value of R?

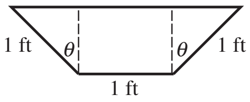
54. Connecting Trigonometry and Geometry A rhombus is a quadrilateral with equal sides. The diagonals of a rhombus bisect the angles of the rhombus and are perpendicular bisectors of each other. Let $\angle ABC = \theta$, $d_1 =$ length of AC, and $d_2 =$ length of BD.



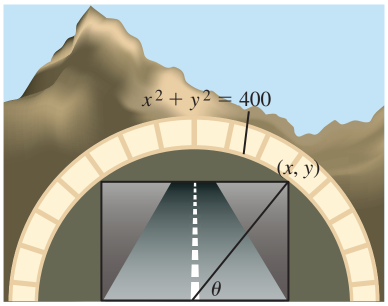
(a) Show that $\cos \frac{\theta}{2} = \frac{d_2}{2x}$ and $\sin \frac{\theta}{2} = \frac{d_1}{2x}$.

(b) Show that $\sin \theta = \frac{d_1 d_2}{2x^2}$.

55. Group Activity Maximizing Volume The ends of a 10-foot-long water trough are isosceles trapezoids as shown in the figure. Find the value of θ that maximizes the volume of the trough and the maximum volume.



56. Group Activity Tunnel Problem A rectangular tunnel is cut through a mountain to make a road. The upper vertices of the rectangle are on the circle $x^2 + y^2 = 400$, as illustrated in the figure.



(a) Show that the cross-sectional area of the end of the tunnel is $400 \sin 2\theta$.

(b) Find the dimensions of the rectangular end of the tunnel that maximizes its cross-sectional area.

Extending the Ideas

In Exercises 57–61, prove the double-angle formulas.

57. $\csc 2u = \frac{1}{2} \csc u \sec u$

58. $\cot 2u = \frac{\cot^2 u - 1}{2 \cot u}$

59. $\sec 2u = \frac{\csc^2 u}{\csc^2 u - 2}$

60. $\sec 2u = \frac{\sec^2 u}{2 - \sec^2 u}$

61. $\sec 2u = \frac{\sec^2 u \csc^2 u}{\csc^2 u - \sec^2 u}$

62. Writing to Learn Explain why

$$\sqrt{\frac{1 - \cos 2x}{2}} = |\sin x|$$

is an identity but

$$\sqrt{\frac{1 - \cos 2x}{2}} = \sin x$$

is not an identity.

63. Hawaiian Sunset Table 5.2 gives the time of day for sunset in Honolulu, HI, on the first day of each month of 2009.

Table 5.2 Sunset in Honolulu, 2009

Date	Day	Time	18:30 +
Jan 1	1	18:01	−29
Feb 1	32	18:22	−8
Mar 1	60	18:36	6
Apr 1	91	18:46	16
May 1	121	18:57	27
Jun 1	152	19:10	40
Jul 1	182	19:17	47
Aug 1	213	19:10	40
Sep 1	244	18:47	17
Oct 1	274	18:19	−11
Nov 1	305	17:55	−35
Dec 1	335	17:48	−42

Source: www.timeanddate.com

The second column gives the date as the day of the year, and the fourth column gives the time as the number of minutes past 18:30.

(a) Enter the numbers in column 2 (day) into list L1 and the numbers in column 4 (minutes past 18:30) into list L2. Make a scatter plot with x-coordinates from L1 and y-coordinates from L2.

(b) Using sine regression, find the regression curve through the points and store its equation in Y1. Superimpose the graph of the curve on the scatter plot. Is it a good fit?

(c) Make a new column showing the *residuals* (the difference between the actual y-value at each point and the y-value predicted by the regression curve) and store them in list L3. Your calculator might have a list called RESID among the NAMES in the LIST menu, in which case the command RESID → L3 will perform this operation. You could also enter L2 − Y1(L1) → L3.

(d) Make a scatter plot with x-coordinates from L1 and y-coordinates from L3. Find the sine regression curve through *these* points and superimpose it on the scatter plot.

(e) Writing to Learn Interpret what the two regressions seem to indicate about the periodic behavior of sunset as a function of time. This is not an unusual phenomenon in astronomical data, and it kept astronomers baffled for centuries.