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On a generalized equation of Smarandache and its integer solutions

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Abstract Let $a \neq 0$ be any given real number. If the variables $x_1, x_2, \cdots, x_n$ satisfy $x_1 x_2 \cdots x_n = 1$, the equation

$$\frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_n}a^{x_n} = na$$

has one and only one nonnegative real number solution $x_1 = x_2 = \cdots = x_n = 1$. This generalized the problem of Smarandache in book [1].

Keywords Equation of Smarandache, real number solutions.

§1. Introduction

Let Q denotes the set of all rational numbers, $a \in Q \setminus \{-1, 0, 1\}$. In problem 50 of book [1], Professor F. Smarandache asked us to solve the equation

$$xa^{\frac{1}{x}} + \frac{1}{x}a^x = 2a. \tag{1}$$

Professor Zhang [2] has proved that the equation has one and only one real number solution $x = 1$. In this paper, we generalize the equation (1) to

$$\frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_n}a^{x_n} = na, \tag{2}$$

and use the elementary method and analysis method to prove the following conclusion:

Theorem. *For any given real number $a \neq 0$, if the variables $x_1, x_2, \cdots, x_n$ satisfy $x_1 x_2 \cdots x_n = 1$, then the equation*

$$\frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_n}a^{x_n} = na$$

has one and only one nonnegative real number solution $x_1 = x_2 = \cdots = x_n = 1$.

§2. Proof of the theorem

In this section, we discuss it in two cases $a > 0$ and $a < 0$.

1) For the case $a > 0$, we let

$$f(x_1, x_2, \cdots, x_{n-1}, x_n) = \frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_{n-1}}a^{x_{n-1}} + \frac{1}{x_n}a^{x_n} - na,$$

If we take x_n as the function of the variables $x_1, x_2, \cdots, x_{n-1}$, we have

$$f(x_1, x_2, \cdots, x_{n-1}, x_n) = \frac{1}{x_1}a^{x_1} + \frac{1}{x_2}a^{x_2} + \cdots + \frac{1}{x_{n-1}}a^{x_{n-1}} + x_1x_2\cdots x_{n-1}a^{\frac{1}{x_1x_2\cdots x_{n-1}}} - na.$$

Then the partial differential of f for every $x_i \quad (i = 1, 2, \cdots, n-1)$ is

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \frac{1}{x_i}a^{x_i}\left(\log a - \frac{1}{x_i}\right) + \frac{1}{x_i}a^{\frac{1}{x_1x_2\cdots x_{n-1}}}(x_1x_2\cdots x_{n-1} - \log a) \\ &= \frac{1}{x_i}\left(a^{x_i}\left(\log a - \frac{1}{x_i}\right) + a^{x_n}\left(\frac{1}{x_n} - \log a\right)\right). \end{aligned}$$

Let

$$g(x_1, x_2, \cdots, x_{n-1}, x_n) = a^{x_i}\left(\log a - \frac{1}{x_i}\right) + a^{x_n}\left(\frac{1}{x_n} - \log a\right), \tag{3}$$

the partial differential quotient of g is

$$\begin{aligned} \frac{\partial g}{\partial x_i} &= a^{x_i}\left(\log^2 a - \frac{\log a}{x_i} + \frac{1}{x_i^2} + \frac{a^{x_n}}{x_ix_n}\left(x_n^2\log^2 a - x_n\log a + 1\right)\right) \\ &= \frac{a^{x_i}}{x_i^2}\left(\left(x_i\log a - \frac{1}{2}\right)^2 + \frac{3}{4}\right) + \frac{a^{x_n}}{x_ix_n}\left(\left(x_n\log a - \frac{1}{2}\right)^2 + \frac{3}{4}\right) > 0. \end{aligned}$$

It's easy to prove that the function $u(x) = a^x(\log a - \frac{1}{x})$ is increasing for the variable x when $x > 0$. From (3) we have:

i) if $x_i > x_n$, $g > 0$, $\frac{\partial f}{\partial x_i} > 0$, and f is increasing for the variable x_i;
ii) if $x_i < x_n$, $g < 0$, $\frac{\partial f}{\partial x_i} < 0$, and f is decreasing for the variable x_i;
iii) if $x_i = x_n$, $g = 0$, $\frac{\partial f}{\partial x_i} = 0$, and we get the minimum value of f.

We have

$$f \geq f_{x_1=x_n} \geq f_{x_1=x_2=x_n} \geq \cdots \geq f_{x_1=x_2=\cdots=x_n} \geq f_{x_1=x_2=\cdots=x_n=1} = 0,$$

and we prove that the equation (2) has only one integer solution $x_1 = x_2 = \cdots = x_n = 1$.

2) For the case $a < 0$, the equation (2) can be written as

$$\frac{1}{x_1}(-1)^{x_1}|a|^{x_1} + \frac{1}{x_2}(-1)^{x_2}|a|^{x_2} + \cdots + \frac{1}{x_n}(-1)^{x_n}|a|^{x_n} = -n|a|, \tag{4}$$

so we know that $x_i \quad (i = 1, 2, \cdots, n)$ is not an irrational number.

Let $x_i = \frac{q_i}{p_i}$ (q_i is coprime to p_i), then p_i must be an odd number because negative number has no real square root. From $x_1x_2\cdots x_n = 1$, we have $p_1p_2\cdots p_n = q_1q_2\cdots q_n$, so q_i is odd number and $(-1)^{x_i} = -1$ $(i = 1, 2, \cdots, n)$. In this case, the equation (4) become the following equation:

$$\frac{1}{x_1}|a|^{x_1} + \frac{1}{x_2}|a|^{x_2} + \cdots + \frac{1}{x_n}|a|^{x_n} = n|a|.$$

From the conclusion of case 1) we know that the theorem is also holds. This completes the proof of the theorem.

References

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[3] Tom M.Apostol, Introduction to analytic number theory, Springer-Verlag, New York, 1976.

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[5] "Smarandache Diophantinc Equations" at http://www.gallup.unm.edu/ smarandache/ Dioph-Eq.txt.