A Triple Inequality with Series and Improper Integrals

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Abstract.

As a consequence of the Integral Test we find a triple inequality which bounds up and down both a series with respect to its corresponding improper integral, and reciprocally an improper integral with respect to its corresponding series.

2000 MSC: 26D15, 40-xx, 65Dxx

1. Introduction.

Before going in details to this triple inequality, we recall the well-known
Integral Test that applies to positive term series:
For all $x \geq 1$ let $f(x)$ be a positive continuous and decreasing function such that $f(n) = a_n$ for $n \geq 1$. Then:

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx \tag{1}$$

either both converge or both diverge.

Following the proof of the Integral Test one easily deduces our inequality.

2. Triple Inequality with Series and Improper Integrals.

Let's first make the below notations:

$$S = \sum_{n=1}^{\infty} a_n \ , \tag{2}$$

$$I = \int_1^{\infty} f(x)dx \ . \tag{3}$$

We have the following
Theorem (Triple Inequality with Series and Improper Integrals):

For all $x \geq 1$ let $f(x)$ be a positive continuous and decreasing function such that $f(n) = a_n$ for $n \geq 1$. Then:

$$S - f(1) \leq I \leq S \leq I + f(1) \qquad (4)$$

Proof.

We consider the closed interval [1, n] the function f is defined on split into n-1 unit subintervals [1, 2], [2, 3], …, [n-1, n], and afterwards the total area of the rectangles of width 1 and length f(k), for $2 \leq k \leq n$, inscribed into the surface generated by the function f and limited by the x-axis and the vertical lines x = 1 and x = n:

$$S_{inf} = \sum_{k=2}^{n} f(k) = f(2) + f(3) + \ldots + f(n) \quad \text{[inferior sum]} \qquad (5)$$

and respectively the total area of the rectangles of width 1 and length f(k), for $1 \leq k \leq n-1$, inscribed into the surface generated by the function f and limited by the x-axis and the vertical lines x = 1 and x = n:

$$S_{sup} = \sum_{k=1}^{n-1} f(k) = f(1) + f(3) + \ldots + f(n-1) \quad \text{[superior sum]} \qquad (6)$$

But the value of the improper integral $\int_1^{\infty} f(x)dx$ is in between these two summations:

$$S_n - f(1) = S_{inf} \leq \int_1^n f(x)dx \leq S_{sup} = S_{n-1} \qquad (7)$$

where

$$S_n = \sum_{k=1}^{n} f(k) . \qquad (8)$$

Now in (7) computing the limit when $n \mapsto \infty$ one gets a double inequality which bounds up and down an improper integral with respect to its corresponding series:

$$S - f(1) \leq I \leq S \qquad (9)$$

And from this one has

$$S \leq I + f(1) \qquad (10)$$

Therefore, combining (9) and (10) we obtain our triple inequality:

$$S - f(1) \leq I \leq S \leq I + f(1)$$

As a consequence of this, one has a double inequality which bounds up and down a series with respect to its corresponding improper integral, similarly to (9):

$$I \leq S \leq I + f(1) \tag{11}$$

Another approximation will be:

$$S_n \leq S \leq S_n + I_n \tag{12}$$

where

$$I_n = \int_n^{\infty} f(x)dx \ \text{ for } n \geq 1 \tag{13}$$

and $I_1 = I$, $S_1 = a_1 = f(1)$.
The bigger is n the more accurate bounding for S.

These inequalities hold even if both the series S and improper integral I are divergent (their values are infinite). According to the Integral Test if one is infinite the other one is also infinite.

3. An Application.

Apply the Triple Inequality to bound up and down the series:

$$S = \sum_{k=1}^{\infty} \frac{1}{k\hat{}2+4} \tag{14}$$

The function $f(x) = \frac{1}{x\hat{}2+4}$ is positive continuous and decreasing for $x \geq 1$. Its corresponding improper integral is:

$$I = \int_1^{\infty} \frac{1}{x\hat{}2+4} dx = \lim_{b \mapsto \infty} \int_1^{b} \frac{1}{x\hat{}2+4} dx = \lim_{b \mapsto \infty} [\frac{1}{2} \arctan \frac{x}{2}]_1^b$$

$$= \frac{1}{2} \lim_{b \mapsto \infty} (\arctan \frac{b}{2} - \arctan \frac{1}{2}) = \frac{1}{2}(\frac{\pi}{2} - \arctan 0.5) \approx 0.553574.$$

Hence:

$$0.553574 = I \leq S \leq I + f(1) = 0.553574 + 1/(1\hat{}2 + 4) = 0.753574$$

or

$$0.553574 \leq S \leq 0.753574.$$

With a TI-92 calculator we approximate series (14) summing its first 1,000 terms and we get:

$$S_{1000} = \sum(1/(x^2+4),x,1,1000) = 0.659404.$$

Sure the more terms we take the better approach for the series we obtain.

In a similar way one can bound up and down an improper integral with respect to its corresponding series.

Reference:

R. Larson, R. P. Hostetler, B. H. Edwards, with assistance of D. E. Heyd, Calculus / Early Transcendental Functions, Houghton Mifflin Co., Boston, New York, 1999.