

Student Thinking

Lesson 1. Misconceptions in mathematics

Objectives

1. What is a misconception?
2. Why study misconceptions?
3. Misconceptions in algebra
4. Misconceptions about math - in general
5. References
6. Student thinking - TEKS connections
7. Discussion Questions

What is a misconception?

1. Student thinking consists of many things. Formulas, relevance, tedium, and enjoyment are part of their attitudes and thinking about mathematics. One problem that leads to very serious learning difficulties in mathematics is those misconceptions student may have from previous inadequate teaching, informal thinking, or poor remembrance. It may be best to begin with a definition. From the Encarta online dictionary, a misconception is “a mistaken idea or view resulting from a misunderstanding of something.” Paraphrasing from the educational literature [Pines, 1985] , we offer,

Certain conceptual relations that are acquired may be inappropriate within a certain context. We terms such relations as "misconceptions." A misconception does not exist independently, but is contingent upon a certain existing conceptual framework. Misconceptions can change or disappear with the framework changes.

Changing the conceptual framework of students is one of the keys goals in repairing mathematics and science misconceptions. That is to say, it is not usually successful to merely inform (e.g. lecture) the student on a misconception. The misconception must be changed internally partly through the student's belief systems and partly through their own cognition.

In another misconceptions framework, we may say many students do not come to the classroom as "blank slates" (Resnick, 1983). Rather, they come with informal theories constructed from everyday experiences. These theories have been actively constructed. They provide an everyday functionality to make sense of the world but are often incomplete half-truths (Mestre, 1987). They are misconceptions.

In this module, we consider student misconceptions in mathematics, particularly those that impact algebra and algebraic thinking. Yet, misconceptions are but one facet of **faulty**, **inaccurate**, or **incorrect** thinking. These are all intertwined causing students unlimited trouble in grasping with mathematics from the most elementary concepts through calculus. In turn, student misconceptions cause teachers immense frustration about why their teaching isn't "getting through."

Why study misconceptions?

Our first thought would be that misconceptions, once rooted in the student's memory, are hard to erase. The situation is somewhat more complex. Researchers' interest in student conceptions has been provoked by numerous studies indicating that

1. Before formal study, persons have firmly held, descriptive, and explanatory systems for scientific and logico-mathematical phenomena, that is, systems of belief about mathematics.
2. These systems of belief differ from what is incorporated into the standard curriculum.
3. Certain constellations of these belief systems show remarkable consistency across ages, abilities, and nationalities.
4. Belief systems are resistant to change through traditional instruction.
See (Champagne, Gunstone, & Klopfer, 1983; Osborne & Wittrock, 1983)
See also (Confrey, Jere, 1990).

This research also suggests that repeating a lesson or making it clearer will not help students who base their reasoning on strongly held misconceptions. (Champagne, Klopfer & Gunstone, 1982; McDermott, 1984; Resnick, 1983). Students tend to be emotionally and intellectually attached to their misconceptions, partly because they have actively constructed them and partly because they give ready methodologies for solving various problems. They definitely interfere with learning when students use them to interpret new experiences.

It is very important to recognize student misconceptions and to re-educate students to correct mathematical thinking.

Although the results apply more to children younger than high school age, Ginsburg [1977] offers a number of observations about errors:

1. Errors result from organized strategies and rules.
2. Faulty rules underlying errors have sensible origins.
3. Too often children see arithmetic as an activity isolated from their ordinary concerns. (As you will note below, many misconceptions and faulty thinking in algebra are related to misconceptions and faulty thinking with arithmetic (e.g. fractions).
4. Children often demonstrate a gap between formal and informal knowledge.

The last point on formal vs. informal knowledge requires definition. Usually, formal knowledge refers to that which is taught in an organized, structured, educational institution. It refers to a system of interrelated definitions and proofs, experiments and arguments. It usually is linked with written methods. On the other hand, informal knowledge refers to more tentative intuitive conjectures and mental strategies. Informal knowledge is generated or learned through one's personal actions. That is, informal knowledge refers to routines that are carried out mechanically, or by habit, or by tradition.

A body of research has also developed connections of misconceptions to math anxiety, as well contributions of acceptance of misconceptions about mathematics, mathematical self-concept, and arithmetic skills to mathematics anxiety. In a study of 92 adult students aged 18 to 57 with a median age of 27, (16 males and 76 females) taking a statistics course, results showed that acceptance of misconceptions and mathematical self-concept were significantly related to mathematics anxiety. The combination of misconceptions, mathematical self-concept and arithmetic skills was significantly related to statistics course performance. Older students returning to school after several years' absence were the ones most debilitated

by negative attitudes toward mathematics. It was concluded that mathematics anxiety involves a mechanistic, nonconceptual approach to math, a low level of confidence and a tendency to give up easily when answers are not immediately apparent.

How to re-educate away from misconceptions?

As mentioned, misconceptions must be deconstructed, and teachers must help students reconstruct correct conceptions. Lohead & Mestre (1988) describe an effective inductive technique for these purposes. There are three steps.

1. Probe for and determine qualitative understanding.
2. Probe for and determine quantitative understanding.
3. Probe for and determine conceptual reasoning.

In addition, it is helpful to confront students with counterexamples to their misconceptions. A self-discovered counterexample will have a far stronger and lasting effect. Incorrect beliefs can be loosened somewhat when so confronted.

Misconceptions in Algebra

Many misconceptions apparent in algebra are rooted in misconceptions of arithmetic.

1. Arithmetic.
 - a. Relations between percentages (%), decimals, and fractions is not well understood. Finding 9.5% or 123% of a number may be a problem for many students. Converting decimal to percent and the inverse are also misunderstood.
 - b. The pseudo equation $-4 + ? = -10$ causes many students trouble early on. This misconception lies with the improper understanding of negative numbers. Some teachers use expressions as *owning* when the quantity is positive and *owing* if it negative. Is this wise? In algebra, there are numerous problems of solving equations for which this lack of understanding/misconception is the source of the difficulty.
2. Number sense. Students do not understand well the difference between rational and irrational numbers. Some think that $\frac{3\pi}{5\pi}$ is irrational, and many think that repeating decimals such as 23.454545... are irrational.
3. Exact vs. approximate. If we compute on our calculator $\frac{\sqrt{3}+\sqrt{7}}{\sqrt{2}} = 3.09557356478$, many students will assume this answer is exact. Similarly calculating $\frac{1}{7}$ as 0.142857142857 will be interpreted as exact. This misunderstanding may well arise from overreliance of calculators or improper teaching of the meaning of numbers generated by the calculator.
4. Fractions. For most of mathematics through calculus, it happens that misconceptions about fractions provide the root source of many student difficulties. Many of these problems come from fractions not being properly understood.
 - a. Example. Operations. Incorrect cancelling of $\frac{ab+c}{b}$ to obtain $a+c$
 - b. Example. Working with large numerators and denominators rather than reducing: when multiplying fractions such as $\frac{13}{7} \cdot \frac{14}{65}$ to get $\frac{13 \cdot 14}{7 \cdot 65} = \frac{182}{455} = \frac{2}{5}$ Instead of cancelling common factors as $\frac{13}{7} \cdot \frac{14}{65} = \frac{1}{1} \cdot \frac{2}{5} = \frac{2}{5}$. This is a bad habit because

working with large numbers increases chances of making a mistake dramatically and, even more importantly, because not seeing numbers as products in arithmetic makes it hard to see the same in working with algebraic expressions. This example is one of faulty procedures leading to algebraic difficulties.

- c. Example. Understanding the "invert and multiply" rule for dividing fractions escapes the comprehension of the vast majority of students. Explanation: The misconception here lies at the nature of the notation and exactly what the division means. In fact

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \left(\frac{c}{d}\right)^{-1} \left(\frac{a}{b}\right) = \frac{d}{c} \cdot \frac{a}{b} = \frac{ad}{bc}$$

So, the invert and multiply rule is little more than writing the multiplicative inverse in place of division and then noting that $\left(\frac{c}{d}\right)^{-1} = \frac{d}{c}$ because

$$1 = \left(\frac{c}{d}\right)^{-1} \left(\frac{c}{d}\right) = \left(\frac{d}{c}\right) \left(\frac{c}{d}\right) = \frac{cd}{cd}$$

- d. Example. Addition of fractions generates a number of errors, most particularly with treating different denominators incorrectly. There is the notational conflict between adding multiplying fractions. In multiplying fractions the numerators and denominators, respectively, are multiplied to achieve the numerator and denominator of the product. It is natural to *want* to do this for addition. In a sense, the student wants to do something correct and proper, but not understanding the absolute need to determine a common denominator at the first step, uses the default, but incorrect, analog method for multiplying fractions.
- e. Example. The very term "cancel" or "cancelling" can cause confusion because both the addition and division operations are involved while the results are respectively, say, zero or one.
5. Magnitudes for negative numbers. For example, which number is larger -8 or -5 ? The problem here is how the student perceives "large." Some teachers mistakenly use expressions such as the "larger negative value." This causes confusion to many students.
6. Order of operations. This is perhaps the most common misconception.
- a. Example. Many students will compute $4 + 3x^2$ as $7x^2$, effectively interpreting $4 + 3x$ as $7x$.
- b. Example. Students often misuse the distributive rule. We may see $x^2 - 2(x - 3)$ written as $x^2 - 2x - 6$. The -2 is not correctly distributed.
7. Powers. Students have trouble with precedence of operations.
- a. Example. Compute $-4^2 = -16$. Many will incorrectly compute $(-4)^2 = 16$.
- b. Example. Compute $16^{(-1/4)}$. Any number of answers may be reported including -2 and 2 rather than correctly as $\frac{1}{2}$.
- c. Example. Is $a \times a \times a$ the same as $3a$? Similarly is a^3 the same as $a \times 3$? Some students are not secure in correct mathematical meaning of various notations. This is partly a fault of there being multiple representations of the same thing. It is partly the misconceptions of not fully understanding and being functionally fluent in making computations.
8. Square roots - definition. Many students have trouble with the correct definition.

$$\left(\sqrt{a^2}\right) = |a|$$

More generally, students have trouble with

$$\sqrt[n]{a^n} = \begin{cases} a & \text{if } n \text{ is odd} \\ |a| & \text{if } n \text{ is even} \end{cases}$$

Even more generally, students have trouble when the definition of a mathematical object is divided by cases.

a. Example. $(\sqrt{(-8)})^2 = |-8| = 8$. Many will give -8 as the answer. In more generality many students will want to write $(x - 5)$ as equal to $\sqrt{(x - 5)^2}$ rather than correctly writing $|x - 5|$.

9. Square roots - with sums. A typical error many students make is to write

$$\begin{aligned} \sqrt{a+b} &= \sqrt{a} + \sqrt{b} \\ \sqrt{a-b} &= \sqrt{a} - \sqrt{b} \end{aligned}$$

This persists well into college.

10. Simplification/factorization of algebraic expressions. Students typically abandon the rules or misinterpret them in many types of simplification problems.

a. Example. Simplifying a rational expression like $\frac{x+2}{x}$ students often think this expression simplifies to 2.

b. Example. Expanding perfect square binomials. Many students compute $(x + 3)^2$ as $x^2 + 9$ or even worse $x^2 + 6$.

c. Example. More generally, we often see $(a + b)^n = a^n + b^n$.

d. Factorization. Many students do not see a common factor in an expression such as $2 \cdot 3 \cdot 4 \cdot 25 + 5$. This could be why they have difficulty seeing the common factor in $2x^2y^3 + 3x$.

11. Using the definition of the absolute value function, particularly for negative numbers. The definition

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

just causes many students to pause. They understand the idea of the absolute value.

However, it is here the actual mathematics notation that causes trouble. It is important for students to use this definition for several problem examples. It is particularly important in many algebra problems, particularly those with solving inequalities.

12. Inequalities. Students have trouble solving various types of inequalities. The worst offenders are inequalities involving quadratic terms and inequalities with an absolute value.

a. Example. Solve $x^2 - 8 > 0$.

b. Example. Solve $|x - 7| > 6$.

students need to be skilled at fundamental facts about factoring and what the terms actually mean. In the first example, the one procedure is factoring $(x - \sqrt{8})(x + \sqrt{8})$. Then, noting the product of two values is positive only if the multiplicands are either both positive or both negative, we solve $x - \sqrt{8} > 0$ and $x + \sqrt{8} > 0$. This also gives the solutions for the terms being negative. We could also write $x^2 > 8$ and take square roots. In this situation, the student must take both roots. This procedure also involves the expression $\sqrt{x^2}$ discussed above. The second example, Solve $|x - 7| > 6$, requires the

student to have a working knowledge of the definition of absolute value.

13. Expansion of algebraic expressions. In some views, the only things the students seem to be able to do correctly are FOIL and add/subtract a constant from both sides of an equation.
14. Exponentials - Properties.. Students are not firm with negative and fractional exponents. Students have trouble with correctly simplifying

$$\frac{a^m}{a^{-n}}, a^m a^{-n}, a^{-m} a^{-n}$$

15. Exponential functions. Students do not work well with exponents. Once misconceptions take root, they become hard to eliminate.
- a. Example. Students will often linearize exponential rules such as writing e^{a+b} as $e^a + e^b$.
- b. Similarly, we often see e^{ab} written as $e^a e^b$.
16. Logarithm - properties.
- a. Often students will write $\log x - \log y = \frac{\log x}{\log y}$ instead of the correct expression $\ln \frac{x}{y}$.
- b. Students will also linearize rules and produce such as $\log \ln(a + b) = \ln a + \ln b$, and $\ln(2x) = 2 \ln x$
17. Logarithms - solving equations. When solving a logarithmic equation, students forget to check if the answer is in the domain, or if they get two answers and the first one checks, they tend to automatically eliminate the second choice.
- a. Example. Solve $\log_2(x - 4) = 3 - \log_2(x + 3)$.

Solution.

$$\log_2(x - 4) = 3 - \log_2(x + 3)$$

$$\log_2(x - 4) + \log_2(x + 3) = 3$$

$$\log_2((x - 4)(x + 3)) = 3$$

$$(x - 4)(x + 3) = 2^3 = 8$$

$$x^2 - x - 12 = 8$$

$$x^2 - x - 20 = 0$$

$$(x - 5)(x + 4) = 0$$

$$x = 5, -4$$

The solution $x = 5$ is valid, but the solution $x = -4$ is not. Students often do not check this. This serves as the type of misconception or misbelief that if an algorithm is followed correctly, only correct answers will result.

- b. Example. Solve $\log(x^2 - 7) = \log(x - 5)$.

Solution.

$$\begin{aligned} \log(x^2 - 7) &= \log(x - 6) \\ \log(x^2 - 7) - \log(x - 6) &= 0 \\ \log \frac{(x^2 - 7)}{(x - 6)} &= 0 \\ \frac{(x^2 - 7)}{(x - 6)} &= 1 \\ (x^2 - 7) - (x - 6) &= 1 \\ x^2 - x - 2 &= 0 \\ x &= 2, -1 \end{aligned}$$

Note when substituted into the original equation, both left and right side are undefined.

18. Functions. Finding the domain of a rational function when a common factor is present in the numerator and denominator.
- a. Example. What is the domain of

$$\frac{x - 3}{x^2 - 4x + 3}$$

Clearly, the domain is $\{x \mid x \neq 3 \text{ and } x \neq 1\}$. Students tend to cancel the common factor and work with what remains. (Cancellation only works when the factor cancelled is nonzero.)

19. Functions - asymptotes. Vertical and horizontal asymptotes are often confused because students poorly understand the reasons making which one vertical and which one horizontal. Also, when asked to count how many asymptotes a function has, the student will frequently count only the vertical ones.
- a. Example. The rational function $f(x) = \frac{x-3}{x^2-1}$ has how many asymptotes.
20. Translational errors. Consider the problem given to freshmen college engineering majors. (Clement, 1982).

Example. Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there are five people who ordered strudel." Let C represent the number of cheesecakes and S represent the number of strudels ordered. The correct answer is $5C = 4S$. This question was missed answered correctly by only 27% of the students. The typical wrong answer was $4C = 5S$. Students were also given the hint: "Be careful. Some students put a number in the wrong place in the equation." This hint improved correctness of this answer by only 6%.

The translational error here is the incorrect conversion of the words to symbols. Clearly this problem is rather nonstandard in appearance. One way to solve it is to consider the equation $\frac{S}{5} = \frac{C}{4}$, which equates groups of five strudel orders with groups of four cheesecake orders. The correct equation follows from this.

For students, the misconception arises from two factors. First, students translate the words of the problem from left to right. Second, they confuse the idea of variables and labels. Using a left-to-right strategy, students interpret the C and the S in the equation as labels

for the terms "cheesecake" and "strudel." They apparently fail to apply the idea that variables stand for numerical expressions.

21. Algebra word problems are interpreted through misconceptions.
 - a. Example. Original and sale prices. Students often mistake the way in which an original price and a sale price relate to one another. They may incorrectly calculate the original price from a sale price by applying the discount to the known sale price, rather than to an unknown original price.

Online resources:

1. <http://www.mathsyear2000.co.uk/resources/misconceptions/index.shtml>
2. <http://www.ericdigests.org/pre-9213/hispanic.htm>

Misconceptions about math — in general

Society has many misconceptions about mathematics and its role in our world.

1. Mathematics is incorrectly viewed as a collection of rigid rules and mysterious procedures that seem to be unrelated to each other and require total mastery with little or no understanding.
2. Mathematics is perceived by many to be difficult and demanding and is considered to be a subject in which it is socially acceptable to do poorly.
3. Mathematical thinking is regarded as essentially unimportant to people that do not actually "do" mathematics in their employment capacity.
4. The pervasive role of mathematics is underestimated in the world of everyday living.

References

1. Gourgey, Annette F, (not published), The relationship of misconceptions about math and mathematical self-concept to math anxiety and statistics performance.. General information on misconceptions.
2. Champagne, A. B., Gunstone, R. E, & Klopfer, L. E. (1983). Naive knowledge and science learning. *Research in Science and Technological Education*, 1(2), 173-183.
3. Clement, John. (1982) Algebra Word Problem Solutions: Thought Processes underlying a common misconception, *Journal for Research in Mathematics Education*, 13, 16-30.
4. Confrey, Jere. (1982). A Review of the Research on Student Conceptions in Mathematics, Science, and Programming, *Review of Research in Education*, Vol. 16, pp. 3-56.
5. Ginsberg, H. (1977). *Childrens's arithmetic: How they learn it and how you teach it.* Austin, TX: Pro-Ed.
6. McDermott, L. (1984). Research on conceptual understanding of physics. *Physics Today*, 37, 24-32.
7. Mestre, J. (1987, Summer). Why should mathematics and science teachers be interested in cognitive research findings? *Academic Connections*, pp. 3-5, 8-11. New York: The College Board.
8. Osborne, R. J., & Wittrock, M. C. (1983). Learning science: A generative process. *Science Education*, 67(4), 498-508.
9. Pines (Eds.), A. L. (1985). Towards a taxonomy of conceptual relations. In L. West and A. L. Pines (Eds.) *Cognitive structure and conceptual change* (pp.101-116). New York,

Academic Press.

10. Resnick, L. (1983). Mathematics and science learning: A new conception. *Science*, 220, 477-478.

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Student Thinking and Misconceptions — TEKS Connections

In Algebra I,

1. Multiple Representation is a(3).
The students are going to represent and model problem situations through analyzing and interpreting relationships between two quantities in the function.
2. Student Thinking (misconceptions) is a(4): students are supposed to use a variety of methods to solve equations. Different approaches in solving the problem is the way to look for any misconception that the students might have. The teacher should conquer this misconception for the students through exposing them to many methods in solving the same problem. Students work in many situations to set up equations and inequalities and use different methods to solve them.
3. Problem-solving is a(6): the students use the problem solving at the end to apply what have been learned in the class in the real life (Meaningful mathematics).
4. Algebraic thinking is a(5) & a(2): the students are solving the meaningful problems in algebra after their experience with different algebraic tools they have learned. Symbolic reasoning is powerful in expressing an algebraic thinking.

In Algebra II,

1. Multiple Representation is a(4).
The students perceive the connection between Algebra and Geometry and try to use the tools of one to solve problems in the other. Also, the relationship between the function and equation is powerful tool in expressing generalization.
2. Student Thinking (misconceptions) is a(3):
The students are supplied with more mathematical experiences to overcome any misconception in their algebraic thinking. Understanding and analyzing a broad variety of relationships between functions and equations is to express generalizations.
3. Problem-solving is a(6): the students use the problem solving at the end to apply what have been learned in the class in the real life.
4. Algebraic thinking is a(5) & a(2): students model mathematical situations to solve meaningful problems in Algebra.

In Pre-calculus,

1. In the introduction section b(1) & b(2), we can see all the items achieved in them.
2. Multiple representations are used to solve different real life problems in order to overcome

all students' misconceptions at that particular level of algebraic thinking.

Misconceptions in Math — Discussion Questions and Activities

1. Develop strategies to re-educate students for misconceptions about the solution of inequalities described in #13 above.
2. Determine counterexamples that can help students resolve misconceptions about cancellation of variables in algebraic quotients.
3. Discuss any misconceptions you may have discovered in your own algebraic thinking. What mistakes have you commonly made? Are they misconceptions or perhaps poorly understood ideas.?
4. Develop a brief diagnostic test to determine common misconceptions.
5. How can you determine whether students are using formal or informal knowledge in making algebraic calculations? How can reliance on informal knowledge lead to continued misconceptions? Make an argument for students having a solid formal knowledge for continued correct calculations.
6. Rank the algebra misconceptions in order of importance. Give reasons for your ranking.

Misconceptions in Math - Survey

Directions. For problems 1-4, please circle the answer that is best reflected by the Misconceptions in Math readings. For 5-7, give your impressions about the topic.

Use the back of this paper if you need more room.

1. Misconceptions are linked to
 - A. The belief system.
 - B. Incorrect thinking.
 - C. Algebraic Thinking.
 - D. Blank slates.

2. The best thing to do when you discover your students have a misconception is
 - A. Re-teach the lesson.
 - B. Re-teach the lesson in a clearer way.
 - C. Inform the students of their misconception.
 - D. Re-educate the belief causing the misconception.

3. Student misconceptions can cause
 - A. Sleepless nights.
 - B. Math anxiety.
 - C. A stronger desire to learn correctly.
 - D. Hostility toward mathematics.

4. Which of the following is not an observation about student errors in mathematics?
 - A. Errors result from organized strategies and rules.
 - B. Faulty rules underlying errors have sensible origins.
 - C. Children often demonstrate a gap between formal and informal knowledge.
 - D. Inaccurate concept maps for mathematical thinking.

5. When I teach mathematics and notice students have a misconception about some topic, my best strategy will be to

6. The most surprising aspects about student misconceptions I learned from this reading was

7. I found the reading very useful and plan to use this knowledge in my teaching.
 Strongly Agree Agree Neutral Disagree Strongly Disagree

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