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AN INTRODUCTION TO NONASSOCIATIVE ALGEBRAS

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Transcriber's notes

This e-text was created from scans of the multilithed book published by the Department of Mathematics at Oklahoma State University in 1961. The book was prepared for multilithing by Ann Caskey.

The original was typed rather than typeset, which somewhat limited the symbols available; to assist the reader we have here adopted the convention of denoting algebras etc by fraktur symbols, as followed by the author in his substantially expanded version of the work published under the same title by Academic Press in 1966.

Minor corrections to punctuation and spelling and minor modifications to layout are documented in the LaTeX source.

These are notes for my lectures in July, 1961, at the Advanced Subject Matter Institute in Algebra which was held at Oklahoma State University in the summer of 1961.

Students at the Institute were provided with reprints of my paper, *Structure and representation of nonassociative algebras* (Bulletin of the American Mathematical Society, vol. 61 (1955), pp. 469–484), together with copies of a selective bibliography of more recent papers on nonassociative algebras. These notes supplement §§3–5 of the 1955 Bulletin article, bringing the statements there up to date and providing detailed proofs of a selected group of theorems. The proofs illustrate a number of important techniques used in the study of nonassociative algebras.

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Stillwater, Oklahoma
July 26, 1961

I. Introduction

By common consent a ring $\mathfrak{R}$ is understood to be an additive abelian group in which a multiplication is defined, satisfying

$$(1) \qquad (xy)z = x(yz) \qquad \text{for all } x, y, z \text{ in } \mathfrak{R}$$

and

$$(2) \qquad (x+y)z = xz + yz, \qquad z(x+y) = zx + zy \qquad \text{for all } x, y, z \text{ in } \mathfrak{R},$$

while an algebra $\mathfrak{A}$ over a field F is a ring which is a vector space over F with

$$(3) \qquad \alpha(xy) = (\alpha x)y = x(\alpha y) \qquad \text{for all } \alpha \text{ in } F,\ x, y \text{ in } \mathfrak{A},$$

so that the multiplication in $\mathfrak{A}$ is bilinear. Throughout these notes, however, the associative law (1) will fail to hold in many of the algebraic systems encountered. For this reason we shall use the terms "ring" and "algebra" for more general systems than customary.

We define a *ring* $\mathfrak{R}$ to be an additive abelian group with a second law of composition, multiplication, which satisfies the distributive laws (2). We define an *algebra* $\mathfrak{A}$ over a field F to be a vector space over F with a bilinear multiplication (that is, a multiplication satisfying (2) and (3)). We shall use the name *associative ring* (or *associative algebra*) for a ring (or algebra) in which the associative law (1) holds.

In the general literature an algebra (in our sense) is commonly referred to as a *nonassociative algebra* in order to emphasize that (1) is not being assumed. Use of this term does not carry the connotation that (1) fails to hold, but only that (1) is not assumed to hold. If (1) is actually not satisfied in an algebra (or ring), we say that the algebra (or ring) is *not associative*, rather than nonassociative.

As we shall see in II, a number of basic concepts which are familiar from the study of associative algebras do not involve associativity in any way, and so may fruitfully be employed in the study of nonassociative algebras. For example, we say that two algebras $\mathfrak{A}$ and $\mathfrak{A}'$ over F are *isomorphic* in case there is a vector space isomorphism $x \leftrightarrow x'$ between them with

$$(4) \qquad (xy)' = x'y' \qquad \text{for all } x, y \text{ in } \mathfrak{A}.$$

Although we shall prove some theorems concerning rings and infinite-dimensional algebras, we shall for the most part be concerned with finite-dimensional algebras. If $\mathfrak{A}$ is an algebra of dimension n over F, let $u_1, \ldots, u_n$ be a basis for $\mathfrak{A}$ over F. Then the bilinear multiplication in $\mathfrak{A}$ is completely determined by the n^3 *multiplication constants* γ_{ijk} which appear in the products

$$(5) \qquad u_i u_j = \sum_{k=1}^{n} \gamma_{ijk} u_k, \qquad \gamma_{ijk} \text{ in } F.$$

We shall call the n^2 equations (5) a *multiplication table*, and shall sometimes have occasion to arrange them in the familiar form of such a table:

$$\begin{array}{c|ccccc}
 & u_1 & \ldots & u_j & \ldots & u_n \\
\hline
u_1 & & & \vdots & & \\
\vdots & & & \vdots & & \\
u_i & & \ldots & \sum \gamma_{ijk} u_k & \ldots & \\
\vdots & & & \vdots & & \\
u_n & & & \vdots & &
\end{array}$$

The multiplication table for a one-dimensional algebra $\mathfrak{A}$ over F is given by $u_1^2 = \gamma u_1 (\gamma = \gamma_{111})$. There are two cases: $\gamma = 0$ (from which it follows that every product xy in $\mathfrak{A}$ is 0, so that $\mathfrak{A}$ is called a *zero algebra*), and $\gamma \neq 0$. In the latter case the element $e = \gamma^{-1} u_1$ serves as a basis for $\mathfrak{A}$ over F, and in the new multiplication table we have $e^2 = e$. Then $\alpha \leftrightarrow \alpha e$ is an isomorphism between F and this one-dimensional algebra $\mathfrak{A}$. We have seen incidentally that any one-dimensional algebra is associative. There is considerably more variety, however, among the algebras which can be encountered even for such a low dimension as two.

Other than associative algebras the best-known examples of algebras are the Lie algebras which arise in the study of Lie groups. A *Lie algebra* $\mathfrak{L}$ over F is an algebra over F in which the multiplication is *anticommutative*, that is,

$$(6) \qquad x^2 = 0 \qquad \text{(implying } xy = -yx\text{)},$$

and the *Jacobi identity*

$$(7) \qquad (xy)z + (yz)x + (zx)y = 0 \qquad \text{for all } x, y, z \text{ in } \mathfrak{L}$$

is satisfied. If $\mathfrak{A}$ is any associative algebra over F, then the *commutator*

$$[x,y] = xy - yx \tag{8}$$

satisfies

$$[x,x] = 0 \tag{6$'$}$$

and

$$\big[[x,y],z\big] + \big[[y,z],x\big] + \big[[z,x],y\big] = 0. \tag{7$'$}$$

Thus the algebra $\mathfrak{A}^-$ obtained by defining a new multiplication (8) in the same vector space as $\mathfrak{A}$ is a Lie algebra over F. Also any subspace of $\mathfrak{A}$ which is closed under commutation (8) gives a subalgebra of $\mathfrak{A}^-$, hence a Lie algebra over F. For example, if $\mathfrak{A}$ is the associative algebra of all $n \times n$ matrices, then the set $\mathfrak{L}$ of all skew-symmetric matrices in $\mathfrak{A}$ is a Lie algebra of dimension $\frac{1}{2}n(n-1)$. The Birkhoff-Witt theorem states that any Lie algebra $\mathfrak{L}$ is isomorphic to a subalgebra of an (infinite-dimensional) algebra $\mathfrak{A}^-$ where $\mathfrak{A}$ is associative. In the general literature the notation $[x,y]$ (without regard to (8)) is frequently used, instead of xy, to denote the product in an arbitrary Lie algebra.

In these notes we shall not make any systematic study of Lie algebras. A number of such accounts exist (principally for characteristic 0, where most of the known results lie). Instead we shall be concerned upon occasion with relationships between Lie algebras and other nonassociative algebras which arise through such mechanisms as the *derivation algebra*. Let $\mathfrak{A}$ be any algebra over F. By a *derivation* of $\mathfrak{A}$ is meant a linear operator D on $\mathfrak{A}$ satisfying

$$(xy)D = (xD)y + x(yD) \qquad \text{for all } x, y \text{ in } \mathfrak{A}. \tag{9}$$

The set $\mathfrak{D}(\mathfrak{A})$ of all derivations of $\mathfrak{A}$ is a subspace of the associative algebra $\mathfrak{E}$ of all linear operators on $\mathfrak{A}$. Since the commutator $[D, D']$ of two derivations D, D' is a derivation of $\mathfrak{A}$, $\mathfrak{D}(\mathfrak{A})$ is a subalgebra of $\mathfrak{E}^-$; that is, $\mathfrak{D}(\mathfrak{A})$ is a Lie algebra, called the *derivation algebra* of $\mathfrak{A}$.

Just as one can introduce the commutator (8) as a new product to obtain a Lie algebra $\mathfrak{A}^-$ from an associative algebra $\mathfrak{A}$, so one can introduce a symmetrized product

$$x * y = xy + yx \tag{10}$$

in an associative algebra $\mathfrak{A}$ to obtain a new algebra over F where the vector space operations coincide with those in $\mathfrak{A}$ but where multiplication is defined by the commutative product $x * y$ in (10). If one is content to restrict attention to fields F of characteristic not two (as we shall be in many places in these notes) there is a certain advantage in writing

$$(10') \qquad x \cdot y = \tfrac{1}{2}(xy + yx)$$

to obtain an algebra $\mathfrak{A}^+$ from an associative algebra $\mathfrak{A}$ by defining products by (10′) in the same vector space as $\mathfrak{A}$. For $\mathfrak{A}^+$ is isomorphic under the mapping $a \to \frac{1}{2}a$ to the algebra in which products are defined by (10). At the same time powers of any element x in $\mathfrak{A}^+$ coincide with those in $\mathfrak{A}$: clearly $x \cdot x = x^2$, whence it is easy to see by induction on n that $x \cdot x \cdot \cdots \cdot x$ (n factors) $= (x \cdot \cdots \cdot x) \cdot (x \cdot \cdots \cdot x) = x^i \cdot x^{n-i} = \frac{1}{2}(x^i x^{n-i} + x^{n-i} x^i) = x^n$.

If $\mathfrak{A}$ is associative, then the multiplication in $\mathfrak{A}^+$ is not only commutative but also satisfies the identity

$$(11) \qquad (x \cdot y) \cdot (x \cdot x) = x \cdot [y \cdot (x \cdot x)] \qquad \text{for all } x, y \text{ in } \mathfrak{A}^+.$$

A (commutative) *Jordan algebra* $\mathfrak{J}$ is an algebra over a field F in which products are *commutative*:

$$(12) \qquad xy = yx \qquad \text{for all } x, y \text{ in } \mathfrak{J},$$

and satisfy the *Jordan identity*

$$(11') \qquad (xy)x^2 = x(yx^2) \qquad \text{for all } x, y \text{ in } \mathfrak{J}.$$

Thus, if $\mathfrak{A}$ is associative, then $\mathfrak{A}^+$ is a Jordan algebra. So is any subalgebra of $\mathfrak{A}^+$, that is, any subspace of $\mathfrak{A}$ which is closed under the symmetrized product (10′) and in which (10′) is used as a new multiplication (for example, the set of all $n \times n$ symmetric matrices). An algebra $\mathfrak{J}$ over F is called a *special Jordan algebra* in case $\mathfrak{J}$ is isomorphic to a subalgebra of $\mathfrak{A}^+$ for some associative $\mathfrak{A}$. We shall see that not all Jordan algebras are special.

Jordan algebras were introduced in the early 1930's by a physicist, P. Jordan, in an attempt to generalize the formalism of quantum mechanics. Little appears to have resulted in this direction, but unanticipated relationships between these algebras and Lie groups and the foundations of geometry have been discovered.

The study of Jordan algebras which are not special depends upon knowledge of a class of algebras which are more general, but in a certain sense only slightly more general, than associative algebras. These are the *alternative* algebras $\mathfrak{A}$ defined by the identities

$$x^2y = x(xy) \qquad \text{for all } x, y \text{ in } \mathfrak{A} \tag{13}$$

and

$$yx^2 = (yx)x \qquad \text{for all } x, y \text{ in } \mathfrak{A}, \tag{14}$$

known respectively as the *left* and *right alternative laws*. Clearly any associative algebra is alternative. The class of 8-dimensional *Cayley algebras* (or *Cayley-Dickson algebras*, the prototype having been discovered in 1845 by Cayley and later generalized by Dickson) is, as we shall see, an important class of alternative algebras which are not associative.

To date these are the algebras (Lie, Jordan and alternative) about which most is known. Numerous generalizations have recently been made, usually by studying classes of algebras defined by weaker identities. We shall see in II some things which can be proved about completely arbitrary algebras.

II. Arbitrary Nonassociative Algebras

Let $\mathfrak{A}$ be an algebra over a field F. (The reader may make the appropriate modifications for a ring $\mathfrak{R}$.) The definitions of the terms *subalgebra*, *left ideal*, *right ideal*, (two-sided) *ideal* $\mathfrak{I}$, *homomorphism*, *kernel* of a homomorphism, *residue class algebra* $\mathfrak{A}/\mathfrak{I}$ (*difference algebra* $\mathfrak{A} - \mathfrak{I}$), *anti-isomorphism*, which are familiar from a study of associative algebras, do not involve associativity of multiplication and are thus immediately applicable to algebras in general. So is the notation $\mathfrak{B}\mathfrak{C}$ for the subspace of $\mathfrak{A}$ spanned by all products bc with b in $\mathfrak{B}$, c in $\mathfrak{C}$ ($\mathfrak{B}$, $\mathfrak{C}$ being arbitrary nonempty subsets of $\mathfrak{A}$); here we must of course distinguish between $(\mathfrak{A}\mathfrak{B})\mathfrak{C}$ and $\mathfrak{A}(\mathfrak{B}\mathfrak{C})$, etc.

We have the *fundamental theorem of homomorphism for algebras*: If $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{I}$ is a homomorphic image of $\mathfrak{A}$ under the natural homomorphism

$$(1) \qquad a \to \bar{a} = a + \mathfrak{I}, \qquad a \text{ in } \mathfrak{A},\ a + \mathfrak{I} \text{ in } \mathfrak{A}/\mathfrak{I}.$$

Conversely, if $\mathfrak{A}'$ is a homomorphic image of $\mathfrak{A}$ (under the homomorphism

$$(2) \qquad a \to a', \qquad a \text{ in } \mathfrak{A},\ a' \text{ in } \mathfrak{A}'),$$

then $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}/\mathfrak{I}$ where $\mathfrak{I}$ is the kernel of the homomorphism.

If $\mathfrak{S}'$ is a subalgebra (or ideal) of a homomorphic image $\mathfrak{A}'$ of $\mathfrak{A}$, then the *complete inverse image* of $\mathfrak{S}'$ under the homomorphism (2)—that is, the set $\mathfrak{S} = \{s \in \mathfrak{A} \mid s' \in \mathfrak{S}'\}$—is a subalgebra (or ideal) of $\mathfrak{A}$ which contains the kernel $\mathfrak{I}$ of (2). If a class of algebras is defined by identities (as, for example, Lie, Jordan or alternative algebras), then any subalgebra or any homomorphic image belongs to the same class.

We have the customary isomorphism theorems:

(i) If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are ideals of $\mathfrak{A}$ such that $\mathfrak{I}_1$ contains $\mathfrak{I}_2$, then $(\mathfrak{A}/\mathfrak{I}_2)/(\mathfrak{I}_1/\mathfrak{I}_2)$ and $\mathfrak{A}/\mathfrak{I}_1$ are isomorphic.

(ii) If $\mathfrak{I}$ is an ideal of $\mathfrak{A}$ and $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$, then $\mathfrak{I} \cap \mathfrak{S}$ is an ideal of $\mathfrak{S}$, and $(\mathfrak{I} + \mathfrak{S})/\mathfrak{I}$ and $\mathfrak{S}/(\mathfrak{I} \cap \mathfrak{S})$ are isomorphic.

Suppose that $\mathfrak{B}$ and $\mathfrak{C}$ are ideals of an algebra $\mathfrak{A}$, and that as a vector space $\mathfrak{A}$ is the direct sum of $\mathfrak{B}$ and $\mathfrak{C}$ ($\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$, $\mathfrak{B} \cap \mathfrak{C} = 0$). Then $\mathfrak{A}$ is called the *direct sum* $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$ of $\mathfrak{B}$ and $\mathfrak{C}$ as algebras. The vector space properties insure that in a direct sum $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$ the components b, c of $a = b + c$ (b in $\mathfrak{B}$, c in $\mathfrak{C}$) are uniquely determined, and that addition and multiplication by scalars are performed componentwise. It is the assumption that $\mathfrak{B}$ and $\mathfrak{C}$ are ideals in $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$ that gives componentwise multiplication as well:

$$(3) \quad (b_1 + c_1)(b_2 + c_2) = b_1 b_2 + c_1 c_2, \qquad b_i \text{ in } \mathfrak{B},\ c_i \text{ in } \mathfrak{C}.$$

For $b_1 c_2$ is in both $\mathfrak{B}$ and $\mathfrak{C}$, hence in $\mathfrak{B} \cap \mathfrak{C} = 0$. Similarly $c_1 b_2 = 0$, so (3) holds, (Although $\oplus$ is commonly used to denote vector space direct sum, it has been reserved in these notes for direct sum of ideals; where appropriate the notation $\perp$ has been used for orthogonal direct sum relative to a symmetric bilinear form.)

Given any two algebras $\mathfrak{B}$, $\mathfrak{C}$ over a field F, one can construct an algebra $\mathfrak{A}$ over F such that $\mathfrak{A}$ is the direct sum $\mathfrak{A} = \mathfrak{B}' \oplus \mathfrak{C}'$ of ideals $\mathfrak{B}'$, $\mathfrak{C}'$ which are isomorphic respectively to $\mathfrak{B}$, $\mathfrak{C}$. The construction of $\mathfrak{A}$ is familiar: the elements of $\mathfrak{A}$ are the ordered pairs (b, c) with b in $\mathfrak{B}$, c in $\mathfrak{C}$; addition, multiplication by scalars, and multiplication are defined componentwise:

$$(4) \quad \begin{aligned} (b_1, c_1) + (b_2, c_2) &= (b_1 + b_2, c_1 + c_2), \\ \alpha(b, c) &= (\alpha b, \alpha c), \\ (b_1, c_1)(b_2, c_2) &= (b_1 c_1, b_2 c_2). \end{aligned}$$

Then $\mathfrak{A}$ is an algebra over F, the sets $\mathfrak{B}'$ of all pairs $(b, 0)$ with b in $\mathfrak{B}$ and $\mathfrak{C}'$ of all pairs $(0, c)$ with c in $\mathfrak{C}$ are ideals of $\mathfrak{A}$ isomorphic respectively to $\mathfrak{B}$ and $\mathfrak{C}$, and $\mathfrak{A} = \mathfrak{B}' \oplus \mathfrak{C}'$. By the customary identification of $\mathfrak{B}$ with $\mathfrak{B}'$, $\mathfrak{C}$ with $\mathfrak{C}'$, we can then write $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$, the direct sum of $\mathfrak{B}$ and $\mathfrak{C}$ as algebras.

As in the case of vector spaces, the notion of direct sum extends to an arbitrary (indexed) set of summands. In these notes we shall have occasion to use only finite direct sums $\mathfrak{A} = \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \cdots \oplus \mathfrak{B}_t$. Here $\mathfrak{A}$ is the direct sum of the vector spaces $\mathfrak{B}_i$, and multiplication in $\mathfrak{A}$ is given by

$$(5) \quad (b_1 + b_2 + \cdots + b_t)(c_1 + c_2 + \cdots + c_t) = b_1 c_1 + b_2 c_2 + \cdots + b_t c_t$$

for b_i, c_i in $\mathfrak{B}_i$. The $\mathfrak{B}_i$ are ideals of $\mathfrak{A}$. Note that (in the case of a vector space direct sum) the latter statement is equivalent to the fact that the $\mathfrak{B}_i$ are subalgebras of $\mathfrak{A}$ such that

$$(6) \qquad \mathfrak{B}_i\mathfrak{B}_j = 0 \qquad \text{for } i \neq j.$$

An element e (or f) in an algebra $\mathfrak{A}$ over F is called a *left* (or *right*) *identity* (sometimes *unity element*) in case $ea = a$ (or $af = a$) for all a in $\mathfrak{A}$. If $\mathfrak{A}$ contains both a left identity e and a right identity f, then $e = f$ $(= ef)$ is a (two-sided) *identity* 1. If $\mathfrak{A}$ does not contain an identity element 1, there is a standard construction for obtaining an algebra $\mathfrak{A}_1$ which does contain 1, such that $\mathfrak{A}_1$ contains (an isomorphic copy of) $\mathfrak{A}$ as an ideal, and such that $\mathfrak{A}_1/\mathfrak{A}$ has dimension 1 over F. We take $\mathfrak{A}_1$ to be the set of all ordered pairs (α, a) with α in F, a in $\mathfrak{A}$; addition and multiplication by scalars are defined componentwise; multiplication is defined by

$$(7) \qquad (\alpha, a)(\beta, b) = (\alpha\beta,\ \beta a + \alpha b + ab), \qquad \alpha, \beta \text{ in } F,\ a, b \text{ in } \mathfrak{A}.$$

Then $\mathfrak{A}_1$ is an algebra over F with identity element $1 = (1, 0)$. The set $\mathfrak{A}'$ of all pairs $(0, a)$ in $\mathfrak{A}_1$ with a in $\mathfrak{A}$ is an ideal of $\mathfrak{A}_1$ which is isomorphic to $\mathfrak{A}$. As a vector space $\mathfrak{A}_1$ is the direct sum of $\mathfrak{A}'$ and the 1-dimensional space $F1 = \{\alpha 1 \mid \alpha \text{ in } F\}$. Identifying $\mathfrak{A}'$ with its isomorphic image $\mathfrak{A}$, we can write every element of $\mathfrak{A}_1$ uniquely in the form $\alpha 1 + a$ with α in F, a in $\mathfrak{A}$, in which case the multiplication (7) becomes

$$(7') \qquad (\alpha 1 + a)(\beta 1 + b) = (\alpha\beta)1 + (\beta a + \alpha b + ab).$$

We say that we have *adjoined a unity element* to $\mathfrak{A}$ to obtain $\mathfrak{A}_1$. (If $\mathfrak{A}$ is associative, this familiar construction yields an associative algebra $\mathfrak{A}_1$ with 1. A similar statement is readily verifiable for (commutative) Jordan algebras and for alternative algebras. It is of course not true for Lie algebras, since $1^2 = 1 \neq 0$.)

Let $\mathfrak{B}$ and $\mathfrak{A}$ be algebras over a field F. The *Kronecker product* $\mathfrak{B} \otimes_F \mathfrak{A}$ (written $\mathfrak{B} \otimes \mathfrak{A}$ if there is no ambiguity) is the tensor product $\mathfrak{B} \otimes_F \mathfrak{A}$ of the vector spaces $\mathfrak{B}$, $\mathfrak{A}$ (so that all elements are sums $\sum b \otimes a$, b in $\mathfrak{B}$, a in $\mathfrak{A}$, multiplication being defined by distributivity and

$$(8) \qquad (b_1 \otimes a_1)(b_2 \otimes a_2) = (b_1 b_2) \otimes (a_1 a_2), \qquad b_i \text{ in } \mathfrak{B},\ a_i \text{ in } \mathfrak{A}.$$

If $\mathfrak{B}$ contains 1, then the set of all $1 \otimes a$ in $\mathfrak{B} \otimes \mathfrak{A}$ is a subalgebra of $\mathfrak{B} \otimes \mathfrak{A}$ which is isomorphic to $\mathfrak{A}$, and which we can identify with $\mathfrak{A}$ (similarly, if $\mathfrak{A}$ contains 1, then $\mathfrak{B} \otimes \mathfrak{A}$ contains $\mathfrak{B}$ as a subalgebra). If $\mathfrak{B}$ and $\mathfrak{A}$ are finite-dimensional over F, then $\dim(\mathfrak{B} \otimes \mathfrak{A}) = (\dim \mathfrak{B})(\dim \mathfrak{A})$.

We shall on numerous occasions be concerned with the case where $\mathfrak{B}$ is taken to be a field (an arbitrary extension K of F). Then K does contain 1, so $\mathfrak{A}_K = K \otimes_F \mathfrak{A}$ contains $\mathfrak{A}$ (in the sense of isomorphism) as a subalgebra over F. Moreover, $\mathfrak{A}_K$ is readily seen to be an algebra over K, which is called the *scalar extension* of $\mathfrak{A}$ to an algebra over K. The properties of a tensor product insure that any basis for $\mathfrak{A}$ over F is a basis for $\mathfrak{A}_K$ over K. In case $\mathfrak{A}$ is finite-dimensional over F, this gives an easy representation for the elements of $\mathfrak{A}_K$. Let $u_1, \ldots, u_n$ be any basis for $\mathfrak{A}$ over F. Then the elements of $\mathfrak{A}_K$ are the linear combinations

$$(9) \qquad \sum \alpha_i u_i \quad \left(= \sum \alpha_i \otimes u_i\right), \qquad \alpha_i \text{ in } K,$$

where the coefficients α_i in (9) are uniquely determined. Addition and multiplication by scalars are performed componentwise. For multiplication in $\mathfrak{A}_K$ we use bilinearity and the multiplication table

$$(10) \qquad u_i u_j = \sum \gamma_{ijk}\, u_k, \qquad \gamma_{ijk} \text{ in } F.$$

The elements of $\mathfrak{A}$ are obtained by restricting the α_i in (9) to elements of F.

For finite-dimensional $\mathfrak{A}$, the scalar extension $\mathfrak{A}_K$ (K an arbitrary extension of F) may be defined in a non-invariant way (without recourse to tensor products) by use of a basis as above. Let $u_1, \ldots, u_n$ be any basis for $\mathfrak{A}$ over F; multiplication in $\mathfrak{A}$ is given by the multiplication table (10). Let $\mathfrak{A}_K$ be an n-dimensional algebra over K with the same multiplication table (this is valid since the γ_{ijk}, being in F, are in K). What remains to be verified is that a different choice of basis for $\mathfrak{A}$ over F would yield an algebra isomorphic (over K) to this one. (A non-invariant definition of the Kronecker product of two finite-dimensional algebras $\mathfrak{A}$, $\mathfrak{B}$ may similarly be given.)

For the classes of algebras mentioned in the Introduction (Jordan algebras of characteristic $\neq 2$, and Lie and alternative algebras of arbitrary characteristic), one may verify that algebras remain in the same class under scalar extension—a property which is not shared by classes of algebras defined by more general identities (as, for example, in V).

Just as the *commutator* $[x, y] = xy - yx$ measures commutativity (and lack of it) in an algebra $\mathfrak{A}$, the *associator*

$$(x, y, z) = (xy)z - x(yz) \tag{11}$$

of any three elements may be introduced as a measure of associativity (and lack of it) in $\mathfrak{A}$. Thus the definitions of alternative and Jordan algebras may be written as

$$(x, x, y) = (y, x, x) = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{A}$$

and

$$[x, y] = (x, y, x^2) = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{A}.$$

Note that the associator (x, y, z) is linear in each argument. One identity which is sometimes useful and which holds in any algebra $\mathfrak{A}$ is

$$a(x, y, z) + (a, x, y)z = (ax, y, z) - (a, xy, z) + (a, x, yz) \qquad \text{for all } a, x, y, z \text{ in } \mathfrak{A}. \tag{12}$$

The *nucleus* $\mathfrak{G}$ of an algebra $\mathfrak{A}$ is the set of elements g in $\mathfrak{A}$ which associate with every pair of elements x, y in $\mathfrak{A}$ in the sense that

$$(g, x, y) = (x, g, y) = (x, y, g) = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{A}. \tag{13}$$

It is easy to verify that $\mathfrak{G}$ is an associative subalgebra of $\mathfrak{A}$. $\mathfrak{G}$ is a subspace by the linearity of the associator in each argument, and $(g_1g_2, x, y) = g_1(g_2, x, y) + (g_1, g_2, x)y + (g_1, g_2x, y) - (g_1, g_2, xy) = 0$ by (13), etc.

The *center* $\mathfrak{C}$ of $\mathfrak{A}$ is the set of all c in $\mathfrak{A}$ which commute and associate with all elements; that is, the set of all c in the nucleus $\mathfrak{G}$ with the additional property that

$$xc = cx \qquad \text{for all } x \text{ in } \mathfrak{A}. \tag{14}$$

This clearly generalizes the familiar notion of the center of an associative algebra. Note that $\mathfrak{C}$ is a commutative associative subalgebra of $\mathfrak{A}$.

Let a be any element of an algebra $\mathfrak{A}$ over F. The *right multiplication* R_a of $\mathfrak{A}$ which is determined by a is defined by

$$R_a : x \to xa \qquad \text{for all } x \text{ in } \mathfrak{A}. \tag{15}$$

Clearly R_a is a linear operator on $\mathfrak{A}$. Also the set $R(\mathfrak{A})$ of all right multiplications of $\mathfrak{A}$ is a subspace of the associative algebra $\mathfrak{E}$ of all linear operators on $\mathfrak{A}$, since $a \to R_a$ is a linear mapping of $\mathfrak{A}$ into $\mathfrak{E}$. (In the familiar case of an associative algebra, $R(\mathfrak{A})$ is a subalgebra of $\mathfrak{E}$, but this is not true in general.) Similarly the *left multiplication* L_a defined by

$$(16) \qquad L_a : x \to ax \qquad \text{for all } x \text{ in } \mathfrak{A}$$

is a linear operator on $\mathfrak{A}$, the mapping $a \to L_a$ is linear, and the set $L(\mathfrak{A})$ of all left multiplications of $\mathfrak{A}$ is a subspace of $\mathfrak{E}$.

We denote by $\mathfrak{M}(\mathfrak{A})$, or simply $\mathfrak{M}$, the enveloping algebra of $R(\mathfrak{A}) \cup L(\mathfrak{A})$; that is, the (associative) subalgebra of $\mathfrak{E}$ generated by right and left multiplications of $\mathfrak{A}$. $\mathfrak{M}(\mathfrak{A})$ is the intersection of all subalgebras of $\mathfrak{E}$ which contain both $R(\mathfrak{A})$ and $L(\mathfrak{A})$. The elements of $\mathfrak{M}(\mathfrak{A})$ are of the form $\sum S_1 \cdots S_n$ where S_i is either a right or left multiplication of $\mathfrak{A}$. We call the associative algebra $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$ the *multiplication algebra* of $\mathfrak{A}$.

It is sometimes useful to have a notation for the enveloping algebra of the right and left multiplications (of $\mathfrak{A}$) which correspond to the elements of any subset $\mathfrak{B}$ of $\mathfrak{A}$; we shall write $\mathfrak{B}^*$ for this subalgebra of $\mathfrak{M}(\mathfrak{A})$. That is, $\mathfrak{B}^*$ is the set of all $\sum S_1 \cdots S_n$, where S_i is either R_{b_i}, the right multiplication of $\mathfrak{A}$ determined by b_i in $\mathfrak{B}$, or L_{b_i}. Clearly $\mathfrak{A}^* = \mathfrak{M}(\mathfrak{A})$, but note the difference between $\mathfrak{B}^*$ and $\mathfrak{M}(\mathfrak{B})$ in case $\mathfrak{B}$ is a proper subalgebra of $\mathfrak{A}$—they are associative algebras of operators on different spaces ($\mathfrak{A}$ and $\mathfrak{B}$ respectively).

An algebra $\mathfrak{A}$ over F is called *simple* in case 0 and $\mathfrak{A}$ itself are the only ideals of $\mathfrak{A}$, and $\mathfrak{A}$ is not a zero algebra (equivalently, in the presence of the first assumption, $\mathfrak{A}$ is not the zero algebra of dimension 1). Since an ideal of $\mathfrak{A}$ is an invariant subspace under $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$, and conversely, it follows that $\mathfrak{A}$ is simple if and only if $\mathfrak{M} \neq 0$ is an irreducible set of linear operators on $\mathfrak{A}$. Since $\mathfrak{A}^2$ ($= \mathfrak{A}\mathfrak{A}$) is an ideal of $\mathfrak{A}$, we have $\mathfrak{A}^2 = \mathfrak{A}$ in case $\mathfrak{A}$ is simple.

An algebra $\mathfrak{A}$ over F is a *division algebra* in case $\mathfrak{A} \neq 0$ and the equations

$$(17) \qquad ax = b, \quad ya = b \qquad (a \neq 0,\ b \text{ in } \mathfrak{A})$$

have unique solutions x, y in $\mathfrak{A}$; this is equivalent to saying that, for any $a \neq 0$ in $\mathfrak{A}$, L_a and R_a have inverses L_a^{-1} and R_a^{-1}. Any division algebra is simple. For, if $\mathfrak{I} \neq 0$ is merely a left ideal of $\mathfrak{A}$, there is an element $a \neq 0$ in $\mathfrak{I}$ and $\mathfrak{A} \subseteq \mathfrak{A}a \subseteq \mathfrak{I}$ by (17), or $\mathfrak{I} = \mathfrak{A}$; also clearly $\mathfrak{A}^2 \neq 0$. (Any associative division algebra $\mathfrak{A}$ has an identity 1, since (17) implies that the non-zero elements form a multiplicative group. In general, a division algebra need not contain an identity 1.) If $\mathfrak{A}$ has finite dimension $n \geq 1$ over F, then $\mathfrak{A}$ is a division algebra if and only if $\mathfrak{A}$ is *without zero divisors* ($x \neq 0$ and $y \neq 0$ in $\mathfrak{A}$ imply $xy \neq 0$), inasmuch as the finite-dimensionality insures that L_a (and similarly R_a), being (1–1) for $a \neq 0$, has an inverse.

In order to make the observation that any simple ring is actually an algebra, so the study of simple rings reduces to that of (possibly infinite-dimensional) simple algebras, we take for granted that the appropriate definitions for rings are apparent and we digress to consider any simple ring $\mathfrak{R}$. The (associative) multiplication ring $\mathfrak{M} = \mathfrak{M}(\mathfrak{R}) \neq 0$ is irreducible as a ring of endomorphisms of $\mathfrak{R}$. Thus by Schur's Lemma the centralizer $\mathfrak{C}'$ of $\mathfrak{M}$ in the ring $\mathfrak{E}$ of all endomorphisms of $\mathfrak{R}$ is an associative division ring. Since $\mathfrak{M}$ is generated by left and right multiplications of $\mathfrak{R}$, $\mathfrak{C}'$ consists of those endomorphisms T in $\mathfrak{E}$ satisfying $R_yT = TR_y$, $L_xT = TL_x$, or

$$(18) \qquad (xy)T = (xT)y = x(yT) \qquad \text{for all } x, y \text{ in } \mathfrak{R}.$$

Hence S, T in $\mathfrak{C}'$ imply $(xy)ST = ((xS)y)T = (xS)(yT) = (x(yS))T = (xT)(yS)$. Interchanging S and T, we have $(xy)ST = (xy)TS$, so that $zST = zTS$ for all z in $\mathfrak{R}^2 = \mathfrak{R}$. That is, $ST = TS$ for all S, T in $\mathfrak{C}'$; $\mathfrak{C}'$ is a field which we call the *multiplication centralizer* of $\mathfrak{R}$. Now the simple ring $\mathfrak{R}$ may be regarded in a natural way as an algebra over the field $\mathfrak{C}'$. Denote T in $\mathfrak{C}'$ by α, and write $\alpha x = xT$ for any x in $\mathfrak{R}$. Then $\mathfrak{R}$ is a (left) vector space over $\mathfrak{C}'$. Also (18) gives the defining relations $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for an algebra over $\mathfrak{C}'$. As an algebra over $\mathfrak{C}'$ (or any subfield F of $\mathfrak{C}'$), $\mathfrak{R}$ is simple since any ideal of $\mathfrak{R}$ as an algebra is *a priori* an ideal of $\mathfrak{R}$ as a ring.

Moreover, $\mathfrak{M}$ is a dense ring of linear transformations on $\mathfrak{R}$ over $\mathfrak{C}'$ (Jacobson, Lectures in Abstract Algebra, vol. II, p. 274), so we have proved

Theorem 1. Let $\mathfrak{R}$ be a simple ring, and $\mathfrak{M}$ be its multiplication ring. Then the multiplication centralizer $\mathfrak{C}'$ of $\mathfrak{M}$ is a field, and $\mathfrak{R}$ may be regarded as a simple algebra over any subfield F of $\mathfrak{C}'$. $\mathfrak{M}$ is a dense ring of linear transformations on $\mathfrak{R}$ over $\mathfrak{C}'$.

Returning now to any simple algebra $\mathfrak{A}$ over F, we recall that the multiplication algebra $\mathfrak{M}(\mathfrak{A})$ is irreducible as a set of linear operators on the vector space $\mathfrak{A}$ over F. But (Jacobson, ibid) this means that $\mathfrak{M}(\mathfrak{A})$ is irreducible as a set of endomorphisms of the additive group of $\mathfrak{A}$, so that $\mathfrak{A}$ is a simple ring. That is, the notions of simple algebra and simple ring coincide, and Theorem 1 may be paraphrased for algebras as

Theorem 1′. Let $\mathfrak{A}$ be a simple algebra over F, and $\mathfrak{M}$ be its multiplication algebra. Then the multiplication centralizer $\mathfrak{C}'$ of $\mathfrak{M}$ is a field (containing F), and $\mathfrak{A}$ may be regarded as a simple algebra over $\mathfrak{C}'$. $\mathfrak{M}$ is a dense ring of linear transformations on $\mathfrak{A}$ over $\mathfrak{C}'$.

Suppose that $\mathfrak{A}$ has finite dimension n over F. Then $\mathfrak{E}$ has dimension n^2 over F, and its subalgebra $\mathfrak{C}'$ has finite dimension over F. That is, the field $\mathfrak{C}'$ is a finite extension of F of degree $r = (\mathfrak{C}' : F)$ over F. Then $n = mr$, and $\mathfrak{A}$ has dimension m over $\mathfrak{C}'$. Since $\mathfrak{M}$ is a dense ring of linear transformations on (the finite-dimensional vector space) $\mathfrak{A}$ over $\mathfrak{C}'$, $\mathfrak{M}$ is the set of *all* linear operators on $\mathfrak{A}$ over $\mathfrak{C}'$. Hence $\mathfrak{C}'$ is contained in $\mathfrak{M}$ in the finite-dimensional case. That is, $\mathfrak{C}'$ is the center of $\mathfrak{M}$ and is called the *multiplication center* of $\mathfrak{A}$.

Corollary. Let $\mathfrak{A}$ be a simple algebra of finite dimension over F, and $\mathfrak{M}$ be its multiplication algebra. Then the center $\mathfrak{C}'$ of $\mathfrak{M}$ is a field, a finite extension of F. $\mathfrak{A}$ may be regarded as a simple algebra over $\mathfrak{C}'$. $\mathfrak{M}$ is the algebra of all linear operators on $\mathfrak{A}$ over $\mathfrak{C}'$.

An algebra $\mathfrak{A}$ over F is called *central simple* in case $\mathfrak{A}_K$ is simple for every extension K of F. Every central simple algebra is simple (take $K = F$).

We omit the proof of the fact that any simple algebra $\mathfrak{A}$ (of arbitrary dimension), regarded as an algebra over its multiplication centralizer $\mathfrak{C}'$ (so that $\mathfrak{C}' = F$) is central simple. The idea of the proof is to show that, for any extension K of F, the multiplication algebra $\mathfrak{M}(\mathfrak{A}_K)$ is a dense ring of linear transformations on $\mathfrak{A}_K$ over K, and hence is an irreducible set of linear operators.

Theorem 2. The center $\mathfrak{C}$ of any simple algebra $\mathfrak{A}$ over F is either 0 or a field. In the latter case $\mathfrak{A}$ contains 1, the multiplication centralizer $\mathfrak{C}' = \mathfrak{C}^* = \{R_c \mid c \in \mathfrak{C}\}$, and $\mathfrak{A}$ is a central simple algebra over $\mathfrak{C}$.

Proof: Note that c is in the center of any algebra $\mathfrak{A}$ if and only if $R_c = L_c$ and $[L_c, R_y] = R_c R_y - R_{cy} = R_y R_c - R_{yc} = 0$ for all y in $\mathfrak{A}$ or, more compactly,

$$(19) \qquad R_c = L_c, \quad R_c R_y = R_y R_c = R_{cy} \quad \text{for all } y \text{ in } \mathfrak{A}.$$

Hence (18) implies that

$$(20) \qquad cT \text{ is in } \mathfrak{C} \qquad \text{for all } c \text{ in } \mathfrak{C},\ T \text{ in } \mathfrak{C}'.$$

For (18) may be written as

$$(18') \qquad R_y T = T R_y = R_{yT} \qquad \text{for all } y \text{ in } \mathfrak{A}$$

or, equivalently, as

$$(18'') \qquad L_x T = L_{xT} = T L_x \qquad \text{for all } x \text{ in } \mathfrak{A}.$$

Then (18′) and (18″) imply $R_{cT} = TR_c = TL_c = L_{cT}$, together with $R_{cT} R_y = R_c T R_y = R_c R_{yT} = R_{c(yT)} = R_{(cT)y}$ and $R_y R_{cT} = R_y R_c T = R_c R_y T = R_c T R_y$ $(= R_{(cT)y})$, That is, (20) holds. Note also that (19) implies

$$(21) \qquad L_x R_c = R_c L_x \qquad \text{for all } c \text{ in } \mathfrak{C},\ x \text{ in } \mathfrak{A}.$$

Since $R_{c_1} R_{c_2} = R_{c_1 c_2}$ (c_i in C) by (19), the subalgebra $\mathfrak{C}^*$ of $\mathfrak{M}(\mathfrak{A})$ is just $\mathfrak{C}^* = \{R_c \mid c \in \mathfrak{C}\}$, and the mapping $c \to R_c$ is a homomorphism of $\mathfrak{C}$ onto $\mathfrak{C}^*$. Also (19) and (21) imply that R_c commutes with every element of $\mathfrak{M}$ so that $\mathfrak{C}^* \subseteq \mathfrak{C}'$. Moreover, $\mathfrak{C}^*$ is an ideal of the (commutative) field $\mathfrak{C}'$ since (18′) and (20) imply that $TR_c = R_{cT}$ is in $\mathfrak{C}^*$ for all T in $\mathfrak{C}'$, c in $\mathfrak{C}$. Hence either $\mathfrak{C}^* = 0$ or $\mathfrak{C}^* = C'$.

Now $\mathfrak{C}^* = 0$ implies $R_c = 0$ for all c in $\mathfrak{C}$; hence $\mathfrak{C} = 0$. For, if there is $c \neq 0$ in $\mathfrak{C}$, then $\mathfrak{I} = Fc \neq 0$ is an ideal of $\mathfrak{A}$ since $\mathfrak{I}\mathfrak{A} = \mathfrak{A}\mathfrak{I} = 0$. Then $\mathfrak{I} = \mathfrak{A}$, $\mathfrak{A}^2 = 0$, a contradiction.

In the remaining case $\mathfrak{C}^* = \mathfrak{C}'$, the identity operator $1_{\mathfrak{A}}$ on $\mathfrak{A}$ is in $\mathfrak{C}' = \mathfrak{C}^*$. Hence there is an element e in $\mathfrak{C}$ such that $R_e = L_e = 1_{\mathfrak{A}}$, or $ae = ea = a$ for all a in $\mathfrak{A}$; $\mathfrak{A}$ has a unity element $1 = e$. Then $c \to R_c$ is an isomorphism between $\mathfrak{C}$ and the field $\mathfrak{C}'$. $\mathfrak{A}$ is an algebra over the field $\mathfrak{C}$, and as such is central simple.

For any algebra $\mathfrak{A}$ over F, one obtains a *derived series* of subalgebras $\mathfrak{A}^{(1)} \supseteq \mathfrak{A}^{(2)} \supseteq \mathfrak{A}^{(3)} \supseteq \cdots$ by defining $\mathfrak{A}^{(1)} = \mathfrak{A}$, $\mathfrak{A}^{(i+1)} = (\mathfrak{A}^{(i)})^2$. $\mathfrak{A}$ is called *solvable* in case $\mathfrak{A}^{(r)} = 0$ for some integer r.

Proposition 1. If an algebra $\mathfrak{A}$ contains a solvable ideal $\mathfrak{I}$, and if $\overline{\mathfrak{A}} = \mathfrak{A}/\mathfrak{I}$ is solvable, then $\mathfrak{A}$ is solvable.

Proof: Since (1) is a homomorphism, it follows that $\overline{\mathfrak{A}^2} = \overline{\mathfrak{A}}^2$ and that $\overline{\mathfrak{A}^{(i)}} = \overline{\mathfrak{A}}^{(i)}$. Then $\overline{\mathfrak{A}}^{(r)} = 0$ implies $\overline{\mathfrak{A}^{(r)}} = 0$, or $\mathfrak{A}^{(r)} \subseteq \mathfrak{I}$. But $\mathfrak{I}^{(s)} = 0$ for some s, so $\mathfrak{A}^{(r+s)} = (\mathfrak{A}^{(r)})^{(s)} \subseteq \mathfrak{I}^{(s)} = 0$. Hence $\mathfrak{A}$ is solvable.

Proposition 2. If $\mathfrak{B}$ and $\mathfrak{C}$ are solvable ideals of an algebra $\mathfrak{A}$, then $\mathfrak{B} + \mathfrak{C}$ is a solvable ideal of $\mathfrak{A}$. Hence, if $\mathfrak{A}$ is finite-dimensional, $\mathfrak{A}$ has a unique maximal solvable ideal $\mathfrak{N}$. Moreover, the only solvable ideal of $\mathfrak{A}/\mathfrak{N}$ is 0.

Proof: $\mathfrak{B} + \mathfrak{C}$ is an ideal because $\mathfrak{B}$ and $\mathfrak{C}$ are ideals. By the second isomorphism theorem $(\mathfrak{B} + \mathfrak{C})/\mathfrak{C} \cong \mathfrak{B}/(\mathfrak{B} \cap \mathfrak{C})$. But $\mathfrak{B}/(\mathfrak{B} \cap \mathfrak{C})$ is a homomorphic image of the solvable algebra $\mathfrak{B}$, and is therefore clearly solvable. Then $\mathfrak{B} + \mathfrak{C}$ is solvable by Proposition 1. It follows that, if $\mathfrak{A}$ is finite-dimensional, the solvable ideal of maximum dimension is unique (and contains every solvable ideal of $\mathfrak{A}$). Let $\mathfrak{N}$ be this maximal solvable ideal, and $\overline{\mathfrak{G}}$ be any solvable ideal of $\overline{\mathfrak{A}} = \mathfrak{A}/\mathfrak{N}$. The complete inverse image $\mathfrak{G}$ of $\overline{\mathfrak{G}}$ under the natural homomorphism of $\mathfrak{A}$ onto $\overline{\mathfrak{A}}$ is an ideal of $\mathfrak{A}$ such that $\mathfrak{G}/\mathfrak{N} = \overline{\mathfrak{G}}$. Then $\mathfrak{G}$ is solvable by Proposition 1, so $\mathfrak{G} \subseteq \mathfrak{N}$. Hence $\mathfrak{G}/\mathfrak{N} = \overline{\mathfrak{G}} = 0$.

An algebra $\mathfrak{A}$ is called *nilpotent* in case there exists an integer t such that any product $z_1 z_2 \cdots z_t$ of t elements in $\mathfrak{A}$, no matter how associated, is 0. This clearly generalizes the concept of nilpotence as defined for associative algebras. Also any nilpotent algebra is solvable.

Theorem 3. An ideal $\mathfrak{B}$ of an algebra $\mathfrak{A}$ is nilpotent if and only if the (associative) subalgebra $\mathfrak{B}^*$ of $\mathfrak{M}(\mathfrak{A})$ is nilpotent.

Proof: Suppose that every product of t elements of $\mathfrak{B}$, no matter how associated, is 0. Then the same is true for any product of more than t elements of $\mathfrak{B}$. Let $T = T_1 \cdots T_t$ be any product of t elements of $\mathfrak{B}^*$. Then T is a sum of terms each of which is a product of at least t linear operators S_i, each S_i being either L_{b_i} or R_{b_i} (b_i in $\mathfrak{B}$).

Since $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, xS_1 is in $\mathfrak{B}$ for every x in $\mathfrak{A}$. Hence xT is a sum of terms, each of which is a product of at least t elements in $\mathfrak{B}$. Hence $xT = 0$ for all x in $\mathfrak{A}$, or $T = 0$, $\mathfrak{B}^*$ is nilpotent. For the converse we need only that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. We show by induction on n that any product of at least 2^n elements in $\mathfrak{B}$, no matter how associated, is of the form $bS_1 \cdots S_n$ with b in $\mathfrak{B}$, S_i in $\mathfrak{B}^*$. For $n = 1$, we take any product of at least 2 elements in $\mathfrak{B}$. There is a final multiplication which is performed. Since $\mathfrak{B}$ is a subalgebra, each of the two factors is in $\mathfrak{B}$: $bb_1 = bR_{b_1} = bS_1$. Similarly in any product of at least 2^{n+1} elements of $\mathfrak{B}$, no matter how associated, there is a final multiplication which is performed. At least one of the two factors is a product of at least 2^n elements of $\mathfrak{B}$, while the other factor b' is in $\mathfrak{B}$. Hence by the assumption of the induction we have either $b'(bS_1 \cdots S_n) = bS_1 \cdots S_n L_{b'} = bS_1 \cdots S_{n+1}$ or $(bS_1 \cdots S_n)b' = bS_1 \cdots S_n R_{b'} = bS_1 \cdots S_{n+1}$, as desired. Hence, if any product $S_1 \cdots S_t$ of t elements in $\mathfrak{B}^*$ is 0, any product of 2^t elements of $\mathfrak{B}$, no matter how associated, is 0. That is, $\mathfrak{B}$ is nilpotent.

III. Alternative Algebras

As indicated in the Introduction, an *alternative algebra* $\mathfrak{A}$ over F is an algebra in which

$$(1) \qquad x^2y = x(xy) \qquad \text{for all } x, y \text{ in } \mathfrak{A}$$

and

$$(2) \qquad yx^2 = (yx)x \qquad \text{for all } x, y \text{ in } \mathfrak{A}.$$

In terms of associators, (1) and (2) are equivalent to

$$(1') \qquad (x, x, y) = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{A}$$

and

$$(2') \qquad (y, x, x) = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{A}.$$

In terms of left and right multiplications, (1) and (2) are equivalent to

$$(1'') \qquad L_{x^2} = {L_x}^2 \qquad \text{for all } x \text{ in } \mathfrak{A}$$

and

$$(2'') \qquad R_{x^2} = {R_x}^2 \qquad \text{for all } x \text{ in } \mathfrak{A}.$$

The associator (x_1, x_2, x_3) "alternates" in the sense that, for any permutation σ of $1, 2, 3$, we have $(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = (\operatorname{sgn}\sigma)(x_1, x_2, x_3)$. To establish this, it is sufficient to prove

$$(3) \qquad (x, y, z) = -(y, x, z) \qquad \text{for all } x, y, z \text{ in } \mathfrak{A}$$

and

$$(4) \qquad (x, y, z) = (z, x, y) \qquad \text{for all } x, y, z \text{ in } \mathfrak{A}.$$

Now $(1')$ implies that $(x + y, x + y, z) = (x, x, z) + (x, y, z) + (y, x, z) + (y, y, z) = (x, y, z) + (y, x, z) = 0$, implying (3). Similarly $(2')$ implies $(x, y, z) = -(x, z, y)$ which gives $(x, z, y) = (y, x, z)$. Interchanging y and z, we have (4). The fact that the associator alternates is equivalent to

$$(5) \qquad \begin{gathered} R_xR_y - R_{xy} = L_{xy} - L_yL_x = L_yR_x - R_xL_y = \\ L_xL_y - L_{yx} = R_yL_x - L_xR_y = R_{yx} - R_yR_x \end{gathered}$$

for all x, y in $\mathfrak{A}$. It follows from (1″), (2″) and (5) that any scalar extension $\mathfrak{A}_K$ of an alternative algebra $\mathfrak{A}$ is alternative.

Now (3) and (2′) imply

$$(6) \qquad (x, y, x) = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{A};$$

that is,

$$(6') \qquad (xy)x = x(yx) \qquad \text{for all } x, y \text{ in } \mathfrak{A},$$

or

$$(6'') \qquad L_x R_x = R_x L_x \qquad \text{for all } x \text{ in } \mathfrak{A}.$$

Identity (6′) is called the *flexible* law. All of the algebras mentioned in the Introduction (Lie, Jordan and alternative) are flexible. The linearized form of the flexible law is

$$(6''') \qquad (x, y, z) + (z, y, x) = 0 \qquad \text{for all } x, y, z \text{ in } \mathfrak{A}.$$

We shall have occasion to use the Moufang identities

$$(7) \qquad (xax)y = x\,[a(xy)]\,,$$

$$(8) \qquad y(xax) = [(yx)a]\,x,$$

$$(9) \qquad (xy)(ax) = x(ya)x$$

for all x, y, a in an alternative algebra $\mathfrak{A}$ (where we may write xax unambiguously by (6′)). Now $(xax)y - x\,[a(xy)] = (xa, x, y) + (x, a, xy) = (-x, xa, y) - (x, xy, a) = -\,[x(xa)]\,y + x\,[(xa)y] - [x(xy)]\,a + x\,[(xy)a] = -(x^2a)y - (x^2y)a + x\,[(xa)y + (xy)a] = -(x^2, a, y) - (x^2, y, a) - x^2(ay) - x^2(ya) + x\,[(xa)y + (xy)a] = x\,[-x(ay) - x(ya) + (xa)y + (xy)a] = x\,[(x, a, y) + (x, y, a)] = 0$, establishing (7). Identity (8) is the reciprocal relationship (obtained by passing to the anti-isomorphic algebra, which is alternative since the defining identities are reciprocal). Finally (7) implies $(xy)(ax) - x(ya)x = (x, y, ax) + x\,[y(ax) - (ya)x] = -(x, ax, y) - x(y, a, x) = -(xax)y + x\,[(ax)y - (y, a, x)] = -x\,[a(xy) - (ax)y + (y, a, x)] = -x\,[-(a, x, y) + (y, a, x)] = 0$, or (9) holds.

Theorem of Artin. The subalgebra generated by any two elements x, y of an alternative algebra $\mathfrak{A}$ is associative.

Proof: Define powers of a single element x recursively by $x^1 = x$, $x^{i+1} = xx^i$. Show first that the subalgebra $F[x]$ generated by a single element x is associative by proving

$$(10) \qquad x^i x^j = x^{i+j} \qquad \text{for all } x \text{ in } \mathfrak{A}\ (i, j = 1, 2, 3, \dots).$$

We prove this by induction on i, but shall require the case $j = 1$:

$$(11) \qquad x^i x = x x^i \qquad \text{for all } x \text{ in } \mathfrak{A} \ (i = 1, 2, \dots).$$

Proving (11) by induction, we have $x^{i+1}x = (xx^i)x = x(x^i x) = x(xx^i) = xx^{i+1}$ by flexibility and the assumption of the induction. We have (10) for $i = 1, 2$ by definition and (1). Assuming (10) for $i \geq 2$, we have $x^{i+1}x^j = (xx^i)x^j = [x(xx^{i-1})]\, x^j = [x(x^{i-1}x)]\, x^j = x\,[x^{i-1}(xx^j)] = x(x^{i-1}x^{j+1}) = xx^{i+j} = x^{i+j+1}$ by (11), (7) and the assumption of the induction. Hence $F[x]$ is associative.

Next we prove that

$$(12) \qquad x^i(x^j y) = x^{i+j} y \qquad \text{for all } x, y \text{ in } \mathfrak{A} \ (i, j = 1, 2, 3, \dots).$$

First we prove the case $j = 1$:

$$(13) \qquad x^i(xy) = x^{i+1} y \qquad \text{for all } x, y \text{ in } \mathfrak{A} \ (i = 1, 2, 3, \dots).$$

The case $i = 1$ of (13) is given by (1); the case $i = 2$ is $x^2(xy) = x\,[x(xy)] = (xxx)y = x^3 y$ by (1) and (7). Then for $i \geq 2$, write the assumption (13) of the induction with xy for y and i for $i + 1$: $x^{i-1}\,[x(xy)] = x^i(xy)$. Then $x^{i+1}(xy) = (xx^{i-1}x)(xy) = x\,[x^{i-1}\,\{x(xy)\}] = x\,[x^i(xy)] = (xx^i x)y = x^{i+2}y$ by (7). We have proved the case $j = 1$ of (12). Then with xy written for y in (12), the assumption of the induction is $x^{i+j}(xy) = x^i\,[x^j(xy)]$. It follows that $x^i(x^{j+1}y) = x^i\,[x^j(xy)] = x^{i+j}(xy) = x^{i+j+1}y$ by (13). Now (12) holds identically in y. Hence

$$(14) \qquad x^i(x^j y^k) = (x^i x^j) y^k.$$

Reciprocally

$$(15) \qquad (y^k x^j) x^i = y^k (x^j x^i).$$

Since the distributive law holds in $\mathfrak{A}$, it is sufficient now to show that

$$(16) \qquad (x^i y^k) x^j = x^i (y^k x^j)$$

in order to show that the subalgebra generated by x, y is associative. But (14) implies $(x^i, y^k, x^j) = -(x^i, x^j, y^k) = 0$.

An algebra $\mathfrak{A}$ over F is called *power-associative* in case the subalgebra $F[x]$ of $\mathfrak{A}$ generated by any element x in $\mathfrak{A}$ is associative. Any alternative algebra is power-associative; the Theorem of Artin also implies

$$(17) \qquad R_x{}^j = R_{x^j}, \qquad L_x{}^j = L_{x^j} \qquad \text{for all } x \text{ in } \mathfrak{A}.$$

An element x in a power-associative algebra $\mathfrak{A}$ is called *nilpotent* in case there is an integer r such that $x^r = 0$. An algebra (ideal) consisting only of nilpotent elements is called a *nilalgebra* (*nilideal*).

Theorem 4. Any alternative nilalgebra $\mathfrak{A}$ of finite dimension over F is nilpotent.

Proof: Any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is generated by a finite number of elements (for example, the elements in a basis for $\mathfrak{B}$ over F). We prove by induction on the number of generators of $\mathfrak{B}$ that $\mathfrak{B}^*$ is nilpotent for all subalgebras $\mathfrak{B}$; hence, in particular, for $\mathfrak{B} = \mathfrak{A}$. If $\mathfrak{B}$ is generated by one element x, then by $(6'')$ and (17) any T in $\mathfrak{B}^*$ is a linear combination of operators of the form

$$(18) \qquad R_x{}^{j_1},\ L_x{}^{j_2},\ R_x{}^{j_3}L_x{}^{j_4} \qquad \text{for } j_i \geq 1.$$

Then, if $x^j = 0$, we have $T^{2j-1} = 0$, $\mathfrak{B}^*$ is nilpotent. Hence, by the assumption of the induction, we may take a maximal proper subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and know that $\mathfrak{B}^*$ is nilpotent. But then there exists an element x not in $\mathfrak{B}$ such that

$$(19) \qquad x\mathfrak{B}^* \subseteq \mathfrak{B}.$$

For $\mathfrak{B}^{*r} = 0$ implies that $\mathfrak{A}\mathfrak{B}^{*r} = 0 \subseteq \mathfrak{B}$, and there exists a smallest integer $m \geq 1$ such that $\mathfrak{A}\mathfrak{B}^{*m} \subseteq \mathfrak{B}$. If $m = 1$, take x in $\mathfrak{A}$ but not in $\mathfrak{B}$; if $m > 1$, take x in $\mathfrak{A}\mathfrak{B}^{*m-1}$ but not in $\mathfrak{B}$. Then (19) is satisfied. Since $\mathfrak{B}$ is maximal, the subalgebra generated by $\mathfrak{B}$ and x is $\mathfrak{A}$ itself. It follows from (19) that $\mathfrak{A} = \mathfrak{B} + F[x]$ so that $\mathfrak{M} = \mathfrak{A}^* = (\mathfrak{B} + Fx)^*$. Put $y = b$ in (5) for any b in $\mathfrak{B}$. Then (19) implies that

$$(20) \qquad \begin{aligned} R_xR_b &= R_{b_1} - R_bR_x, \quad & R_xL_b &= L_bR_x + R_bR_x - R_{b_2}, \\ L_xR_b &= R_bL_x + L_bL_x - L_{b_3}, \quad & L_xL_b &= L_{b_1} - L_bL_x \end{aligned}$$

for b_i in $\mathfrak{B}$. Equations (20) show that, in each product of right and left multiplications in $\mathfrak{B}^*$ and $(Fx)^*$, the multiplication R_x or L_x may be systematically passed from the left to the right of R_b or L_b in a fashion which, although it may change signs and introduce new terms, preserves the number of factors from $\mathfrak{B}^*$ and does not increase the number of factors from $(Fx)^*$. Hence any T in $\mathfrak{A}^* = (\mathfrak{B} + Fx)^*$ may be written as a linear combination of terms of the form (18) and others of the form

$$B_1, \quad B_2{R_x}^{m_1}, \quad B_3{L_x}^{m_2}, \quad B_4{R_x}^{m_3}{L_x}^{m_4}$$

for B_i in $\mathfrak{B}^*$, $m_i \geq 1$. Then if $\mathfrak{B}^{*r} = 0$ and $x^j = 0$, we have $T^{r(2j-1)} = 0$; for every term in the expansion of $T^{r(2j-1)}$ contains either an uninterrupted sequence of at least $2j-1$ factors from $(Fx)^*$ or at least r factors B_i. In the latter case the R_x or L_x may be systematically passed from the left to the right of B_i (as above) preserving the number of factors from $\mathfrak{B}^*$, resulting in a sum of terms each containing a product $B_1B_2 \cdots B_r = 0$. Hence every element T of the finite-dimensional associative algebra $\mathfrak{A}^*$ is nilpotent. Hence $\mathfrak{A}^*$ is nilpotent (Albert, Structure of Algebras, p. 23). Hence $\mathfrak{A}$ is nilpotent by Theorem 3.

Any nilpotent algebra is solvable, and any solvable (power-associative) algebra is a nilalgebra. By Theorem 4 the concepts of nilpotent algebra, solvable algebra, and nilalgebra coincide for finite-dimensional alternative algebras. Hence there is a unique maximal nilpotent ideal $\mathfrak{N}$ (= solvable ideal = nilideal) in any finite-dimensional alternative algebra $\mathfrak{A}$; we call $\mathfrak{N}$ the *radical* of $\mathfrak{A}$. We have seen that the radical of $\mathfrak{A}/\mathfrak{N}$ is 0.

We say that $\mathfrak{A}$ is *semisimple* in case the radical of $\mathfrak{A}$ is 0, and omit the proof that any finite-dimensional semisimple alternative algebra $\mathfrak{A}$ is the direct sum $\mathfrak{A} = \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_t$ of simple algebras $\mathfrak{S}_i$. The proof is dependent upon the properties of the *Peirce decomposition* relative to an idempotent e.

An element e of an (arbitrary) algebra $\mathfrak{A}$ is called an *idempotent* in case $e^2 = e \neq 0$.

Proposition 3. Any finite-dimensional power-associative algebra, which is not a nilalgebra, contains an idempotent e ($\neq 0$).

Proof: $\mathfrak{A}$ contains an element x which is not nilpotent. The subalgebra $F[x]$ of $\mathfrak{A}$ generated by x is a finite-dimensional associative algebra which is not a nilalgebra. Then $F[x]$ contains an idempotent e ($\neq 0$) (Albert, ibid), and therefore $\mathfrak{A}$ does.

By $(1'')$ and $(2'')$ L_e and R_e are idempotent operators on $\mathfrak{A}$ which commute by $(6'')$ ("commuting projections"). It follows that $\mathfrak{A}$ is the vector space direct sum

$$\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{10} + \mathfrak{A}_{01} + \mathfrak{A}_{00} \tag{21}$$

where $\mathfrak{A}_{ij}$ $(i, j = 0, 1)$ is the subspace of $\mathfrak{A}$ defined by

$$\mathfrak{A}_{ij} = \{x_{ij} \mid ex_{ij} = ix_{ij},\ x_{ij}e = jx_{ij}\} \qquad i, j = 0, 1. \tag{22}$$

Just as in the case of associative algebras, the decomposition of any element x in $\mathfrak{A}$ according to the *Peirce decomposition* (21) is

$$x = exe + (ex - exe) + (xe - exe) + (x - ex - xe + exe). \tag{23}$$

We derive a few of the properties of the Peirce decomposition as follows:

$$\begin{aligned}(x_{ij}y_{ji})e &= (x_{ij}, y_{ji}, e) + x_{ij}(y_{ji}e)\\ &= -(x_{ij}, e, y_{ji}) + x_{ij}(y_{ji}e)\\ &= -jx_{ij}y_{ji} + jx_{ij}y_{ji} + ix_{ij}y_{ji}\\ &= ix_{ij}y_{ji}\end{aligned}$$

and similarly $e(x_{ij}y_{ji}) = ix_{ij}y_{ji}$, so

$$\mathfrak{A}_{ij}\mathfrak{A}_{ji} \subseteq \mathfrak{A}_{ii}, \qquad i, j = 0, 1. \tag{24}$$

That is, $\mathfrak{A}_{11}$ and $\mathfrak{A}_{00}$ are subalgebras of $\mathfrak{A}$, while $\mathfrak{A}_{10}\mathfrak{A}_{01} \subseteq \mathfrak{A}_{11}$, $\mathfrak{A}_{01}\mathfrak{A}_{10} \subseteq \mathfrak{A}_{00}$. Also $x_{11}y_{00} = (ex_{11}e)y_{00} = e\,[x_{11}(ey_{00})] = 0$ by (7), and similarly $y_{00}x_{11} = 0$. Hence $\mathfrak{A}_{11}$ and $\mathfrak{A}_{00}$ are orthogonal subalgebras of $\mathfrak{A}$. Similarly $\mathfrak{A}_{ii}\mathfrak{A}_{ij} \subseteq \mathfrak{A}_{ij}$, $\mathfrak{A}_{ij}\mathfrak{A}_{jj} \subseteq \mathfrak{A}_{ij}$, etc.

We wish to define the class of Cayley algebras mentioned in the Introduction. We construct these algebras in the following manner. The procedure works slightly more smoothly if we assume that F has characteristic $\neq 2$, so we make this restriction here although it is not necessary.

An algebra $\mathfrak{A}$ with 1 over F is called a *quadratic algebra* in case $\mathfrak{A} \neq F1$ and for each x in $\mathfrak{A}$ we have

$$x^2 - t(x)x + n(x)1 = 0, \qquad t(x),\ n(x) \text{ in } F. \tag{25}$$

If x is not in $F1$, the scalars $t(x)$, $n(x)$ in (25) are uniquely determined; set $t(\alpha 1) = 2\alpha$, $n(\alpha 1) = \alpha^2$ to make the *trace* $t(x)$ linear and the *norm* $n(x)$ a quadratic form.

An *involution* (*involutorial anti-isomorphism*) of an algebra $\mathfrak{A}$ is a linear operator $x \to \overline{x}$ on $\mathfrak{A}$ satisfying

$$(26) \qquad \overline{xy} = \overline{y}\,\overline{x}, \quad \overline{\overline{x}} = x \qquad \text{for all } x, y \text{ in } \mathfrak{A}.$$

Here we are concerned with an involution satisfying

$$(27) \qquad x + \overline{x} \in F1, \quad x\overline{x} (= \overline{x}x) \in F1 \qquad \text{for all } x \text{ in } \mathfrak{A}.$$

Clearly (27) implies (25) with

$$(27') \qquad x + \overline{x} = t(x)1, \quad x\overline{x} (= \overline{x}x) = n(x)1 \qquad \text{for all } x \text{ in } \mathfrak{A}$$

(since $\overline{1} = 1$, we have $t(\alpha 1) = 2\alpha$, $n(\alpha 1) = \alpha^2$ from (27)).

Let $\mathfrak{B}$ be an algebra with 1 having dimension n over F and such that $\mathfrak{B}$ has an involution $x \to \overline{x}$ satisfying (27). We construct an algebra $\mathfrak{A}$ of dimension $2n$ over F with the same properties and having $\mathfrak{B}$ as subalgebra (with $1 \in \mathfrak{B}$) as follows: $\mathfrak{A}$ consists of all ordered pairs $x = (b_1, b_2)$, b_i in $\mathfrak{B}$, addition and multiplication by scalars defined componentwise, and multiplication defined by

$$(28) \qquad (b_1, b_2)(b_3, b_4) = (b_1 b_3 + \mu b_4 \overline{b_2},\ \overline{b_1} b_4 + b_3 b_2)$$

for all b_i in $\mathfrak{B}$ and some $\mu \neq 0$ in F. Then $1 = (1, 0)$ is a unity element for $\mathfrak{A}$, $\mathfrak{B}' = \{(b, 0) \mid b \in \mathfrak{B}\}$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\mathfrak{B}$, $v = (0, 1)$ is an element of $\mathfrak{A}$ such that $v^2 = \mu 1$ and $\mathfrak{A}$ is the vector space direct sum $\mathfrak{A} = \mathfrak{B}' + v\mathfrak{B}'$ of the n-dimensional vector spaces $\mathfrak{B}'$, $v\mathfrak{B}'$. Identifying $\mathfrak{B}'$ with $\mathfrak{B}$, the elements of $\mathfrak{A}$ are of the form

$$(29) \qquad x = b_1 + v b_2 \qquad (b_1 \text{ in } \mathfrak{B} \text{ uniquely determined by } x),$$

and (28) becomes

$$(28') \qquad (b_1 + v b_2)(b_3 + v b_4) = (b_1 b_3 + \mu b_4 \overline{b_2}) + v(\overline{b_1} b_4 + b_3 b_2)$$

for all b_i in $\mathfrak{B}$ and some $\mu \neq 0$ in F. Defining

$$(30) \qquad \overline{x} = \overline{b_1} - v b_2,$$

we have $\overline{xy} = \overline{y}\,\overline{x}$ by (28′) since $b \to \bar{b}$ is an involution of $\mathfrak{B}$; hence $x \to \overline{x}$ is an involution of $\mathfrak{A}$. Also

$$x + \overline{x} = t(x)1, \qquad x\overline{x}(= \overline{x}x) = n(x)1$$

where, for x in (29), we have

$$(31) \qquad t(x) = t(b_1), \qquad n(x) = n(b_1) - \mu n(b_2).$$

Assume that the norm on $\mathfrak{B}$ is a nondegenerate quadratic form; that is, the associated symmetric bilinear form

$$(32) \qquad (a, b) = \tfrac{1}{2}\left[n(a + b) - n(a) - n(b)\right] \quad (= \tfrac{1}{2}t(a\bar{b}))$$

is nondegenerate (if $(a, b) = 0$ for all b in $\mathfrak{B}$, then $a = 0$). Then the norm $n(x)$ on $\mathfrak{A}$ defined by (31) is nondegenerate. For $y = b_3 + vb_4$ implies that $(x, y) = \frac{1}{2}\left[n(x + y) - n(x) - n(y)\right] = \frac{1}{2}\left[n(b_1 + b_3) - \mu n(b_2 + b_4) - n(b_1) + \mu n(b_2) - n(b_3) + \mu n(b_4)\right] = (b_1, b_3) - \mu(b_2, b_4)$. Hence $(x, y) = 0$ for all $y = b_3 + vb_4$ implies $(b_1, b_3) = \mu(b_2, b_4)$ for all b_3, b_4 in $\mathfrak{B}$. Then $b_4 = 0$ implies $(b_1, b_3) = 0$ for all b_3 in $\mathfrak{B}$, or $b_1 = 0$ since $n(b)$ is nondegenerate on $\mathfrak{B}$; similarly $b_3 = 0$ implies $(b_2, b_4) = 0$ (since $\mu \neq 0$) for all b_4 in $\mathfrak{B}$, or $b_2 = 0$. That is, $x = 0$; $n(x)$ is nondegenerate on $\mathfrak{A}$.

When is $\mathfrak{A}$ alternative? Since $\mathfrak{A}$ is its own reciprocal algebra, it is sufficient to verify the left alternative law (1′), which is equivalent to $(x, \overline{x}, y) = 0$ since $(x, \overline{x}, y) = (x, t(x)1 - x, y) = -(x, x, y)$. Now $(x, \overline{x}, y) = n(x)y - (b_1 + vb_2)\Big[(\overline{b_1}b_3 - \mu b_4\overline{b_2}) + v(b_1b_4 - b_3b_2)\Big] = n(x)y - \Big[b_1(\overline{b_1}b_3) - \mu b_1(b_4\overline{b_2}) + \mu(b_1b_4)\overline{b_2} - \mu(b_3b_2)\overline{b_2}\Big] - v\Big[\overline{b_1}(b_1b_4) - \overline{b_1}(b_3b_2) + (\overline{b_1}b_3)b_2 - \mu(b_4\overline{b_2})b_2\Big] = n(x)y - \Big[n(b_1) - \mu n(b_2)\Big](b_3 + vb_4) - \mu(b_1, b_4, \overline{b_2}) - v(\overline{b_1}, b_3, b_2) = -\mu(b_1, b_4, \overline{b_2}) - v(\overline{b_1}, b_3, b_2)$ by a trivial extension of the Theorem of Artin. Hence $\mathfrak{A}$ is alternative if and only if $\mathfrak{B}$ is associative.

The algebra $F1$ is not a quadratic algebra, but the identity operator on $F1$ is an involution satisfying (27); also $n(\alpha 1)$ is nondegenerate on $F1$. Hence we can use an iterative process (beginning with $\mathfrak{B} = F1$) to obtain by the above construction algebras of dimension 2^t over F; these depend completely upon the t nonzero scalars $\mu_1, \mu_2, \ldots, \mu_t$ used in the successive steps. The norm on each algebra is a nondegenerate quadratic form. The 2-dimensional algebras $\mathfrak{Z} = F1 + v_1(F1)$ are either quadratic fields over F (μ_1 a nonsquare in F) or isomorphic to

$F \oplus F$ (μ_1 a square in F). The 4-dimensional algebras $\mathfrak{Q} = \mathfrak{Z} + v_2\mathfrak{Z}$ are associative central simple algebras (called *quaternion algebras*) over F; any $\mathfrak{Q}$ which is not a division algebra is (by Wedderburn's theorem on simple associative algebras) isomorphic to the algebra of all 2×2 matrices with elements in F.

We are concerned with the 8-dimensional algebras $\mathfrak{C} = \mathfrak{Q} + v_3\mathfrak{Q}$ which are called *Cayley algebras* over F. Since any $\mathfrak{Q}$ is associative, Cayley algebras are alternative. However, no Cayley algebra is associative. For $\mathfrak{Q}$ is not commutative and there exist q_1, q_2 in $\mathfrak{Q}$ such that $[q_1, q_2] \neq 0$; hence $(v_3, q_2, q_1) = (v_3q_2)q_1 - v_3(q_2q_1) = v_3[q_1, q_2] \neq 0$ by (28′). Thus this iterative process of constructing alternative algebras stops after three steps. The quadratic form $n(x)$ is nondegenerate; also it *permits composition* in the sense that

$$n(xy) = n(x)n(y) \qquad \text{for all } x, y \text{ in } \mathfrak{C}. \tag{33}$$

For $n(xy)1 = (xy)(\overline{xy}) = xy\overline{y}\,\overline{x} = n(y)x\overline{x} = n(x)n(y)1$. Also

$$t\left((xy)z\right) = t\left(x(yz)\right) \qquad \text{for all } x, y, z \text{ in } \mathfrak{A}. \tag{34}$$

For $(x, y, z) = -(z, y, x) = (\overline{z}, \overline{y}, \overline{x})$ implies $(xy)z + \overline{z}(\overline{y}\,\overline{x}) = x(yz) + (\overline{z}\,\overline{y})\overline{x}$, so that (34) holds.

Theorem 5. Two Cayley algebras $\mathfrak{C}$ and $\mathfrak{C}'$ are isomorphic if and only if their corresponding norm forms $n(x)$ and $n'(x')$ are equivalent (that is, there is a linear mapping $x \to xH$ of $\mathfrak{C}$ into $\mathfrak{C}'$ such that

$$n'(xH) = n(x) \qquad \text{for all } x \text{ in } \mathfrak{C}; \tag{35}$$

H is necessarily (1–1) since $n(x)$ is nondegenerate).

Proof: Suppose $\mathfrak{C}$ and $\mathfrak{C}'$ are isomorphic, the isomorphism being H. Then (25) implies $(xH)^2 - t(x)(xH) + n(x)1' = 0$ where $1' = 1H$ is the unity element of $\mathfrak{C}'$. But also $(xH)^2 - t'(xH)(xH) + n'(xH)1' = 0$. Hence $[t'(xH) - t(x)]\,(xH) + [n(x) - n'(xH)]\,1' = 0$. If $x \notin F1$, then $xH \notin F1'$ and $n(x) = n'(xH)$. On the other hand $n(\alpha 1) = \alpha^2 = n'(\alpha 1')$, and we have (35) for all x in $\mathfrak{C}$.

For the converse we need to establish the fact that, if $\mathfrak{B}$ is a proper subalgebra of a Cayley algebra $\mathfrak{C}$, if $\mathfrak{B}$ contains the unity element 1 of $\mathfrak{C}$, and if (relative to the nondegenerate symmetric bilinear form

(x, y) defined on $\mathfrak{C}$ by (32)) $\mathfrak{B}$ is a non-isotropic subspace of $\mathfrak{C}$ (that is, $\mathfrak{B} \cap \mathfrak{B}^\perp = 0$), then there is a subalgebra $\mathfrak{A} = \mathfrak{B} + v\mathfrak{B}$ (constructed as above). For the involution $x \to \bar{x}$ on $\mathfrak{C}$ induces an involution on $\mathfrak{B}$, since $\bar{b} = t(b)1 - b$ is in $\mathfrak{B}$ for all b in $\mathfrak{B}$. Also $\mathfrak{B}$ non-isotropic implies $\mathfrak{C} = \mathfrak{B} \perp \mathfrak{B}^\perp$ with $\mathfrak{B}^\perp$ non-isotropic (Jacobson, Lectures in Abstract Algebra, vol. II, p. 151; Artin, Geometric Algebra, p. 117). Hence there is a non-isotropic vector v in $\mathfrak{B}^\perp$, $n(v) = -\mu \neq 0$. Since $t(v) = t(v\bar{1}) = 2(v, 1) = 0$, we have

$$v^2 = \mu 1, \qquad \mu \neq 0 \text{ in } F. \tag{36}$$

Now $v\mathfrak{B} \subseteq \mathfrak{B}^\perp$ since (34) implies $(va, b) = \frac{1}{2}t\left((va)\bar{b}\right) = \frac{1}{2}t\left(v(a\bar{b})\right) = (v, b\bar{a}) = 0$ for all a, b in $\mathfrak{B}$. Hence $\mathfrak{B} \perp v\mathfrak{B}$. Also $v\mathfrak{B}$ has the same dimension as $\mathfrak{B}$ since $b \to vb$ is (1–1). Suppose $vb = 0$; then $v(vb) = v^2 b = \mu b = 0$, implying $b = 0$. In order to show that $\mathfrak{A} = \mathfrak{B} \perp v\mathfrak{B}$ is the algebra constructed above, it remains to show that

$$a(vb) = v(\bar{a}b), \tag{37}$$

$$(va)b = b(va), \tag{38}$$

$$(va)(vb) = \mu b\bar{a} \tag{39}$$

for all a, b in $\mathfrak{B}$. Now $t(v) = 0$ implies $\bar{v} = -v$; hence v in $\mathfrak{B}^\perp$ implies $0 = 2(v, b) = t(v\bar{b}) = v\bar{b} + b\bar{v} = v\bar{b} - bv$, or

$$bv = v\bar{b} \qquad \text{for all } b \text{ in } \mathfrak{B}. \tag{40}$$

Hence $(v, \bar{a}, b) + (\bar{a}, v, b) = 0 = (v\bar{a})b - v(\bar{a}b) + (\bar{a}v)b = (v\bar{a})b - v(\bar{a}b) + (va)b - \bar{a}(vb) = [t(a)1 - \bar{a}]\, vb - v(\bar{a}b)$, establishing (37). Applying the involution to $\bar{b}(va) = v(ba)$, and using (40), we have (38). Finally $(va)(vb) = (va)(\bar{b}v) = v(a\bar{b})v = v^2(b\bar{a}) = \mu b\bar{a}$ by the Moufang identity (9). Hence $\mathfrak{A} = \mathfrak{B} \perp \mathfrak{B}^\perp$ is the subalgebra specified. Since $\mathfrak{B}$ and $\mathfrak{B}^\perp$ are non-isotropic, so is $\mathfrak{A}$. [Remark: we have shown incidentally that if $\mathfrak{Q}$ is any quaternion subalgebra containing 1 in a Cayley algebra $\mathfrak{C}$, then $\mathfrak{Q}$ may be used in the construction of $\mathfrak{C}$ as $\mathfrak{C} = \mathfrak{Q} + v\mathfrak{Q}$.]

Now let $\mathfrak{C}$ and $\mathfrak{C}'$ have equivalent norm forms $n(x)$ and $n'(x')$. Let $\mathfrak{B}$ (and $\mathfrak{B}'$) be as above. If $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic under H_0, then the restrictions of $n(x)$ and $n'(x')$ to $\mathfrak{B}$ and $\mathfrak{B}'$ are equivalent. Then by Witt's theorem (Jacobson, ibid, p. 162; Artin, ibid, p. 121), since $n(x)$ and $n'(x')$ are equivalent, the restrictions of $n(x)$ and $n'(x')$ to $\mathfrak{B}^\perp$ and

$\mathfrak{B}'^{\perp}$ are equivalent. Choose v in $\mathfrak{B}^{\perp}$ with $n(v) \neq 0$; correspondingly we have v' in $\mathfrak{B}'^{\perp}$ such that $n'(v') = n(v)$. Then $a + vb \to aH_0 + v'(bH_0)$ is an isomorphism of $\mathfrak{B} \perp v\mathfrak{B}$ onto $\mathfrak{B}' \perp v'\mathfrak{B}'$ by the construction above. Hence if we begin with $\mathfrak{B} = F1$, $\mathfrak{B}' = F1'$, repetition of the process gives successively isomorphisms between $\mathfrak{Z}$ and $\mathfrak{Z}'$, $\mathfrak{Q}$ and $\mathfrak{Q}'$, $\mathfrak{C}$ and $\mathfrak{C}'$.

A Cayley algebra $\mathfrak{C}$ is a division algebra if and only if $n(x) \neq 0$ for every $x \neq 0$ in $\mathfrak{C}$. For $x \neq 0$, $n(x) = 0$ imply $x\overline{x} = n(x)1 = 0$, $\mathfrak{C}$ has zero divisors. Conversely, if $n(x) \neq 0$, then $\overline{x}(xy) = (\overline{x}x)y = n(x)y$ for all y implies $\frac{1}{n(x)} L_x L_{\overline{x}} = 1_{\mathfrak{C}}$, $L_x^{-1} = \frac{1}{n(x)} L_{\overline{x}}$ and similarly $R_x^{-1} = \frac{1}{n(x)} R_{\overline{x}}$; hence if $n(x) \neq 0$ for all $x \neq 0$, then $\mathfrak{C}$ is a division algebra.

[Remark: If F is the field of all real numbers, the norm form $n(x) = \sum {\alpha_i}^2$ for $x = \sum \alpha_i u_i$ clearly has the property above. Also there are alternative algebras $F1$, $\mathfrak{Z}$, $\mathfrak{Q}$, $\mathfrak{C}$ with this norm form (take $\mu_i = -1$ at each step). Hence there are real alternative division algebras of dimensions 1, 2, 4, 8. It has recently been proved (see reference [12] of the appended bibliography of recent papers) that finite-dimensional real division algebras can have only these dimensions. It is not true, however, that the only finite-dimensional real division algebras are the four listed above; they are the only alternative ones. For other examples of finite-dimensional real division algebras (necessarily of these specified dimensions of course) see reference [23] in the bibliography of the 1955 Bulletin article.]

Corollary. Any two Cayley algebras $\mathfrak{C}$ and $\mathfrak{C}'$ with divisors of zero are isomorphic.

Proof: Show first that $\mathfrak{C}$ has divisors of zero if and only if there is $w \notin F1$ such that $w^2 = 1$. For $1 - w \neq 0$, $1 + w \neq 0$ imply $(1 - w)(1 + w) = 1 - w^2 = 0$ (note $t(w) = 0$ implies $\overline{1 \pm w} = 1 \mp w$ so that $n(1 \pm w) = 0$). Conversely, if $\mathfrak{C}$ has divisors of zero, there exists $x \neq 0$ in $\mathfrak{C}$ with $n(x) = 0$. Then $x = \alpha 1 + u$, $u \in (F1)^{\perp} = \{u \mid t(u) = 0\}$ implies $0 = n(x)1 = x\overline{x} = (\alpha 1 + u)(\alpha 1 - u) = \alpha^2 1 - u^2$. If $\alpha \neq 0$, then $w = \alpha^{-1}u$ satisfies $w^2 = 1$ ($w \notin F1$). If $\alpha = 0$, then $n(u) = 0$ so that u is an isotropic vector in the non-isotropic space $(F1)^{\perp}$. Hence there exists w in $(F1)^{\perp}$ with $n(w) = -1$ (Jacobson, ibid, p. 154, ex. 3), or $w^2 = t(w)w - n(w)1 = 1$ ($w \notin F1$).

Now let $e_1 = \frac{1}{2}(1 - w)$, $e_2 = 1 - e_1 = \frac{1}{2}(1 + w)$. Then ${e_1}^2 = e_1$, ${e_2}^2 = e_2$, $e_1 e_2 = e_2 e_1 = 0$ (e_1 and e_2 are *orthogonal idempotents*). Also $n(e_i) = 0$ for $i = 1, 2$. Hence every vector in $e_i\mathfrak{C}$ is isotropic since

$n(e_i x) = n(e_i)n(x) = 0$. This means that $e_i\mathfrak{C}$ is a totally isotropic subspace ($e_i\mathfrak{C} \subseteq (e_i\mathfrak{C})^\perp$). Hence $\dim(e_i\mathfrak{C}) \leq \frac{1}{2}\dim\mathfrak{C} = 4$ (Jacobson, p. 170; Artin, p. 122). But $x = 1x = e_1x + e_2x$ for all x in $\mathfrak{C}$, so $\mathfrak{C} = e_1\mathfrak{C} + e_2\mathfrak{C}$. Hence $\dim(e_i\mathfrak{C}) = 4$, and $n(x)$ has maximal Witt index $= 4 = \frac{1}{2}\dim\mathfrak{C}$. Similarly $n'(x')$ has maximal Witt index $= 4$. Hence $n(x)$ and $n'(x')$ are equivalent (Artin, ibid). By Theorem 5, $\mathfrak{C}$ and $\mathfrak{C}'$ are isomorphic.

Over any field F there is a Cayley algebra without divisors of zero (take $\mu = 1$ so $v^2 = 1$). This unique Cayley algebra over F is called the *split Cayley algebra* over F.

$F1$ is both the nucleus and center of any Cayley algebra. Also any Cayley algebra is simple (hence central simple over F). (This is obvious for all but the split Cayley algebra.) For, if $\mathfrak{I}$ is any nonzero ideal of $\mathfrak{C}$, there is $x \neq 0$ in $\mathfrak{I}$. But x is contained is some quaternion subalgebra $\mathfrak{Q}$ of $\mathfrak{C}$. Then $\mathfrak{Q} \times \mathfrak{Q}$ is an ideal of the simple algebra $\mathfrak{Q}$. Hence $1 \in \mathfrak{Q} = \mathfrak{Q} \times \mathfrak{Q} \subseteq \mathfrak{I}$, and $\mathfrak{I} = \mathfrak{C}$.

We omit the proof of the fact that the only alternative central simple algebras of finite dimension which are not associative are Cayley algebras. (Actually the following stronger result is known: any simple alternative ring, which is not a nilring and which is not associative, is a Cayley algebra over its center; in the finite-dimensional case the restriction eliminating nilalgebras is not required since Theorem 4 implies that $\mathfrak{A}^2 \neq \mathfrak{A}$ for a finite-dimensional alternative nilalgebra). Hence the simple components $\mathfrak{S}_i$ in a finite-dimensional semisimple alternative algebra are either associative or Cayley algebras over their centers.

The derivation algebra $\mathfrak{D}(\mathfrak{C})$ of any Cayley algebra of characteristic $\neq 3$ is a central simple Lie algebra of dimension 14, called an *exceptional Lie algebra of type G* (corresponding to the 14-parameter complex exceptional simple Lie group G_2). The related subject of automorphisms of Cayley algebras is studied in [33].

IV. Jordan Algebras

In the Introduction we defined a (commutative) Jordan algebra $\mathfrak{J}$ over F to be a commutative algebra in which the *Jordan identity*

$$(1) \qquad (xy)x^2 = x(yx^2) \qquad \text{for all } x, y \text{ in } \mathfrak{J}$$

is satisfied. Linearization of (1) requires that we assume F has characteristic $\neq 2$; we make this assumption throughout IV. It follows from (1) and the identities (2), (3) below that any scalar extension $\mathfrak{J}_K$ of a Jordan algebra $\mathfrak{J}$ is a Jordan algebra.

Replacing x in

$$(1') \qquad (x, y, x^2) = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{J}$$

by $x + \lambda z$ ($\lambda \in F$), the coefficient of λ is 0 since F contains at least three distinct elements, and we have

$$(2) \qquad 2(x, y, zx) + (z, y, x^2) = 0 \qquad \text{for all } x, y, z \text{ in } \mathfrak{J}.$$

Replacing x in (2) by $x + \lambda w$ ($\lambda \in F$), we have similarly (after dividing by 2) the multilinear identity

$$(3) \quad (x, y, wz) + (w, y, zx) + (z, y, xw) = 0 \quad \text{for all } w, x, y, z \text{ in } \mathfrak{J}.$$

Recalling that $L_a = R_a$ since $\mathfrak{J}$ is commutative, we see that (3) is equivalent to

$$(3') \quad [R_x, R_{wz}] + [R_w, R_{zx}] + [R_z, R_{xw}] = 0 \qquad \text{for all } w, x, z \text{ in } \mathfrak{J}$$

and to

$$(3'') \quad R_z R_{xy} - R_z R_y R_x + R_y R_{zx} - R_{y(zx)} + R_x R_{zy} - R_x R_y R_z = 0 \qquad \text{for all } x, y, z \text{ in } \mathfrak{J}.$$

Interchange x and y in $(3'')$ and subtract to obtain

$$(4) \quad [R_z, [R_x, R_y]] = R_{(x,z,y)} = R_{z[R_x, R_y]} \qquad \text{for all } x, y, z \text{ in } \mathfrak{J}.$$

Now (4) says that, for all x, y in $\mathfrak{J}$, the operator $[R_x, R_y]$ is a derivation of $\mathfrak{J}$, since the defining condition for a derivation D of an arbitrary algebra $\mathfrak{A}$ may be written as

$$[R_z, D] = R_{zD} \qquad \text{for all } z \text{ in } \mathfrak{A}.$$

Our first objective is to prove that any Jordan algebra $\mathfrak{J}$ is power-associative. As in III we define powers of x by $x^1 = x$, $x^{i+1} = xx^i$, and prove

$$(5) \qquad x^i x^j = x^{i+j} \qquad \text{for all } x \text{ in } \mathfrak{J}.$$

For any x in $\mathfrak{J}$, write $\mathfrak{G}_x = R_x \cup R_{x^2}$. Then the enveloping algebra $\mathfrak{G}_x^*$ is commutative, since the generators R_x, R_{x^2} commute by (1). For $i \geq 2$, we put $y = x$, $z = x^{i-1}$ in (3″) to obtain

$$(6) \qquad R_{x^{i+1}} = R_{x^{i-1}} R_{x^2} - R_{x^{i-1}} R_x^2 - R_x^2 R_{x^{i-1}} + 2R_x R_{x^i}.$$

By induction on i we see from (6) that R_{x^i} is in $\mathfrak{G}^*$ for $i = 3, 4, \ldots$. Hence

$$(7) \qquad R_{x^i} R_{x^j} = R_{x^j} R_{x^i} \qquad \text{for } i, j = 1, 2, 3, \ldots$$

Then, in a proof of (5) by induction on i, we can assume that $x^i x^{j+1} = x^{i+j+1}$; then $x^{i+1} x^j = (xx^i)x^j = xR_{x^i}R_{x^j} = xR_{x^j}R_{x^i} = x^{j+1}x^i = x^{i+j+1}$ as desired.

One can prove, by a method similar to the proof of Theorem 4 in III (only considerably more complicated since the identities involved are more complicated), that any finite-dimensional Jordan nilalgebra is nilpotent. We omit the proof, which involves also a proof of the fact that

$$(8) \qquad R_x \text{ is nilpotent for any nilpotent } x \text{ in a finite-dimensional Jordan algebra.}$$

As in III, this means that there is a unique maximal nilpotent (= solvable = nil) ideal $\mathfrak{N}$ which is called the *radical* of $\mathfrak{J}$. Defining $\mathfrak{J}$ to be *semisimple* in case $\mathfrak{N} = 0$, we have seen that $\mathfrak{J}/\mathfrak{N}$ is semisimple. The proof that any semisimple Jordan algebra $\mathfrak{S}$ is a direct sum $\mathfrak{S} = \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_t$ of simple $\mathfrak{S}_i$ is quite complicated for arbitrary F; we shall use a trace argument to give a proof for F of characteristic 0.

Let e be an idempotent in a Jordan algebra $\mathfrak{J}$. Put $i = 2$ and $x = e$ in (6) to obtain

$$(9) \qquad 2R_e^3 - 3R_e^2 + R_e = 0;$$

that is, $f(R_e) = 0$ where $f(\lambda) = (\lambda - 1)(2\lambda - 1)\lambda$. Hence the minimal polynomial for R_e divides $f(\lambda)$, and the only possibilities for characteristic roots of R_e are 1, $\frac{1}{2}$, 0 (1 must occur since e is a characteristic vector belonging to the characteristic root 1: $eR_e = e^2 = e \neq 0$). Also the minimal polynomial for R_e has simple roots. Hence $\mathfrak{J}$ is the vector space direct sum

$$(10) \qquad \mathfrak{J} = \mathfrak{J}_1 + \mathfrak{J}_{1/2} + \mathfrak{J}_0$$

where

$$(11) \qquad \mathfrak{J}_i = \{x_i \mid x_i e = i x_i\}, \qquad i = 1, 1/2, 0.$$

Taking a basis for $\mathfrak{J}$ adapted to the *Peirce decomposition* (10), we see that R_e has for its matrix relative to this basis the diagonal matrix $\operatorname{diag}\{1, 1, \ldots, 1, 1/2, 1/2, \ldots, 1/2, 0, 0, \ldots, 0\}$ where the number of 1's is $\dim \mathfrak{J}_1 > 0$ and the number of 1/2's is $\dim \mathfrak{J}_{1/2}$. Hence

$$(12) \qquad \operatorname{trace} R_e = \dim \mathfrak{J}_1 + \tfrac{1}{2} \dim \mathfrak{J}_{1/2}.$$

If F has characteristic 0, then $\operatorname{trace} R_e \neq 0$.

A symmetric bilinear form (x, y) defined on an arbitrary algebra $\mathfrak{A}$ is called a *trace form* (*associative* or *invariant* symmetric bilinear form) on $\mathfrak{A}$ in case

$$(13) \qquad (xy, z) = (x, yz) \qquad \text{for all } x, y, z \text{ in } \mathfrak{A}.$$

If $\mathfrak{I}$ is any ideal of an algebra $\mathfrak{A}$ on which such a bilinear form is defined, then $\mathfrak{I}^\perp$ is also an ideal of $\mathfrak{A}$: for x in $\mathfrak{I}$, y in $\mathfrak{I}^\perp$, a in $\mathfrak{A}$ imply that xa and ax are in $\mathfrak{I}$ so that $(x, ay) = (xa, y) = 0$ and $(x, ya) = (ya, x) = (y, ax) = 0$ by (13). In particular, the radical $\mathfrak{A}^\perp = \{x \mid (x, y) = 0 \text{ for all } y \in \mathfrak{A}\}$ of the trace form is an ideal of $\mathfrak{A}$.

We also remark that it follows from (13) that $(xR_y, z) = (x, zL_y)$ and $(xL_y, z) = (z, yx) = (zy, x) = (x, zR_y)$ so that, for right (or left) multiplications S_i determined by b_i,

$$(14) \qquad (xS_1 S_2 \cdots S_h, y) = (x, yS'_h \cdots S'_2 S'_1)$$

where S_i' is the left (or right) multiplication determined by b_i; then, if $\mathfrak{B}$ is any subset of $\mathfrak{A}$,

$$(15) \qquad (xT, y) = (x, yT') \qquad \text{for all } x, y \text{ in } \mathfrak{A},\ T \text{ in } \mathfrak{B}^*,$$

where T' is in $\mathfrak{B}^*$.

Theorem 6. The radical $\mathfrak{N}$ of any finite-dimensional Jordan algebra $\mathfrak{J}$ over F of characteristic 0 is the radical $\mathfrak{J}^\perp$ of the trace form

$$(16) \qquad (x, y) = \operatorname{trace} R_{xy} \qquad \text{for all } x, y \text{ in } \mathfrak{J}.$$

Proof: Without any assumption on the characteristic of F it follows from (4) that (x, y) in (16) is a trace form: $(xy, z) - (x, yz) = \operatorname{trace} R_{(x,y,z)} = 0$ since the trace of any commutator is 0. Hence $\mathfrak{J}^\perp$ is an ideal of $\mathfrak{J}$. If $\mathfrak{J}$ were not a nilideal, then (by Proposition 3) $\mathfrak{J}^\perp$ would contain an idempotent e ($\neq 0$) and, assuming characteristic 0, $(e, e) = \operatorname{trace} R_e \neq 0$ by (12), a contradiction. Hence $\mathfrak{J}^\perp$ is a nilideal and $\mathfrak{J}^\perp \subseteq \mathfrak{N}$. Conversely, if x is in $\mathfrak{N}$, then xy is in $\mathfrak{N}$ for every y in $\mathfrak{A}$, and R_{xy} is nilpotent by (8). Hence $(x, y) = \operatorname{trace} R_{xy} = 0$ for all y in $\mathfrak{A}$; that is, x is in $\mathfrak{J}^\perp$. Hence $\mathfrak{N} \subseteq \mathfrak{J}^\perp$, $\mathfrak{N} = \mathfrak{J}^\perp$.

Theorem 7. Let $\mathfrak{A}$ be a finite-dimensional algebra over F (of arbitrary characteristic) satisfying

(i) there is a nondegenerate (associative) trace form (x, y) defined on $\mathfrak{A}$, and

(ii) $\mathfrak{I}^2 \neq 0$ for every ideal $\mathfrak{I} \neq 0$ of $\mathfrak{A}$.

Then $\mathfrak{A}$ is (uniquely) expressible as a direct sum $\mathfrak{A} = \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_t$ of simple ideals $\mathfrak{S}_i$.

Proof: Let $\mathfrak{S}$ ($\neq 0$) be a minimal ideal of $\mathfrak{A}$. Since (x, y) is a trace form, $\mathfrak{S}^\perp$ is an ideal of $\mathfrak{A}$. Hence the intersection $\mathfrak{S} \cap \mathfrak{S}^\perp$ is either 0 or $\mathfrak{S}$, since $\mathfrak{S}$ is minimal. We show that $\mathfrak{S}$ totally isotropic ($\mathfrak{S} \subseteq \mathfrak{S}^\perp$) leads to a contradiction.

For, since $\mathfrak{S}^2 \neq 0$ by (ii), we know that the ideal of $\mathfrak{A}$ generated by $\mathfrak{S}^2$ must be the minimal ideal $\mathfrak{S}$. Thus $\mathfrak{S} = \mathfrak{S}^2 + \mathfrak{S}^2\mathfrak{M}$ where $\mathfrak{M}$ is the multiplication algebra of $\mathfrak{A}$. Any element s in $\mathfrak{S}$ may be written in the form $s = \sum (a_i b_i) T_i$ for a_i, b_i in $\mathfrak{S}$, where $T_i = T_i'$ is the identity operator $1_{\mathfrak{A}}$ or T_i is in $\mathfrak{M}$. For every y in $\mathfrak{A}$ we have by

(15) that $(s, y) = \sum ((a_i b_i)T_i, y) = \sum(a_i b_i, yT_i') = \sum (a_i, b_i(yT_i')) = 0$ since $b_i(yT_i') \in \mathfrak{S} \subseteq \mathfrak{S}^\perp$. Then $s = 0$ since (x, y) is nondegenerate; $\mathfrak{S} = 0$, a contradiction. Hence $\mathfrak{S} \cap \mathfrak{S}^\perp = 0$; that is, $\mathfrak{S}$ is non-isotropic. Hence $\mathfrak{A} = \mathfrak{S} \perp \mathfrak{S}^\perp$ and $\mathfrak{S}^\perp$ is non-isotropic. That is, $\mathfrak{A} = \mathfrak{S} \oplus \mathfrak{S}^\perp$, the direct sum of ideals $\mathfrak{S}$, $\mathfrak{S}^\perp$, and the restriction of (x, y) to $\mathfrak{S}^\perp$ is a nondegenerate (associative) trace form defined on $\mathfrak{S}^\perp$. That is, (i) holds for $\mathfrak{S}^\perp$ as well as $\mathfrak{A}$. Moreover, any ideal of the direct summand $\mathfrak{S}$ or $\mathfrak{S}^\perp$ is an ideal of $\mathfrak{A}$; hence $\mathfrak{S}$ is simple and (ii) holds for $\mathfrak{S}^\perp$. Induction on the dimension of $\mathfrak{A}$ completes the proof.

Corollary. Any (finite-dimensional) semisimple Jordan algebra $\mathfrak{J}$ over F of characteristic 0 is (uniquely) expressible as a direct sum $\mathfrak{J} = \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_t$ of simple ideals $\mathfrak{S}_i$.

Proof: By Theorem 6 the (associative) trace form (16) is nondegenerate; hence (i) is satisfied. Also any ideal $\mathfrak{I}$ such that $\mathfrak{I}^2 = 0$ is nilpotent; hence $\mathfrak{I} = 0$, establishing (ii).

As mentioned above, the corollary is actually true for F of characteristic $\neq 2$. What remains then, as far as the structure of semisimple Jordan algebras is concerned, is a determination of the central simple algebras. The first step in this is to show that every semisimple $\mathfrak{J}$ (hence every simple $\mathfrak{J}$) has a unity element 1. Again the argument we use here is valid only for characteristic 0, whereas the theorem is true in general.

We begin by returning to the Peirce decomposition (10) of any Jordan algebra $\mathfrak{J}$ relative to an idempotent e. The subspaces $\mathfrak{J}_1$ and $\mathfrak{J}_0$ are orthogonal subalgebras of $\mathfrak{J}$ which are related to the subspace $\mathfrak{J}_{1/2}$ as follows:

$$(17) \qquad \mathfrak{J}_{1/2}\mathfrak{J}_{1/2} \subseteq \mathfrak{J}_1 + \mathfrak{J}_0, \qquad \mathfrak{J}_1\mathfrak{J}_{1/2} \subseteq \mathfrak{J}_{1/2}, \qquad \mathfrak{J}_0\mathfrak{J}_{1/2} \subseteq \mathfrak{J}_{1/2}.$$

For we may put $x = e$, $z = x_i \in \mathfrak{J}_i$, $y = y_j \in \mathfrak{J}_j$ in (2) to obtain $2i(e, y_j, x_i) + (x_i, y_j, e) = 0$, or $(1 - 2i)\,[(x_i y_j)e - j(x_i y_j)] = 0$, so that

$$(18) \qquad \mathfrak{J}_i\mathfrak{J}_j \subseteq \mathfrak{J}_j \qquad \text{if } i \neq 1/2.$$

Hence ${\mathfrak{J}_1}^2 \subseteq \mathfrak{J}_1$, ${\mathfrak{J}_0}^2 \subseteq \mathfrak{J}_0$, $\mathfrak{J}_1\mathfrak{J}_0 = \mathfrak{J}_0\mathfrak{J}_1 \subseteq \mathfrak{J}_0 \cap \mathfrak{J}_1 = 0$, so $\mathfrak{J}_1$ and $\mathfrak{J}_0$ are orthogonal subalgebras by (18), and also the last two inclusions in (17) hold. Put $x = x_{1/2}$, $z = y_{1/2}$, $y = w = e$ in (3) and write $x_{1/2}y_{1/2} = a = a_1 + a_{1/2} + a_0$ to obtain $\frac{1}{2}(x_{1/2}, e, y_{1/2}) + (e, e, a) + \frac{1}{2}(y_{1/2}, e, x_{1/2}) =$

$(e, e, a) = 0$. Hence $ea - e(ea) = a_1 + \frac{1}{2}a_{1/2} - e(a_1 + \frac{1}{2}a_{1/2}) = a_1 + \frac{1}{2}a_{1/2} - a_1 - \frac{1}{4}a_{1/2} = \frac{1}{4}a_{1/2} = 0$. Hence $x_{1/2}y_{1/2} = a_1 + a_0 \in \mathfrak{J}_1 + \mathfrak{J}_0$, establishing (17).

Now

$$\text{(19)} \qquad \operatorname{trace} R_b = 0 \qquad \text{for all } b \in \mathfrak{J}_{1/2}.$$

For b in $\mathfrak{J}_{1/2}$ implies $\operatorname{trace} R_b = 2 \operatorname{trace} R_{eb} = 2(e, b) = 2(e^2, b) = 2(e, eb) = (e, b)$ by (16) and (13), so $\operatorname{trace} R_b = (e, b) = 0$. Writing $x = x_1 + x_{1/2} + x_0$, $y = y_1 + y_{1/2} + y_0$ in accordance with (10), we have $xy = (x_1y_1 + x_{1/2}y_{1/2} + x_0y_0) + (x_1y_{1/2} + x_{1/2}y_1 + x_{1/2}y_0 + x_0y_{1/2})$ with the last term in parentheses in $\mathfrak{J}_{1/2}$ by (17). Hence (19) implies that

$$\text{(20)} \qquad (x, y) = \operatorname{trace} R_{x_1y_1 + x_{1/2}y_{1/2} + x_0y_0}.$$

Now $x_{1/2}y_{1/2} = c = c_1 + c_0$ (c_i in $\mathfrak{J}_i$) implies $\operatorname{trace} R_{c_1} + \operatorname{trace} R_{c_0} = \operatorname{trace} R_c = (x_{1/2}, y_{1/2}) = 2(ex_{1/2}, y_{1/2}) = 2(e, x_{1/2}y_{1/2}) = 2 \operatorname{trace} R_{e(c_1+c_0)} = 2 \operatorname{trace} R_{c_1}$, so that $\operatorname{trace} R_{c_1} = \operatorname{trace} R_{c_0}$. Then (20) may be written as

$$\text{(20}'\text{)} \qquad (x, y) = \operatorname{trace} R_{x_1y_1 + z_0}, \qquad z_0 = 2c_0 + x_0y_0 \text{ in } \mathfrak{J}_0.$$

In any algebra $\mathfrak{A}$ over F an idempotent e is called *principal* in case there is no idempotent u in $\mathfrak{A}$ which is orthogonal to e ($u^2 = u \neq 0$, $ue = eu = 0$); that is, there is no idempotent u in the subspace $\mathfrak{A}_0 = \{x_0 \mid x_0 \in \mathfrak{A},\ ex_0 = x_0e = 0\}$. In a finite-dimensional Jordan algebra $\mathfrak{J}$, this means that e is a principal idempotent of $\mathfrak{J}$ if and only if the subalgebra $\mathfrak{J}_0$ (in the Peirce decomposition (10) relative to e) is a nilalgebra.

Now any finite-dimensional Jordan algebra $\mathfrak{J}$ which is not a nilalgebra contains a principal idempotent. For $\mathfrak{J}$ contains an idempotent e by Proposition 3. If e is not principal, there is an idempotent u in $\mathfrak{J}_0$, $e' = e + u$ is idempotent, and $\mathfrak{J}_{1,e'}$ (the $\mathfrak{J}_1$ relative to e') contains properly $\mathfrak{J}_{1,e} = \mathfrak{J}_1$. For x_1 in $\mathfrak{J}_{1,e}$ implies $x_1e' = x_1(e + u) = x_1e + x_1u = x_1$, or x_1 is in $\mathfrak{J}_{1,e'}$. That is, $\mathfrak{J}_{1,e} \subseteq \mathfrak{J}_{1,e'}$. But $u \in \mathfrak{J}_{1,e'}$, $u \notin \mathfrak{J}_{1,e}$. Then $\dim \mathfrak{J}_{1,e} < \dim \mathfrak{J}_{1,e'}$, and this process of increasing dimensions must terminate, yielding a principal idempotent.

Theorem 8. Any semisimple (hence any simple) Jordan algebra $\mathfrak{J}$ of finite dimension over F of characteristic 0 has a unity element 1.

Proof: $\mathfrak{J}$ has a principal idempotent e. Then $\mathfrak{J}_0$ is a nilalgebra, so that $(x, y) = \operatorname{trace} R_{x_1 y_1}$ by (20′) since $\operatorname{trace} R_{z_0} = 0$ by (8). Hence x in $\mathfrak{J}_{1/2} + \mathfrak{J}_0$ implies $x_1 = 0$, $(x, y) = 0$ for all y in $\mathfrak{J}$, so x is in $\mathfrak{J}^{\perp}$. That is, $\mathfrak{J}_{1/2} + \mathfrak{J}_0 \subseteq \mathfrak{J}^{\perp} = \mathfrak{N} = 0$, or $\mathfrak{J} = \mathfrak{J}_1$, $e = 1$.

If $\mathfrak{J}$ contains 1 and $e_1 \neq 1$, then $e_2 = 1 - e_1$, is an idempotent, and the Peirce decompositions relative to e_1 and e_2 coincide (with differing subscripts). We introduce a new notation: $\mathfrak{J}_{11} = \mathfrak{J}_{1,e_1}$ $(= \mathfrak{J}_{0,e_2})$, $\mathfrak{J}_{12} = \mathfrak{J}_{1/2,e_1}$ $(= \mathfrak{J}_{1/2,e_2})$, $\mathfrak{J}_{22} = \mathfrak{J}_{0,e_1}$ $(= \mathfrak{J}_{1,e_2})$. More generally, if $1 = e_1 + e_2 + \cdots + e_t$ for pairwise orthogonal idempotents e_i, we have the refined Peirce decomposition

$$\text{(21)} \qquad \mathfrak{J} = \sum_{i \leq j} \mathfrak{J}_{ij}$$

of $\mathfrak{J}$ as the vector space direct sum of subspaces $\mathfrak{J}_{ii} = \mathfrak{J}_{1,e_i}$ $(1 \leq i \leq t)$, $\mathfrak{J}_{ij} = \mathfrak{J}_{1/2,e_i} \cap \mathfrak{J}_{1/2,e_j}$ $(1 \leq i < j \leq t)$; that is,

$$\text{(22)} \qquad \begin{aligned} &\mathfrak{J}_{ii} = \{x \mid x \in \mathfrak{J},\ xe_i = x\}, \\ &\mathfrak{J}_{ij} = \mathfrak{J}_{ji} = \{x \mid x \in \mathfrak{J},\ xe_i = \tfrac{1}{2}x = xe_j\}, \quad i \neq j. \end{aligned}$$

Multiplicative relationships among the $\mathfrak{J}_{ij}$ are consequences of (17) and the statement preceding it.

An idempotent e in $\mathfrak{J}$ is called *primitive* in case e is the only idempotent in $\mathfrak{J}_1$ (that is, e cannot be written as the sum $e = u + v$ of orthogonal idempotents), and *absolutely primitive* in case it is primitive in any scalar extension $\mathfrak{J}_K$ of $\mathfrak{J}$. A central simple Jordan algebra $\mathfrak{J}$ is called *reduced* in case $1 = e_1 + \cdots + e_t$ for pairwise orthogonal absolutely primitive idempotents e_i in $\mathfrak{J}$. In this case it can be shown that the subalgebras $\mathfrak{J}_{ii}$ in the Peirce decomposition (22) are 1-dimensional ($\mathfrak{J}_{ii} = Fe_i$) and that the subspaces $\mathfrak{J}_{ij}$ $(i \neq j)$ all have the same dimension. If $\mathfrak{J}$ is a central simple algebra over F, there is a scalar extension $\mathfrak{J}_K$ which is reduced (for example, take K to be the algebraic closure of F), and it can be shown that the number t of pairwise orthogonal absolutely primitive idempotents e_i in $\mathfrak{J}_K$ such that $1 = e_1 + \cdots + e_t$ is unique; t is called the *degree* of $\mathfrak{J}$.

We list without proof all (finite-dimensional) central simple Jordan algebras $\mathfrak{J}$ of degree t over F of characteristic $\neq 2$. Recall from the Introduction that $\mathfrak{J}$ is a *special Jordan algebra* in case $\mathfrak{J}$ is isomorphic to a subalgebra of an algebra $\mathfrak{A}^+$ where $\mathfrak{A}$ is associative and multiplication in $\mathfrak{A}^+$ is defined by

$$x \cdot y = \tfrac{1}{2}(xy + yx). \tag{23}$$

We say that each algebra is of *type* A, B, C, D, or E listed below.

$\mathrm{A_I}$. $\mathfrak{J} \cong \mathfrak{A}^+$ with $\mathfrak{A}$ any central simple associative algebra (necessarily of dimension t^2 over F).

$\mathrm{A_{II}}$. Let $\mathfrak{A}$ be any involutorial simple associative algebra over F, the involution being of the second kind (so that the center $\mathfrak{Z}$ of $\mathfrak{A}$ is a quadratic extension of F and the involution induces a non-trivial automorphism on $\mathfrak{Z}$ (Albert, Structure of Algebras, p. 153)). Then $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{A})$, the t^2-dimensional subalgebra of self-adjoint elements in the $2t^2$-dimensional algebra $\mathfrak{A}^+$. If $\mathfrak{J}$ is of type A_{I} or A_{II}, and if K is the algebraic closure of F, then $\mathfrak{J}_K \cong K_t^+$ where K_t is the algebra of all $t \times t$ matrices with elements in K.

B, C. Let $\mathfrak{A}$ be any involutorial central simple associative algebra over F (so the involution is of the first kind). Then $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{A})$, the subalgebra of self-adjoint elements in $\mathfrak{A}^+$. There are two types (B and C) which may be distinguished by passing to the algebraic closure K of F, so that $\mathfrak{A}_K$ is a total matrix algebra. In case B the (extended) involution on $\mathfrak{A}_K$ is transposition ($a \to a'$) so that $\mathfrak{A}$ has dimension t^2 and $\mathfrak{J}$ has dimension $\frac{1}{2}t(t+1)$ over F. In case C the (extended) involution on $\mathfrak{A}_K$ is $a \to g^{-1}a'g$ where $g = \begin{pmatrix} 0 & 1_t \\ -1_t & 0 \end{pmatrix}$ so that $\mathfrak{A}$ has dimension $4t^2$ and $\mathfrak{J}$ has dimension $2t^2 - t$ over F.

D. Let (x, y) be any nondegenerate symmetric bilinear form on a vector space $\mathfrak{M}$ of dimension $n \geq 2$. Then $\mathfrak{J}$ is the vector space direct sum $\mathfrak{J} = F1 + \mathfrak{M}$, multiplication in the $(n+1)$-dimensional algebra $\mathfrak{J}$ being defined by $xy = (x, y)1$ for all x, y in $\mathfrak{M}$. Here $t = 2$ ($\dim J \geq 3$).

E. The algebra $\mathfrak{C}_3$ of all 3×3 matrices with elements in a Cayley algebra $\mathfrak{C}$ over F has the *standard involution* $x \to \overline{x'}$ (conjugate transpose). The 27-dimensional subspace $\mathfrak{H}(\mathfrak{C}_3)$ of self-adjoint elements

$$\begin{pmatrix} \xi_1 & c & \bar{b} \\ \bar{c} & \xi_2 & a \\ b & \bar{a} & \xi_3 \end{pmatrix}, \qquad \xi_i \text{ in } F,\ a, b, c \text{ in } \mathfrak{C}, \tag{24}$$

is a (central simple) Jordan algebra of degree $t = 3$ under the multiplication (23) where xy is the multiplication in $\mathfrak{C}_3$ (which is not associative). Then $\mathfrak{J}$ is any algebra such that some scalar extension $\mathfrak{J}_K \cong \mathfrak{H}(\mathfrak{C}_3)_K$ $(= \mathfrak{H}((\mathfrak{C}_K)_3))$.

The possibility of additional algebras of degree 1, mentioned in the 1955 Bulletin article, has been eliminated in reference [32] of the bibliography of more recent papers. The proof involves use of a two-variable identity which is easily seen to be true for special Jordan algebras. But any such identity is then true for arbitrary Jordan algebras since it has been proved that the free Jordan algebra with two generators is special [71; 38], a result which is false for three generators [9]. The identity in question is

$$(25) \qquad \{aba\}^2 = \left\{a\{ba^2b\}a\right\} \qquad \text{for all } a, b \text{ in } \mathfrak{J},$$

where $\{abc\}$ is defined in a Jordan algebra $\mathfrak{J}$ by $\{abc\} = (ab)c + (bc)a - (ac)b$, so that $\{aba\} = b(2{R_a}^2 - R_{a^2})$. Hence in $\mathfrak{A}^+$ ($\mathfrak{A}$ associative) we have $\{aba\} = 2(b \cdot a) \cdot a - b \cdot a^2 = aba$, so that $\{aba\}^2 = aba^2ba = \{a\{ba^2b\}a\}$. Then (25) is satisfied in any special Jordan algebra (in particular, the free Jordan algebra with two generators) and thus in any Jordan algebra.

Therefore all (finite-dimensional) separable Jordan algebras are known, and the Wedderburn decomposition theorem stated in the 1955 Bulletin article is valid without restriction. Some of the computations employed in the original proof may be eliminated [79].

A central simple Jordan algebra of degree 2 (that is, of type D) is a commutative quadratic algebra with 1 ($a^2 - t(a)a + n(a)1 = 0$) having nondegenerate norm form $n(a)$, and conversely. For $a = \alpha 1 + x$, $x \in \mathfrak{M}$, implies $a^2 - t(a)a + n(a)1 = 0$ where $t(a) = 2\alpha$, $n(a) = \alpha^2 - (x, x)$, and $n(a)$ is nondegenerate if and only if (x, y) is.

The algebras of types A, B, C are special Jordan algebras by definition. An algebra of type D is a subalgebra of $\mathfrak{A}^+$, where $\mathfrak{A}$ is the (associative) Clifford algebra of (x, y) (Artin, *Geometric Algebra*, p. 186). But algebras of type E are not special (as we show below), and are therefore called *exceptional* central simple Jordan algebras. Exceptional Jordan division algebras exist (over suitable fields F; but not, for example, over a finite field or the field of all real numbers) [2]. If an exceptional central simple Jordan algebra $\mathfrak{J}$ is not a division algebra, then it is reduced, and $\mathfrak{J}$ is isomorphic to an algebra $\mathfrak{H}(\mathfrak{C}_3, \Gamma)$ of self-adjoint elements in $\mathfrak{C}_3$ under a *canonical involution* $x \to \Gamma^{-1}\overline{x'}\Gamma$ where $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_i \neq 0$ in F. Isomorphism of reduced exceptional simple Jordan algebras is studied in [8].

The unifying feature in the list of central simple Jordan algebras above is that, for $t > 2$, a reduced central simple Jordan algebra is isomorphic to the algebra $\mathfrak{H}(\mathfrak{D}_t, \Gamma)$ defined as follows: $\mathfrak{D}$ is an alternative algebra (of dimension 1, 2, 4, or 8) with unity element u and involution $d \to \overline{d}$ satisfying $d + \overline{d} \in Fu$, $d\overline{d} = n(d)u$, $n(d)$ nondegenerate on $\mathfrak{D}$; $\mathfrak{D}_t$ is the algebra of all $t \times t$ matrices with elements in $\mathfrak{D}$; $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$, $\gamma_i \neq 0$ in F. Then $x \to \Gamma^{-1}\overline{x'}\Gamma$ is a canonical involution in $\mathfrak{D}_t$, and the set $\mathfrak{H}(\mathfrak{D}_t, \Gamma)$ of all self-adjoint elements in $\mathfrak{D}_t$ is a subalgebra of $\mathfrak{D}_t^+$ (that is, we do not need $\mathfrak{A}$ associative to define $\mathfrak{A}^+$ by (23)). If $\mathfrak{D}$ is associative, then $\mathfrak{D}_t = \mathfrak{D} \otimes_F F_t$ is associative, and $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{D}_t, \Gamma)$ is a special Jordan algebra. If $\mathfrak{D}$ is not associative, then $\mathfrak{J} \cong \mathfrak{H}(\mathfrak{D}_t, \Gamma)$ is not a Jordan algebra unless $t = 3$. Hence we have $\mathfrak{J}$ of type B if $\mathfrak{D} = F1$; $\mathfrak{J}$ of type A if $\mathfrak{D} = \mathfrak{Z}$ (type $\mathrm{A_I}$ if $\mathfrak{Z} = F \oplus F$; type $\mathrm{A_{II}}$ if $\mathfrak{Z}$ is a quadratic field over F); $\mathfrak{J}$ of type C if $\mathfrak{D} = \mathfrak{Q}$; $\mathfrak{J}$ of type E if $t = 3$ and $\mathfrak{D} = \mathfrak{C}$. The corresponding dimensions for $\mathfrak{J}$ are clearly $t + \frac{1}{2}t(t-1)(\dim \mathfrak{D})$; that is, $\frac{1}{2}t(t+1)$ for type B, t^2 for type A, $2t^2 - t$ for type C, and 27 for type E.

Theorem 9. Any central simple Jordan algebra $\mathfrak{J}$ of type E is exceptional (that is, is not a special Jordan algebra).

Proof: It is sufficient to prove that $\mathfrak{H}(\mathfrak{C}_3)$ is not special. For, if $\mathfrak{J}$ were special, then $\mathfrak{J} \cong \mathfrak{J}' \subseteq \mathfrak{A}^+$ with $\mathfrak{A}$ associative implies $\mathfrak{J}_K = K \otimes \mathfrak{J} \cong K \otimes \mathfrak{J}' \subseteq K \otimes \mathfrak{A}^+ = (K \otimes \mathfrak{A})^+ = \mathfrak{A}_K{}^+$ so that $\mathfrak{H}((\mathfrak{C}_K)_3) \cong \mathfrak{J}_K$ is special, a contradiction.

Suppose that $\mathfrak{H}(\mathfrak{C}_3)$ is special. There is an associative algebra $\mathfrak{A}$ (of possibly infinite dimension over F) such that U is an isomorphism of $\mathfrak{H}(\mathfrak{C}_3)$ into $\mathfrak{A}^+$. For $i = 1, 2, 3$ define elements e_i in $\mathfrak{A}$ and 8-dimensional subspaces

$$\mathfrak{S}_i = \{d_i \mid d \in \mathfrak{C}\}$$

of $\mathfrak{A}$ by

$$xU = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + a_1 + b_2 + c_3 \tag{26}$$

for x in (24); that is, for ξ_i in F and a, b, c in $\mathfrak{C}$. (Note that our notation is such that we will never use e for an element of $\mathfrak{C}$). Then

$$\mathfrak{S} = Fe_1 + Fe_2 + Fe_3 + \mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 = \mathfrak{H}(\mathfrak{C}_3)U \tag{27}$$

is a 27-dimensional subspace of $\mathfrak{A}$. $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}^+$. The mapping $V = U^{-1}$ defined on $\mathfrak{S}$ (not on all of $\mathfrak{A}$) is an isomorphism of $\mathfrak{S}$ onto $\mathfrak{H}(\mathfrak{C}_3)$:

$$(xU \cdot yU)V = x \cdot y \qquad \text{for all } x, y \text{ in } \mathfrak{H}(\mathfrak{C}_3). \tag{28}$$

For our proof we do not need all of the products in $\mathfrak{A}^+$ of elements of $\mathfrak{S}$. However, performing the multiplications in $\mathfrak{H}(\mathfrak{C}_3)$, we see that (28) yields

$$e_i{}^2 = e_i \ (\neq 0), \qquad i = 1, 2, 3; \tag{29}$$

$$e_i \cdot e_j = 0, \qquad i \neq j; \tag{30}$$

$$e_i \cdot a_i = 0, \qquad a \text{ in } \mathfrak{C},\ i = 1, 2, 3; \tag{31}$$

$$e_i \cdot a_j = \tfrac{1}{2} a_j, \qquad a \text{ in } \mathfrak{C},\ i \neq j; \tag{32}$$

$$u_i{}^2 = e_j + e_k, \qquad i, j, k \text{ distinct}, \tag{33}$$

where u is the unity element in $\mathfrak{C}$; and

$$2a_i \cdot b_j = (\bar{b}\,\bar{a})_k, \qquad a, b \text{ in } \mathfrak{C},\ i, j, k \text{ a cyclic permutation of } 1, 2, 3. \tag{34}$$

Now (29) and (30) imply that e_i ($i = 1, 2, 3$) are pairwise orthogonal idempotents. For $\mathfrak{A}$ is associative, so $e_ie_j + e_je_i = 0$ for $i \neq j$ implies $e_i{}^2e_j + e_ie_je_i = 0 = e_ie_je_i + e_je_i{}^2$, or $e_ie_j = e_je_i$; hence $e_ie_j = 0$ for $i \neq j$. By an identical proof it follows from (31) that

$$e_ia_i = a_ie_i = 0, \qquad i = 1, 2, 3. \tag{31$'$}$$

For i, j, k distinct, (32) implies $e_ia_j + a_je_i = a_j = e_ka_j + a_je_k$; then $fa_j + a_jf = 2a_j$ for the idempotent $f = e_i + e_k$. Hence $f^2a_j + fa_jf = 2fa_j$, so $fa_jf = fa_j$ and similarly $fa_jf = a_jf$; that is, $fa_j = a_jf = a_j$:

$$(e_i + e_k)a_j = a_j = a_j(e_i + e_k), \qquad i, j, k \text{ distinct}. \tag{35}$$

Also (32) implies $e_ia_j = a_j - a_je_i$, so $e_ia_je_i = a_je_i - a_je_i{}^2 = 0$:

$$(36) \qquad e_ia_je_i = 0, \qquad\qquad i \neq j.$$

For any a in $\mathfrak{C}$, define

$$(37) \qquad a' = e_1a_3u_3 \qquad\qquad \text{in } \mathfrak{A}.$$

Then $(ab)' = e_1(ab)_3u_3 = e_1(\overline{b_1}\,\overline{a_2} + \overline{a_2}\,\overline{b_1})u_3 = e_1\overline{a_2}\,\overline{b_1}u_3$ by (34) and (31′). Also (34) implies $a_3u_1 + u_1a_3 = (\overline{u}\,\overline{a})_2 = \overline{a_2}$ and

$$(38) \qquad u_2b_3 + b_3u_2 = \overline{b_1}.$$

Hence $(ab)' = e_1(a_3u_1 + u_1a_3)(u_2b_3 + b_3u_2)u_3 = e_1a_3u_1(u_2b_3 + b_3u_2)u_3$ by (31′). Now $b_3u_2u_3 = b_3u_2(e_1 + e_3)u_3 = b_3u_2e_1u_3 = (\overline{b_1} - u_2b_3)e_1u_3 = -u_2b_3e_1u_3 = -u_2(e_1 + e_3)b_3e_1u_3 = -u_2e_1b_3e_1u_3 = 0$ by (35), (31′), (38), and (36). Also $u_1u_2b_3 = u_1u_2(e_1 + e_2)b_3 = u_1u_2e_1b_3 = (u_3 - u_2u_1)e_1b_3 = u_3e_1b_3$ by (35), (31′), (34). Hence $(ab)' = e_1a_3u_1u_2b_3u_3 = e_1a_3u_3e_1b_3u_3 = a'b'$.

Clearly the mapping $a \to a'$ is linear; hence it is a homomorphism of $\mathfrak{C}$ onto the subalgebra $\mathfrak{C}'$ of $\mathfrak{A}$ consisting of all a'. Since $\mathfrak{C}$ is simple, the kernel of this homomorphism is either 0 or $\mathfrak{C}$; in the latter case $0 = u' = e_1u_3{}^2 = e_1(e_1 + e_2) = e_1 \neq 0$ by (33), and we have a contradiction. Hence $a \to a'$ is an isomorphism. But $\mathfrak{C}'$ is associative, whereas $\mathfrak{C}$ is not. Hence $\mathfrak{H}(\mathfrak{C}_3)$ is an exceptional Jordan algebra.

Any central simple exceptional Jordan algebra $\mathfrak{J}$ over F is a *cubic algebra*: for any x in $\mathfrak{J}$,

$$(39) \qquad x^3 - T(x)x^2 + Q(x)x - N(x)1 = 0, \qquad T(x),\ Q(x),\ N(x) \text{ in } F.$$

Here $x^2x = xx^2$ $(= x^3)$ since $\mathfrak{J}$ is commutative. It is sufficient to show (39) for $\mathfrak{H}(\mathfrak{C}_3)$. But x in (24) implies (39) where

$$(40) \qquad T(x) = \xi_1 + \xi_2 + \xi_3,$$

$$(41) \qquad Q(x) = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 - n(a) - n(b) - n(c) = \tfrac{1}{2}\left[(T(x))^2 - T(x^2)\right],$$

$$(42) \qquad N(x) = \xi_1\xi_2\xi_3 - \xi_1n(a) - \xi_2n(b) - \xi_3n(c) + t(abc);$$

if F has characteristic $\neq 3$ (as well as $\neq 2$), formula (42) may be written as

$$(42') \qquad N(x) = \tfrac{1}{6}\left[(T(x))^3 - 3T(x)T(x^2) + 2T(x^3)\right].$$

The cubic norm form (42) satisfies $N(\{xyx\}) = [N(x)]^2 N(y)$; that is, $N(x)$ permits a new type of composition [35; 69].

In the 1955 Bulletin article, only passing mention is made in §7 of the relationships between exceptional central simple Jordan algebras $\mathfrak{J}$ and exceptional simple Lie algebras and groups; relationships which stem from the fact that the derivation algebra $\mathfrak{D}(\mathfrak{J})$ is a central simple Lie algebra of dimension 52, called an *exceptional Lie algebra of type* F (corresponding to the 52-parameter complex exceptional Lie group F_4)—a proof of this for F of characteristic $\neq 2$ appears in [36] (characteristic $\neq 2,\ 3$ in [70]). Since 1955 the relationships, including a characterization of Cayley planes by means of $\mathfrak{H}(\mathfrak{C}_3, \Gamma)$, have been vigorously exploited [18; 19; 20; 35; 36; 37; 39; 70; 76; 78; 81; 82].

V. Power-associative Algebras

We recall that an algebra $\mathfrak{A}$ over F is called *power-associative* in case the subalgebra $F[x]$ generated by any element x of $\mathfrak{A}$ is associative. We have seen that this is equivalent to defining, for any x in $\mathfrak{A}$,

$$x^1 = x, \qquad x^{i+1} = xx^i \qquad \text{for } i = 1, 2, 3, \ldots,$$

and requiring

$$(1) \qquad x^i x^j = x^{i+j} \qquad \text{for } i, j = 1, 2, 3, \ldots$$

All algebras mentioned in the Introduction are power-associative (Lie algebras trivially, since $x^2 = 0$ implies $x^i = 0$ for $i = 2, 3, \ldots$). We shall encounter in V new examples of power-associative algebras.

The most important tool in the study of noncommutative power-associative algebras $\mathfrak{A}$ is the passage to the commutative algebra $\mathfrak{A}^+$. Let F have characteristic $\neq 2$ throughout V; we shall also require that F contains at least four distinct elements. The algebra $\mathfrak{A}^+$ is the same vector space as $\mathfrak{A}$ over F, but multiplication in $\mathfrak{A}^+$ is defined by

$$(2) \qquad x \cdot y = \tfrac{1}{2}(xy + yx) \qquad \text{for } x, y \text{ in } \mathfrak{A},$$

where xy is the (nonassociative) product in $\mathfrak{A}$. If $\mathfrak{A}$ is power-associative, then (as in the Introduction) powers in $\mathfrak{A}$ and $\mathfrak{A}^+$ coincide, and it follows that $\mathfrak{A}^+$ is a commutative power-associative algebra.

Let $\mathfrak{A}$ be power-associative. Then (2) implies

$$(3) \qquad x^2 x = xx^2 \qquad \text{for all } x \text{ in } \mathfrak{A}$$

and

$$(4) \qquad x^2 x^2 = x(xx^2) \qquad \text{for all } x \text{ in } \mathfrak{A}.$$

In terms of associators, we have

$$(3') \qquad (x, x, x) = 0 \qquad \text{for all } x \text{ in } \mathfrak{A}$$

and

$$(4') \qquad (x, x, x^2) = 0 \qquad \text{for all } x \text{ in } \mathfrak{A}.$$

Also (3) may be written in terms of a commutator as

$$(3'') \qquad [x^2, x] = 0 \qquad \text{for all } x \text{ in } \mathfrak{A}.$$

Using the linearization process employed in IV, we obtain from (3″), by way of the intermediate identity

$$(3''') \qquad 2[x \cdot y, x] + [x^2, y] = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{A},$$

the multilinear identity

$$(3\text{m}) \qquad [x \cdot y, z] + [y \cdot z, x] + [z \cdot x, y] = 0 \qquad \text{for all } x, y, z \text{ in } \mathfrak{A}.$$

Similarly, assuming four distinct elements in F, (4) is equivalent to

$$(4'') \qquad 2(x, x, x \cdot y) + (x, y, x^2) + (y, x, x^2) = 0 \quad \text{for all } x, y \text{ in } \mathfrak{A},$$

to

$$(4''') \qquad \begin{aligned} &2(x, y, x \cdot z) + 2(x, z, x \cdot y) + 2(y, x, x \cdot z) + 2(x, x, y \cdot z) \\ &\quad + 2(z, x, x \cdot y) + (y, z, x^2) + (z, y, x^2) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } x, y, z \text{ in } \mathfrak{A}, \end{aligned}$$

and finally to the multilinear identity

$$(4\text{m}) \qquad \begin{aligned} &(x, y, z \cdot w) + (z, y, w \cdot x) + (w, y, x \cdot z) \\ &+(y, x, z \cdot w) + (z, x, w \cdot y) + (w, x, y \cdot z) \\ &+(z, w, x \cdot y) + (x, w, y \cdot z) + (y, w, z \cdot x) \\ &+(w, z, x \cdot y) + (x, z, y \cdot w) + (y, z, w \cdot x) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } x, y, z, w \text{ in } \mathfrak{A}, \end{aligned}$$

where in each row of the formula (4m) we have left one of the four elements x, y, z, w fixed in the middle position of the associator and permuted the remaining three cyclically.

We omit the proof of the fact that, if F has characteristic 0, then identities (3) and (4) are sufficient to insure that an algebra is power-associative; the proof involves inductions employing the multilinear identities (3m) and (4m). We omit similarly a proof of the fact that, if F has characteristic $\neq 2, 3, 5$, the single identity (4) in a commutative algebra implies power-associativity. One consequence of this latter fact is that in a number of proofs concerning power-associative algebras separate consideration has to be given to the characteristic 3 or 5 case by bringing in associativity of fifth or sixth powers and an assumption that F contains at least 6 distinct elements. We shall omit these details, simply by assuming characteristic $\neq 3, 5$ upon occasion.

An algebra $\mathfrak{A}$ over F is called *strictly power-associative* in case every scalar extension $\mathfrak{A}_K$ is power-associative. If $\mathfrak{A}$ is a commutative power-associative algebra over F of characteristic $\neq 2, 3, 5$, then $\mathfrak{A}$ is strictly power-associative. The assumption of strict power-associativity is employed in the noncommutative case, and in the commutative case of characteristic 3 or 5, when one wishes to use the method of extension of the base field.

Let $\mathfrak{A}$ be a finite-dimensional power-associative algebra over F. Just as in the proofs of Propositions 1 and 2, one may argue that $\mathfrak{A}$ has a unique maximal nilideal $\mathfrak{N}$, and that $\mathfrak{A}/\mathfrak{N}$ has maximal nilideal 0. For if $\mathfrak{A}$ is a power-associative algebra which contains a nilideal $\mathfrak{I}$ such that $\mathfrak{A}/\mathfrak{I}$ is a nilalgebra, then $\mathfrak{A}$ is a nilalgebra. [If x is in $\mathfrak{A}$, then $\overline{x^s} = \overline{x}^s = 0$ for some s, so that $x^s = y \in \mathfrak{I}$ and $x^{rs} = (x^s)^r = y^r = 0$ for some r.] Since any homomorphic image of a nilalgebra is a nilalgebra, it follows from the second isomorphism theorem that, if $\mathfrak{B}$ and $\mathfrak{C}$ are nilideals, then so is $\mathfrak{B} + \mathfrak{C}$. For $(\mathfrak{B} + \mathfrak{C})/\mathfrak{C} \cong \mathfrak{B}/(\mathfrak{B} \cap \mathfrak{C})$ is a nilalgebra, so $\mathfrak{B} + \mathfrak{C}$ is. This establishes the uniqueness of $\mathfrak{N}$. It follows as in the proof of Proposition 2 that 0 is the only nilideal of $\mathfrak{A}/\mathfrak{N}$. $\mathfrak{N}$ is called the *nilradical* of $\mathfrak{A}$, and $\mathfrak{A}$ is called *semisimple* in case $\mathfrak{N} = 0$. Of course any anticommutative algebra (for example, any Lie algebra) is a nilalgebra, and hence is its own nilradical. Hence this concept of semisimplicity is trivial for anticommutative algebras.

For the moment let $\mathfrak{A}$ be a commutative power-associative algebra, and let e be an idempotent in $\mathfrak{A}$. Putting $x = e$ in (4″) and using commutativity, we have $y(2{R_e}^3 - 3{R_e}^2 + R_e) = 0$ for all y in $\mathfrak{A}$, or

$$(5) \qquad 2{R_e}^3 - 3{R_e}^2 + R_e = 0$$

for any idempotent e in a commutative power-associative algebra $\mathfrak{A}$. As we have seen in the case of Jordan algebras in IV, this gives a Peirce decomposition

$$(6) \qquad \mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_{1/2} + \mathfrak{A}_0$$

of $\mathfrak{A}$ as a vector space direct sum of subspaces $\mathfrak{A}_i$ defined by

$$(7) \qquad \mathfrak{A}_i = \{x_i \mid x_i e = i x_i\}, \qquad i = 1, 1/2, 0;\ \mathfrak{A} \text{ commutative.}$$

Now if $\mathfrak{A}$ is any power-associative algebra, the algebra $\mathfrak{A}^+$ is a commutative power-associative algebra. Hence we have the Peirce decomposition (6) where

$$\mathfrak{A}_i = \{x_i \mid ex_i + x_i e = 2ix_i\}, \qquad i = 1, 1/2, 0. \tag{7'}$$

Put $y = z = e$ in (3m) and let $x = x_i \in \mathfrak{A}_i$ as in (7′) to obtain $(2i-1)[x_i, e] = 0$; that is, $x_i e = ex_i$ if $i \neq 1/2$. Hence (7′) becomes

$$\begin{aligned} \mathfrak{A}_i &= \{x_i \mid ex_i = x_i e = ix_i\}, \qquad i = 1, 0; \\ \mathfrak{A}_{1/2} &= \{x_{1/2} \mid ex_{1/2} + x_{1/2}e = x_{1/2}\} \end{aligned} \tag{7''}$$

in the Peirce decomposition (6) of any power-associative algebra $\mathfrak{A}$. As we have just seen, the properties of commutative power-associative algebras may be used (via $\mathfrak{A}^+$) to obtain properties of arbitrary power-associative algebras.

Let $\mathfrak{A}$ be a commutative power-associative algebra with Peirce decomposition (6), (7) relative to an idempotent e. Then $\mathfrak{A}_1$ and $\mathfrak{A}_0$ are orthogonal subalgebras of $\mathfrak{A}$ which are related to $\mathfrak{A}_{1/2}$ as follows:

$$\begin{aligned} \mathfrak{A}_{1/2}\mathfrak{A}_{1/2} &\subseteq \mathfrak{A}_1 + \mathfrak{A}_0, \\ \mathfrak{A}_1\mathfrak{A}_{1/2} &\subseteq \mathfrak{A}_{1/2} + \mathfrak{A}_0, \\ \mathfrak{A}_0\mathfrak{A}_{1/2} &\subseteq \mathfrak{A}_1 + \mathfrak{A}_{1/2}. \end{aligned} \tag{8}$$

Note that the last two inclusion relations of (8) are weaker than for Jordan algebras in IV. The proofs are similar to those in IV, and are given by putting $x = e$, $y = y_j \in \mathfrak{A}_j$, $z = x_i \in \mathfrak{A}_i$ in (4‴). We omit the details except to note that the characteristic 3 case of orthogonality requires associativity of fifth powers.

For $x \in \mathfrak{A}_1$, $w \in \mathfrak{A}_{1/2}$, we have $wx = (wx)_{1/2} + (wx)_0 \in \mathfrak{A}_{1/2} + \mathfrak{A}_0$ by (8). Then $w \to (wx)_{1/2}$ is a linear operator on $\mathfrak{A}_{1/2}$ which we denote by S_x:

$$wS_x = (wx)_{1/2} \qquad \text{for } x \in \mathfrak{A}_1,\ w \in \mathfrak{A}_{1/2}. \tag{9}$$

If $\mathfrak{H}$ is the (associative) algebra of all linear operators on $\mathfrak{A}_{1/2}$, then $x \to 2S_x$ is a homomorphism of $\mathfrak{A}_1$ into the special Jordan algebra $\mathfrak{H}^+$, for $x \to S_x$ is clearly linear and we verify

$$S_{xy} = S_xS_y + S_yS_x \qquad \text{for all } x, y \text{ in } \mathfrak{A}_1 \tag{10}$$

as follows: put $x \in \mathfrak{A}_1$, $y \in \mathfrak{A}_1$, $z = e$, $w \in \mathfrak{A}_{1/2}$ in (4m) to obtain

$$(11) \quad e\,[y(wx) + x(wy) + w(xy)] + w(xy) - 2x(yw) - 2y(xw) = 0,$$

since $e(yw) = \frac{1}{2}(yw)_{1/2}$ implies $x\,[e(yw)] = \frac{1}{2}x(yw)_{1/2} = \frac{1}{2}x(yw)$ and $y\,[e(xw)] = \frac{1}{2}y(xw)$ by interchange of x and y. By taking the $\mathfrak{A}_{1/2}$ component in (11), we have (10) after dividing by 3. Similarly, defining the linear operator T_z on $\mathfrak{A}_{1/2}$ for any z in $\mathfrak{A}_0$ by

$$(12) \qquad wT_z = (wz)_{1/2} \qquad \text{for } z \in \mathfrak{A}_0,\ w \in \mathfrak{A}_{1/2},$$

we have

$$(13) \qquad T_{zy} = T_zT_y + T_yT_z \qquad \text{for all } z, y \text{ in } \mathfrak{A}_0,$$

and

$$(14) \qquad S_xT_z = T_zS_x \qquad \text{for all } x \text{ in } \mathfrak{A}_1,\ z \text{ in } \mathfrak{A}_0.$$

This is part of the basic machinery used in developing the structure theory for commutative power-associative algebras as reported in the 1955 Bulletin article. The result that simple algebras (actually rings) of degree greater than 2 are Jordan has been extended by the same technique to flexible power-associative rings (the conclusion being that $\mathfrak{A}^+$ is Jordan) [58]. All semisimple commutative power-associative algebras of characteristic 0 are Jordan algebras [51]. The determination of all simple commutative power-associative algebras of degree 2 and characteristic $p > 0$ is still not complete [1; 24].

Here we shall develop only as much of the technique as will be required in the proof of the following generalization of Wedderburn's theorem that every finite associative division ring is a field (Artin, Geometric Algebra, p. 37). In IV it was mentioned that exceptional Jordan division algebras do exist over suitable fields F; however, F cannot be finite in that event and we assume this particular case of the following theorem (as well as Wedderburn's theorem) in the proof of

Theorem 10. Let $\mathfrak{D}$ be a finite power-associative division ring of characteristic $\neq 2, 3, 5$. Then $\mathfrak{D}$ is a field.

For the proof we require an exercise and a lemma.

Exercise. If u and v are orthogonal idempotents in a commutative power-associative algebra $\mathfrak{A}$, then

$$(15) \qquad R_uR_v = R_vR_u.$$

(Hint: put $x = u$, $y = v$ in $(4''')$ and use (5). After considerable manipulation, and use of characteristic $\neq 3, 5$, this (together with what one obtains by interchanging u and v) will yield (15).)

Lemma. Let e be a principal idempotent in a commutative power-associative algebra $\mathfrak{A}$ (that is, $\mathfrak{A}_0$ in (7) is a nilalgebra). Then every element in $\mathfrak{A}_{1/2}$ is nilpotent.

Proof: To obtain $(18')$ below, one does not need to assume that e is principal. For any $w \in \mathfrak{A}_{1/2}$ put $x = w$, $y = e$ in $(4'')$ to obtain

$$(16) \qquad w^3 - w(ew^2) - ew^3 = 0 \qquad \text{for } w \text{ in } \mathfrak{A}_{1/2}$$

Let $x = (w^2)_1 \in \mathfrak{A}_1$, $z = (w^2)_0 \in \mathfrak{A}_0$, so that $w^2 = x + z$, $ew^2 = x$, $w(ew^2) = wx = (wx)_{1/2} + (wx)_0$. Also $w^3 = wx + wz = (wx)_{1/2} + (wx)_0 + (wz)_{1/2} + (wz)_1$ so that $ew^3 = \frac{1}{2}(wx)_{1/2} + \frac{1}{2}(wz)_{1/2} + (wz)_1$. Then (16) implies $(wx)_{1/2} = (wz)_{1/2}$, or

$$(17) \qquad wS_x = wT_z \qquad \text{for any } w \text{ in } \mathfrak{A}_{1/2}$$

where $w^2 = x + z$, $x = (w^2)_1$, $z = (w^2)_0$. Now

$$(18) \qquad w{S_x}^k = w{T_z}^k \qquad \text{for } k = 1, 2, 3, \ldots$$

For (17) is the case $k = 1$ of (18) and, assuming (18), we have $w{S_x}^{k+1} = w{S_x}^k S_x = w{T_z}^k S_x = wS_x{T_z}^k = w{T_z}^{k+1}$ by (14). But (10) and (13) imply $S_{x^k} = 2^{k-1}{S_x}^k$, $T_{z^k} = 2^{k-1}{T_z}^k$, so (18) yields

$$(18') \qquad wS_{x^k} = wT_{z^k} \qquad \text{for } k = 1, 2, 3, \ldots$$

where w is any element of $\mathfrak{A}_{1/2}$ and $w^2 = x + z$, $x \in \mathfrak{A}_1$, $z \in \mathfrak{A}_0$.

Now let e be a principal idempotent in $\mathfrak{A}$. Then every element w in $\mathfrak{A}_{1/2}$ is nilpotent. For $z = (w^2)_0$ is nilpotent, and $z^k = 0$ for some k. By $(18')$ we have $w^{2k+1} = w(w^2)^k = w(x + z)^k = w(x^k + z^k) = wx^k = (wx^k)_{1/2} + (wx^k)_0 = wS_{x^k} + (wx^k)_0 = (wx^k)_0$ in $\mathfrak{A}_0$ is nilpotent; hence w is nilpotent.

We use the Lemma to prove that any finite-dimensional power-associative division algebra $\mathfrak{D}$ over a field F has a unity element 1. For $\mathfrak{D}^+$ is a (finite-dimensional) commutative power-associative algebra without nilpotent elements $\neq 0$, so $\mathfrak{D}^+$ contains a principal idempotent e (as in IV, $\mathfrak{D}^+$ contains an idempotent e; if e is not principal, there is an idempotent $u \in \mathfrak{D}_0^+ = \mathfrak{D}_{0,e}^+$, $e' = e + u$ is idempotent, $\dim \mathfrak{D}_{1,e}^+ < \dim \mathfrak{D}_{1,e'}^+$, and the increasing dimensions must terminate). By the Lemma, since 0 is the only nilpotent element in $\mathfrak{D}^+$, we have $\mathfrak{D}_{1/2}^+ = \mathfrak{D}_0^+ = 0$, $\mathfrak{D}^+ = \mathfrak{D}_1^+$, e is a unity element for $\mathfrak{D}^+$. By (7″) e is a unity element for $\mathfrak{D}$.

Proof of Theorem 10: We are assuming that $\mathfrak{D}$ is a finite power-associative division ring. We have seen in II that this means that $\mathfrak{D}$ is a (finite-dimensional) division algebra over a (finite) field. Hence we have just seen that $\mathfrak{D}$ has a unity element 1, so that $\mathfrak{D}$ is an algebra over its center. Thus we may as well take $\mathfrak{D}$ to be an algebra over its center F, a finite field. Hence F is perfect (Zariski-Samuel, Commutative Algebra, vol. I, p. 65).

Now $\mathfrak{D}^+$ is a Jordan algebra over F. For let x, y be any elements of $\mathfrak{D}^+$. If $x \in F1$, the Jordan identity

$$(x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2) \qquad \text{for all } x, y \text{ in } \mathfrak{D}^+ \tag{19}$$

holds trivially. Otherwise the (commutative associative) subalgebra $F[x]$ of $\mathfrak{D}^+$ is a finite (necessarily separable) extension of F, so there is an extension K of F such that $F[x]_K = K \oplus K \oplus \cdots \oplus K$, x is a linear combination $x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n$ of pairwise orthogonal idempotents e_i in $F[x]_K \subseteq (\mathfrak{D}^+)_K$ with coefficients in K. In order to establish (19), it is sufficient to establish

$$(e_i \cdot y) \cdot (e_j \cdot e_k) = e_i \cdot [y \cdot (e_j \cdot e_k)], \qquad i, j, k = 1, \ldots, n. \tag{19′}$$

For $j \neq k$, (19′) is obvious; for $j = k$, (19′) reduces to (15).

Now the radical of $\mathfrak{D}^+$ (consisting of nilpotent elements) is 0. Although our proof of the Corollary to Theorem 7 is valid only for characteristic 0, we remarked in IV that the conclusion is valid for characteristic $\neq 0$. Hence $\mathfrak{D}^+$ is a direct sum $\mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_r$ of r simple ideals $\mathfrak{S}_i$, each with unity element e_i. The existence of an idempotent $e \neq 1$ in $\mathfrak{D}^+$ is sufficient to give zero divisors in $\mathfrak{D}$, a contradiction, since the product $e(1 - e) = 0$ in $\mathfrak{D}$. Hence $r = 1$ and $\mathfrak{D}^+$ is a simple Jordan algebra over F. Let C be the center of $\mathfrak{D}^+$. Then C is a finite separable extension of F, $C = F[z]$, $z \in C$ (Zariski-Samuel, ibid, p. 84). If $\mathfrak{D}^+ = C = F[z]$, then $\mathfrak{D} = F[z]$ is a field, and the theorem is established. Hence we may assume that $\mathfrak{D}^+ \neq C$, so $\mathfrak{D}^+$ is a central simple Jordan algebra of degree $t \geq 2$ over the finite field C and is of one of the types A–E listed in IV. We are assuming that type E is known not to occur. The other types are eliminated as follows.

Wedderburn's theorem implies that, over any finite field, there are no associative central division algebras of dimension > 1. Hence, by Wedderburn's theorem on simple associative algebras, every associative central simple algebra over a finite field is a total matric algebra. Thus we have the following possibilities:

A_{I}. $\mathfrak{D}^+ \cong {C_t}^+$, $t \geq 2$. Then ${C_t}^+$ contains an idempotent $e_{11} \neq 1$, a contradiction.

A_{II}. $\mathfrak{D}^+$ is the set $\mathfrak{H}(\mathfrak{Z}_t)$ of self-adjoint elements in $\mathfrak{Z}_t$, $\mathfrak{Z}$ a quadratic extension of C, where the involution may be taken to be $a \to g^{-1}\overline{a'}g$ with g a diagonal matrix. Hence $\mathfrak{H}(\mathfrak{Z}_t)$ contains $e_{11} \neq 1$, a contradiction.

B. $\mathfrak{D}^+ \cong \mathfrak{H}(C_t)$, the involution being $a \to g^{-1}a'g$ with g diagonal; hence $\mathfrak{H}(C_t)$ contains $e_{11} \neq 1$, a contradiction.

C. $\mathfrak{D}^+ \cong \mathfrak{H}(C_{2t})$, the involution being $a \to g^{-1}a'g$, $g = \begin{pmatrix} 0 & 1_t \\ -1_t & 0 \end{pmatrix}$; $\mathfrak{H}(C_{2t})$ contains the idempotent $e_{11} + e_{t+1,t+1} \neq 1$, a contradiction.

There remains the possibility that $\mathfrak{D}^+$ might be of type D (where the dimension is necessarily ≥ 3). The basis $u_1, \ldots, u_n$ for $\mathfrak{M}$ may be normalized so that $(u_i, u_j) = 0$ for $i \neq j$, $(u_i, u_i) = \alpha_i \neq 0$ in C; that is, ${u_i}^2 = \alpha_i 1$, $u_i \cdot u_j = 0$ for $i \neq j$. Each of the fields $C[u_i]$ is a quadratic extension of C. But in the sense of isomorphism there is only one quadratic extension of the finite field C (Zariski-Samuel, ibid, pp. 73, 83); hence all α_i may be taken to be the same nonsquare α in C. Let β be any element of C. Then $w = \beta u_1 + u_2 \notin F1$ implies $F[w]$ is isomorphic to $F[u_1]$, so $w^2 = (\beta^2 + 1)\alpha 1 = \gamma^2 \alpha 1$, γ in C; that is, for any $\beta \in C$, there is $\gamma \in C$ satisfying

$$\gamma^2 = \beta^2 + 1. \tag{20}$$

Now let P be the prime field of characteristic p contained in C. It follows from (20) that all elements in P are squares of elements in C. For 1 (also 0) satisfies this condition, and it follows by induction from (20) that all elements in P do. In particular, $-1 = \beta^2$ for some $\beta \in C$. Then $w^2 = (\beta u_1 + u_2)^2 = 0$, a contradiction.

Theorem 11. Let $\mathfrak{A}$ be a finite-dimensional power-associative algebra over F satisfying the following conditions:

(i)	there is an (associative) trace form (x, y) defined on $\mathfrak{A}$;	
(ii)	$(e, e) \neq 0$	for any idempotent e in $\mathfrak{A}$;
(iii)	$(x, y) = 0$	if $x \cdot y$ is nilpotent, x, y in $\mathfrak{A}$.

Then the nilradical $\mathfrak{N}$ of $\mathfrak{A}$ coincides with the nilradical of $\mathfrak{A}^+$, and is the radical $\mathfrak{A}^\perp$ of the trace form (x, y). The semisimple power-associative algebra $\mathfrak{S} = \mathfrak{A}/\mathfrak{N}$ satisfies (i)–(iii) with (x, y) nondegenerate. For any such $\mathfrak{S}$ we have

(a) $\mathfrak{S} = \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_t$ for simple $\mathfrak{S}_i$;

(b) $\mathfrak{S}$ is flexible.

If F has characteristic $\neq 5$, then

(c) $\mathfrak{S}^+$ is a semisimple Jordan algebra;

(d) $\mathfrak{S}_i^+$ is a simple (Jordan) algebra, $i = 1, \ldots, t$.

Proof: By (i) we know from IV that $\mathfrak{A}^\perp$ is an ideal of $\mathfrak{A}$. If there were an idempotent e in $\mathfrak{A}^\perp$, then (ii) would imply $(e, e) \neq 0$, a contradiction. Hence $\mathfrak{A}^\perp$ is a nilideal: $\mathfrak{A}^\perp \subseteq \mathfrak{N}$. Conversely, x in $\mathfrak{N}$ implies $x \cdot y$ is in $\mathfrak{N}$ for all y in $\mathfrak{A}$, so that $(x, y) = 0$ for all y in $\mathfrak{N}$ by (iii), or x is in $\mathfrak{A}^\perp$. Hence $\mathfrak{N} \subseteq \mathfrak{A}^\perp$, $\mathfrak{N} = \mathfrak{A}^\perp$. Any ideal of $\mathfrak{A}$ is clearly an ideal of $\mathfrak{A}^+$; hence any nilideal of $\mathfrak{A}$ is a nilideal of $\mathfrak{A}^+$, and $\mathfrak{N}$ is contained in the nilradical $\mathfrak{N}_1$ of $\mathfrak{A}^+$. But x in $\mathfrak{N}_1$ implies $x \cdot y$ is in $\mathfrak{N}_1$ for all y in $\mathfrak{A}^+$, or $(x, y) = 0$ by (iii) and we have $\mathfrak{N}_1 \subseteq \mathfrak{A}^\perp = \mathfrak{N}$.

Now (x, y) induces a nondegenerate symmetric bilinear form $(\overline{x}, \overline{y})$ on $\mathfrak{A}/\mathfrak{A}^\perp = \mathfrak{A}/\mathfrak{N}$ where $\overline{x} = x + \mathfrak{N}$, etc.; that is, $(\overline{x}, \overline{y}) = (x, y)$. Then $(\overline{x}\,\overline{y}, \overline{z}) = (\overline{xy}, \overline{z}) = (xy, z) = (x, yz) = (\overline{x}, \overline{y}\,\overline{z})$, so $(\overline{x}, \overline{y})$ is a trace form. To show (ii) we take any idempotent $\overline{e}$ in $\mathfrak{A}/\mathfrak{N}$ and use the power-associativity of $\mathfrak{A}$ to "lift" the idempotent to $\mathfrak{A}$: $F[e]$ is a subalgebra of $\mathfrak{A}$ which is not a nilalgebra, so there is an idempotent $u \in F[e] \subseteq Fe + \mathfrak{N}$, and $\overline{u} = \overline{e}$. Then $(\overline{e}, \overline{e}) = (\overline{u}, \overline{u}) = (u, u) \neq 0$. Suppose $\overline{x} \cdot \overline{y} = \overline{x \cdot y}$ is nilpotent. Then some power of $x \cdot y$ is in $\mathfrak{N}$, $x \cdot y$ is nilpotent, and $(\overline{x}, \overline{y}) = (x, y) = 0$, establishing (iii).

Now let $\mathfrak{S}$ satisfy (i)–(iii) with (x, y) nondegenerate. Then the nilradical of $\mathfrak{S}$ is 0, and the hypotheses of Theorem 7 apply. For if

$\mathfrak{I}^2 = 0$ for an ideal $\mathfrak{I}$ of $\mathfrak{S}$, then $\mathfrak{I}$ is a nilideal, $\mathfrak{I} = 0$. We have $\mathfrak{S} = \mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_t$ for simple $\mathfrak{S}_i$; also we know that the $\mathfrak{S}_i$ are not nilalgebras (for then they would be nilideals of $\mathfrak{S}$), but this will follow from (d).

Now (3‴) implies that $a \cdot z = 0$ where $a = 2[x \cdot y, x] + [x^2, y]$. Since $a \cdot z$ is nilpotent, (iii) implies $(a, z) = ((xy)x, z) + ((yx)x, z) - (x(xy), z) - (x(yx), z) + (x^2 y, z) - (yx^2, z) = 0$ for all x, y, z in $\mathfrak{S}$. The properties of a trace form imply that

$$(21) \qquad (xy + yx, xz - zx) = (x^2, zy - yz).$$

Interchange z and y to obtain $(xz + zx, xy - yx) = (x^2, yz - zy) = (xy + yx, zx - xz)$ by (21). Add $(xy + yx, xz + zx)$ to both sides of this to obtain $(xy, xz + zx) = (xy + yx, zx)$. Then $(xy, xz) = (yx, zx)$, so that

$$(22) \qquad ((xy)x, z) = (x(yx), z) \qquad \text{for all } x, y, z \text{ in } \mathfrak{S}.$$

Since (x, y) is nondegenerate on $\mathfrak{S}$, (22) implies $(xy)x = x(yx)$; that is, $\mathfrak{S}$ is flexible.

To prove (c) we note first that (x, y) is a trace form on $\mathfrak{S}^+$:

$$(23) \qquad (x \cdot y, z) = (x, y \cdot z) \qquad \text{for all } x, y, z \text{ in } \mathfrak{S}.$$

Also it follows from (23), just as in formula (14) of IV, that

$$(24) \qquad (yS_1 S_2 \cdots S_h, z) = (y, zS_h \cdots S_2 S_1)$$

where S_i are right multiplications of the commutative algebra $\mathfrak{S}^+$. In the commutative power-associative algebra $\mathfrak{S}^+$ formula (4″) becomes

$$(25) \qquad 4x^2 \cdot (x \cdot y) - 2x \cdot [x \cdot (x \cdot y)] - x \cdot (y \cdot x^2) - y \cdot x^3 = 0.$$

Applying the same procedure as above, we write a for the left side of (25), have $a \cdot z = 0$ for all z in $\mathfrak{S}^+$, so (iii) implies $4\,(x^2 \cdot (x \cdot y), z) - 2\,(x \cdot [x \cdot (x \cdot y)]\,, z) - (x \cdot (y \cdot x^2), z) - (y \cdot x^3, z) = 0$ or

$$(26) \quad (y \cdot z, x^3) + 2\,(x \cdot [x \cdot (x \cdot y)], z) = 4(x \cdot y, x^2 \cdot z) - (y \cdot x^2, x \cdot z).$$

By (24) the left-hand side of (26) is unaltered by interchange of y and z. Hence $4(x \cdot y, x^2 \cdot z) - (y \cdot x^2, x \cdot z) = 4(x \cdot z, x^2 \cdot y) - (z \cdot x^2, x \cdot y)$ so that (after dividing by 5) we have $(x \cdot y, x^2 \cdot z) = (y \cdot x^2, x \cdot z)$. Hence $((x \cdot y) \cdot x^2, z) = (x \cdot (y \cdot x^2), z)$ for all z in $\mathfrak{S}$, or $(x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2)$, $\mathfrak{S}^+$ is a Jordan algebra. We know from IV that, since the nilradical of $\mathfrak{S}^+$ is 0, $\mathfrak{S}^+$ is a direct sum of simple ideals, but it is conceivable that these are not the ${\mathfrak{S}_i}^+$ given by (a). To see that the simple components of $\mathfrak{S}^+$ are the ${\mathfrak{S}_i}^+$ given by (a), we need to establish (d).

Let $\mathfrak{T}$ be an ideal of ${\mathfrak{S}_i}^+$; we need to show that $\mathfrak{T}$ is an ideal of $\mathfrak{S}_i$. It follows from (a) that $\mathfrak{T}$ is an ideal of $\mathfrak{S}^+$, and is therefore by (c) a direct sum of simple ideals of $\mathfrak{S}^+$ each of which has a unity element. The sum of these pairwise orthogonal idempotents in $\mathfrak{S}^+$ is the unity element e of $\mathfrak{T}$. Now e is an idempotent in $\mathfrak{S}^+$ (and $\mathfrak{S}$), and the Peirce decomposition ($7''$) characterizes $\mathfrak{T}$ as

$$(27) \qquad \mathfrak{T} = \mathfrak{S}_{1,e} = \{t \in \mathfrak{S} \mid et = te = t\}.$$

Let s be any element of $\mathfrak{S}$. Then flexibility implies $(s, t, e) + (e, t, s) = 0$, or

$$(28) \qquad (st)e - st + ts = e(ts) \qquad \text{for all } t \in \mathfrak{T},\ s \in \mathfrak{S}.$$

But $\mathfrak{T}$ an ideal of $\mathfrak{S}^+$ implies that $s \cdot t \in \mathfrak{T}$, so that $st + ts = e(st + ts) = e(st) + (st)e - st + ts$ by (27) and (28); that is, $e(st) + (st)e = 2st$, and st is in $\mathfrak{T} = \mathfrak{S}_{1,e}$ by ($7'$). But then $s \cdot t$ in $\mathfrak{T}$ implies ts is in $\mathfrak{T}$ also; $\mathfrak{T}$ is an ideal of $\mathfrak{S}$. Then $\mathfrak{T} \subseteq \mathfrak{S}_i$ is an ideal of $\mathfrak{S}_i$. Hence the only ideals of ${\mathfrak{S}_i}^+$ are 0 and ${\mathfrak{S}_i}^+$. Since ${\mathfrak{S}_i}^+$ cannot be a zero algebra, ${\mathfrak{S}_i}^+$ is simple.

We list without proof the central simple flexible algebras $\mathfrak{A}$ over F which are such that $\mathfrak{A}^+$ is a (central) simple Jordan algebra. These are the algebras which (over their centers) can appear as the simple components $\mathfrak{S}_i$ in (a) above:

1. $\mathfrak{A}$ is a central simple (commutative) Jordan algebra.

2. $\mathfrak{A}$ is a *quasiassociative* central simple algebra. That is, there is a scalar extension $\mathfrak{A}_K$, K a quadratic extension of F, such that $\mathfrak{A}_K$ is isomorphic to an algebra $\mathfrak{B}(\lambda)$ defined as follows: $\mathfrak{B}$ is a central simple associative algebra over K, $\lambda \neq \frac{1}{2}$ is a fixed element of K, and $\mathfrak{B}(\lambda)$ is the same vector space over K as $\mathfrak{B}$ but multiplication in $\mathfrak{B}(\lambda)$ is defined by

$$(29) \qquad x * y = \lambda xy + (1 - \lambda) yx \qquad \text{for all } x, y \text{ in } \mathfrak{B}$$

where xy is the (associative) product in $\mathfrak{B}$.

3. $\mathfrak{A}$ is a flexible quadratic algebra over F with nondegenerate norm form.

Note that, except for Lie algebras, all of the central simple algebras which we have mentioned in these notes are listed here (associative algebras are the case $\lambda = 1$ (also $\lambda = 0$) in 2; Cayley algebras are included in 3).

We should remark that, if an algebra $\mathfrak{A}$ contains 1, any trace form (x, y) on $\mathfrak{A}$ may be expressed in terms of a linear form $T(x)$. That is, we write

$$T(x) = (1, x) \qquad \text{for all } x \text{ in } \mathfrak{A}, \tag{30}$$

and have

$$(x, y) = T(xy) \qquad \text{for all } x, y \text{ in } \mathfrak{A} \tag{31}$$

since $(x1, y) = (1, xy)$. The symmetry and the associativity of the trace form (x, y) are equivalent to the vanishing of $T(x)$ on commutators and associators:

$$\begin{aligned} T(xy) &= T(yx), \\ T((xy)z) &= T(x(yz)) \end{aligned} \qquad \text{for all } x, y, z \text{ in } \mathfrak{A}. \tag{32}$$

If $\mathfrak{A}$ is power-associative, hypotheses (ii) and (iii) of Theorem 11 become

$$T(e) \neq 0 \qquad \text{for any idempotent } e \text{ in } \mathfrak{A}, \tag{33}$$

and

$$T(z) = 0 \qquad \text{for any nilpotent } z \text{ in } \mathfrak{A}, \tag{34}$$

the latter being evident as follows: (34) implies that, if $x \cdot y$ is nilpotent, then $0 = T(x \cdot y) = (1, x \cdot y) = \frac{1}{2}(1, xy) + \frac{1}{2}(1, yx) = \frac{1}{2}(x, y) + \frac{1}{2}(y, x) = (x, y)$ and, conversely, if $z = 1 \cdot z$ is nilpotent, then (iii) implies $T(z) = (1, z) = 0$.

A natural generalization to noncommutative algebras of the class of (commutative) Jordan algebras is the class of algebras $\mathfrak{J}$ satisfying the Jordan identity

$$(xy)x^2 = x(yx^2) \qquad \text{for all } x, y \text{ in } \mathfrak{J}. \tag{35}$$

As in IV, we can linearize (35) to obtain

$$(35') \qquad (x, y, w \cdot z) + (w, y, z \cdot x) + (z, y, x \cdot w) = 0 \qquad \text{for all } w, x, y, z \text{ in } \mathfrak{J}.$$

If $\mathfrak{J}$ contains 1, then $w = 1$ in (35′) implies

$$(36) \qquad (x, y, z) + (z, y, x) = 0 \qquad \text{for all } x, y, z \text{ in } \mathfrak{J};$$

that is, $\mathfrak{J}$ is *flexible*:

$$(37) \qquad (xy)x = x(yx) \qquad \text{for all } x, y \text{ in } \mathfrak{J}.$$

If a unity element 1 is adjoined to $\mathfrak{J}$ as in II, then a necessary and sufficient condition that (35′) be satisfied in the algebra with 1 adjoined is that both (35′) and (36) be satisfied in $\mathfrak{J}$. Hence we define a *noncommutative Jordan algebra* to be an algebra satisfying both (35) and (37).

Exercise. Prove: A flexible algebra $\mathfrak{J}$ is a noncommutative Jordan algebra if and only if any one of the following is satisfied:

$$(38) \qquad (x^2y)x = x^2(yx) \qquad \text{for all } x, y \text{ in } \mathfrak{J};$$

$$(39) \qquad x^2(xy) = x(x^2y) \qquad \text{for all } x, y \text{ in } \mathfrak{J};$$

$$(40) \qquad (yx)x^2 = (yx^2)x \qquad \text{for all } x, y \text{ in } \mathfrak{J};$$

(41) $\qquad \mathfrak{J}^+$ is a (commutative) Jordan algebra.

We see from (41) that any semisimple algebra (of characteristic $\neq 5$) satisfying the hypotheses of Theorem 11 is a noncommutative Jordan algebra.

Since (35′) and (36) are multilinear, any scalar extension $\mathfrak{A}_K$ of a noncommutative Jordan algebra is a noncommutative Jordan algebra. It may be verified directly that any noncommutative Jordan algebra is power-associative (hence strictly power-associative).

Let $\mathfrak{J}$ be any noncommutative Jordan algebra. By (41) $\mathfrak{J}^+$ is a (commutative) Jordan algebra, and we have seen in IV that a trace form on $\mathfrak{J}^+$ may be given in terms of right multiplications of $\mathfrak{J}^+$. Our application of this to the situation here works more smoothly if there is a unity element 1 in $\mathfrak{J}$, so (if necessary) we adjoin one to $\mathfrak{J}$ to obtain a noncommutative Jordan algebra $\mathfrak{J}_1$ with 1 and having $\mathfrak{J}$ as a subalgebra (actually ideal). Then by the proof of Theorem 6 we know that

$$(x,y) = \operatorname{trace} R^+_{x\cdot y} = \tfrac{1}{2}\operatorname{trace}(R_{x\cdot y} + L_{x\cdot y}) \qquad \text{for all } x,y \text{ in } \mathfrak{J}_1 \tag{42}$$

is a trace form on $\mathfrak{J}_1{}^+$ where R^+ indicates the right multiplication in $\mathfrak{J}_1{}^+$; hence (23) holds for all x, y, z in $\mathfrak{J}_1$, where (x, y) is the symmetric bilinear form (42). In terms of $T(x)$ defined by (30), we see that (23) is equivalent to

$$T\left((x\cdot y)\cdot z\right) = T\left(x\cdot(y\cdot z)\right) \qquad \text{for all } x,y,z \text{ in } \mathfrak{J}_1. \tag{43}$$

Now (36) implies

$$L_{xy} - L_yL_x + R_yR_x - R_{yx} = 0 \qquad \text{for all } x,y \text{ in } \mathfrak{J}_1. \tag{44}$$

Interchanging x and y in (44), and subtracting, we have

$$R_{[x,y]} + L_{[x,y]} = [R_y, R_x] + [L_x, L_y] \qquad \text{for all } x,y \text{ in } \mathfrak{J}_1. \tag{45}$$

Hence $T([x,y]) = (1,[x,y]) = \frac{1}{2}\operatorname{trace}(R_{[x,y]} + L_{[x,y]}) = 0$ by (42) and (45). Then $xy = x\cdot y + \frac{1}{2}[x,y]$ implies $T(xy) = T(x\cdot y) = \frac{1}{2}T(xy) + \frac{1}{2}T(yx)$, or

$$T(xy) = T(yx) = (x,y) \qquad \text{for all } x,y \text{ in } \mathfrak{J}_1 \tag{46}$$

since $T(x\cdot y) = (1, x\cdot y) = (x,y)$ by (23). Now (43) and (46) imply that (x,y) is a trace form on $\mathfrak{J}_1$. For $0 = 4T\left[(x\cdot y)\cdot z - x\cdot(y\cdot z)\right] = T\left[(xy)z + (yx)z + z(xy) + z(yx) - x(yz) - x(zy) - (yz)x - (zy)x\right] = 2T\left[(xy)z - x(yz) - (zy)x + z(yx)\right] = 4T\left[(xy)z - x(yz)\right]$ by (36), so $T\left((xy)z\right) = T\left(x(yz)\right)$, or $(xy,z) = (x,yz)$ as desired. Then (42) induces a trace form on the subalgebra $\mathfrak{J}$ of $\mathfrak{J}_1$.

Corollary to Theorem 11. Modulo its nilradical, any finite-dimensional noncommutative Jordan algebra of characteristic 0 is (uniquely) expressible as a direct sum $\mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_t$ of simple ideals $\mathfrak{S}_i$. Over their centers these $\mathfrak{S}_i$ are central simple algebras of the following types: (commutative) Jordan, quasiassociative, or flexible quadratic.

Proof: Only the verification for (x, y) in (42) of hypotheses (ii) and (iii) remains. But these are (12) and (8) of IV.

It was remarked in IV that, although proof was given only for commutative Jordan algebras of characteristic 0, the results were valid for arbitrary characteristic ($\neq 2$). The same statement cannot be made here. The trace argument in Theorem 11 can be modified to give the direct sum decomposition for semisimple algebras [58]. But new central simple algebras occur for characteristic p [52; 55]; central simple algebras which are not listed in the Corollary above are necessarily of degree one [58] and are *ramified* in the sense of [35].

A finite-dimensional power-associative algebra $\mathfrak{A}$ with 1 over F is called a *nodal algebra* in case every element of $\mathfrak{A}$ is of the form $\alpha 1 + z$ where $\alpha \in F$ and z is nilpotent, and $\mathfrak{A}$ is not of the form $\mathfrak{A} = F1 + \mathfrak{N}$ for $\mathfrak{N}$ a nil subalgebra of $\mathfrak{A}$. There are no such algebras which are alternative (of arbitrary characteristic), commutative Jordan (of characteristic $\neq 2$) [32], or noncommutative Jordan of characteristic 0. But nodal noncommutative Jordan algebras of characteristic $p > 0$ do exist. Any nodal algebra has a homomorphic image which is a simple nodal algebra.

Let $\mathfrak{J}$ be a nodal noncommutative Jordan algebra over F. Since the commutative Jordan algebra $\mathfrak{J}^+$ is not a nodal algebra, $\mathfrak{J}^+ = F1 + \mathfrak{N}^+$ where $\mathfrak{N}^+$ is a nil subalgebra of $\mathfrak{J}^+$; that is, $\mathfrak{J} = F1 + \mathfrak{N}$, where $\mathfrak{N}$ is a subspace of $\mathfrak{J}$ consisting of all nilpotent elements of $\mathfrak{J}$, and $x \cdot y \in \mathfrak{N}$ for all $x, y \in \mathfrak{N}$. For any elements $x, y \in \mathfrak{N}$ we have

$$xy = \lambda 1 + z, \qquad \lambda \in F,\ z \in \mathfrak{N}. \tag{47}$$

There must exist x, y in $\mathfrak{N}$ with $\lambda \neq 0$ in (47). Since $\mathfrak{N}^+$ is a nilpotent commutative Jordan algebra, the powers of $\mathfrak{N}^+$ lead to 0; equivalently, the subalgebra $(\mathfrak{N}^+)^*$ of $\mathfrak{M}(\mathfrak{J}^+)$ is nilpotent. Now (47) implies $yx = -\lambda 1 + (2x \cdot y - z)$ and $(xy)x = \lambda x + zx = x(yx) = -\lambda x + 2x(x \cdot y) - xz$, or

$$x(x \cdot y) = \lambda x + x \cdot z. \tag{48}$$

Now $0 = (x, x, y) + (y, x, x) = x^2 y - x(\lambda 1 + z) + (-\lambda 1 + 2x \cdot y - z)x - yx^2 = 2x^2 y - 2\lambda x - 2x \cdot z + 4(x \cdot y) \cdot x - 2x(x \cdot y) - 2x^2 \cdot y$ implies

$$x^2 y = 2\lambda x + 2x \cdot z - 2(x \cdot y) \cdot x + x^2 \cdot y \tag{49}$$

by (48). Defining

$$(47') \qquad x_i y = \lambda_i 1 + z_i, \qquad \lambda_i \in F,\ z_i \in \mathfrak{N},$$

linearization of (49) gives

$$(49') \qquad (x_1 \cdot x_2)y = \lambda_1 x_2 + \lambda_2 x_1 + x_1 \cdot z_2 + x_2 \cdot z_1 - (x_1 \cdot y) \cdot x_2 - (x_2 \cdot y) \cdot x_1 + (x_1 \cdot x_2) \cdot y.$$

Theorem 12. Let $\mathfrak{J}$ be a simple nodal noncommutative Jordan algebra over F. Then F has characteristic p, $\mathfrak{J}^+$ is the p^n-dimensional (commutative) associative algebra $\mathfrak{J}^+ = F[1, x_1, \ldots, x_n]$, $x_i^p = 0$, $n \geq 2$, and multiplication in $\mathfrak{J}$ is given by

$$(50) \qquad fg = f \cdot g + \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij}, \quad c_{ij} = -c_{ji},$$

where at least one of the c_{ij} $(= -c_{ji})$ has an inverse.

Proof: Since $\mathfrak{J} = F1 + \mathfrak{N}$, every element a in $\mathfrak{J}$ is of the form

$$(51) \qquad a = \alpha 1 + x, \qquad \alpha \in F,\ x \in \mathfrak{N}.$$

By (51) every associator relative to the multiplication in $\mathfrak{J}^+$ is an associator

$$(52) \qquad [x_1, x_2, x_3] = (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3), \qquad x_i \in \mathfrak{N}.$$

We shall first show that $\mathfrak{J}^+$ is associative by showing that the subspace $\mathfrak{B}$ spanned by all of the associators (52) is 0. For any y in $\mathfrak{N}$, (49′) implies that $(x_1 \cdot x_2)y$ is in $\mathfrak{N}$, so $[(x_1 \cdot x_2) \cdot x_3]\, y = \lambda_3 x_1 \cdot x_2 + (x_1 \cdot x_2) \cdot z_3 + x_3 \cdot [\lambda_1 x_2 + \lambda_2 x_1 + x_1 \cdot z_2 + x_2 \cdot z_1 - (x_1 \cdot y) \cdot x_2 - (x_2 \cdot y) \cdot x_1 + (x_1 \cdot x_2) \cdot y] - [(x_1 \cdot x_2) \cdot y] \cdot x_3 - (x_3 \cdot y) \cdot (x_1 \cdot x_2) + [(x_1 \cdot x_2) \cdot x_3] \cdot y$ by (47′) and (49′). Interchange subscripts 1 and 3, and subtract, to obtain $[x_1, x_2, x_3]y = [x_1, x_2, x_3] + [x_1, z_2, x_3] + [z_1, x_2, x_3] - [x_1 \cdot y, x_2, x_3] - [x_1, x_2 \cdot y, x_3] + [x_3 \cdot y, x_2, x_1] + [x_1, x_2, x_3] \cdot y$, so that we have the first inclusion in

$$(53) \qquad \mathfrak{B}\mathfrak{N} \subseteq \mathfrak{B} + \mathfrak{B} \cdot \mathfrak{N}, \qquad \mathfrak{N}\mathfrak{B} \subseteq \mathfrak{B} + \mathfrak{B} \cdot \mathfrak{N}.$$

The second part of (53) follows from $nb = -bn + 2b \cdot n$ for b in $\mathfrak{B}$, n in $\mathfrak{N}$.

Define an ascending series $\mathfrak{C}_0 \subseteq \mathfrak{C}_1 \subseteq \mathfrak{C}_2 \cdots$ of subspaces $\mathfrak{C}_i$ of $\mathfrak{J}$ by

$$(54) \qquad \mathfrak{C}_0 = \mathfrak{B}, \qquad \mathfrak{C}_{i+1} = \mathfrak{C}_i + \mathfrak{C}_i \cdot \mathfrak{N}.$$

Note that all the $\mathfrak{C}_i$ are contained in $\mathfrak{N}$ (actually in $\mathfrak{N} \cdot \mathfrak{N} \cdot \mathfrak{N}$, since $\mathfrak{B}$ is). Since $(\mathfrak{N}^+)^*$ is nilpotent, there is a positive integer k such that $\mathfrak{C}_{k+1} = \mathfrak{C}_k$. We prove by induction on i that

$$(55) \qquad \mathfrak{C}_i\mathfrak{N} \subseteq \mathfrak{C}_{i+1}, \qquad \mathfrak{N}\mathfrak{C}_i \subseteq \mathfrak{C}_{i+1}.$$

The case $i = 0$ of (55) is (53). We assume (55) and prove that $\mathfrak{C}_{i+1}\mathfrak{N} \subseteq \mathfrak{C}_{i+2}$ as follows: by the assumption of the induction it is sufficient to show

$$(56) \qquad (\mathfrak{C}_i \cdot \mathfrak{N})\mathfrak{N} \subseteq \mathfrak{C}_{i+1} + \mathfrak{C}_{i+1} \cdot \mathfrak{N}.$$

Now the flexible law (36) is equivalent to

$$(57) \qquad (x \cdot y)z = (yz) \cdot x + (yx) \cdot z - (y \cdot z)x \text{ for all } x, y, z \text{ in } \mathfrak{J}.$$

Put x in $\mathfrak{C}_i$, y and z in $\mathfrak{N}$ into (57), and use $yz = \mu 1 + w$, $\mu \in F$, $w \in \mathfrak{N}$, to see that each term of the right-hand side of (57) is in $\mathfrak{C}_{i+1} + \mathfrak{C}_{i+1} \cdot \mathfrak{N}$ by the assumption (55) of the induction. We have established (56), and therefore $\mathfrak{C}_{i+1}\mathfrak{N} \subseteq \mathfrak{C}_{i+2}$. Then, as above, $\mathfrak{N}\mathfrak{C}_{i+1} \subseteq \mathfrak{C}_{i+1}\mathfrak{N} + \mathfrak{C}_{i+1} \cdot \mathfrak{N} \subseteq \mathfrak{C}_{i+2}$, and we have established (55). For the positive integer k such that $\mathfrak{C}_{k+1} = \mathfrak{C}_k$, we have $\mathfrak{C}_k$ an ideal of $\mathfrak{J}$. For $\mathfrak{C}_k\mathfrak{J} = \mathfrak{C}_k(F1 + \mathfrak{N}) \subseteq \mathfrak{C}_k$ by (55), and similarly $\mathfrak{J}\mathfrak{C}_k \subseteq \mathfrak{C}_k$. The ideal $\mathfrak{C}_k$, being contained in $\mathfrak{N}$, is not $\mathfrak{J}$. Hence $\mathfrak{C}_k = 0$, since $\mathfrak{J}$ is simple. But $\mathfrak{B} \subseteq \mathfrak{C}_k$, so $\mathfrak{B} = 0$, $\mathfrak{J}^+$ is associative.

An ideal $\mathfrak{I}$ of an algebra $\mathfrak{A}$ is called a *characteristic* ideal (or $\mathfrak{D}$-ideal) in case $\mathfrak{I}$ is mapped into itself by every derivation of $\mathfrak{A}$. $\mathfrak{A}$ is called $\mathfrak{D}$-*simple* if 0 and $\mathfrak{A}$ are the only characteristic ideals of $\mathfrak{A}$.

We show next that the commutative associative algebra $\mathfrak{J}^+$ is $\mathfrak{D}$-simple. Interchange x and y in (36) to obtain

$$(36') \qquad (y, x, z) + (z, x, y) = 0 \qquad \text{for all } x, y, z \text{ in } \mathfrak{J};$$

interchange y and z in (36) to obtain

$$(36'') \qquad (x, z, y) + (y, z, x) = 0 \qquad \text{for all } x, y, z \text{ in } \mathfrak{J};$$

adding (36) and (36′), and subtracting (36″), we obtain the identity

$$[x \cdot y, z] = [x, z] \cdot y + x \cdot [y, z] \qquad \text{for all } x, y, z \text{ in } \mathfrak{J}, \tag{58}$$

which is valid in any flexible algebra. Identity (58) is equivalent to the statement that

$$D = R_z - L_z \qquad \text{for any } z \text{ in } \mathfrak{J} \tag{59}$$

is a derivation of $\mathfrak{J}^+$. If $\mathfrak{I}$ is an ideal of $\mathfrak{J}^+$, then $x \cdot z$ is in $\mathfrak{I}$ for all x in $\mathfrak{I}$, z in $\mathfrak{J}$. If, furthermore, $\mathfrak{I}$ is characteristic, then $[x, z] = xD$ is in $\mathfrak{I}$, since D in (59) is a derivation of $\mathfrak{J}^+$. Hence $xz = x \cdot z + \frac{1}{2}[x, z]$ and $zx = x \cdot z - \frac{1}{2}[x, z]$ are in $\mathfrak{I}$ for all x in $\mathfrak{I}$, z in $\mathfrak{J}$; that is, $\mathfrak{I}$ is an ideal of $\mathfrak{J}$. Hence $\mathfrak{J}$ simple implies that the commutative associative algebra $\mathfrak{J}^+$ is $\mathfrak{D}$-simple.

It is a recently proved result in the theory of commutative associative algebras (see [24]) that, if $\mathfrak{A}$ is a finite-dimensional $\mathfrak{D}$-simple commutative associative algebra of the form $\mathfrak{A} = F1 + \mathfrak{R}$ where $\mathfrak{R}$ is the radical of $\mathfrak{A}$, then (except for the trivial case $\mathfrak{A} = F1$ which may occur at characteristic 0, and which does not give a nodal algebra) F has characteristic p and $\mathfrak{A}$ is the p^n-dimensional algebra $\mathfrak{A} = F[1, x_1, \ldots, x_n]$, ${x_i}^p = 0$.

Now any derivation D of such an algebra has the form

$$f \to fD = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot a_i, \qquad a_i \in \mathfrak{A}, \tag{60}$$

where the a_i of course depend on the derivation D. Then (59) implies that $f \to [f, g]$ is a derivation of $\mathfrak{J}^+$ for any g in $\mathfrak{J}$. By (60) we have

$$[f, g] = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot a_i(g), \qquad a_i(g) \in \mathfrak{J}. \tag{61}$$

To evaluate the $a_i(g)$, note that $x_i D = [x_i, g] = a_i(g)$ and

$$[g, x_i] = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j} \cdot a_j(x_i). \tag{62}$$

Then $a_j(x_i) = [x_j, x_i]$ implies $a_i(g) = -[g, x_i] = -\sum_{j=1}^{n} \frac{\partial g}{\partial x_j} \cdot [x_j, x_i]$, or

$$[f, g] = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot [x_i, x_j] \tag{63}$$

by (61), so that $fg = f \cdot g + \frac{1}{2}[f, g]$ implies (50) where $c_{ij} = \frac{1}{2}[x_i, x_j]$. If every c_{ij} were in $\mathfrak{N}$, then $\mathfrak{N}$ would be a subalgebra of $\mathfrak{A}$, a contradiction. Hence at least one of the c_{ij} is of the form (51) with $\alpha \neq 0$, so it has an inverse, and $n \geq 2$.

Not every algebra described in the conclusion of Theorem 12 is simple (see [55]). However, all such algebras of dimension p^2 are, and for every even n there are simple algebras of dimension p^n. There are relationships between the derivation algebras of nodal noncommutative Jordan algebras and recently discovered (non-classical) simple Lie algebras of characteristic p [7; 11; 17; 68]. For a general discussion of Lie algebras of characteristic p, see [61].

The following list of papers on nonassociative algebras is intended to bring up to date (May 1961) the selective bibliography which appears at the end of the expository article [64].

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