DURBAN UNIVERSITY OF TECHNOLOGY

South Africa

Irrationality of the Euler- Mascheroni Constant

Andile Mabaso

2012

Abstract

In this paper we prove that the Euler- Mascheroni constant is irrational and transcendental.

Introduction

The Euler- Mascheroni constant was firstly introduced by Leonhard Euler (1707-1783) in 1734. Euler defined it as

$$\gamma = \lim_{n\to\infty}\left[\sum_{1}^{n}\frac{1}{k} - \ln(n)\right] \tag{1}$$

And its approximate value is 0.577218. Whether this constant is rational or irrational or transcendental has never been proved up to this day. In order to prove that this constant is irrational and transcendental we need to understand what it means to say a number is rational, irrational and transcendental. We do this by stating some definitions.

Definitions

Rational numbers

A number x is rational if $x = a/b$ where a and b are whole numbers.

Irrational numbers

These are the opposite for rational numbers. Irrational numbers cannot be represented as a ratio a/b where a and b are whole numbers and they cannot be represented as repeating decimals.

Algebraic numbers

Algebraic numbers are real numbers that can occur as roots of polynomial equations that have integer coefficients. For example, all rational numbers are algebraic. So are all surds such as $\sqrt{7}$, as well as numbers built from surds such as $\frac{42+\sqrt[3]{15.2}}{\sqrt{4-\sqrt{3}}}$. .

Transcendental numbers

Real numbers which are not algebraic are known as transcendental numbers meaning that they cannot be represented using operations of arithmetic and radicals as algebraic numbers.

We use the above definitions in our proof to prove that γ is irrational.

Theorem 1: *The sum of two or more different numbers is irrational if one of those numbers is irrational.*

The above theorem is applicable if and only if the following conditions are satisfied.

1. In the summation process there should be at least one irrational number.
2. That irrational number should not disappear in the equation or add up with another one equal to it but different in sign. Otherwise **theorem 1** will be invalid

Below is a couple of examples.

Example 1:

$$e = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Clearly in the above example all the figures or values we see are rational and therefore **theorem 1** is not applicable.

Example 2:

$$F = 1 + \pi + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

$$= e + \pi$$

In this example conditions 1 and 2 are satisfied since π is irrational, therefore **theorem 1** is applicable. According to **theorem 1** knowing the fact that π is irrational is enough to deduce that $e + \pi$ is irrational, even if we didn't know whether e is irrational or rational. We now prove **theorem 1**.

Proof:

Let $A = a/b$ be <u>rational</u> and $B \neq c/d$ be irrational, where a, b, c and d are whole numbers such that $A \neq -B$, Then

$$A + B = a/b + B = \frac{a + bB}{b} \qquad (2)$$

$a + bB$ is not a whole number since B is irrational and therefore

$\frac{a + bB}{b}$ is not a ratio of whole numbers and so it is irrational.

Now let $A \neq a/b$ be <u>irrational</u> and $B \neq c/d$ be irrational as well, where a, b, c and d are whole numbers such that $A \neq -B$, Then

$$A + B \neq a/b + c/d \qquad (3)$$

$$\neq \frac{ad + bc}{bd}$$

Since a, b, c and d are whole numbers then $ad + bc$ adds up to a whole number or integer and the product bd is a whole number.

$\therefore \frac{ad + bc}{bd}$ is rational.

But $A + B \neq \frac{ad + bc}{bd}$

That is, the sum of two numbers $A + B$ is not equal to a rational number, therefore by definition, $A + B$ is irrational. We now prove that the Euler- Mascheroni constant irrational.

Claim: The Euler- Mascheroni constant is irrational.

Proof:

$$\gamma = \lim_{n\to\infty}\left[\sum_{1}^{n}\frac{1}{k} - \ln(n)\right] \qquad (1)$$

$\gamma = \lim_{n\to\infty}\left[\sum_{1}^{n}\frac{1}{k} - \right] - \lim_{n\to\infty}[\ln(n)]$, because the limit of the sum is the sum of the limits

Now we check if conditions 1 and 2 are satisfied.

We note that

$\lim_{n\to\infty}\left[\sum_{1}^{n}\frac{1}{k}\right] - \lim_{n\to\infty}[\ln(n)] \neq 0$ Since $\gamma \neq 0$

$therefore \lim_{n\to\infty}\left[\sum_{1}^{n}\frac{1}{k}\right] \neq \lim_{n\to\infty}[\ln(n)]$- Condition 2 is satisfied.

It is known that $\ln(n)$ is irrational (refer to **reference no.1**) for $n > 1$ where n is an integer. Since"$n > 1$" includes infinitely large integers, we conclude that

$\lim_{n\to\infty}[\ln(n)]$is irrational, thus condition 1 is satisfied.

Therefore according to ***theorem 1*** proved above we conclude that the Euler- Mascheroni constant is irrational.

Q.E.D

We further prove that γ is transcendental.

Theorem 2: *The sum of two or more different numbers is transcendental if one of those numbers is transcendental.*

Proof: Let A be algebraic and B be transcendental. Then

A+B cannot be written as a solution for polynomial equations with integer coefficients, since B is transcendental, and therefore A+B is transcendental.

Let both A and B be transcendental then

A+B cannot be written as a solution for polynomial equations with integer coefficients, since both A and B are transcendental. Therefore A+B is transcendental.

Q.E.D

We are now ready to prove that γ is transcendental.

Claim: The Euler- Mascheroni constant is transcendental.

Proof:

$$\gamma = \lim_{n\to\infty}\left[\sum_{1}^{n}\frac{1}{k} - \ln(n)\right] \qquad (1)$$

$$\gamma = \lim_{n\to\infty}\left[\sum_{1}^{n}\frac{1}{k} - \right] - \lim_{n\to\infty}[\ln(n)]$$

Because the limit of the sum is the sum of the limits

Since $\ln(n)$ is transcendental (refer to **reference no.4**) and according to **Theorem 2** above, we conclude that the Euler- Mascheroni constant is transcendental.

Q.E.D

References

1. Feldvoss, J. May 19, 2008, *LOGARITHMS OF INTEGERS ARE IRRATIONAL*, Department of Mathematics and Statistics, University of South Alabama
2. Dunham W. EULER THE MASTER OF US ALL. No. 22. The Mathematical Association of America
3. http://www.mathwords.com/a/algebraic_numbers.htm
4. http://www.phengkimving.com/calc_of_one_real_var/07_the_exp_and_log_func/07_08_transcendency_of_the_exp_and_log_func.htm