Overview

There is no new material for this lesson, we just apply our knowledge from the previous lesson to some (admittedly complicated) word problems. Recall that given a first-order linear differential equation in the form

$$y' + p(t)y = q(t),$$

the general solution is found via

$$y\mu(t) = \int \mu(t)q(t)\,dt + C, \tag{1}$$

where $\mu(t) = e^{\int p(t)\,dt}$.

Examples

Example 1. Find the general solution to the differential equation

$$(x-4)y' + y = x^2 + 20.$$

Solution. As discussed in the previous lesson, we'll assume that $x > 4$ (so that $x - 4 > 0$). Notice that if $x = 4$, then we just have $y = 36$ and we would be done. We get the differential equation in the form that we want.

$$y' + \frac{1}{x-4}y = \frac{x^2+20}{x-4}.$$

Now calculating our integrating factor,

$$\mu(x) = e^{\int \frac{1}{x-4}\,dx} = e^{\ln(x-4)} = x - 4.$$

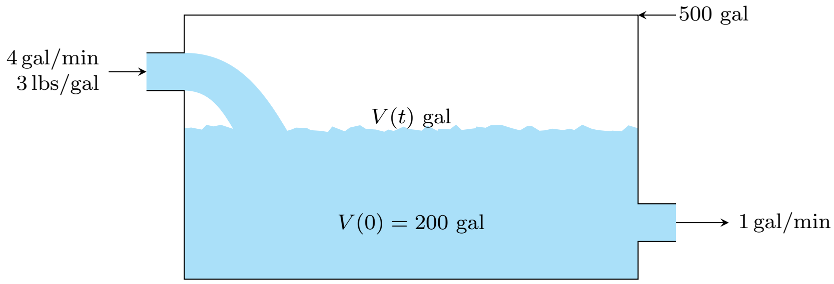
Then the general solution is found by

$$\begin{aligned} y(x-4) &= \int \frac{x^2+20}{x-4} \cdot (x-4)\,dx + C \\ y(x-4) &= \int \left(x^2+20\right)\,dx + C \\ y(x-4) &= \frac{1}{3}x^3 + 20x + C \\ y &= \frac{x^3}{3(x-4)} + \frac{20x}{x-4} + \frac{C}{x-4}. \end{aligned}$$

□

Example 2. A 500-gallon tank initially contains 200 gallons of brine containing 80 pounds of dissolved salt. Brine containing 3 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 1 gallon per minute. How much salt is in the tank when it is full?

Solution. We first notice that the question is asking for the *amount of salt* in the tank, and we are given an initial condition for the *amount of salt* in the tank. So let $A(t)$ represent the amount of salt in pounds at time t. Then we can represent the situation with the following picture.



Then the differential equation we're after is

$$\begin{aligned}\frac{dA}{dt} &= (\text{Rate in}) - (\text{Rate out}) \\ &= \frac{4 \text{ gal}}{\text{min}} \cdot \frac{3 \text{ lbs}}{\text{gal}} - \frac{1 \text{ gal}}{\text{min}} \cdot \frac{A(t) \text{ lbs}}{V(t) \text{ gal}} \\ &= 12 - \frac{A}{V(t)}.\end{aligned}$$

This is a first-order linear equation, so adding the $\frac{A}{V(t)}$ to both sides gets it in the desired form.

$$\frac{dA}{dt} + \frac{A}{V(t)} = 12 \tag{2}$$

But what is $V(t)$? Solution is pouring into the tank at a rate of 4 gal/min and out of the tank at a rate of 1 gal/min. So $dV/dt = 4 - 1 = 3$. Thus

$$V(t) = \int_0^t 3\,dx + V(0) = 3t + 200.$$

Now (2) becomes

$$\frac{dA}{dt} + \frac{A}{3t + 200} = 12. \tag{3}$$

Now to we compute the integrating factor $\mu(t)$ for (3).

$$\begin{aligned}\mu(t) &= e^{\int \frac{1}{3t+200}\,dt} \\ &= e^{\frac{1}{3}\ln(3t+200)} \\ &= e^{\ln(3t+200)^{1/3}} \\ &= (3t + 200)^{1/3}.\end{aligned}$$

Now using (1), the solution is given by

$$\begin{aligned}
A(3t+200)^{1/3} &= \int 12(3t+200)^{1/3}\,dt + C \\
&= 12\cdot\frac{1}{3}\int u^{1/3}\,du + C && \begin{aligned} u &= 3t+200 \\ du &= 3\,dt \end{aligned} \\
&= 4\cdot\frac{3}{4}u^{4/3} + C \\
&= 3u^{4/3} + C \\
&= 3(3t+200)^{4/3} + C. && (4)
\end{aligned}$$

Now dividing both sides by $(3t+200)^{1/3}$,

$$A = 3(3t+200) + C(3t+200)^{-1/3}. \tag{5}$$

We can solve for C using the fact that $A(0) = 80$ (recall that A represents the amount of salt in the tank) and plugging that into (4). When we do that, we get

$$\begin{aligned}
80(200)^{1/3} &= 3(200)^{4/3} + C \\
C &= 80(200)^{1/3} - 3(200)^{4/3} \\
C &= (80 - 3\cdot 200)\,(200)^{1/3} \\
C &= -520(200)^{1/3}.
\end{aligned}$$

Using the C we just found in (5), we find that

$$A(t) = 3(3t+200) - 520(200)^{1/3}(3t+200)^{-1/3}. \tag{6}$$

Now with (6), we can say how much salt is in the tank at any given time t. The time we are concerned about is when the tank is full. Since the tank holds 500 gallons, this occurs when $500 = V(t) = 3t + 200$. Solving this equation for t shows that $t = 100$ when $V = 500$. The question at hand is now as simple as plugging in 100 for t in (6), and

$$A(100) \approx 1116.$$

□

While these tank-style problems are ubiquitous, not all differential equations word problems amount to (rate in) − (rate out), as the next two examples illustrate.

Example 3. You have an oversized trench coat and decide that its best use is to sell chocolate bars. The trench coat is capable of holding 80 chocolate bars, and you currently have 50. Using your mad 16020 math skills, you have determined that you sell the chocolate bars at a daily rate equal to 14% of the available chocolate bar-capacity of your trench coat. When will you sell out of chocolate bars?

Solution. Here we want to answer a question about when we sell out of chocolate bars. There are two natural guesses for the function that we want to represent with a differential equation: the number of chocolate bars we have sold, or the number we have left to sell. Since our initial condition is about the number we have left to sell, this should be our function. So let $A(t)$ be the number of chocolate bars we have left to sell. Then the problem states

$$\frac{dA}{dt} = -.14(80 - A).$$

Why the negative sign? If A is the number of chocolate bars we have, then since we are losing chocolate bars, the rate of change should be negative. From here there are two options to continue. We can distribute the $-.14$ on the right hand side and get this differential equation in the desired form for first-order linear equations. Or we can notice that this equation is separable and use separation of variables. Both will lead to the exact same answer.

Using separation of variables, we have

$$\begin{aligned} \frac{dA}{80 - A} &= -.14\,dt \\ \int \frac{dA}{80 - A} &= \int -.14\,dt \\ -\ln|80 - A| &= -.14t + C \\ \ln(80 - A) &= .14t + C, \end{aligned} \tag{7}$$

where in the last line we have used the fact that $A < 80$ and relabeled $-C$ by C. Now we can use $A(0) = 50$ to easily see that $C = \ln(80 - 50) = \ln 30$. Using this in (7), and solving for A,

$$\begin{aligned} \ln(80 - A(t)) &= .14t + \ln 30 \\ 80 - A(t) &= e^{.14t + \ln 30} \\ A(t) &= 80 - 30e^{.14t}. \end{aligned}$$

We didn't really need to do this last part although it is good practice. The question asks for the time when you sell out of chocolate bars. This happens when $A = 0$. So using (7) with the C we found and plugging 0 for A,

$$\begin{aligned} \ln(80) &= .14t + \ln 30 \\ .14t &= \ln 80 - \ln 30 \\ .14t &= \ln\left(\frac{8}{3}\right) \\ t &= \frac{1}{.14}\ln\left(\frac{8}{3}\right) \\ &\approx 2.32 \text{ days.} \end{aligned}$$

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Example 4. Aliens are invading West Lafayette at an alarming rate of

$$y' = ty + t$$

aliens per day, after an initial invasion of 40 aliens. How long will it take the alien population to grow to 750?

Solution. This is the best kind of word problem because we don't have to figure out what the differential equation is. We are told $y(0) = 40$ and we want to know for what time t

$y(t) = 750$. The given differential equation is first-order linear, so we start by subtracting ty from both sides to get it into the desired form.

$$y' - ty = t.$$

Then finding an integrating factor:

$$\mu(t) = e^{\int -t\,dt} = e^{-t^2/2}.$$

Now we use (1) to find a solution.

$$\begin{aligned} ye^{-t^2/2} &= \int te^{-t^2/2}\,dt + C \\ ye^{-t^2/2} &= -e^{-t^2/2} + C \\ y &= -1 + Ce^{t^2/2}. \end{aligned}$$

Now $y(0) = 40$ implies that $40 = -1 + C$ so that $C = 41$. So

$$y(t) = -1 + 41e^{t^2/2}.$$

Then solving $y(t) = 750$,

$$\begin{aligned} 750 &= -1 + 41e^{t^2/2} \\ 751 &= 41e^{t^2/2} \\ \frac{751}{41} &= e^{t^2/2} \\ t^2/2 &= \ln\left(\frac{751}{41}\right) \\ t^2 &= 2\ln\left(\frac{751}{41}\right) \\ t &= \sqrt{2\ln\left(\frac{751}{41}\right)} \\ t &\approx 2.41 \text{ days.} \end{aligned}$$

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