HYBRID SIMULATION OF A CONTROLLED SYSTEM

Understand differences among controllers, even if they were not part of those selected to be performed for report.

PC (LabVIEW) is the controller:

Digital approximation to analog controller – PID (proportional integral derivative)
- may be inefficient

True digital controller with difference equation & Z.O.H. on output
- more efficient but requires a digital model of the analog plant

Procedure covers:

Four digital implementations of analog PID controllers
- 3 PID controllers
- 1 Analog state-space (state feedback using pole placement) controller

A true digital Finite Settling Time, FST (Ripple Free) controller
- FST for a plant without delay and FST for a plant with delay

A Digital state-space (state feedback using pole placement) controller

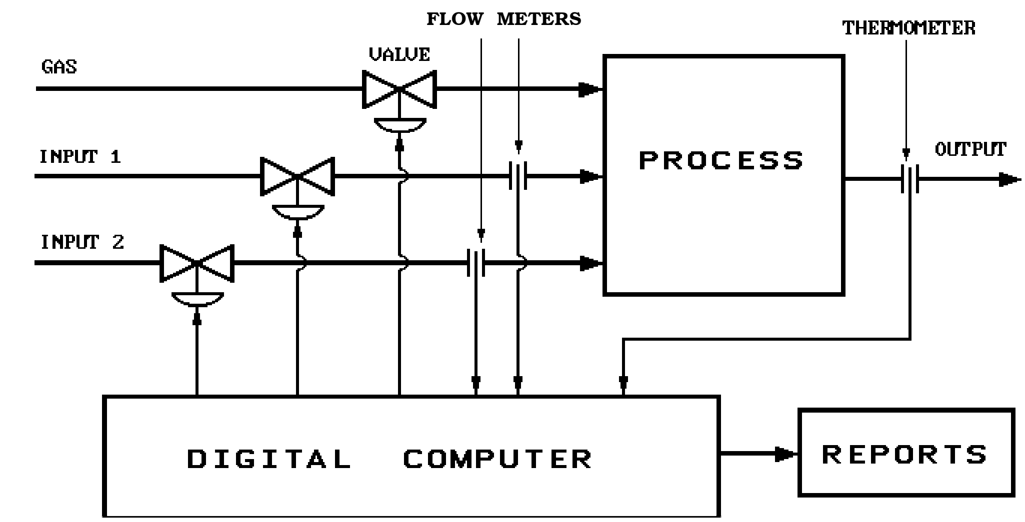
Required for report:

2 of the 3 analog PID controllers

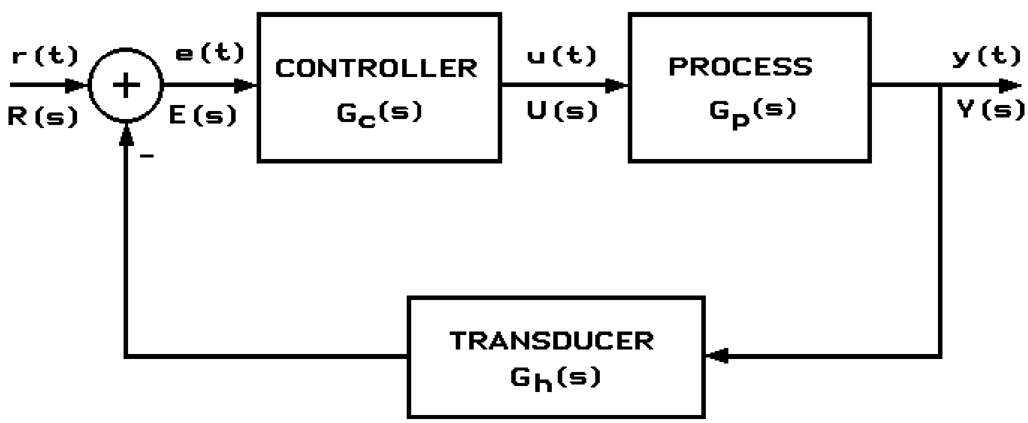
1 FST controller (either with or without delay)

1 State-Space controller (either analog or digital)

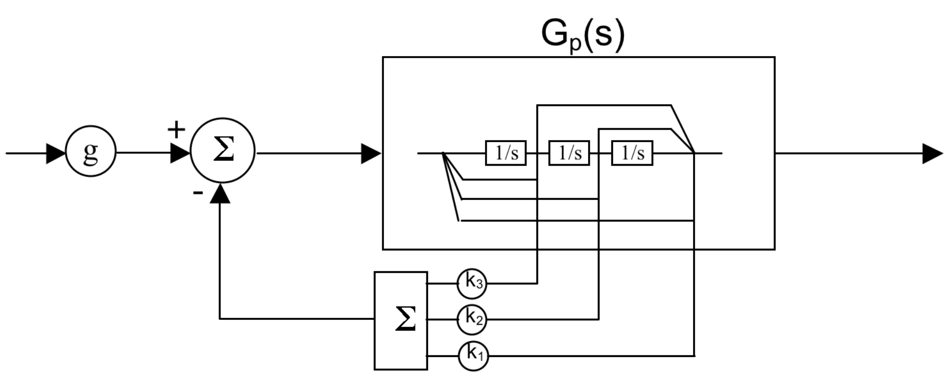
NOTE:
It should be noted here that all process time delays will be implemented on the PC. It is done this way because it is much easier to create a process with delays digitally. In practice, process time delays are harmful and adversely affect the stability of the controlled system. A controller should never purposely insert a delay. Unavoidable delays could be caused by slow processes, converters, or remote sensing devices (e.g. Earth-Mars remote control signals).



Typical Direct Digital Control Application.



***t* and *s* Domain Block Diagram of a Closed Loop System.**



State Feedback Control

PID CONTROLLER $G_c(s)$

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s$$

where:

K_p is the proportional gain (initial feedback control, error forces change)
K_i is the integral gain (integrator forces steady-state error to zero)
K_d is the derivative gain (derivative slows down fast changing response to reduce overshoot)

Variations:

1) Proportional (**P**) controller: the K_i and K_d terms are zero.
2) Proportional + Integral (**PI**) controller: the K_d term is zero.
3) Proportional + Derivative (**PD**) controller: the K_i term is zero.
4) Proportional + Integral + Derivative (**PID**) controller: all terms are non-zero.

$$\dot{x}(t) = \frac{dx}{dt} = \lim_{T \to 0} \frac{x(t+T) - x(t)}{T} \approx \frac{x(t+T) - x(t)}{T}$$

The difference equation for the digital PID controller is:

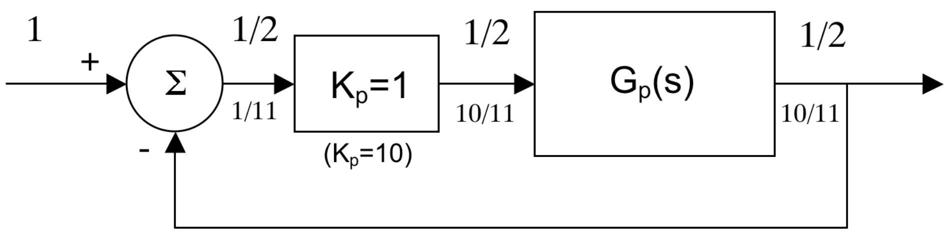
$$u(k) = K_p e_1(k) + K_i e_{22}(k) + K_d e_{23}(k)$$

where:

$U(k)$ is the control at the k^{th} sample instant,
$e_1(k) = r(k) - y(k)$, the error term,
$e_{22}(k) = e_{22}(k - 1) + Te_1(k)$ the integral approximation,
$e_{23}(k) = [e_1(k) - e_1(k - 1)]/T$ the derivative term,
T is the sampling time in *seconds*, $r(k)$ is the reference signal and $y(k)$ is the output signal.

Steady State Error with Pure Proportional Feedback

For a plant $G_p(s)$ with no free integrator (1/s term that can be factored out of the numerator)



Pure Proportional Control with Steady State Error

With K_p = 1, steady state error will be 50% $\left(1 - \frac{1}{2} = 50\%\right)$

With K_p = 10, steady state error will be 9% $\left(1 - \frac{10}{11} = 9\%\right)$

Increasing K_p decreases steady state error, but will not make it go to zero

Note: increasing K_p will increase the overshoot and settling time

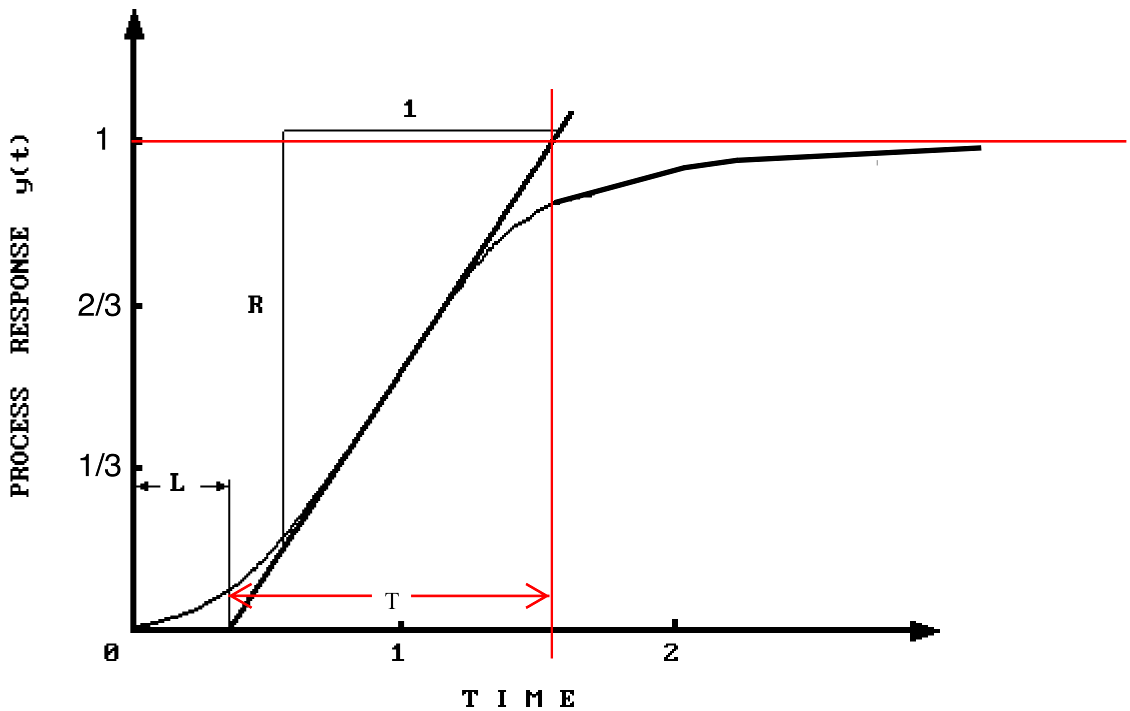
Need a K_i term to drive steady state error to zero

Output of controller block will keep adjusting itself until input is zero
(no steady state error)

Ziegler-Nichols Design

Optimize: fast response time but with significant overshoot.

Uses only 2 parameters derived from the system step response, R & L



Example of a Unit Step Response.

For the P control:

$$K_p = 1/RL = T/L$$

For the PI control:

$$K_p = 0.9/RL, \quad K_i = 0.3K_p/L = 0.27/(RL^2)$$

For the PID control:

$$K_p = 1.2/RL, \quad K_i = 0.5K_p/L = 0.6/(RL^2), \quad K_d = 0.5LK_p = 0.6/R$$

Gallier-Otto Design

Optimize: $IAE = \int_0^\infty |e(t)|dt$

Assumes the model for the plant matches the 2nd order equation below, need to have approximations for T_1, T_2, K, & T_D

$$G_p(s) = \frac{Ke^{-T_D s}}{(T_1 s+1)(T_2 s+1)}$$

where:

K is the process gain,

T_1 is the smaller time constant,

T_2 is the larger time constant,

T_D is the time delay.

Define:

$$\lambda = T_1/T_2$$

$$T_n = T_1 + T_2 + T_D$$

$$t_D = T_D/T_n$$

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s\right)$$

found for various values of λ and t_D.

Gallier and Otto compiled graphs relating the controller parameters to the process parameters l and t_D, for both PI and PID control. In particular, these graphs give optimal values for the normalized parameters:

$K_0 = K_p K$, the loop gain of the closed loop system,

$t_i = T_i/T_n$, the normalized reset time,

$t_d = T_d/T_n$, the normalized derivative time.

$$G_c(s) = K_p + K_i/s + K_d s$$

three coefficients become:

$$K_p = \frac{K_0}{K}$$

$$K_i = \frac{K_p}{T_i} = \frac{K_p}{t_i T_n}$$

$$K_d = K_p T_d = K_p t_d T_n$$

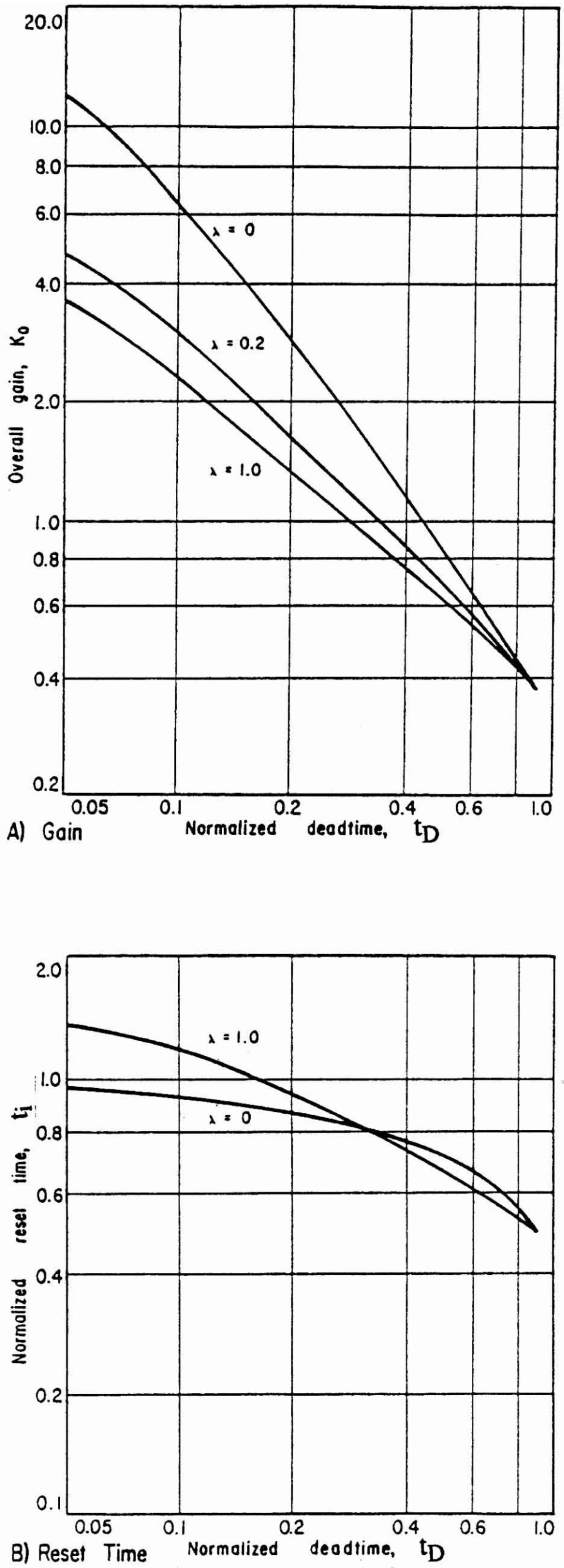


Figure 2. Tuning coefficients for two-mode control with minimum IAE $(\lambda = T_1/T_2)$

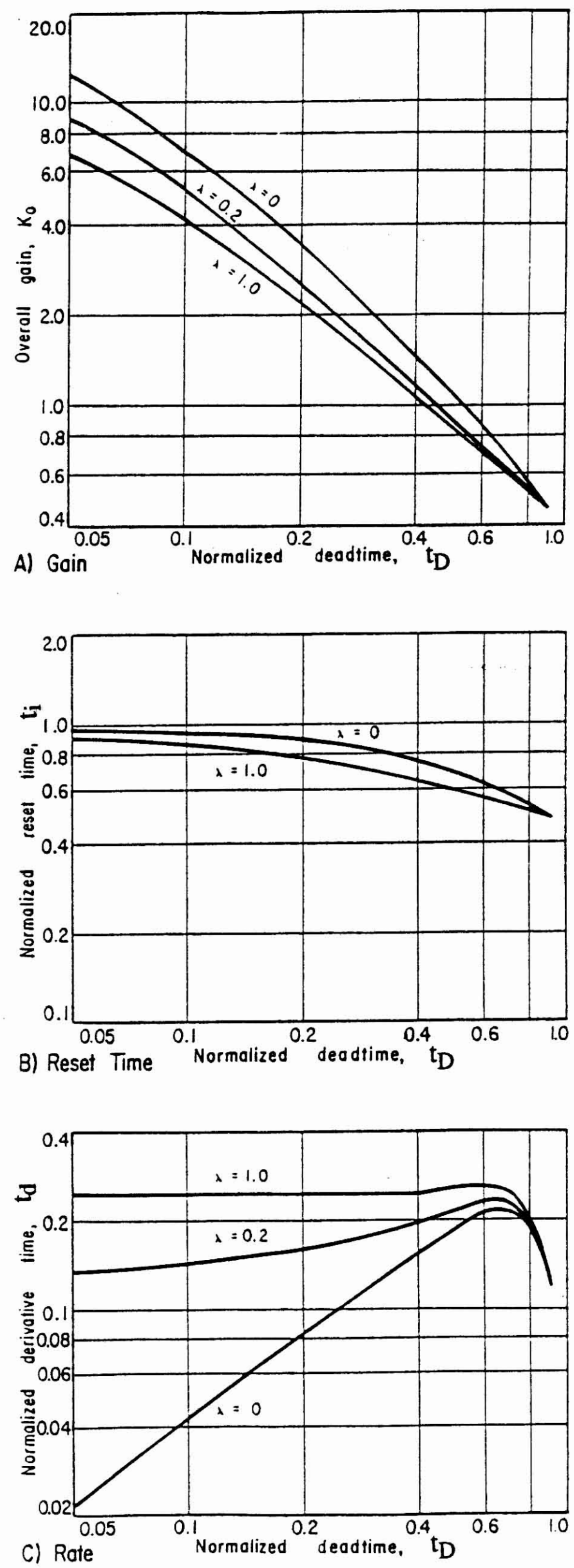


Figure 3. Tuning coefficients for three-mode control with minimum IAE $(\lambda = T_1/T_2)$

Gallier-Otto Gains for PI and PID Control.

Graham-Lathrop

Optimize: $ITAE = \int_0^\infty t\,|e(t)|\,dt$

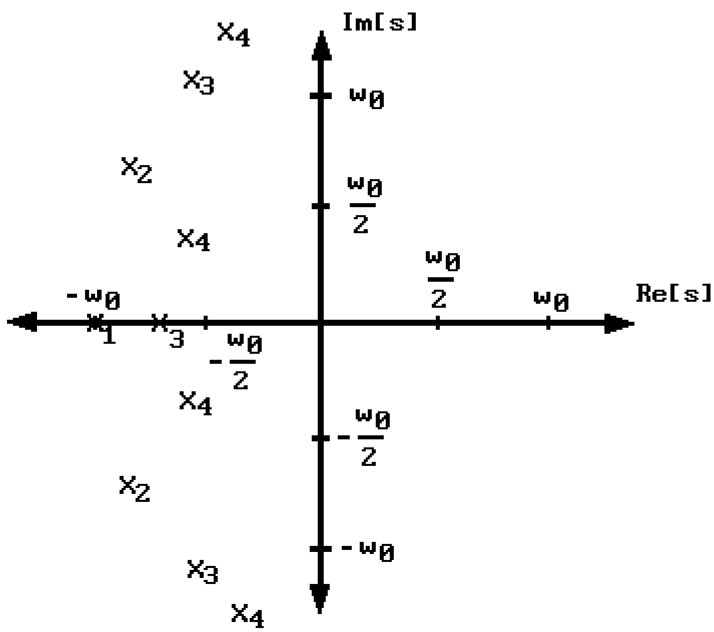
Assumes model for the controlled system N(s)/D(s) and all coefficients are known, here

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \text{ with } G_c(s) = K_p \text{ or } \frac{K_p s + K_i}{s} \text{ or } \frac{K_d s^2 + K_p s + K_i}{s}$$

Want the characteristic equation of a closed loop system (denominator of the **C**losed **L**oop **T**ransfer **F**unction) to have its roots match those of a standard form:

	Closed Loop Characteristic Equations as derived by Graham & Lathrop
1	$s + \omega_0$
2	$s^2 + 1.4\omega_0 s + \omega_0^2$
3	$s^3 + 1.75\omega_0 s^2 + 2.15\omega_0^2 s + \omega_0^3$
4	$s^4 + 2.1\omega_0 s^3 + 3.4\omega_0^2 s^2 + 2.7\omega_0^3 s + \omega_0^4$
5	$s^5 + 2.8\omega_0 s^4 + 5.0\omega_0^2 s^3 + 5.5\omega_0^3 s^2 + 3.4\omega_0^4 s + \omega_0^5$
6	$s^6 + 3.25\omega_0 s^5 + 6.6\omega_0^2 s^4 + 8.6\omega_0^3 s^3 + 7.45\omega_0^4 s^2 + 3.95\omega_0^5 s + \omega_0^6$
7	$s^7 + 4.47\omega_0 s^6 + 10.42\omega_0^2 s^5 + 15.08\omega_0^3 s^4 + 15.54\omega_0^4 s^3 + 10.64\omega_0^5 s^2 + 4.58\omega_0^6 s + \omega_0^7$
8	$s^8 + 5.2\omega_0 s^7 + 12.8\omega_0^2 s^6 + 21.6\omega_0^3 s^5 + 25.75\omega_0^4 s^4 + 22.22\omega_0^5 s^3 + 13.3\omega_0^6 s^2 + 5.15\omega_0^7 s + \omega_0^8$

Graham-Lathrop Standard Forms Table.



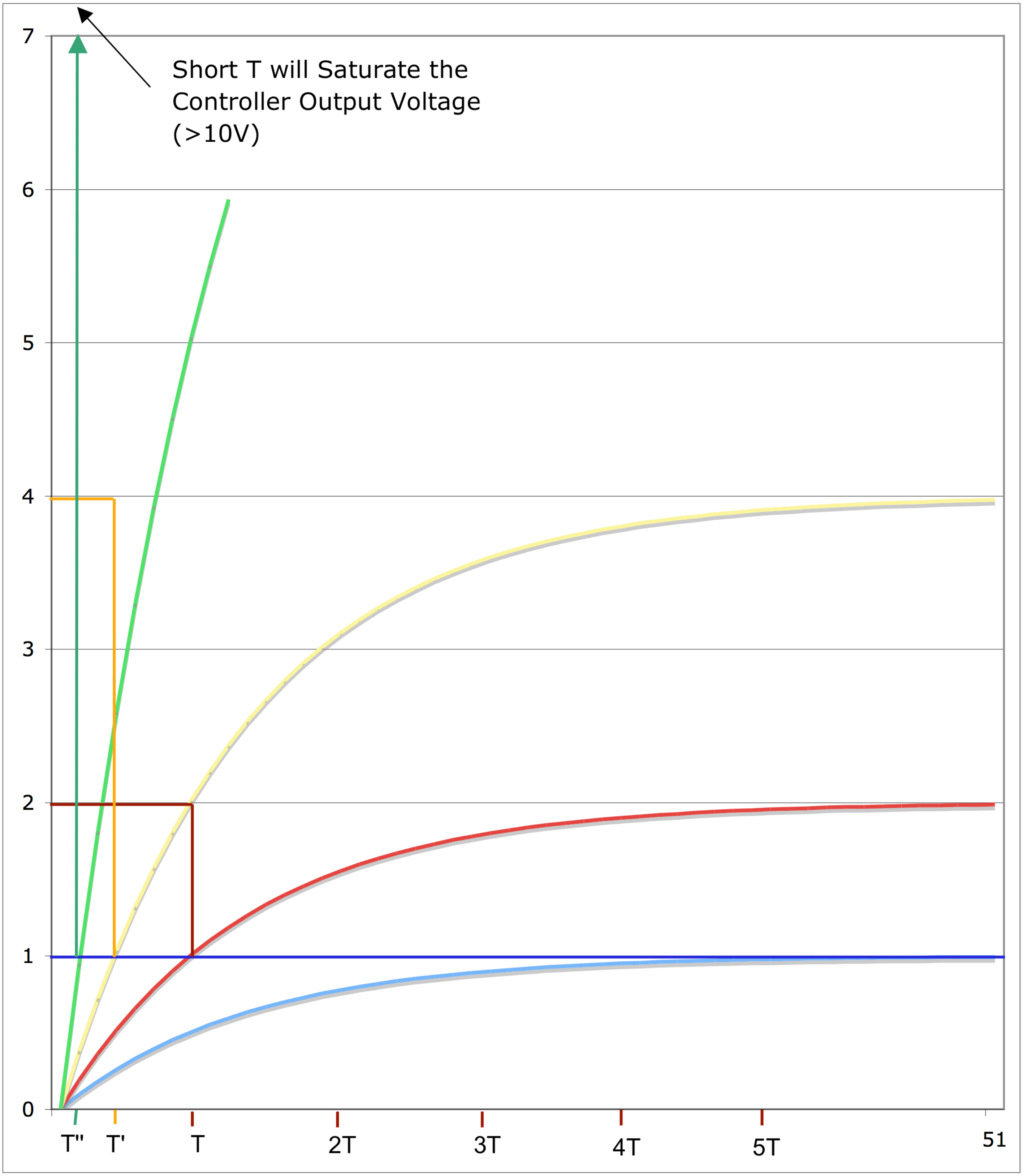
X_i is the root location for the i^{th} order equation

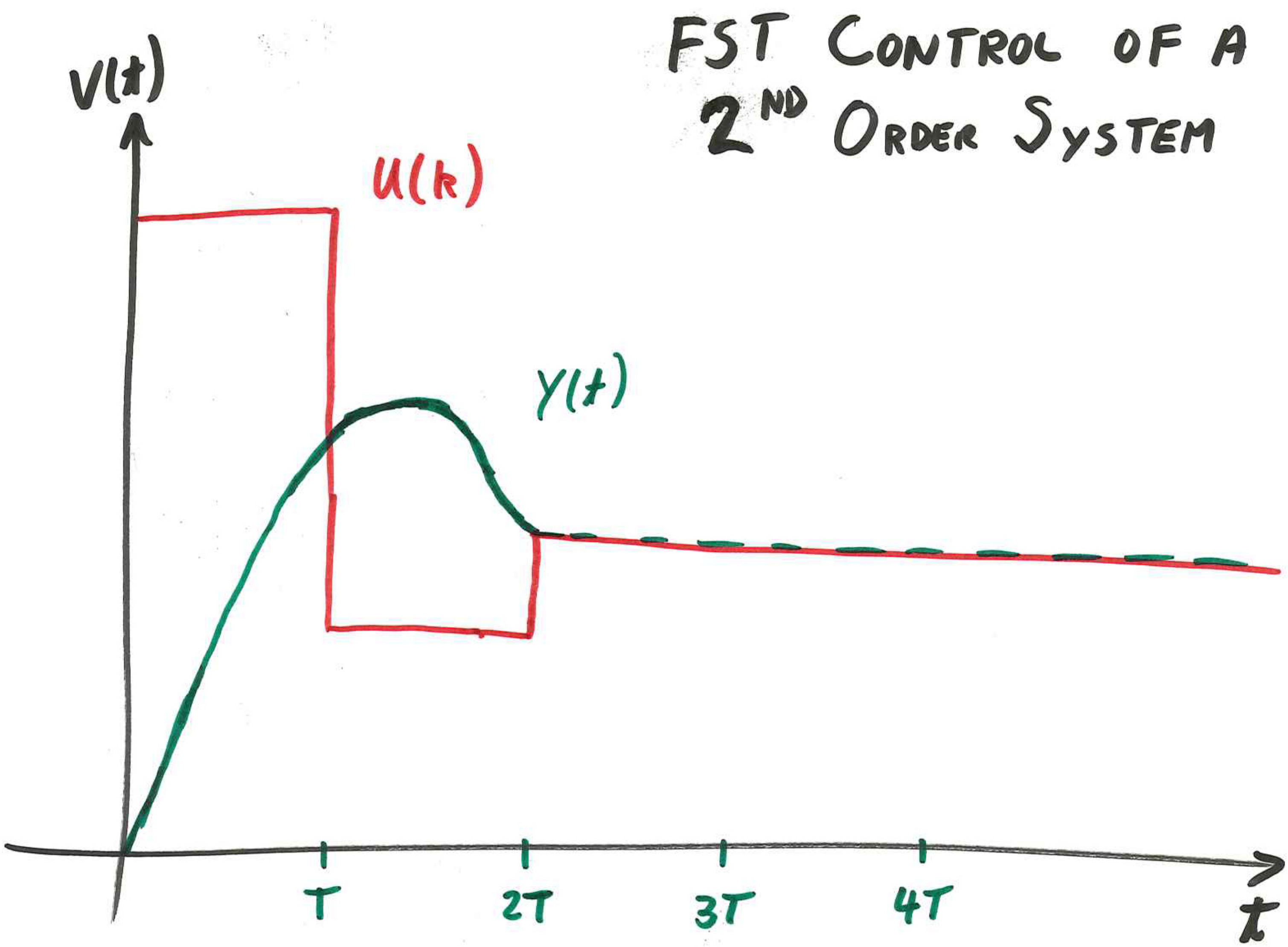
w_0 is the natural frequency response of the CL system

Roots for 1st, 2nd, 3rd, and 4th order Graham-Lathrop Forms.

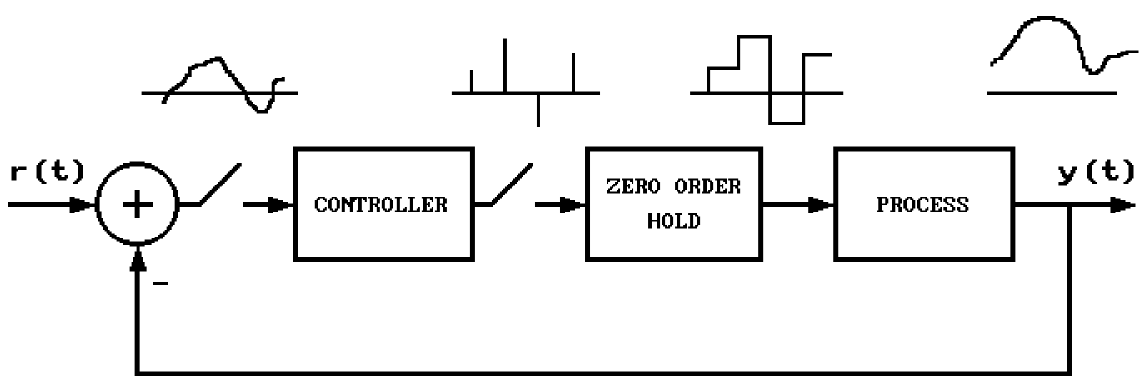
FINITE SETTLING TIME (FST) CONTROLLER

FST Controller for a 1st Order System

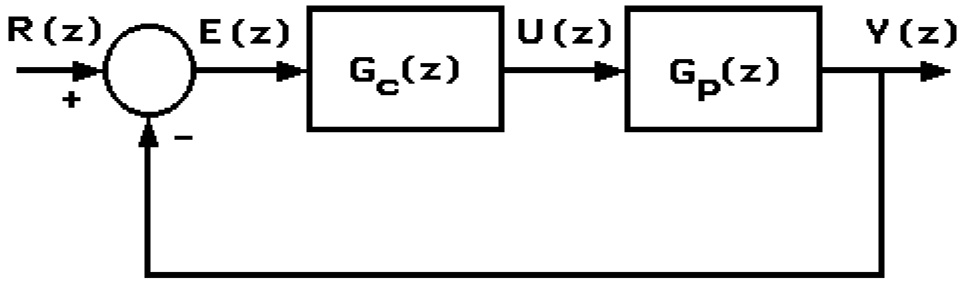




FST Controller for a 2nd Order Process Without Time Delay (Ripple Free Controller)



Sampled Data Version of the Continuous System



Typical *z*-Plane Block Diagram.

Assume:

$$G_p(s) = \frac{1}{(T_1 s + 1)(T_2 s + 1)}$$

The process transfer function with a zero order hold on its input is given by:

$$G_p^*(s) = G_p(s)\frac{1 - e^{-Ts}}{s} = \frac{1 - e^{-Ts}}{s(T_1 s + 1)(T_2 s + 1)}$$

$$\uparrow \text{Z.O.H.}$$

The process pulse transfer function is given by $\mathcal{Z}\{G_p^*(s)\}$:

$$G_p^*(z) = \frac{[1 - (1 + d_1)p_1 - (1 + d_2)p_2]z^{-1} + [p_1 p_2 + d_1 p_1 + d_2 p_2]z^{-2}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

where:

$$p_1 = e^{-T/T_1}$$
$$p_2 = e^{-T/T_2}$$
$$d_1 = \frac{-T_2}{T_2 - T_1}$$
$$d_2 = \frac{T_1}{T_2 - T_1}$$

The process pulse transfer function can be transformed into the general form:

$$G_p^*(z) = \frac{c_1 z^{-1} + c_2 z^{-2}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

where:

$$c_1 = 1 - (1 + d_1)p_1 - (1 + d_2)p_2$$
$$c_2 = p_1 p_2 + d_1 p_1 + d_2 p_2$$

FST controller difference equation

$$u(k) = \sum_{i=0}^{N_e} K_{ei} e(k - i) + \sum_{i=1}^{N_u} K_{ui} u(k - i)$$

where:

$e(k - i)$ is the error term at the k^{th} sample instant
$u(k - i)$ is the control term at the k^{th} sample instant
K_e is the vector of gains for the error terms
K_u is the vector of gains for the control terms
N_e is the order of the error sum
N_u is the order of the control sum

Ragazzini and Franklin:

$$N_e = 2$$
$$N_u = 2$$
$$K_{e0} = \frac{1}{c_1 + c_2}$$
$$K_{e1} = \frac{-p_1 - p_2}{c_1 + c_2}$$
$$K_{e2} = \frac{p_1 p_2}{c_1 + c_2}$$
$$K_{u1} = \frac{c_1}{c_1 + c_2}$$
$$K_{u2} = \frac{c_2}{c_1 + c_2}$$

Controller difference equation:

$$u(k) = K_{e0} e(k) + K_{e1} e(k - 1) + K_{e2} e(k - 2) + K_{u1} u(k - 1) + K_{u2} u(k - 2)$$

All controller coefficients are function of the sample period T. Note that as T gets smaller, the K_e terms increase. For T < **1 second** the control values that are proportional to the K_e terms will need to exceed their maximum value of **10 Volts** to properly control the system, unless the input step size is very small. For the purposes of this experiment, sampling time T for the FST controller should be greater than 1 second.

FST Controller for a 2^{nd} Order Process With Time Delay

The process with time delay may be approximated by the transfer function

$$\boxed{G_p(s) = \frac{e^{-T_D s}}{(T_1 s + 1)(T_2 s + 1)}} \qquad \text{[Numerator} \neq 1]$$

where

T_1 and T_2 are the process time constants.

T_D is the time delay implemented as an integral number of sample periods (T_D = *MT using a stack in the PC*).

The process transfer function with zero order hold is given by:

$$G_p(s) = \frac{(1 - e^{-Ts})e^{-MTs}}{s(T_1 s + 1)(T_2 s + 1)}$$

where

M is the number of sample periods of time delays.

T is the sampling period.

The process pulse transfer function is [Remember $e^{Ts} = z$]

$$G_p(z) = \frac{(c_1 z^{-1} + c_2 z^{-2})z^{-M}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

Controller difference equation:

$$u(k) = K_{e0}e(k) + K_{e1}e(k-1) + K_{e2}e(k-2) + K_{u1}u(k-1-M) + K_{u2}u(k-2-M)$$

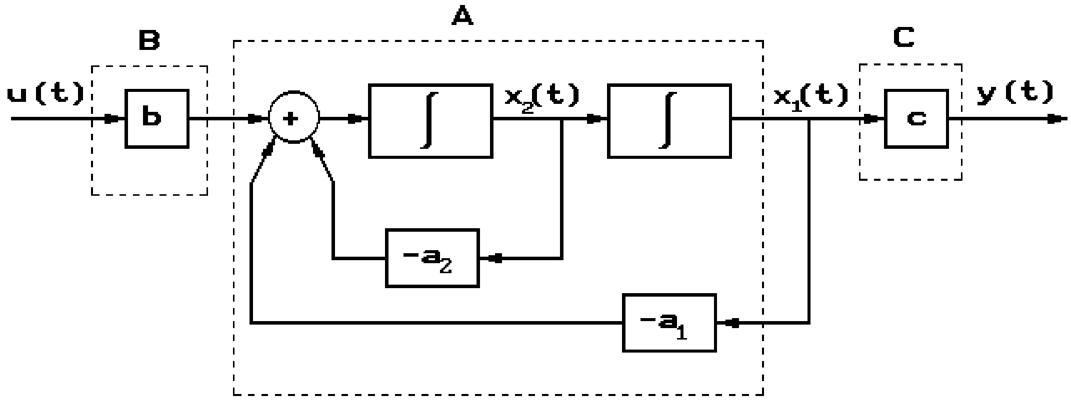
Note change: ↑ ↑

STATE FEEDBACK CONTROLLER USING THE POLE PLACEMENT METHOD

Introduction to State Variable and Pole Placement

Uses canonic form of state variables for plant $G_p(s)$ in matrix form and assumes all the states can be observed (measured)

This requires the analog computer be rewired for the plant
(Use 2nd analog computer for Hybrid Part D)



Block Diagram of a Typical 2nd order Plant.

$$\dot{x}_1(t) = x_2(t) \qquad x_1(0) = x_{10}$$
$$\dot{x}_2(t) = -a_1x_1(t) - a_2x_2(t) + bu(t) \qquad x_2(0) = x_{20}$$
$$y(t) = cx_1(t)$$

or in matrix form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A_c x}(t) + \boldsymbol{B_c u}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$
$$\boldsymbol{y}(t) = \boldsymbol{C_c x}(t)$$

where:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{A}_c = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad \mathbf{C}_c = \begin{bmatrix} c & 0 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = H(s) = \frac{bc}{s^2 + a_2 s + a_1}$$

$\det[s\boldsymbol{I} - \boldsymbol{A_c}] = s^2 + a_2 s + a_1 = 0$ [characteristic equation, denominator of H(s)]

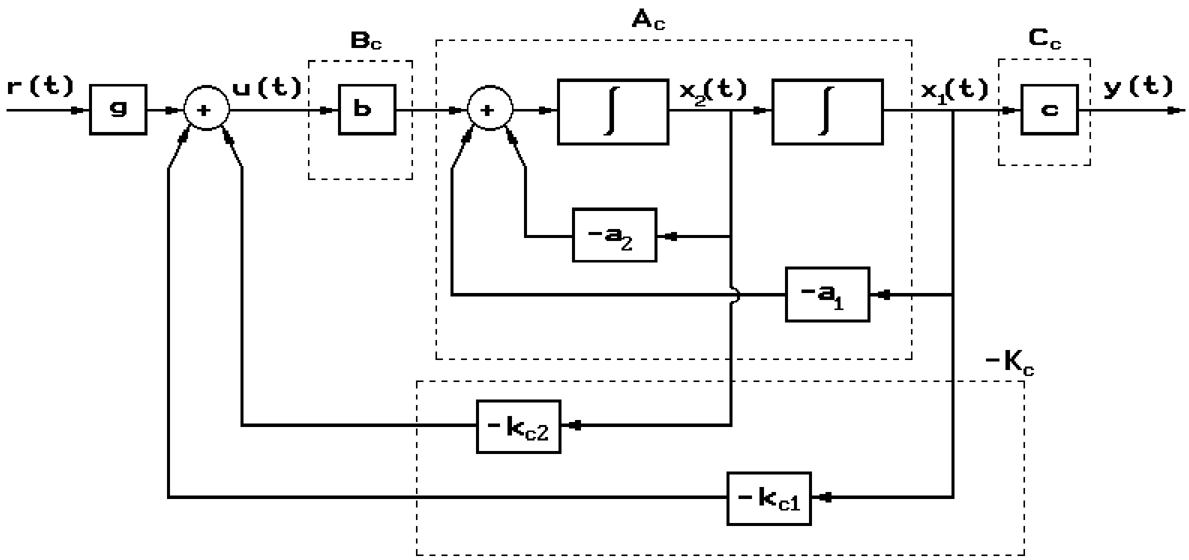
A state space controller feedback signal is only the sum of the system states multiplied by gain constants: $u(k) = k_{c1}x_1(k) + k_{c2}x_2(k)$

Pole Placement for the Continuous Case

Using the same general plant above, the objective is to determine gains k_{c1} & k_{c2} so that the closed loop system has its poles at p_1, and p_2. Thus the desired characteristic polynomial becomes:

$$s^2 + ms + n = 0\,, \quad -m = p_1 + p_2\,, \quad n = p_1 p_2$$

In augmented system with the feedback gains k_{c1}, k_{c2} and a reference input signal $r(t)$ all present, it is necessary to amplify the reference signal with a feedforward gain g (to be found), for zero steady state error (output matching the input) under a step input.



Augmented system with feedback controller and reference.

Assuming zero initial conditions as before, the new state equations are:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -a_1 x_1(t) - a_2 x_2(t) - k_{c1} b x_1(t) - k_{c2} b x_2(t) + g b r(t)$$
$$y(t) = c x_1(t)$$

From direct comparison between equations (30) and (26) it's easy to note that the control law $u(t)$ is given by:

$$u(t) = gr(t) - [k_{c1}x_1(t) + k_{c2}x_2(t)] = gr(t) - \boldsymbol{K_c x}(t)$$

with $\mathbf{K}_c = \begin{bmatrix} k_{c1} & k_{c2} \end{bmatrix}$ which precisely is the formula for the state feedback control. Equations (30) can be written in matrix form as:

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A_c} - \boldsymbol{B_c K_c})\boldsymbol{x}(t) + \boldsymbol{B_c} g r(t)\,, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$
$$\boldsymbol{y}(t) = \boldsymbol{C_c x}(t)$$

The overall transfer function and characteristic equation of the closed loop system are:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{gbc}{s^2 + (a_2 + bk_{c2})s + (a_1 + bk_{c1})}$$

$$\det[\,s\boldsymbol{I} - \boldsymbol{A_c}\,] = s^2 + (a_2 + bk_{c2})s + (a_1 + bk_{c1}) = 0$$

Gains are found by coefficient matching

The unknown feedforward gain *g* is calculated as follows:

$$e_{ss} = \lim_{t\to\infty}(R - y(t)) = R - y_{ss}$$

Applying the **Final Value** theorem and noting that $R(s) = R/s$, we calculate y_{SS}

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{gb\frac{R}{s}}{s^2 + (a_2 + bk_{c2})s + (a_1 + bk_{c1})} = \frac{gbR}{a_1 + bk_{c1}}$$

Want zero steady state error hence y_{SS} must be equal to *R*, which identifies *g* as a function of the gains $\boldsymbol{K_C}$:

$$g = \frac{a_1}{b} + k_{c1}$$

Pole Placement for the Discrete Case

Convert continuous system to discrete using the relationship:

$$z_i = e^{s_i T}$$

where *T* is the sampling time.

As in the continuous case the discrete desired second order polynomial is

$$z^2 + qz + p = 0\,, \quad \text{with} \quad -q = z_1 + z_2\,, \quad p = z_1 z_2$$

Equivalent discrete-time system is:

$$\boldsymbol{x}(k+1) = \boldsymbol{A_d}\boldsymbol{x}(k) + \boldsymbol{B_d}\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = \boldsymbol{C_d}\boldsymbol{x}(k)$$

where $\boldsymbol{A_d}$, $\boldsymbol{B_d}$, and $\boldsymbol{C_d}$ are the discrete counterparts of $\boldsymbol{A_C}$, $\boldsymbol{B_C}$, and $\boldsymbol{C_C}$ determined by:

$$A_d = e^{A_c T}, \quad B_d = \int_0^T e^{A_c \mu} B_c \, d\mu = (A_d - I)(A_c)^{-1} B_c, \quad C_d = C_c$$

May us MATLAB to solve for **A**$_d$, **B**$_d$, and **C**$_d$

Key element is the **State Transition Matrix** $e^{A_c T}$. This is computed using the above either analytically as the inverse Laplace transform of the matrix $[(sI - A_c)^{-1}]$, or numerically from the infinite sum (or using MATLAB):

$$e^{A_c t} \doteq L^{-1}\{(sI - A_c)^{-1}\} \doteq I + A_c t + A_c^2 \frac{t^2}{2!} + A_c^3 \frac{t^3}{3!} + \ldots$$

Compute the K_d gains by coefficient matching with the discrete system closed loop characteristic equation:

$$\det[\, zI - (A_d - B_d K_d)\,] = 0$$

Again the discrete control $u(k)$ has the same form as its continuous counterpart:

$$u(k) = gr(k) - [\, k_{d1} x_1(k) + k_{d2} x_2(k)\,]$$

with $r(k)$, $x_1(k)$, $x_2(k)$ the discrete equivalent of the reference and state signals.

The discrete closed loop transfer function is given by equation (43):

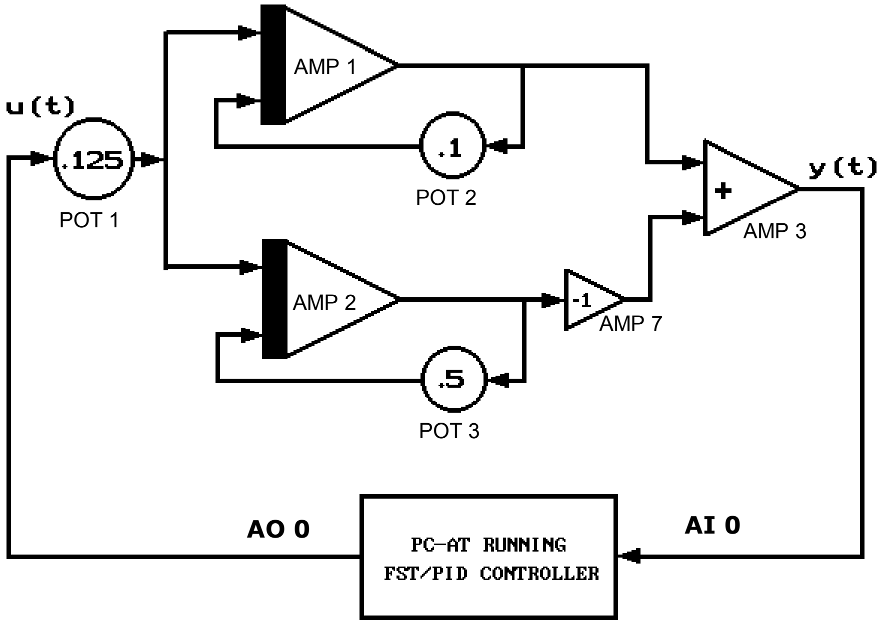
$$H(z) = \frac{Y(z)}{R(z)} = C_d[\, zI - (A_d - B_d K_d)\,]^{-1} B_d$$

The feedforward gain is calculated as before, using the discrete Final Value Theorem, the output $Y(z) = R(z)H(z)$, the transfer function $H(z)$, and the unit step $R(z) = \dfrac{R}{(1 - z^{-1})}$

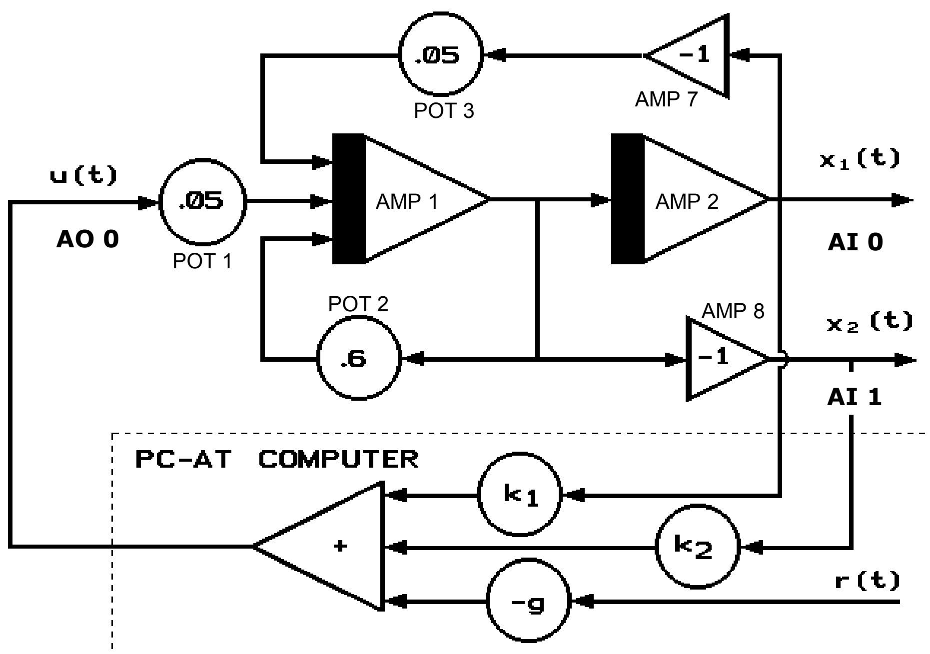
$$y_{ss} = \lim_{k\to\infty} y(k) = \lim_{z\to 1}(1 - z^{-1})Y(z) = \lim_{z\to 1}(1 - z^{-1})R(z)H(z) = \lim_{z\to 1}(1 - z^{-1})\frac{R}{(1 - z^{-1})}H(z) = RH(1)$$

The steady state error $e_{ss} = R - y_{ss}$ is required to be zero, thus $y_{ss} = R$, and from the above formulas we conclude that $H(1) = 1$, which determines the gain g.

$H(1)$ is the $H(z)$ evaluated at $z = 1$, and $C_d = C_c = 1$.



PID & FST Plant Simulation Diagram



See OPTIMAL CONTROL experiment analog closed loop wiring diagram for pure **analog controller**.

Pole Placement Plant Simulation Diagram

$$G_p(s) = \frac{0.05}{(s+0.1)(s+0.5)} = \frac{0.05}{s^2+0.6s+0.05} = \frac{0.125}{s+0.1} + \frac{-0.125}{s+0.5}$$

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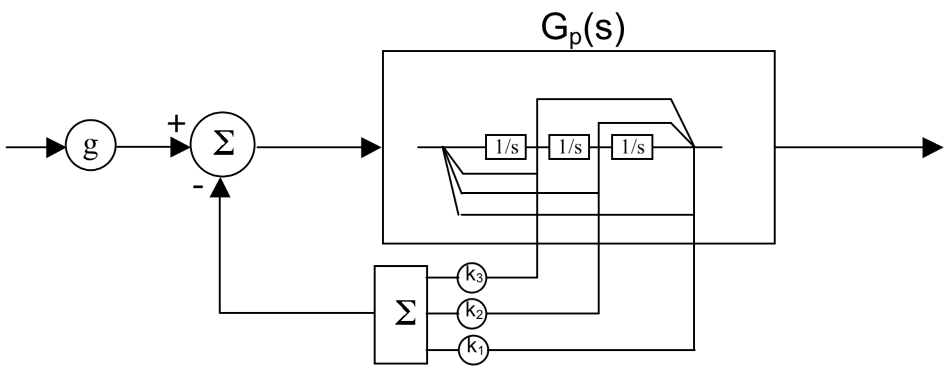
EXPERIMENTS IN OPTIMAL CONTROL

STATE SPACE CONTROLLER, LINEAR QUADRATIC CONTROLLER (LQR)



State Feedback Control

General plant equations:

$$\dot{\mathbf{x}}(t) = \alpha(\mathbf{x}, \mathbf{u})$$
$$\mathbf{x}(t_0) = \mathbf{c} \quad \leftarrow \text{I.C.s are very important}$$

Instead of applying a step input to the system and observing the response, here the system starts with nonzero initial values and the response as it moves toward zero is observed

Objective is get the system states to go from I.C.s to zero as fast as possible while meeting certain conditions and constraints - formally minimizing system performance index J:

$$J = h(\boldsymbol{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), \mathrm{t})\, dt$$

where:

t_f represents the end of the control interval

h and g are user defined penalty expressions.

Lab implements both a continuous and discrete controller

CONTINUOUS LQR CONTROL

Plant is described by the matrix equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad \leftarrow \text{Don't care about sustem output}$$

where:

- $\boldsymbol{x}(t)$ is the (n x 1) state vector,
- $\boldsymbol{x}_0$ is the initial state vector,
- $\boldsymbol{u}(t)$ is the (m x 1) control input vector,
- $\boldsymbol{y}(t)$ is the (r x 1) output vector,
- $\boldsymbol{A}$ is the (n x n) state dynamics matrix,
- $\boldsymbol{B}$ is the (n x m) control dynamics matrix,
- $\boldsymbol{C}$ is the (r x n) state-output matrix,
- $\boldsymbol{D}$ is the (r x m) input-output matrix (for all practical purposes assumed **0** thereafter).

General system performance index (penalty function) in a quadratic form becomes:

$$J(\boldsymbol{u}(t), \boldsymbol{x}(t), t) = \frac{1}{2}\, \boldsymbol{x}'(t)\boldsymbol{H}\boldsymbol{x}(t) + \frac{1}{2}\int_0^{t_f} \{\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\}\, dt$$

where:

- $\boldsymbol{H}$ is the (n x n) terminal state penalty matrix,
- $\boldsymbol{Q}$ is the (n x n) state penalty matrix,
- $\boldsymbol{R}$ is the (m x m) control penalty matrix.

Simplified linear time-invariant form (Linear Quadratic Regulator):

$$J = \frac{1}{2}\int_0^{t_f} \{\mathbf{x}'(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\}\, dt$$

Note: for a 1st order system:

$$J = \frac{1}{2}\int_0^{t_f} \{\mathbf{Q}\mathbf{x}^2(t) + \mathbf{R}\mathbf{u}^2(t)\}\, dt$$

with Q the weight on the state error (penalize position error) and R the weight on the input (penalize fuel use)

Optimal control signal is defined by:

$$\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}'\boldsymbol{P}(t)\boldsymbol{x}(t) = \boldsymbol{G}(t)\boldsymbol{x}(t)$$
$$\boldsymbol{G}(t) \doteq -\boldsymbol{R}^{-1}\boldsymbol{B}'\boldsymbol{P}(t)$$

where:

G(*t*) is the (*m* x *n*) optimal feedback gain matrix,
P(*t*) is an (*n* x *n*) symmetric and positive definite matrix that satisfies the continuous matrix differential **Riccati** equation given by:

And P is the solution to the Riccati Equation:

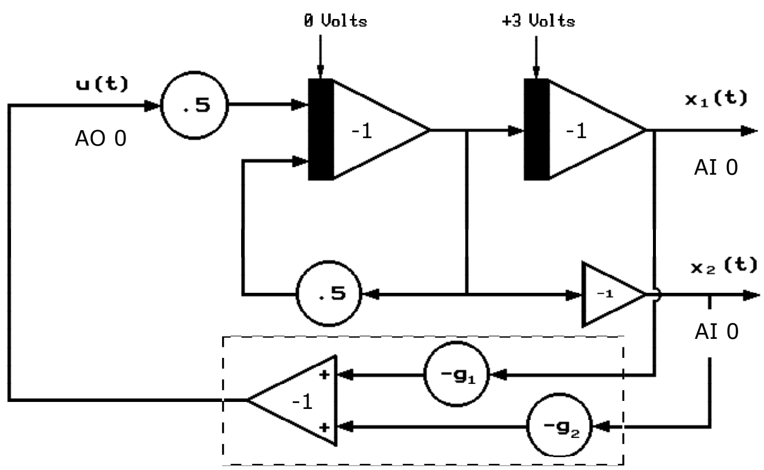
$$\dot{\boldsymbol{P}}(\mathrm{t}) = -\boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{A}'\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}'\boldsymbol{P}(t) - \boldsymbol{Q}\ , \qquad \boldsymbol{P}(t_f) = \mathbf{H}$$

for Steady State conditions:

$$\mathbf{0} = -\boldsymbol{P}\boldsymbol{A} - \boldsymbol{A}'\boldsymbol{P} + \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}'\boldsymbol{P} - \boldsymbol{Q}$$

Once P is found (solution is provided by routines in the experiment), then G can be found from R, B, & P, and the optimal feedback gains can be implemented in the controller

NOTE: as R → 0, K → ∞, meaning fuel savings is completely unimportant



Analog computer simulation of the closed loop system.

DISCRETE LQR CONTROL

Plant is described by the discrete matrix difference equations:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k), \qquad \boldsymbol{x}(0) = \boldsymbol{c}$$
$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) \quad \leftarrow \text{Still don't care about system output}$$

The discrete performance index in a quadratic form is given by:

$$J = \frac{1}{2}\,\boldsymbol{x}'(N)\boldsymbol{H}\boldsymbol{x}(N) + \frac{1}{2}\sum_{k=0}^{N}\{\boldsymbol{x}'(k)\boldsymbol{Q}\boldsymbol{x}(k) + \boldsymbol{u}'(k)\boldsymbol{R}\boldsymbol{u}(k)\}$$

where ***A, B, C, D, H, Q*** and ***R*** are similar to those in the continuous case and N is a fixed number of time intervals

The optimal control feedback gains is found to be:

$$\boldsymbol{u}^*(k) = \boldsymbol{G}(k)\boldsymbol{x}(k)$$

where ***G*** is again the (*m* x *n*) feedback gain matrix given by:

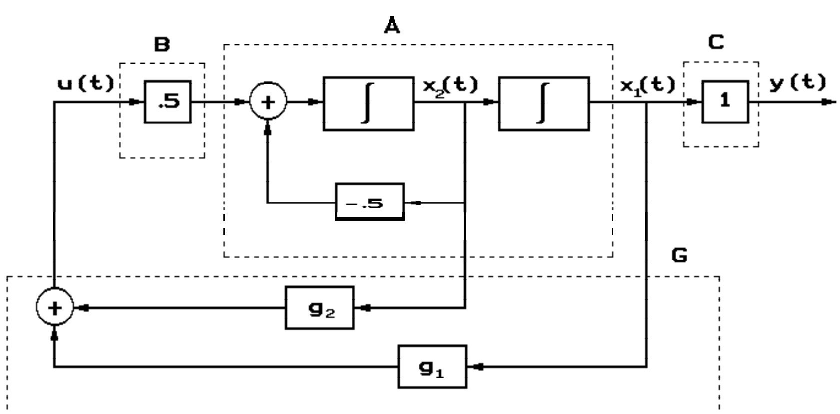
$$\boldsymbol{G}(k) = -\boldsymbol{R}^{-1}\boldsymbol{B}'(\boldsymbol{A}')^{-1}[\boldsymbol{P}(k) - \boldsymbol{Q}\,]$$

P is the (*n* x *n)* real positive definite solution of the discrete matrix difference Riccati equation given by:

$$\boldsymbol{P}(k) = \boldsymbol{Q} + \boldsymbol{A}'[\boldsymbol{P}^{-1}(k+1) + \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}']^{-1}\boldsymbol{A}$$

When N approaches ∞ and the same conditions mentioned in the continuous case apply, then the ***P***(*k*) matrix converges to a constant symmetric, real, positive definite matrix ***P***. Hence:

$$\boldsymbol{P}(k) = \boldsymbol{P}(k+1) = \boldsymbol{P}$$



Block diagram of the closed loop system. Feedback done using LabVIEW on PC.