# A Tutorial on Compressed Sensing and Control Theory: Some Answers and Some Questions Part-2

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# Outline

# Preface

Slides are in two parts and are available at:

- http://www.utdallas.edu/ m.vidyasagar/Talks/Tut-1.pdf
- http://www.utdallas.edu/ m.vidyasagar/Talks/Tut-2.pdf

Please feel free to download and follow along!

# Notation

If $x \in \mathbb{C}^n$, then $[n]$ denotes $\{1, \ldots, n\}$,

$$\mathrm{supp}(x) = \{i \in [n] : x_i \neq 0\}, \|x\|_0 = |\mathrm{supp}(x)|.$$

Three different norms are defined on matrices. Let $\boldsymbol{\sigma}(A) \in \mathbb{R}_+^m$ denote the vector of singular values of $A$. Then

$$\|A\|_S = \|\boldsymbol{\sigma}(A)\|_\infty, \|A\|_F = \|\boldsymbol{\sigma}(A)\|_2, \|A\|_N = \|\boldsymbol{\sigma}(A)\|_1$$

denote respectively the **spectral norm, Frobenius norm, and nuclear norm** of $A$.

# Outline

# Another Hardness Result

**Theorem**

*Given a linear map* $\mathcal{A} : \mathbb{C}^{r \times s} \to \mathbb{C}^m$ *and* $y \in \mathbb{C}^m$*, the problem*

$$\min_{Z \in \mathbb{C}^{r \times s}} \operatorname{rank}(Z) \text{ s.t. } \mathcal{A}(Z) = y$$

*is NP-hard.*

Sketch of proof: Choose $r = s$ and $X = \operatorname{Diag}(x), x \in \mathbb{C}^r$. Then $\operatorname{rank}(X) = \|x\|_0$. So this problem is at least as hard as minimizing $\| \cdot \|_0$ subject to linear constraints. The latter is NP-hard.

# Outline

1. Matrix Recovery
   - Matrix Recovery from Vector Recovery
   - Matrix Completion
2. Partial Realization
   - Preliminaries
   - Solution via Nuclear Norm Minimization
   - Numerical Examples
3. Maximum Hands-Off Control
   - Problem Formulation
   - Hands-Off Control Using CLOT
   - Numerical Examples

# Problem Formulation

Recall the problem: Fix integers $r$ and $s$, and assume without loss of generality that $r \leq s$. Suppose $k < r$, and let $\mathcal{M}(k)$ denote the subset of matrices of $\mathbb{C}^{r \times s}$ of rank $k$ or less. If $X \in \mathbb{C}^{r \times s}$, define its $k$-**rank sparsity index** $\theta_k(X, \| \cdot \|)$ as

$$\theta_k(X, \| \cdot \|) := \min_{Z \in \mathcal{M}(k)} \|X - Z\|.$$

# Problem Formulation (Cont'd)

Suppose $\mathcal{A} : \mathbb{C}^{r \times s} \to \mathbb{C}^m$ is a linear measurement map, and that $\Delta : \mathbb{C}^m \to \mathbb{C}^{r \times s}$ is a decoder map. Then the pair $(\mathcal{A}, \Delta)$ is said to achieve **robust rank recovery of order $k$** if there exist constants $C$ and $D$ such that

$$\|\Delta(\mathcal{A}(X) + \eta) - X\|_F \leq C\theta_k(X, \|\cdot\|_N) + \epsilon$$

whenever $\|\eta\|_2 \leq \epsilon$.

**Challenge**: Choosing the maps $\mathcal{A}$ and $\Delta$.

Because the convex envelope of the rank function $X \mapsto \mathrm{rank}(X)$ (over the unit ball in the spectral norm $\|\cdot\|_S$) is the nuclear norm $\|\cdot\|_N$, let $y = \mathcal{A}(X) + \eta$ and define

$$\hat{X} := \operatorname*{argmin}_{Z} \|Z\|_N \text{ s.t. } \|y - \mathcal{A}(Z)\|_2 \leq \epsilon.$$

# A Meta Theorem and a Corollary

Oymak et al. (2011) present a "meta-theorem" that allows one to convert sufficient conditions for vector recovery into sufficient conditions for matrix recovery. Detailed and specific statement found in my notes. We will state only one useful corollary here.

## Definition

A linear map $\mathcal{A} : \mathbb{C}^{r \times s} \to \mathbb{C}^m$ is said to satisfy the **rank restricted isometry property (RRIP)** of rank $k$ and constant $\delta_k$ if

$$(1 - \delta_k)\|X\|_F^2 \leq \|\mathcal{A}(X)\|_2^2 \leq (1 + \delta_k)\|X\|_F^2, \ \forall X \in \mathcal{M}(k).$$

# A Meta Theorem and a Corollary (Cont'd)

## Theorem

*Suppose* $\mathcal{A} : \mathbb{C}^{r \times s} \to \mathbb{C}^m$ *is linear and satisfies the RRIP of rank* $tk$ *with constant* $\delta_{tk} < \sqrt{(t-1)/t}$*. Suppose* $y = \mathcal{A}(X) + \eta$ *where* $\|\eta\|_2 \leq \epsilon$*, and define*

$$\hat{X} := \operatorname*{argmin}_Z \|Z\|_N \text{ s.t. } \|y - \mathcal{A}(Z)\|_2 \leq \epsilon.$$

*Then there exist constants* $C$ *and* $D$ *such that*

$$\|\Delta(\mathcal{A}(X) + \eta) - X\|_F \leq C\theta_k(X, \|\cdot\|_N) + \epsilon$$

*Moreover, these are the same constants as in the corresponding theorem for vector recovery.*

# A Meta Theorem and a Corollary (Cont'd)

**Theorem**

*(Cai-Zhang(2014)) If $t \geq 4/3$, then the above bound is tight.*

**Advantage:** Approach works even for matrices that are "nearly" of low rank (in contrast with next set of methods).

**Open Problem:** How to construct linear maps that satisfy the RRIP?

# Outline

# Problem Formulation: Reprise

Recall the earlier problem formulation: Suppose $X \in \mathbb{C}^{r \times s}$ and that $\Omega \subseteq [r] \times [s]$. Say $\Omega = \{(i_1, j_1), \ldots, (i_l, j_l)\}$. Define $Q \in \{0,1\}^{r \times s}$ by

$$q_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \Omega, \\ 0, & \text{if } (i,j) \notin \Omega. \end{cases}$$

Then the measurements of $A$ consist of the **Hadamard product** $\mathcal{A}(X) = Q \circ X$ whereby

$$[Q \circ X]_{ij} = q_{ij} x_{ij}.$$

So the measurements consist of specific elements of the unknown matrix $X$, and the objective is to "complete" it.

# Motivation: “Netflix Problem”

Suppose there $r$ clients and $s$ movies. Each client assigns a rating to some (in fact very few) movies. The objective is to infer the entire “rating matrix” of how each client would rate each movie if s/he had the chance.

Not possible in general, but what if the rating matrix is either exactly or nearly of low rank?

Can the unknown matrix be recovered using measurements of some components?

# Matrix Completion via Nuclear Norm Minimization

Problem is

$$\min \operatorname{rank}(Z) \text{ s.t. } \mathcal{A}(Z) = y,$$

where $\mathcal{A} : \mathbb{C}^{r \times s} \to \mathbb{C}^m$ is a linear operator and $y = \mathcal{A}(X)$.

**Approach:** Because $\| \cdot \|_N$ is the convex envelope of the rank function, change the NP-hard problem above to the convex optimization problem

$$\min \|Z\|_N \text{ s.t. } \mathcal{A}(Z) = y.$$

Problem formulation and initial solution by Candès and Recht (2008); follow-up work by Candè and Tao, Keshavan-et-al (2010a, 2010b), Recht, Fazel, and others. Highly recommend survey paper by Davenport and Romberg (2016).

# Alternatives to Nuclear Norm Minimization

While in theory matrix completion can (sometimes) be achieved via nuclear norm minimization, algorithms for this problem are still evolving.

Other algorithms are being developed, not all of them taking advantage of the convexity of the nuclear norm! Example: Method of alternating projections.

# Not All Matrices Can be Completed

Why can (in principle) low rank matrices be recovered from a few observations? Because the low rank property implies algebraic relationships between various elements.

But elements of one row or column need not always tell us much about other rows or columns!

Suppose $X$ is of rank one and has a 1 in position $(1, 1)$ and zeros elsewhere. Then almost all samples of $X$ will equal zero!

There is a need for the singular vectors to have *low coherence!*

# Low Coherence SVDs

Suppose the unknown matrix $X$ is in $\mathbb{C}^{n \times n}$ (is square), has the SVD $X = U \Sigma V^\dagger$ and rank $r$. Let $P_U$ denote the orthogonal projection of $\mathbb{C}^n$ onto the range of $U$. Then define

$$\mu(U) := \frac{n}{r} \max_{i \in [n]} \| P_U \mathbf{e}_i \|_2^2,$$

where $\mathbf{e}_i$ is the $i$-th canonical basis element. Easy to show that

$$1 \leq \mu(U) \leq n/r.$$

Upper bound occurs when some $e_i$ belongs to the range of $U$ (e.g. when $X$ has one element of $1$ and the rest zero).

Matrices with high coherence cannot be completed!

# Typical Results on Matrix Completion

Low coherence SVD + nuclear norm minimization implies matrix recovery.

Candès and Recht (2008), Candès and Tao (2010: For each $X \in \mathbb{C}^{n \times n}$ of rank $r$ and low coherence, it is possible to recover $X$ by choosing a set $S \subseteq [n] \times [n]$ of cardinality $Cn^{1.2} r \log n$ (compare with $n^2$ elements of $X$).

Keshavan-et-al (2010a), Bhojanapalli and Jain (2014) invert the order of the quantifiers. There exists a set $S$ of cardinality $O(rn)$ such that it can *every* matrix of rank $r$ (and sufficiently low coherence) can be recovered from measuring $x_{ij}, (i, j) \in S$.

Keshavan-et-al (2010b) also permit *noisy measurements*.

# Outline

# Outline

# Hankel Matrices

Suppose $\{f_t\}_{t \geq 1}$ is a sequence of real numbers. The associated infinite and finite Hankel matrices are defined as

$$H_{f,\infty} := \begin{bmatrix} f_1 & f_2 & f_3 & \cdots \\ f_2 & f_3 & f_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$H_{f,n} := \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ f_2 & f_3 & \cdots & f_n & f_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_n & f_{n+1} & \cdots & f_{2n-2} & f_{2n-1} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

# An Old Theorem

## Theorem

*(Kronecker (1881)) Suppose $\{f_t\}_{t \geq 1}$ is an $\ell_1$ sequence. Then $\mathrm{rank}(H_{f,\infty})$ is finite if and only if the power series*

$$f(z) = \sum_{t=1}^{\infty} f_t z^{t-1}$$

*defines a rational function of $z$. If so the rank of $H_{f,\infty}$ is the degree of the rational function.*

# A Basic Fact

Consider a linear discrete-time SISO system

$$x_{t+1} = Ax_t + Bu_t, y_t = Cx_t,$$

where the pairs $(A, B)$ and $(C, A)$ are controllable and observable respectively. Define the unit pulse response and transfer function of the system as

$$h_t = CA^{t-1}B, t \geq 1, \tilde{h}(z) = \sum_{t=1}^{\infty} h_t z^{t-1}.$$

Then the dimension of $A$ is the degree of $\tilde{h}(z)$, which is in turn the dimension of $H_{h,\infty}$.

# Partial Realization: Problem Formulation

**Original:** Given a finite sequence $\{h_t\}_{t=1}^m$, find an *infinite sequence* $\{f_t\}_{t \geq 1}$ such that (i) $f(t) = h(t)$ for $t = 1, \ldots, m$, and (ii) $\mathrm{rank}(H_{f,\infty})$ is minimized.

**Realistic:** Given a finite sequence $\{h_t\}_{t=1}^m$, and an integer $n \gg m$, find a *finite sequence* $\{f_t\}_{t=1}^{2n-1}$ such that (i) $f(t) = h(t)$ for $t = 1, \ldots, n$, and (ii) $\mathrm{rank}(H_{f,n})$ is minimized.

# A Simple Observation

Suppose $\{h_t\}_{t=1}^m$ is a subsequence of an infinite sequence $\{h_t\}_{t \geq 1}$ such that $H_{h,\infty}$ has finite rank, say $d$. Then, for each integer $n \geq 2m - 1$,

$$\left\{ \min_{f \in \mathbb{R}^{2n-1}} \text{rank}(H_{f,n}) \text{ s.t. } f_{[1:m]} = h_{[1:m]} \right\} \leq d.$$

Note:

$$\begin{aligned} \left\{ \min_{f \in \mathbb{R}^{2n-1}} \text{rank}(H_{f,n}) \text{ s.t. } f_{[1:m]} = h_{[1:m]} \right\} &\leq \text{rank}(H_{h,n}) \\ &\leq \text{rank}(H_{h,\infty}) = d, \end{aligned}$$

because $H_{h,n}$ is a submatrix of $H_{h,\infty}$.

# Outline

# Modified Problem Formulation

Unfortunately (as we have already seen), minimizing the rank subject to linear constraints is NP-hard. So we replace the rank function by its convex envelope, namely the nuclear norm.

**Modified Problem:** Given $\{h_t\}_{t=1}^{m}$, and $n \gg m$,

$$\min_{f \in \mathbb{R}^{2n-1}} \|H_{f,n}\|_N \text{ s.t. } f_{[1:m]} = h_{[1:m]}.$$

This approach seems to work surprisingly well!

# Nonstandard Partial Realization Problem

Why specify only *first* $m$ elements of the unit pulse response? Why not specify *some* $m$ elements?

Note: Many results break down if specifications are not contiguous.

# Nehari's Theorem

Recall $\mathcal{H}_\infty$ is the Hardy space consisting of functions that are analytic over the open unit disk and essentially bounded on the unit circle. Define

$$\|f\|_\infty := \sup_{\theta \in [0,2\pi]} |f(\exp(\mathbf{i}\theta)|.$$

**Problem:** Given constants $c_0, \ldots, c_m$, find

$$\min_{f \in \mathcal{H}_\infty} \|f\|_\infty \text{ s.t. } f(0) = c_0, \left. \frac{d^j f}{dz^j} \right|_{z=0} = c_i, i = 1, \ldots, m.$$

Equivalently, choose $h \in \mathcal{H}_\infty$ so as to minimize

$$\left\| \sum_{j=0}^{m} c_j z^j + z^{m+1} h(z) \right\|_\infty .$$

# Nehari's Theorem (Cont'd)

**Theorem**

*(Nehari (1951)) Define the Hankel matrix*

$$H = \begin{bmatrix} c_m & c_{m-1} & \ldots & c_1 & c_1 \\ c_{m-1} & c_{m-2} & \ldots & c_0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

*Then*

$$\min_{h \in \mathcal{H}_\infty} \left\| \sum_{j=0}^{m} c_j z^j + z^{m+1} h(z) \right\|_\infty = \|H\|_S.$$

The theorem does not work at all unless *first* $m+1$ values are specified.

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# Example No. 1

A fourth-order system defined by

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0.3528 & 0.0490 & 0.2300 & 0.1000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top, C = \begin{bmatrix} 1 & 3 & 2 & 0 \end{bmatrix},$$

The system poles are at $0.9, -0.8, \pm 0.7\mathbf{i}$.

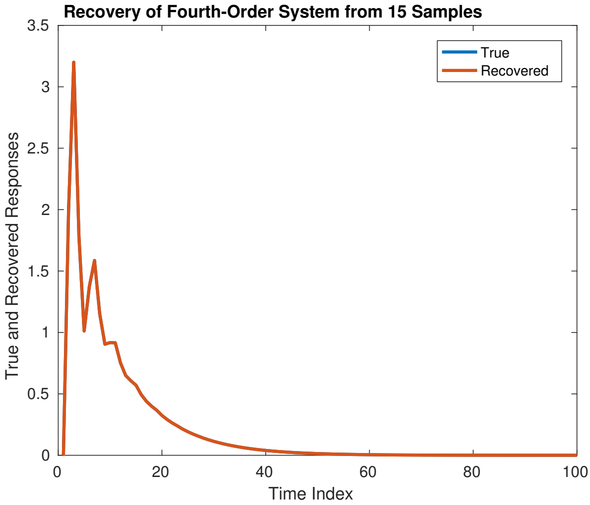
# Recovery Using First $m$ Samples

The system is of order $4$. Using the first $m$ elements of the unit pulse response, identify the rest using nuclear norm minimization.

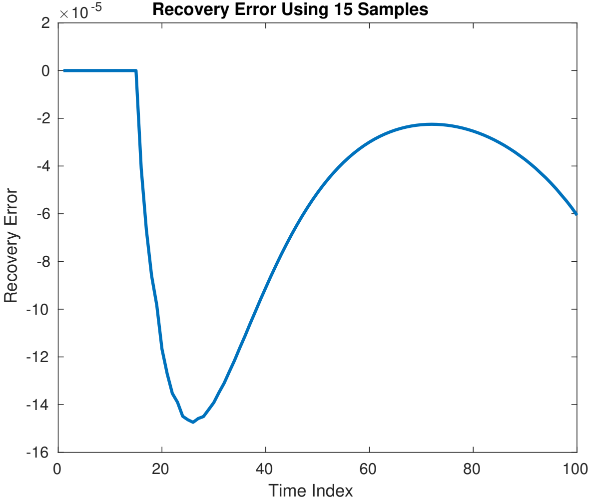
With $n = 50$ (so that $2n - 1 = 99$), the unit pulse response was recovered.

Results are shown on next few slides.

# Recovery Using 15 Samples



# Error in Recovery Using 15 Samples



# Example No. 2

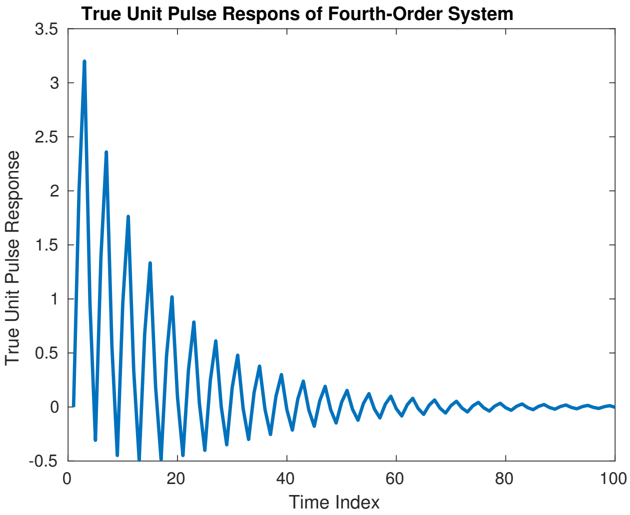
A fourth-order system defined by

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0.6498 & 0.0902 & -0.1825 & 0.1000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top C = \begin{bmatrix} 1 & 3 & 2 & 0 \end{bmatrix}.$$

The system poles are at $0.9, -0.8, \pm 0.95\mathbf{i}$, So the system is stable but highly oscillatory, as shown on next slide.

# True Unit Pulse Response



# Recovery Using First $m$ Samples

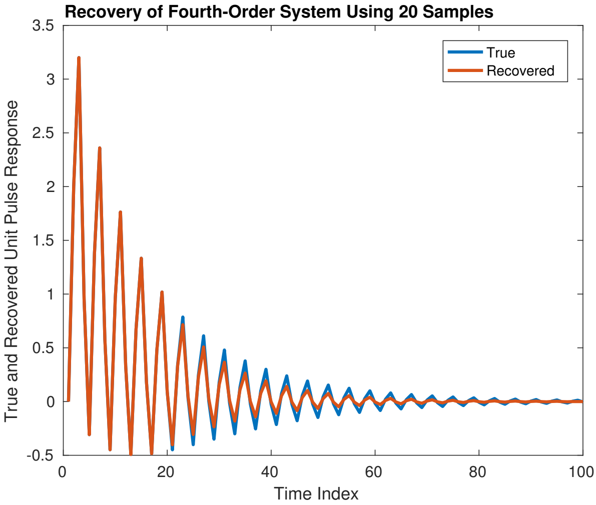
The system is of order $4$. Using the first $m$ elements of the unit pulse response, identify the rest using nuclear norm minimization.

With $n = 50$ (so that $2n - 1 = 99$), the unit pulse response was recovered.

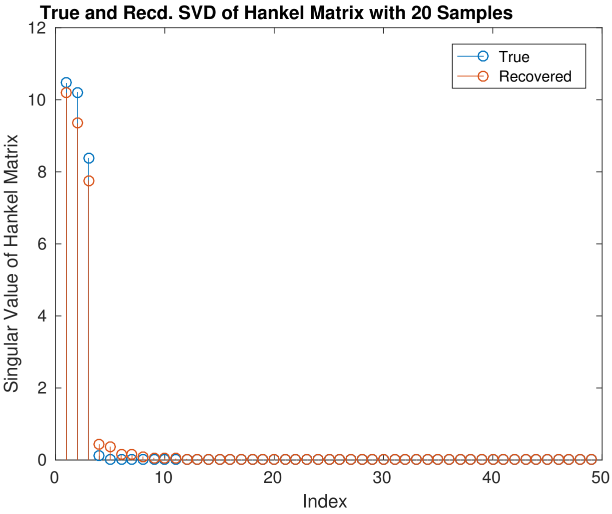
Results are good, and are shown on next few slides.

**Challenge:** How can this approach be put on a firm theoretical foundation?

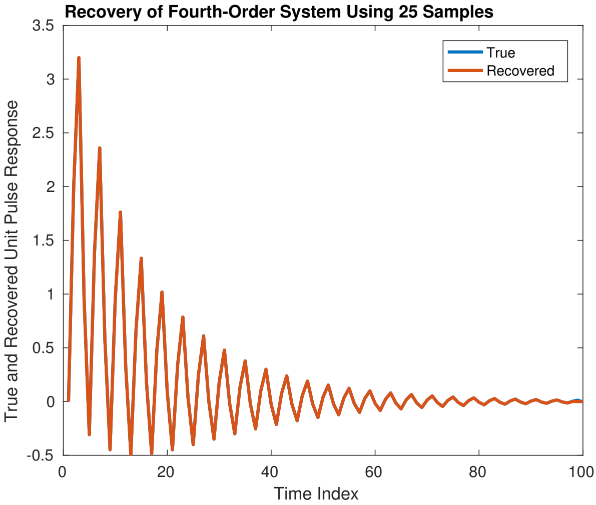
# Recovery Using 20 Samples



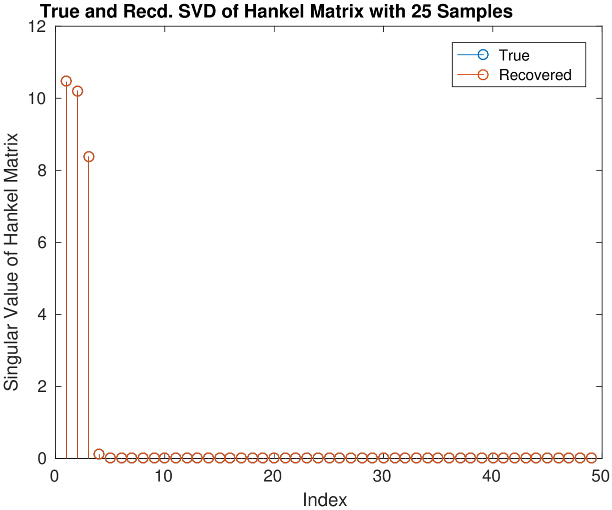
# True and Recovered Singular Values of Hankel Matrix



# Recovery Using 25 Samples



# True and Recovered Singular Values of Hankel Matrix



# Partial Realization with Missing Samples

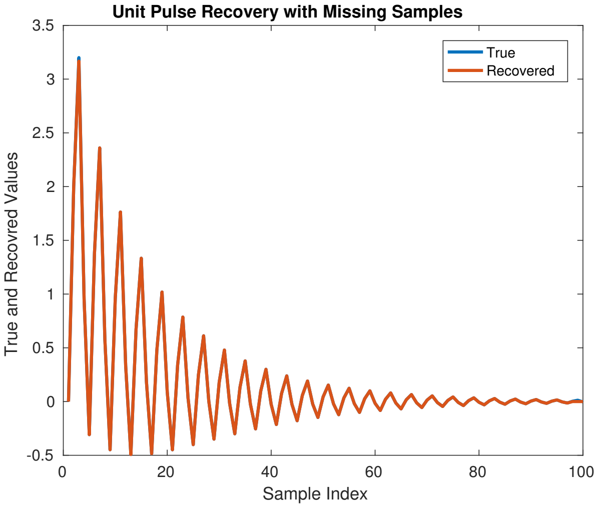
Suppose that out of the first $30$ samples, we miss out samples $3, 9, 12, 19, 22$. Define

$$S := \{1, \ldots, 30\} \setminus \{3, 9, 12, 19, 22\}.$$

We minimize the nuclear norm of $H(f)$ subject to the constraint that $f_S = h_t$ for all $t \in S$.

Results shown on next page.

# Partial Realization with Missing Samples (Cont'd)



# Outline

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# Problem Formulation in Continuous-Time

(Nagahara, Quevedo and Nešić, T-AC March 2016) Consider a system

$$\dot{x} = Ax(t) + Bu(t), x(0) \neq \mathbf{0}.$$

With $T$ specified, choose a control $u(\cdot)$ such that

- $\|u(t)\|_\infty \leq 1$ for all $t \in [0, T]$.
- $x(T) = \mathbf{0}$.
- The "attention span"

$$\mathcal{A}(u(\cdot)) := \lambda[\mathrm{supp}(u)], \mathrm{supp}(u) := \{t \in [0, T] : u(t) \neq \mathbf{0}\}$$

is minimized, where $\lambda(\cdot)$ denotes the Lebesgue measure of a set and $\mathrm{supp}(u)$ is the support of the signal $u(\cdot)$.

# Solution Using Pontryagin's Principle

*Convexify* the problem: Replace $\mathcal{A}(u(\cdot))$ by $\|u(\cdot)\|_1$, and minimize $\|u(\cdot)\|_1$ subject to $\|u(\cdot)\|_\infty \leq 1$ and $x(T) = \mathbf{0}$.

This is a classical "fuel-optimal control" problem that can be solved using Pontryagin's minimum principle.

Therefore, under mild conditions, optimal control $u^*(\cdot)$ satisfies $u^*(t) \in \{-1, 0, 1\}$ for all $t \in [0, T]$.

*Observe:* If a function $f : [0, T] \to \mathbb{R}$ satisfies $f(t) \in \{-1, 0, 1\}$ for all $t \in [0, T]$, then $\mathcal{A}(f(\cdot)) = \|f(\cdot)\|_1$.

Ergo (with some technicalities) minimum *fuel* control is also a minimum *attention* control.

# Smoothening the Solution

Fuel-optimal control is "bang-off-bang" and thus discontinuous. Can we find a suboptimal continuous solution?

**Approach:** Substitute as below:

$$\|u(\cdot)\|_1 \leftarrow (1-\mu)\|u(\cdot)\|_1 + \|u(\cdot)\|_2^2.$$

Resulting optimal control is continuous for all $\mu$ close to zero (Nagahara, Quevedo and Nešić, T-AC March 2016)

One could say that the "LASSO" penalty $\|u(\cdot)\|_1$ is replaced by the "Elastic Net" penalty $(1-\mu)\|u(\cdot)\|_1 + \|u(\cdot)\|_2^2$.

# Outline

# CLOT: An Alternative to LASSO and Elastic Net

CLOT regularizer, proposed in Ahsen, Challapalli and MV (JMLR, to appear):

$$\mathcal{R}_{\text{CLOT}}(v) := (1-\mu)\|v\|_1 + \mu\|v\|_2, \mu \in (0,1).$$

CLOT stands for "Combined L-One and Two." Compare with

$$\mathcal{R}_{\text{LASSO}}(v) := \|v\|_1, \mathcal{R}_{\text{EN}}(v) := (1-\mu)\|v\|_1 + \mu\|v\|_2^2, \mu \in (0,1).$$

# Benefits of CLOT Penalty

It is shown in Ahsen, Challapalli and MV (JMLR, to appear) that CLOT combines the best features of LASSO and EN. Specifically

- EN *does not achieve* robust sparse recovery in compressed sensing.
- For $\mu$ sufficiently small, CLOT achieves robust sparse recovery in compressed sensing, like LASSO but unlike EN.
- CLOT achieves the grouping effect in sparse regression, like EN but unlike LASSO.

Can CLOT be applied to the problem of hands-off control, and if so, does it lead to "lower attention" control signals than EN?

# Problem Formulation: Discrete-Time

Consider a discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t, y_t = Cx_t, x_0 \neq \mathbf{0}.$$

Given a specified final time $T$, we wish to find a control sequence $\{u_t\}_{t=0}^{N-1}$ such that (i) $\|u_t\|_\infty \leq 1$ for $t = 0, \ldots, N-1$, (ii) $X_N = \mathbf{0}$, and (iii) the cardinality of the set

$$\text{supp}(u) = \{t : u_t \neq \mathbf{0}\}$$

is small.

# CLOT Formulation

Rewrite problem as

$$\min_{u \in \mathbb{R}^N} (1-\mu)\|u\|_1 + \mu\|u\|_2$$

subject to the specified constraints, which are all *linear*. This is a convex programming problem that can be solved efficiently.

The approach can also handle constraints on the state $x(\cdot)$, which are not amenable to Pontrygin's principle.

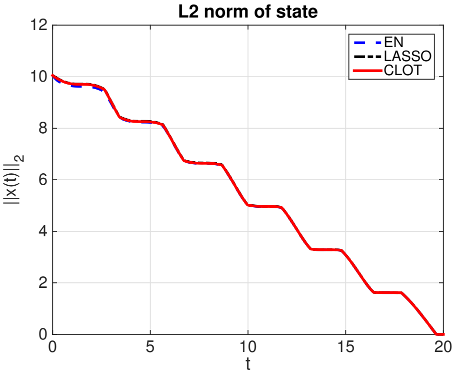
# Outline

# Numerical Studies

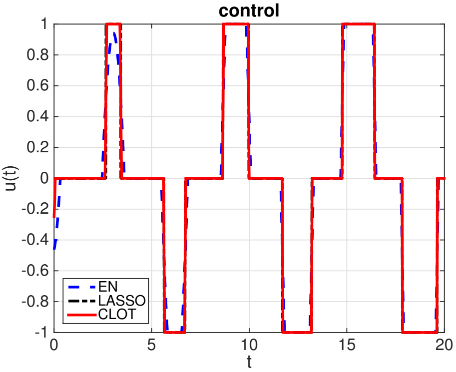
Several systems, both without and state constraints, were studied, with various initial conditions. In all the examples studied, the "sparsity ratio" $\mathcal{A}(u(\cdot)/T$ of the CLOT-optimal control is smaller than that for the EN-optimal control.

A few typical results are shown in next several slides.

# Lightly Damped Harmonic Oscillator: State Trajectories



# Lightly Damped Harmonic Oscillator: Control Trajectories



# Comparison of Sparsity Indices: No State Constraints

| No. | LASSO | EN | CLOT |
|---|---|---|---|
| 1 | 0.1690 | 0.5915 | 0.4475 |
| 2 | 0.1690 | 0.3270 | 0.2480 |
| 3 | 0.0480 | 0.1155 | 0.0830 |
| 4 | 0.4055 | 0.5555 | 0.4225 |
| 5 | 0.1655 | 0.3050 | 0.2180 |
| 6 | 0.0040 | 0.0395 | 0.0805 |
| 7 | 0.0595 | 0.1100 | 0.0845 |
| 8 | 0.0568 | 0.1438 | 0.1125 |
| 9 | 0.0568 | 0.1438 | 0.1125 |

Table: Sparsity densities for optimal controllers produced by various methods

# Maximum Hands-Off Control with State Constraints

Plant: Fourth-order integrator $= 1/s^4$.

Constraints: $|u(t)| \leq 1$, $\|x(t)\|_2 \leq \theta$ for all $t \in [0, T]$ with $T = 20$.

Initial state: $x(0) = [1, 0, 1, 1]^\top$.

Sparsity densities for LASSO, CLOT and EN are shown in the table.

# Sparsity Indices for Various Thresholds

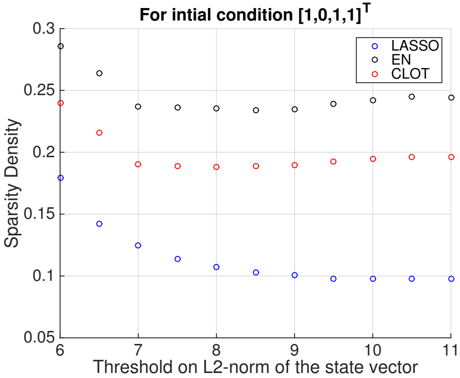



Figure: Sparsity Density vs state threshold for $4^{th}$ order integrator with intial condition $[1, 0, 1, 1]^\top$

# Sparsity Indices with Different Initial Conditions

| Intial State | Sparsity Density | | |
|---|---|---|---|
| | LASSO | EN | CLOT |
| $[1, 0, 0, 1]^\top$ | 0.0820 | 0.2150 | 0.1765 |
| $[1, 0, 0, -1]^\top$ | 0.0790 | 0.2085 | 0.1720 |
| $[1, 0, 1, 1]^\top$ | 0.1795 | 0.2855 | 0.2400 |
| $[1, 0, 1, -1]^\top$ | 0.1075 | 0.2305 | 0.1855 |
| $[1, 0, -1, 1]^\top$ | 0.0660 | 0.1715 | 0.1435 |
| $[1, 0, -1, -1]^\top$ | 0.0640 | 0.1545 | 0.1320 |

# Some Questions Worth Exploring

- Development of *deterministic* algorithms for matrix recovery and matrix completion
- Proofs that the approaches to partial realization and hands-off control actually work

**Thank You!**